EQUIVARIANT BP-COHOMOLOGY FOR FINITE GROUPS

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Dedicated to Professor T. Yamanoshita on the occasion of his 60th birthday

Abstract. The Brown-Peterson cohomology rings of classifying spaces of finite groups are studied, considering relations to the other generalized cohomology theories. In particular, \( BP^*(M) \) are computed for minimal nonabelian \( p \)-groups \( M \). As an application, we give a necessary condition for the existence of nonabelian \( p \)-subgroups of compact Lie groups.

Introduction

The topology of classifying space \( BG \) for a finite group \( G \) is important in algebraic topology. Given generalized cohomology theory \( h^*(-) \), \( h^*(BG) \) plays the central role, e.g., cohomology of a group, completion of the representation ring and the Burnside ring when \( h \) is the ordinary cohomology, the complex \( K \)-theory, and the stable cohomotopy theory, respectively. Recently, the Morava \( K \)-theory of \( BG \) has been studied by Hopkins, Kuhn, and Ravenel [20]. For simplicity, let us denote \( k^*(BG) \) by \( k^*(G) \).

In this paper, we study the Brown-Peterson cohomology \( BP^*(G) \) for a prime \( p \) and the related cohomology \( k^*(G) \) with the coefficient \( k^* = BP^*/(\text{Ideal } S) \), where \( S \) is a set of generators in \( BP^* \).

Landweber showed [3] that \( BP^*(\mathbb{Z}/p^r) \) is a flat \( BP^* \)-module and for an abelian group \( A \), \( BP^*(A) \) is given by the tensor product of \( BP^*(\mathbb{Z}/p^r) \). For nonabelian \( p \)-groups, when \( |G| = p^3 \), \( BP^*(G) \) is determined by Tezuka-Yagita [11] and some relations to the other cohomology theories are given by \( BP^*(G) \otimes_{BP^*} \mathbb{Z}(p) = H_{\text{even}}(G) \) and \( K(n)^*(G) = K(n)^* \otimes_{BP^*} BP^*(G) \).

Consider the map induced from restrictions

\[ r: k^*(G) \to \text{Lim inv } k^*(A) , \]

\( A \subset G \), conjugacy classes of abelian groups. Ravenel conjectured that for \( k = BP \), \( r \) is an isomorphism [8]. Unfortunately, this does not hold, however, we show that for \( k = BP(-; \mathbb{Z}/p) \), \( r \) is an \( F \)-isomorphism by using Quillen's argument, which showed that \( F \)-isomorphy for \( k = HZ/p \), the ordinary mod \( p \) cohomology [6]. Moreover, we show that \( \rho : BP^*(G)_{BP^*} \mathbb{Z}/p \to H^*(G; \mathbb{Z}/p) \) is \( F \)-epic.

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We will compute $BP^*(M)$ for $M$, minimal nonabelian $p$-groups. Then $BP^*(M)$ is a flat $BP^*$-module and the map $r$ is injective for $k^{*} = BP^*$. Moreover, if $G$ is a group whose $p$-Sylow subgroup is a direct product of minimal nonabelian $p$-groups and abelian groups, then $r$ is injective and $BP^*(G) \otimes_{BP^*} P(n)^* \cong P(n)^*(G)$. $BP^*(G) \otimes_{BP^*} K(n)^* \cong K(n)^*(G)$.

In the last section, to see that $BP^*(G)$ is useful, we will study the existence of nonabelian $p$-subgroups of compact Lie groups. For example, we prove that if $G$ is a compact Lie group such that $H^*(G)_{(p)} = \bigwedge(x_1, \ldots, x_n)$ and $G$ contains nonabelian $p$-groups as subgroups, then $p$ divides $|x_i| + 1$ for some $i$.

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1. Cohomology theories

Let $BP^*(-)$ be the Brown-Peterson cohomology theory with the coefficient $BP^* = \mathbb{Z}[v_1, \ldots, |v_i| = -2p^i + 2$ for a prime $p$. Given a set of generators $S = (v_{i_1}, \ldots, v_{i_n}, \ldots)$, ($v_0 = p$), by using Baas-Sullivan theory [2, 13], we can construct cohomology theory $BP(S)^*(-)$ with the coefficient

\begin{equation}
BP(S)^* = BP^*/(\text{Ideal } S) = \mathbb{Z}[v_j \mid j \neq i_k] \quad \text{if } p \notin S,
= \mathbb{Z}/p[v_j \mid j \neq i_k] \quad \text{if } p \in S.
\end{equation}

The cohomology $BP(S)^*(-)$ has a good multiplication, and if $p \geq 3$ it is commutative. A useful result of this theory is the following Sullivan-Bockstein exact sequence; that is, if $v_n$ is not contained in $S$, then

\begin{equation}
BP(S)^*(X) \xrightarrow{\rho} BP(S)^*(X) \xrightarrow{\delta} BP(S, v_n)^*(X) \xrightarrow{\delta} BP(S)^{*+2p^r-1}(x)
\end{equation}

is exact, where $v_n$ is a map of multiplying by $v_n$, $\rho$ is the natural induced map, and $\delta$ is the $v_n$-Bockstein boundary map (for details see [2, 13]).

The examples of $BP(S)^*(-)$ are

\begin{align*}
P(n)^* &= BP(p, v_1, \ldots, v_{n-1})^* = \mathbb{Z}/p[v_n, v_{n+1}, \ldots], \\
k(n)^* &= BP(p, \ldots, v_n, \ldots)^* = \mathbb{Z}/p[v_n], \\
K(n)^* &= [v_n^{-1}] \cdot k(n)^*, \\
BP(n)^* &= BP(v_{n+1}, \ldots)^* = \mathbb{Z}[v_1, \ldots, v_n], \\
HZ^*(p)^* &= BP(v_1, \ldots)^* = \mathbb{Z}_p, \\
HZ/p^* &= \mathbb{Z}/p.
\end{align*}

In this paper we consider these cohomology theories $BP(S)^*(-)$. For simplicity of notation, we write it as $k^*(-)$ and denote by $\#(k)$ the cardinal number of the set $(p, v_1, \ldots) - S$; that is, $\#(k) = n$ if $k^* = \mathbb{Z}/p[v_{i_1}, \ldots, v_{i_n}]$ or $k^* = \mathbb{Z}_p[v_{i_1}, \ldots, v_{i_n-1}]$.

Let $G$ be a compact Lie group and $BG$ be its classifying space. Let $PG$ be a contractible free $G$-space. Then the equivariant cohomology of a $G$-space $X$ is defined by

\begin{equation}
K_G^*(X) = k^*(PG \times_G X) \quad \text{and} \quad k_G^*(pt) = k_G^* = k^*(G) = k^*(BG).
\end{equation}
2. Cohomology of abelian groups

Consider the homomorphism
\[ m: S^1 \times S^1 \to S^1 \]
defined by \( m(x, y) = x + y \) identifying \( S^1 = R/Z \). The induced map of classifying spaces
\[ m: BS^1 \times BS^1 \to BS^1, \quad BS^1 \cong CP^\infty, \]
is the usual product map induced from the tensor bundle. Take two-dimensional elements \( u, u_1, u_2 \) so that \( k^*(CP^\infty \times CP^\infty) \cong k^*[[u_1, u_2]] \) and \( k^*(CP^\infty) \cong k^*[[u]] \). Then the map from (2.1)
\[ m^*(u) = \sum a_{ij} u^1_i u^1_j = u_1 + k u_2 \]
defines the formal group law \([2, 7]\).

The formal group law for \( BP^* \)-theory is the universal group law for group laws over rings which are \( Z(p) \)-modules. It is well known that
\[ u_1 + Bp u_2 = u_1 + u_2 + v_1 \sum_{0 < i < p} \frac{1}{p} u^i_1 u^p_{2-i} + \ldots, \]
\[ [p](u) = pu + v_1 u^p + \ldots + v_n u^{p^n} + \ldots \]
where \([p](u)\) is the \( p \)th sum \( u + Bp \cdots + Bp u \). Given an \( m \times m \)-matrix \( C = (c_{ij}) \) over \( Z \), it induces a map
\[ C: S^1 \times \cdots \times S^1 \to S^1 \times \cdots \times S^1, \]
\[ C^*: k^*[[u_1, \ldots, u_n]] \to k^*[[u_1, \ldots, u_n]], \quad \text{and} \]
\[ C^*(u_j) = \sum_k c_{ij}(u_j). \]

In particular, the short exact sequence
\[ 0 \to Z/p^r \to S^1 \to S^1 \to 0 \]
induces the map of fiber spaces \( S^1 \to BZ/p^r \to BS^1 \). This follows the Gysin exact sequence and we have
\[ k^*(Z/p^r) \cong k^*[[u]]/([p^r](u)). \]
Landweber showed that the Künneth formula holds for \( BP^* \)-cohomology of abelian groups; that is,
\[ \text{BP}^* \left( \bigoplus_{i} Z/p^i \right) \cong \bigotimes_{p^r} \text{BP}^*(Z/p^r) \]
\[ \cong \text{BP}^*[[u_1, \ldots, u_s]]/([p^r](u_1), \ldots, [p^r](u_s)). \]
We always assume that \( \bigotimes \) means the complete tensor product. Of course, the Künneth formula of this type does not hold for \( k = HZ(p) \). By using Stretch's argument \([10]\), we prove the Künneth formula for smaller \( p \)-rank groups.
Lemma 2.7. Let $A$ be an abelian $p$-group with rank $A \leq \#(k)$. Then $k^*(A) \cong \bigotimes_k k^*(Z/p^r)$.

Proof. From the Sullivan-Bockstein exact sequence and (2.6), it is easily seen that $\rho : BP^*(A) \to k^*(A)$ is epic if and only if $k^*(A) \cong \bigotimes_k k^*(Z/p^r)$. Hence we need only prove the result for the case rank $A = \#(k)$.

Let rank $A = n$ and $k^* = Z/p[v_{i_1}, \ldots, v_{i_n}]$. Let us write $s_i = [p^r](u_i)$, $S_n = (s_1, \ldots, s_n)$, and $k^*[U_n] = k^*[[u_1, \ldots, u_n]]$. Then we need to prove that $S_n$ is regular in $k^*[U_n]$. Assume by induction that $S_{n-1}$ is regular in $k'[U_{n-1}]$ for all $k'$ with $\#(k') = n-1$ and $k'^0 = Z/p$.

Suppose the regularity does not hold, namely, there is $a \in k^*[U_n]$ such that $as_n \notin \text{Ideal} S_{n-1}$ for all $k'$. Let us write

$$s_n = v_{i_1}u_n + \cdots$$

with $a_i \notin \text{Ideal} S_{n-1}$. Since

$$as_n = a_1v_{i_1}u_n + \cdots \in \text{Ideal} S_{n-1},$$

we have $a_1v_{i_1} \in \text{Ideal} S_{n-1}$. Therefore there is $b \in k^*[U_{n-1}]$ such that $b \notin \text{Ideal} S_{n-1}$ but $v_{i_1}b \in \text{Ideal} S_{n-1}$.

Let $k'^* = Z/p[v_{i_2}, \ldots, v_{i_n}]$. Then by the inductive assumption, $S_{n-1}$ is regular in $k'[U_{n-1}]$. Therefore $K = k'^*[U_{n-1}] \otimes \bigwedge (e_1, \ldots, e_{n-1})$, $de_i = s_i$, is a Koszul complex and it is an acyclic complex. Let us write $v_{i_1}b = \sum_{i=1}^{n-1} \mu_i s_i$. Then $d(\sum \mu_i e_i) = 0$ in $K$ because $v_{i_1} = 0$ in $k'^*$. By the exactness of $K$, we can take $c_{ij} \in k'[U_{n-1}]$ with

$$\sum \mu_i e_i = d \left( \sum c_{ij} e_i e_j \right).$$

Hence we get in $k^*[U_n]$

$$\mu_i = \sum c_{ij} s_j + v_{i_1} l_i, \quad c_{ij} = -c_{ji},$$

$$v_{i_1} b = \sum c_{ij} s_j s_i + \sum v_{i_1} l_is_i = \sum v_{i_1} l_is_i.$$

Therefore $b = \sum l_is_i \in \text{Ideal} S_{n-1}$. This is a contradiction. Hence we prove the theorem when $k^0 = Z/p$.

When $k^0 = Z/p$, we can prove the theorem by similar arguments, taking $s_i = p^r u + \cdots$ and $k'^* = Z/p[v_{i_1}, \ldots, v_{i_{n-1}}]$. Q.E.D.

Remark 2.8. For $k^* = k(n)$, we can easily see that

$$k(n)^*(Z/p \oplus Z/p) \cong k(n)^*[[y_1, y_2]]/([p](y_1), [p](y_2)) \oplus Z/p[y_1, y_2]$$

where $\rho(\alpha) = Q_\alpha(x_1 x_2)$ in $H^*(Z/p \oplus Z/p; Z/p)$ and $Q_\alpha$ is the Milnor exterior operation, $Q_0(x_i) = y_i$. Moreover, when $\#(k) = n$, we see that $k^*(\bigoplus^{n+1} Z/p)$
$\not\otimes_k k^*(\mathbb{Z}/p)$ because there is an element $\alpha$ such that $Q_{i_1} \cdots Q_{i_n}(x_1 \cdots x_{n+1}) = \rho(\alpha)$ or $Q_{i_1} \cdots Q_{i_{n-1}}Q_0(x_1 \cdots x_{n+1}) = \rho(\alpha)$.

3. The restriction homomorphism

Restriction map $i_A^*: k^*(G) \to k^*(A)$ for all conjugacy classes of abelian subgroups $A$ of $G$ induce the map

$$r : k^*(G) \to \varprojlim \text{inv } k^*(A),$$

$A \subset G$, conjugacy classes of abelian $p$-groups. We will show that (3.1) is an $F$-isomorphism if $(k) \geq \text{rank}_p G$ and $k^0 = \mathbb{Z}/p$. A ring homomorphism $f : A \to B$ is said to be an $F$-isomorphism if $\text{Ker } f \subset \sqrt{0}$ (nilpotent elements) and for all $b \in B$ there is $i$ such that $b^i \in \text{Image } f$. Quillen proved the $F$-isomorphism of $r$ for $k = H\mathbb{Z}/p$ and the conjugacy classes of elementary abelian $p$-groups $eA$ [6]. However, $\text{Ker } i^*$ of the restriction map $i^*: k^*(\mathbb{Z}/p^2) \to k^*(\mathbb{Z}/p)$ is not nilpotent for $(k) \geq 2$, and we consider all abelian $p$-groups. Most arguments of this section are $k^*$-theory versions of Quillen’s arguments [5].

Lemma 3.2. Let $X$ be a compact manifold and $G$ act on $X$ smoothly. If $u \in k^*_G(X)$ restricts to zero on each orbit of $X$, then $u$ is nilpotent.

Proof. The $k^*$-theory version of Lemma 3.9 in [5].

Theorem 3.3. The kernel of $r$ in (3.1) is nilpotent.

Proof. Let $\rho : G \to U$ be a unitary representation and $T$ be a maximal torus of $U$. Consider the map of equivariant cohomologies

$$k^*(G) \cong k^*_G(pt) \to k^*_G(U/T) \to k^*_G(Gx) \cong k^*_G(G/A) \cong k^*(A).$$

The orbits of $G$ on the flag manifold $U/T$ are of the form $G/A$ where $A$ is an abelian group as it is conjugated in $U$ to a subgroup of $T$.

Assume that $u \in k^*(G)$ and $u | A = 0$. The image $pr^*(u)$ restricts to zero on each orbit of $U/T$, hence it is nilpotent by Lemma 3.2. But the map $pr^*$ is injective. Indeed, since $k^*(-)$ is complex oriented, the Leray-Hirsch theorem holds; that is, $k^*(BG) \to k^*(P(\rho^*\xi))$ is injective where $P(\rho^*\xi)$ is a $U/S^1$-bundle induced from $BP : BG \to BU$ of the universal bundle $\xi$. Q.E.D.

Let $H$ be a subgroup of $G$ such that $[G; H] = m$. Let $\Sigma^m$ be the symmetric group of $m$ letters. Then there is the inclusion

$$\Phi : G \hookrightarrow \Sigma^m \ltimes H = \Sigma^m \ltimes (H \times \cdots \times H).$$

Consider the inclusion map of classifying spaces

$$Bi : (BH)^m \hookrightarrow P\Sigma^m \times_{\Sigma^m} (BH)^m = B(\Sigma^m \ltimes H).$$

Denote by $i$, the Gysin map of $Bi$, constructed by Quillen in [7] and for $k = \mathbb{B}P(S)$ in [13].

$$i : k^*((BH)^m) \to k^*(B(\Sigma^m \ltimes H)).$$
Remark 3.8. Let $BG^N$ be an $N$-dimensional skeleton of $BG$. By dimensional reason of the spectral sequence $H^*(BG^n, k^*) \Rightarrow k^*(BG^N)$, we can easily prove

$$\lim_{N \to \infty} k^*(BG^N) = k^*(BG).$$

Here $BG^N$ is a finite complex since $G$ is a finite group. The Gysin map defined above is defined only on finite complexes; however, we can extend this to $BG$ by (3.9).

Define the Evens norm $N[H \hookrightarrow G]: k^*(H) \to k(G)$ by

$$N[H \hookrightarrow G](x) = \Phi^*(i^*x^m).$$

Then we can show that this norm has the following properties by the arguments of Evens for $k^* = HZ$, in §6 in [1], namely transitivity, naturality, multiplicative property, and double coset formula.

Lemma 3.11 ((2.1) in [5]). If $u \in k^*(G')$ is such that $u \not\equiv 1$ for all $G'' \not\subseteq G'$, then we have

$$N[G' \hookrightarrow G](u) | K = \begin{cases} 1 & \text{if } G' \not\hookrightarrow K, \\ \prod_{g \in I} i_g^*u & \text{if } K = G', \end{cases}$$

where $I$ is the set of cosets representative for $G'$ in the normalizer $N_G(G')$, the notation $G' \not\hookrightarrow K$ means $G'$ is not conjugate to a subgroup of $K$, and $i_g^*$ is the conjugation map by $g$.

Let $A = \bigoplus Z/p^r$ and $k^*(A) \cong \bigotimes k^*[[u]]/(p^r[[u]])$. Define an element $e_A \in k^*(A)$ by

$$e_A = \prod_{\lambda \not\in \{\lambda_1, \ldots, \lambda_n\} \in A} ([\lambda_1](u_1) + k \cdots + k [\lambda_n](u_n)).$$

The element is unique except for multiplying units. It is immediate that if $A' \not\subseteq A$ then $e_A | A' = 0$.

Lemma 3.13 (Lemma 2.4 in [5]). Let $\#(k) \geq \text{rank}_p G$ and $k^0 = Z/p$. If $[N_G(A); A] = qh$, $q = p^s$ and $(p, h) = 1$, then there is $v_A \in k^*(G)$ such that

$$v_A | A' = \begin{cases} 0 & \text{if } A \not\hookrightarrow A', \\ e_A^q & \text{if } A = A'. \end{cases}$$

Moreover if $y \in k^*(A)$ is invariant under $N_G(A)$, then there is an $\alpha(y)$ in $k^*(G)$ with $\alpha(y) | A = y^q e_A^q$.

Proof. Set $z = N[A \hookrightarrow G](1 + e_A)$. Then from (3.11) and the property of $e_A$, we have

$$z | A = (1 + e_A)^q h = (1 + e_A^q)^h$$

Taking $1/h$ times the homogeneous component of $z$ of degree $e_A^q$, we have $v_A$. By taking $z = N[A \hookrightarrow G](1 + e_A y)$ we have $\alpha(y)$. Q.E.D.
Theorem 3.14. Let \( \#(k) \geq \text{rank}_p G \) and \( k^0 = Z/p \). Then \( r \) in (3.1) is an F-isomorphism.

Proof. Given \( 0 \neq (\lambda_1, \ldots, \lambda_m) = p^i(\lambda'_1, \ldots, \lambda'_m) \), \( A = \bigoplus Z/p^{r_i} \) with \( \lambda'_i \neq 0 \mod p \) for some \( i \), take \( M \) to be a matrix such that the 1st column is \((\lambda'_1, \ldots, \lambda'_m)\) and \( M \) induces an automorphism of \( A \). Then from (2.4) the kernel of the map in \( k^* \)-theory induced from

\[
Z/p^1 \times Z/p^2 \times \cdots \times Z/p^m \rightarrow A^M/A
\]

is the ideal generated by the following element in \( k^*(A) \):

\[
[\lambda_1](\mu_1) + \cdots + k[\lambda_m](\mu_m).
\]

Therefore if \( x \in k^*(A) \) satisfies \( x \mid A' = 0 \) for all \( A' \nsubseteq A \), then \( x^{p^i} \in \text{Ideal} e_A \).

Given \( x \in \text{Lim inv} k^*(A) \) in (3.1), there is an abelian group \( A \) such that \( x \mid A' = 0 \) for all \( A' \nsubseteq A \) and \( 0 \neq x \mid A \in k^*(A)^{N_G(A)} \). Then \( x^{p^i} \mid A = e_A^\alpha \) and \( \alpha \in k^*(A)^{N_G(A)} \) since \( e_A \in k^*(A)^{\text{Aut}(A)} \) from the definition of \( e_A \). By Lemma 3.13, we have completed the proof. Q.E.D.

4. Relation to \( H^*(G;Z/p) \)

In [11], we see that when \( G \) is an abelian \( p \)-group or \( |G| = p^3 \), there is an isomorphism

\[
(BP^* (G) \otimes_{BP^*} Z(p))/\sqrt{0} \cong H^*(G)/\sqrt{0}.
\]

We consider some extensions of this fact. Restriction maps to elementary abelian \( p \)-groups \( eA \) of \( G \) induce the map

\[
k^*(G) \rightarrow \text{Lim inv} k^*(eA) \rightarrow \bigoplus_{eA \subset G} k^*(eA)
\]

(4.1)

eA \rightarrow G, \text{ conjugacy classes of elementary abelian } p \text{-groups. Let}

\[
J = (iv')^{-1}(\text{Ideal}(p, v_1, \ldots)).
\]

Of course, \( k^*(G)/J \) is a quotient algebra of \( k^*(G) \otimes_{k^*} Z/p \).

Theorem 4.2. If \( \#(k) \geq \text{rank}_p G \), then there is an F-isomorphism \( \rho/J: k^*(G)/J \rightarrow H^*(G;Z/p)/\sqrt{0} \).

Proof. By the definition of \( J \), there is an injection

\[
k^*(G)/J \rightarrow \prod k^*(eA) \otimes_{k^*} Z/p \cong \prod H^*(eA;Z/p)/\sqrt{0}.
\]

Hence \( k^*(G)/J \rightarrow \rho/J \text{ Lim inv } H^*(eA;Z/p)/\sqrt{0} \) is injective. By the same arguments as in the proof of Theorem 3.14, \( r' \) is F-epic. We show

\[
\rho: k^*(eA)^{W_G(A)} \rightarrow H^*(eA;Z/p)^{W_G(A)}
\]
is also $F$-epic. Indeed, given $w \in W_G(A)$ and $\alpha \in H^*(eA; \mathbb{Z}/p)^{(w)}$, $|w| = p^k p'$, $(p, p') = 1$, we take $\hat{\alpha} \in k^*(eA)$ with $\rho(\hat{\alpha}) = \alpha$ and 

$$\hat{\alpha}_w = \frac{1}{p} \sum_{j=0}^{p'-1} w_j \left( \prod_{i=0}^{p'-1} w_i^{*+p'} \hat{\alpha} \right)$$

so that $w^* \hat{\alpha}_w = \hat{\alpha}_w$ and $\rho(\hat{\alpha}_w) = \alpha^{p'}$. Therefore we can prove $\rho r' / J$ is an $F$-isomorphism by the arguments similar to Theorem 3.14.

Quillen's main theorem [6] says that $H^*(G; \mathbb{Z}/p) \to \lim \text{inv} H^*(eA; \mathbb{Z}/p)$ is an $F$-isomorphism. Hence we have the theorem. Q.E.D.

**Corollary 4.3.** There is an $F$-isomorphism

$$\rho : k(n)^*(G) \otimes_{k(n)^*} \mathbb{Z}/p \to H^*(G; \mathbb{Z}/p).$$

**Proof.** Since there is cohomology theory $k^*$ such that $\rho : k^*(-) \to k(n)^*(-)$ and $\#(k) = \infty$, the map is $F$-epic from Theorem 4.2. By the Sullivan-Bockstein exact sequence, it is also injective. Q.E.D.

### 5. Relation to the Morava $K$-theory

Recall $P(n)^* = BP^*/(p, v_1, \ldots, v_{n-1}) \cong \mathbb{Z}/p[v_n, \ldots]$. We see in [11] that when $G$ is an abelian $p$-group or $|G| = p^3$, there is an isomorphism

$$BP^*(G) \otimes_{BP^*} P(n)^* \cong P(n)^*(G)$$

for all $n \geq 1$. By the Sullivan-Bockstein exact sequence, (5.1) is equivalent to

$$v_n : BP^*(G) \otimes_{BP^*} P(n)^* \to BP^*(G) \otimes_{BP^*} P(n)^*$$

is injective for each $n \geq 0$.

The Landweber exact functor theorem [4] says that if $BP^*(G)$ satisfies (5.2), then $BP^*(G) \otimes_{BP^*} -$ is an exact functor for finite $BP^*(BP)$ modules. Moreover, $P(n)^*(G) \otimes_{P(n)^*} -$ is also an exact functor from [12].

**Theorem 5.3.** If a $p$-Sylow subgroup $P$ of $G$ is a direct product of groups which satisfy (5.2), then we have

$$BP^*(G) \otimes_{BP^*} P(n)^* \cong P(n)^*(G),$$

$$BP^*(G) \otimes_{BP^*} K(n)^* \cong K(n)^*(G).$$

**Proof.** Let $P = P_1 \oplus \cdots \oplus P_s$ and $P_i$ satisfies (5.2). By the exact functor theorem for $P(n)^*$-theory

$$P(n)^*(- \wedge BP^*_i) \cong P(n)^*(-) \otimes_{P(n)^*} P(n)^*(P_i)$$

because both are cohomology theories with the same coefficient. Hence $P(n)^*(P) \cong \bigotimes_{P(n)^*} P(n)^*(P_i)$. Therefore we have

$$P(n)^*(P) \cong \left( \bigotimes_{BP^*} BP^*(P_i) \right) \otimes_{BP^*} P(n)^* \cong BP^*(P) \otimes_{BP^*} P(n)^*. $$
Hence $P$ satisfies (5.1) and so (5.2). Since $P(n)^*(G) \hookrightarrow P(n)^*(P)$, multiplying $v_n$ in $P(n)^*(G)$ is injective. By the Conner-Floyd type theorem, $P(n)^*(-) \otimes F(n)$. $K(n)^* \cong K(n)^*(-)$, and we have the theorem. Q.E.D.

6. Minimal nonabelian $p$-groups

For an odd prime $p$, the minimal nonabelian $p$-groups are of two types (Redéi [9])

Type 1. $G_1 = \langle a, b; a^{p^\alpha} = b^{p^\beta} = 1, [a, b] = a^{p^{\alpha-1}} \rangle$.

Type 2. $G_2 = \langle a, b, c; a^{p^\alpha} = b^{p^\beta} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$.

When $p = 2$, there is an isomorphism $G_1 (\alpha = 2, \beta = 1) \cong G_2 (\alpha = 1, \beta = 1)$ and we need to add another type

$$Q = \langle a, b; a^4 = 1, a^2 = b^2 = [a, b] \rangle.$$

For each of the above types, there is an exact sequence

$$1 \rightarrow C \rightarrow G \rightarrow \mathbb{Z}/p \oplus \mathbb{Z}/p \rightarrow 1$$

where $C$ is the center of $G$ and is isomorphic to $\langle a^p, b^p \rangle$ for Type 1 and $\langle c, a^p, b^p \rangle$ for Type 2. The induced Hochschild-Serre spectral sequence is

$$E_2^{*,*} = H^*(\mathbb{Z}/p \oplus \mathbb{Z}/p; \text{BP}^*(C))$$

$$\cong \tilde{Z}/p[y_1, y_2] \otimes \bigwedge (\alpha) \otimes \text{BP}^*(C) \Rightarrow \text{BP}^*(G)$$

where $\tilde{Z}/p[a]$ means $Z[a]/(pa)$ and $|y_1| = |y_2| = 2$ and $|\alpha| = 3$. Let us write

$$\text{BP}^*(C) = \text{BP}^*[[u, u_1, u_2]]/([p](u), [p^{\alpha-1}](u_1), [p^{\beta-1}](u_2))$$

or

$$\text{BP}^*[[u_1, u_2]]/([p^\alpha](u_1), [p^{\beta-1}](u_2)).$$

We will compute the spectra sequence (6.2).

Type 2. Consider the quotient map $q$

$$1 \rightarrow C \rightarrow G \rightarrow \mathbb{Z}/p \oplus \mathbb{Z}/p \rightarrow 1$$

The spectral sequence $\tilde{E}_2^{*,*}$ induced from the lower exact sequence is known from [11]. The differentials are

$$d_3 u = \alpha,$$

$$d_{2p-1} u^{p-1} = y_1^p y_2 - y_1 y_2^p,$$

and we get

$$\tilde{E}_2^{*,*} \cong \tilde{E}_\infty^{*,*} \cong \text{BP}^* \otimes (Z\{pu, \ldots, pu^{p-1}\}) \oplus \mathbb{Z}/p[[y_1, y_2]]/(y_1^p y_2 - y_1 y_2^p)$$

$$\otimes \mathbb{Z}[[u^p]]/([p](u)).$$
There are splitting maps $\langle b \rangle \cong G$, $\langle a \rangle \cong G$ which induced that $u_1$ and $u_2$ are permanent cycles. Since $BP^*(\langle a^p, b^p \rangle)$ is a flat $BP^*$-module for $BP^*(BP)$-modules, we have
\begin{equation}
E_{r}^{*,*} \otimes_{BP^*} BP^*(\langle a^p, b^p \rangle) \cong E_{r}^{*,*}.
\end{equation}

In particular, we get
\begin{equation}
E_{\infty}^{*,*} \cong (6.6) \otimes_{BP^*} BP^*[u_1, u_2]/([p^{\alpha-1}](u_1), [p^{\beta-1}](u_2)).
\end{equation}

Type 1 case. First, we consider the case $\beta = 1$ and denote by $\tilde{E}_{r}^{*,*}$ the induced spectral sequence from (6.1). Consider also the spectral sequence converging to $H^*(G_1; \mathbb{Z})$. Since
\[ H_2(G_1; \mathbb{Z}) = \text{Hom}(G, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}/p^{\alpha-1}, \quad \beta = 1, \]
we get $d_3u = \alpha$. Moreover, considering the spectral sequence converging to $H^*(G_1; \mathbb{Z}/p)$, we also have $d_{2p-1}u_0^{\alpha-1} = y_1^p y_2 - y_1 y_2^p$. Similar results hold in $BP^*$-theory. Therefore $E_{\infty}^{*,*} \cong E_{\infty}^{*,*} \cong (6.6)$. The flatness of $BP^*(\langle b^p \rangle)$ implies $E_{r}^{*,*} \cong (6.6) \otimes_{BP^*} BP^*(\langle b^p \rangle)$,\begin{equation}
E_{\infty}^{*,*} \cong (6.6) \otimes_{BP^*} BP^*[u_2]/([p^{\beta-1}](u_2)).
\end{equation}

**Corollary 6.10.** For $G_1$ or $G_2$, the image of the map
\[ j: BP^*(G) \to BP^*(\langle c \rangle) \otimes_{BP^*} \mathbb{Z}/p \cong \mathbb{Z}/p[u]\]
is $\text{Im } j = \mathbb{Z}/p[u]$. 

**Remark 6.11.** For $G_2$, $\alpha \geq 2$, $\beta \geq 2$, the image of the map
\[ j: H^*(G_2) \to H^*(\langle c \rangle) \cong \tilde{Z}/p[u]\]
is $\text{Im } j = \text{Ideal } u^2$. (See [14, 19] for more details.) We explain here the difference of spectral sequences for $H^*(G_2)$ and $BP^*(G_2)$. Consider the spectral sequence from (6.1) for $\alpha = 2$ and $\beta = 2$:
\begin{align*}
E^*_{2} &= H^*(Z/p \oplus Z/p; H^*(C)) \Rightarrow H^*(G), \\
E^*_{2} &\cong (Z/p \oplus Z/p; H^*(C; Z/p)) \Rightarrow H^*(G; Z/p).
\end{align*}

Then $E^*_{2} = Z/p[y_1, y_2] \otimes (X, x_2, x_3) = E^*_{2} \cong Z/p[u_1, u_2, u_3] \otimes (Z, z_2, z_3)$, and
$E^*_{2} \cong E^*_{2}(Z/p) = E^*_{2} \cong E^*_{2}(Z/p) \otimes E_{2}^{*,*}(Z/p)$. The integral parts are
$E_{2}^{*,*} \cong \tilde{Z}/p[y_1, y_2] \otimes (\beta(x_1, x_2))$ where $\beta$ is the Bockstein operation, $E_{2}^{*,*} \cong \tilde{Z}/p[u_1, u_2, u_3] \otimes (1, \beta(z_2, z_3), \beta(z_1 z_2 z_3))$, and $E_{2}^{*,*} \cong E_{2}^{*,*} \otimes E_{2}^{*,*}(Z/p)$ for $* > 0$. The first differentials are
\[ d_2 z_1 = y_1, \quad d_2 z_2 = y_2, \quad \text{and} \quad d_2 z_3 = x_1 x_2. \]

Then $d_3 u_3 = \beta(x_1 x_2) \neq 0$. However, $d_3 u_3 = 0$. Indeed,
\[ d_2(x_2 \beta(z_1 z_3) - x_1 \beta(z_2 z_3)) = \beta(x_1 x_2)u_3, \]
which is also the image $d_3 u_3^2$. Therefore we see that $u_3^2$ is a permanent cycle.

**Remark 6.12.** From Corollary 6.10 and Remark 6.11, the map $p/J$ in Theorem 4.2 is not epic for $G_2$, $\beta \geq 2$, $\alpha \geq 2$. 

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Theorem 6.13. If a p-Sylow subgroup $P$ of $G$ is a direct product of minimal nonabelian $p$-groups and abelian $p$-groups, then

$$BP^*(G) \otimes_{BP^*} P(n)^* \simeq P(n)^*(G),$$
$$BP^*(G) \otimes_{BP^*} K(n)^* \simeq K(n)^*(G).$$

Proof. From Theorem 5.3, we need to prove (5.1) only in the cases $G_1$ and $G_2$. We will prove here the case $G_2$ only; the other case is proved by a similar argument. Consider the exact sequence

$$1 \to (a^p, b^p) \to G_2 \to E \to 1,$$

where $E = G_2$ ($\alpha = 1, \beta = 1$), and the induced spectral sequences

$$E_2^{*,*} = H^*(E; P(n)^*((a^p, b^p))) \cong H^*(E; P(n)^*) \otimes_{P(n)^*} P(n)^*((a^p, b^p))$$

$$\Rightarrow P(n)^*(G_2),$$

$$\tilde{E}_2^{*,*} = H^*(E; P(n))^P(n)^*(E).$$

Since all elements in $P(n)^*((a^p, b^p))$ are permanent, we have $E_r^{*,*} \cong \tilde{E}_r^{*,*} \otimes_{P(n)^*} P(n)^*((a^p, b^p))$ by flatness and naturality. In particular, $\tilde{E}_\infty^{*,*}$ is generated by even-dimensional elements; so is $E_\infty^{*,*}$. Hence $G_2$ satisfies (5.2) because if $v_n: P(n - 1)^*(G_2) \to P(n - 1)^*(G_2)$ is not injective, then $P(n)^{odd}(G_2) \neq 0$ by the Sullivan-Bockstein spectral sequence. Q.E.D.

Theorem 6.14. If a p-Sylow subgroup $p$ of $G$ is a direct product of minimal nonabelian $p$-groups and abelian groups, then $r$ in (3.1) is injective for $k^* = BP^*$ or $P(n)^*$.

Proof. We need to prove the case $G = G_1$ or $G_2$. We will prove the case $k^* = BP^*$ and the other cases are proved similarly. Let $A$ be a maximal abelian $p$-subgroup of $G$. Let us denote by $E_r^{*,*}(A)$ the spectral sequence induced from

$$1 \to C \to A \to Z/p \to 1.$$

Then the spectral sequence collapses and we have

$$E_\infty^{*,*}(A) \cong BP^*(C) \otimes \tilde{Z}/p[y].$$

Let us denote by $E_r^{*,*}(G)$ the spectral sequence induced from (6.1). Then we know $E_\infty^{*,*}(G)$ by (6.6), (6.8), and (6.9).

We will prove that for each $x \in E_\infty^{*,*}(G)$, there is an abelian subgroup $A$ with $i^*(x) \neq 0$ in $E_\infty^{*,*}(A)$, $i: A \hookrightarrow G$.

When $x \in E_\infty^{*,*}(G)$, this is obvious since $E_\infty^{*,*}(G) \hookrightarrow E_\infty^{*,*}(A)$.

We will prove that $i^*(x) \neq 0$ for $x \in E_r^{*,*}(G)$, $r$ positive. Suppose $x$ is written such that $x = ay_1^r + y_2c$ where $a \neq 0$ in $BP^*/p[[u]]/([p](u))$. Then $x \mid (a) = ay_1^r \neq 0$. Suppose $x$ is written so that

$$x = ay_1^r(\lambda_1y_1^{p-1}y_2 + \lambda_2y_1^{p-2}y_2^2 + \cdots + \lambda_{p-1}y_1y_2^{p-1}) + b(y_1^{s+p}y_2 + \cdots) + \cdots.$$
If $A/C \cong \langle ab^\mu \rangle$, we take a two-dimensional element $y$ so that $i^*(y_1) = y$ and $i^*(y_2) = \mu y$. Hence

$$i^*(x)\left(\lambda_1 \mu + \lambda_2 \mu^2 + \cdots + \lambda_{p-1} \mu^{p-1}\right)y^{s+p} + b'. $$

Since $0 = 1 - \mu^{p-1} = (1 - \mu)(1 + \mu + \cdots + \mu^{p-2})$, if $i^*(x) = 0$ for all $\mu \neq 1$, then $\lambda_1 = \lambda_2 = \cdots = \lambda_{p-1} = 1$. But when $\mu = 1$, $i^*(x) = (p-1)a y^{s+p} + b' \neq 0$.

Q.E.D.

**Proposition 6.15.** For each minimal nonabelian $p$-group $G$, the restriction maps $BP^*(G) \to BP^*(A)^{W_G(A)}$ are epic for all maximal abelian subgroups $A$.

**Proof.** Each maximal abelian subgroup of $G$ is isomorphic to $\langle ab^\mu, c, b^\mu \rangle$ or $\langle b, c \rangle$ (Type 1 case $c = a^\mu$). We will prove the map is epic for the case $A = \langle a, c, b^\mu \rangle$ and Type 2. The other cases are proved similarly.

The map induced from the conjugation on $b$ is given by

$$b^*u = u + [p^\alpha]u_1, \quad b^*y_1 = y_1, \quad b^*y_2 = y_2.$$

We can prove that

$$BP^*(A)^{(b)} \cong \frac{BP^*(\{1, Nu, \ldots, Nu^{p-1}\} \otimes \{U, y_1, y_2\})}{([p](u), [p^{n-1}](y_1), [p^{p-1}](y_2))}$$

where

$$Nu^g = \sum b^i u^g = pu^g + \cdots,$$

$$U = \prod b^i u = u^p + p^{\alpha-1}y_1^p + \cdots.$$

Therefore, from (6.6) and (6.8) we show the epimorphism.

The invariant is computed, for example, as follows. Let

$$x = (u^s + a_1 u^{s+1} + \cdots) y_1^k + by_1^{k+1} + \cdots$$

in $BP^*(\langle a, c \rangle)$, with $s \neq 0 \mod p$ and $a_i \in BP^*[\{y\}]/[p^\alpha](\{y\})$. Then

$$b^*x = ((u + [p^{n-1}](y_1))^s + a_1 (u + [p^{n-1}](y_1))^{s+1} + \cdots) y_1^k + \cdots,$$

$$(1 - b^*)x \equiv p^{\alpha-1}((sv_1^{s-1} y_1 + v_1 s u^{p+s-2} y_1 + \cdots) + a_1 (s + 1) u^s y_1^k + \cdots) y_1^k \mod (p^\alpha, y_1^{k+2}, u^{(p-1)(a-1)+p-1}) ,$$

which is nonzero. Q.E.D.

Ravenel conjectured that $r$ in (3.1) is isomorphic for $k^* = BP$. However this does not correct. Suppose $p \geq 3$ and $G = G_2$ ($\alpha = \beta = 1$). Let $A^\alpha = \langle ab^\mu, c \rangle$ and $A^\beta = \langle b, c \rangle$ be the maximal abelian subgroups in $G$. By the arguments similar to the proof of Proposition 6.15, there is an element $\hat{y}_{\mu} \in BP^2(A^\alpha)^{W_G(A_\alpha)}$ such that $\hat{y}_{\mu}\langle ab^\mu \rangle \neq 0 \mod (p, v_1, \ldots)$ and $\hat{y}_{\mu}\langle c \rangle = 0$. Consider the element

$$y = (0, \hat{y}_1, 0, 0, \ldots, 0) \in BP^*(A^1)^W \times BP^*(A^1)^W \times \cdots \times BP^*(A^p)^W ,$$
which is in \( \text{Lim} \BP^*(A) \) since \( A^\mu \cap A^\lambda = \langle c \rangle \) for \( \mu \neq \lambda \). Recall that \([11]\) \( \BP^*(G)/(p, v_1, \ldots) \) is generated by \( y_1 \) and \( y_2 \) with \( y_1|A^0=y_0 \), \( y_1|A^p=0 \), \( y_2|A^0=0 \) and \( y_2|A^p=y_p \). Hence there is no two-dimensional element \( y \) in \( \BP^*(G) \) such that \( y|A^1=y_1 \) and \( y|A^\mu=0 \) for all \( \mu \neq 1 \).

7. Applications; nonabelian \( p \)-subgroup

In this section we consider the existence of nonabelian \( p \)-subgroups of topological groups by using Corollary 6.10.

**Theorem 7.1.** Let \( G \) be a compact group such that \( H^*(BG)/(p) \) is finitely generated as a ring and \( \rho: \BP^*(BG) \to H^*(BG)/(p, \sqrt{0}) \) is epic. If \( G \) contains nonabelian \( p \)-subgroups, then there is a ring generator \( x \in H^*(G)/(p, \sqrt{0}) \) with \( 2p | x \).

**Proof.** Let \( P \) be a minimal nonabelian \( p \)-subgroup and \( D \cong \mathbb{Z}/p \) be the subgroup generated by \( c \) for Type 2 and \( a^{p^r-1} \) for Type 1. Consider the following commutative diagram:

\[
\begin{array}{ccc}
\BP^*(G) & \longrightarrow & \BP^*(P) \\
\rho_G & & \rho_P \\
\downarrow & & \downarrow \\
H^*(BG)/(p, \sqrt{0}) & \longrightarrow & \BP^*(P)/(p, \sqrt{0}) \\
\iota_H & & \iota_H \\
\end{array}
\]

From Corollary 6.10, \( \text{Im}(\rho_Dj_Bp) = \mathbb{Z}/p[u^p] \). Hence

\[ \text{Im}(j_B^*H^*_G) = \text{Im}(\rho_Dj_B^*\rho_G) \subset \mathbb{Z}/p[u^p]. \]

Since \( \rho_G \) is epic, \( \text{Im}(j_B^*\rho_G) \subset \mathbb{Z}/p[u^p] \). From Quillen's main theorem of equivariant cohomology \([6]\), \( j_B^*\rho_G \neq 0 \) for some \( * > 0 \). Therefore there is a ring generator \( x \in H^*(G)/(p, \sqrt{0}) \) such that \( j_B^*H^*_G(x) = u^p x \). Q.E.D.

**Corollary 7.2.** Let \( G \) be a compact Lie group containing nonabelian \( p \)-subgroups.

(1) If \( H^*(G)/(p) \cong \langle x_1, \ldots, x_n \rangle \), then there is \( i \) with \( 2p | x_i + 1 \).

(2) If \( H^*(BG)/(p, \sqrt{0}) \) is generated by \( c_i^s, \ 1 \leq s \leq n, \ i_s \) th Chern classes of some representations, then there is \( s \) such that \( 2p | i_s \).

**Remark 7.3.** (1) of the above corollary is an immediate consequence of a result of Borel-Serre \([15]\) and its converse also holds. Let \( P \subset G \) be a \( p \)-group. By \([15]\), we may (after conjugation) assume \( P \subset N(T) \), the normalizer of a maximal torus \( T \). If \( P \) is nonabelian, then \( P \not\subset T \) and \( p \) divides the order \( |W| \) of the Wyle group \( W = N(T)/T \). Since \( |W| = \Pi((x_i | i_s + 1)/2 \), we have (1).

Conversely, if \( p \ | |W| \), then \( N(T) \) contains nonabelian \( p \)-subgroups. The extension \( T \to N(T) \to W \) defines an element of \( H^2(W, T) \cong 0 \) or \( \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z}/2 \) \([18]\). So an element of order \( p \) in \( W \) lifts to an element \( x \) of order
Let \( V \subset T \) be the set of solutions of \( t^p = 1 \) (\( t^4 = 1 \) for \( p = 2 \)). Since \( x \) acts nontrivially on \( T \), \( x \) also acts nontrivially on \( V \).

If we consider \( V \) as a vector space over \( F_p \), then the action on it is given by a Jordan decomposition, so we can find a subspace of dimension 2 on which the action is given by \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). This means that there is a subgroup of Type 2 \( \alpha = \beta = 1 \) (nonabelian group of order \( \leq 4^3 \) for \( p = 2 \)).

This remark is due to J. F. Adams. The author is grateful to Professor Adams for his kind comments.

**Example 7.4.** The cohomologies of simply connected simple Lie groups are known and the cohomologies of some cases of their classifying spaces are known. For example,

\[
H^*(BSU(n)) \cong \mathbb{Z}[y_4, \ldots, y_{2n-1}], \\
H^*(BE_7(p)) \cong \mathbb{Z}_p[y_i \mid i = 4, 12, 16, 20, 24, 28, 36] \quad \text{for } p \geq 5.
\]

Thus \( SU(n), (\text{resp. } Sp(n), SO(2n + 1)) \) contains nonabelian \( p \)-subgroup if and only if \( p \leq n \). The exceptional Lie group \( G_2 \) (resp. \( F_4, E_6, E_7, E_8 \)) contains nonabelian \( p \)-subgroups if and only if \( p \leq 3 \) (resp. \( \leq 3, \leq 5, \leq 7, \leq 7 \)).

Let \( G(F_q) \) be the \( F_q \)-rational points of the universal Chevally group of the reductive complex Lie group type \( G \). Let \( q = p^s \) and \( l \neq p \). Then \( H^*(BG; \mathbb{Z}/l) \cong H^*(BG(F_q); \mathbb{Z}/l) \) where \( F_q \) is the algebraic closure of \( F_p \).

The cohomology of the \( F_q \)-rational points is computed by considering the coinvariant under the Frobenius-Adams operation \( \sigma_p \). Let \( r \) be the smallest number such that \( qr = 1 \mod l \). Quillen showed \( H^*(GL_n(F_q))/(l, \sqrt{q}) \cong \mathbb{Z}/l[c_r, c_{2r}, \ldots, c_{r|n/r|}] \); in this case we get \( \sigma_q c_i = q^i c_i \). Hence if \( GL_n(F_q) \) contains nonabelian \( l \)-subgroups, then \( lr \leq n \). Exceptional Lie group types are computed by Kleinerman [17]. For example, in the case \( G = E_7 \), \( \sigma_q y_i = q^{i/2} y_i \). Hence \( H^*(BE_7(F_q))/(l, \sqrt{q}) \cong \mathbb{Z}/l[y_i \mid 2l| i] \). Therefore we see that if \( E_7(F_q) \) contains nonabelian 5-subgroups (resp. 7-subgroups), then \( r = 1, 2, 5, 10 \) (resp. \( r = 1, 2, 7, 14 \)). When \( l = p \), exceptional Lie types always contain nonabelian \( l \)-subgroups, since they contain \( SL_3(F_p) \).

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