HOMOGENEOUS CONTINUA IN EUCLIDEAN \((n + 1)\)-SPACE WHICH CONTAIN AN \(n\)-CUBE ARE \(n\)-MANIFOLDS

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Abstract. Let \(X\) be a homogeneous continuum and let \(E^n\) be Euclidean \(n\)-space. We prove that if \(X\) is properly contained in a connected \((n + 1)\)-manifold, then \(X\) contains no \(n\)-dimensional umbrella (i.e. a set homeomorphic to the set \(\{(x_1, \ldots, x_{n+1}) \in E^{n+1} : x_1^2 + \cdots + x_{n+1}^2 \leq 1 \text{ and } x_{n+1} \leq 0 \text{ and either } x_1 = \cdots = x_n = 0 \text{ or } x_{n+1} = 0\}\)). Combining this fact with an earlier result of the author we conclude that if \(X\) lies in \(E^{n+1}\) and topologically contains \(E^n\), then \(X\) is an \(n\)-manifold.

The main purpose of this paper is to prove the following theorem.

1. Theorem. Each homogeneous proper subcontinuum of a connected \((n + 1)\)-manifold contains no \(n\)-dimensional umbrella.

The results of this paper are related to two classical results: the first one of S. Mazurkiewicz [M], and, the second one of R. H. Bing [B]. Namely, with the help of the result of [P], we give a full generalization of the result of [B] to all finite-dimensional cases (Theorem 7 below, and also, the statement formulated in the title). As it was emphasized in [P], the theorem of [B] may be obtained by combining two other theorems: 1\(^\circ\) each homogeneous locally connected nondegenerate plane continuum is a simple closed curve (this is the result of [M]), 2\(^\circ\) each homogeneous plane continuum that contains an arc is locally connected (this is the step really done in [B]), and thus 3\(^\circ\) each homogeneous plane continuum that contains an arc is a simple closed curve. (Bing's proof did not follow this scheme.) One can easily observe that Theorem 1 implies the result of [M] (for \(n = 1\)). Thus this paper generalizes step 1\(^\circ\). Step 2\(^\circ\) has already been extended in [P] to all finite-dimensional cases. Therefore we get Theorem 7 as a generalization of step 3\(^\circ\).

Finally, let us stress the fact that, similarly as in [P], the \(\varepsilon\)-push property (Theorem 4) plays a crucial role in the argument of the proof of Theorem 1. Probably, this is the real reason that the results of [P] and of this paper have not been earlier found.

Received by the editors June 9, 1987 and, in revised form, June 1, 1988.

1980 Mathematics Subject Classification (1985 Revision). Primary 54F20; Secondary 54C25.

Key words and phrases. Continuum, Euclidean space, homogeneity, \(n\)-dimensional umbrella, \(n\)-manifold.

\(143\)
The author gratefully acknowledges a great deal of help from Professor J. J. Charatonik during the preparation of this paper.

All spaces considered here will be either Euclidean \( n \)-spaces \( E^n \) equipped with the usual Euclidean metric \( d \), or (not necessarily compact) \( n \)-manifolds with a metric also denoted by \( d \). The open ball of the space with center \( c \) and radius \( \varepsilon \) will be denoted by \( B(c, \varepsilon) \). For two subsets \( A \) and \( B \) of the space we put \( d(A, B) = \inf \{d(a, b) : a \in A \text{ and } b \in B\} \). If \( M \) is an \( n \)-manifold, the symbol \( \partial M \) denotes its combinatorial boundary. An arc with end points \( a \) and \( b \) will often be denoted by \( ab \). The symbol \( I \) means the unit segment \([0, 1]\). Let a set \( A \) be homeomorphic to the cube \( I^n \) and let \( ab \) be an arc. If \( A \cap ab = \{a\} \text{ and } a \notin \partial A \), then the union \( A \cup ab \) will be called an \( n \)-dimensional umbrella. A set \( X \) is said to be homogeneous if for given \( x, y \in X \) there is a homeomorphism \( h: X \rightarrow X \) with \( h(x) = y \). A mapping means a continuous function. A mapping (a homeomorphism) \( f: X \rightarrow Y \) between subsets \( X \) and \( Y \) of the same space is called an \( \varepsilon \)-translation (an \( \varepsilon \)-homeomorphism) provided \( d(x, f(x)) < \varepsilon \) for every \( x \in X \). A point \( x \) is said to be accessible from a set \( V \) if there is an arc \( xy \) with \( xy \setminus \{x\} \subseteq V \). A set \( C \) separates a set \( V \) between two points \( p, q \in V \), if \( p \) and \( q \) lie in distinct components of \( V \setminus C \).

We start with two lemmas, which we need to prove Theorem 1.

2. Lemma. If a point \( c \in C \subset E^{n+1} \) has a neighborhood (in a set \( C \)) homeomorphic to \( E^n \), then the number of components of \( E^{n+1} \setminus C \) containing \( c \) in their closures is either one or two. Moreover, \( c \) is accessible from each of these components.

Proof. Let a neighborhood \( A \) of \( c \) in \( C \) be homeomorphic to \( I^n \), and let a ball \( B(c, \xi) \subset E^{n+1} \) be such that \( C \cap B(c, \xi) = A \cap B(c, \xi) \) and \( \partial A \cap B(c, \xi) = \emptyset \). Further, let \( \{A_0, A_1, \ldots\} \) be the family (finite or infinite) of all components of \( A \cap B(c, \xi) \) with \( c \in A_0 \). By Proposition 3 of [P] the set \( A_0 \) separates \( B(c, \xi) \) into exactly two components \( U_1^0 \) and \( U_2^0 \). By the local connectedness of \( A \) there is a ball \( B(c, \tau) \) with \( A \cap B(c, \tau) \subset A_0 \). Since no pair of points of \( U_i^0 \cap B(c, \tau) \) is separated by any \( A_n \) in \( B(c, \xi) \) for \( i \in \{1, 2\} \) and \( B(c, \xi) \) is homeomorphic to \( E^{n+1} \), we see by Proposition 4 of [P] that \( A \) also does not separate \( B(c, \xi) \) between such points. This implies that \( c \in \text{cl} \cup_i \) for the component \( U_i \) of \( B(c, \xi) \setminus A \) containing \( U_i^0 \cap B(c, \tau) \), for \( i \in \{1, 2\} \), and \( c \) does not lie in the closure of any other component of \( B(c, \xi) \setminus A \). Thus the number of components of \( E^{n+1} \setminus C \) containing \( c \) in their closures is at most two, and, in fact, not less than one. Moreover, since \( A_0 \) is an ANR-set, the point \( c \) is accessible from both \( U_1^0 \) and \( U_2^0 \) (see [Bo, p. 217]). Finally, the desired accessibility of \( c \) follows by the previous argument.

3. Lemma. Let a set \( A \subset E^{n+1} \) be homeomorphic to \( I^n \). Given a point \( c \in E^{n+1} \) and a number \( \varepsilon > 0 \) such that \( d(c, A) < \varepsilon < d(c, \partial A) \), let \( A_1 \) denote a component of \( A \cap B(c, \varepsilon) \). For two given points \( p \) and \( q \) of distinct components of \( B(c, \varepsilon) \setminus A_1 \) let \( pa \) and \( aq \) be arcs in \( B(c, \varepsilon) \) such that \( (pa \cup aq) \cap A_1 = \{a\} \)
HOMOGENEOUS CONTINUA IN EUCLIDEAN \((n + 1)\)-SPACE

(\text{compare Proposition 3 of [P] and Lemma 2}). Then for every \(\delta > 0\) such that

\[ \delta < \frac{1}{2} \delta_0 = \frac{1}{2} \min \{ d(\{p, q\}, A) \} \]

and for every \(\delta\)-homeomorphism \(h: A_1 \to h(A_1) \subset E^{n+1}\) the component \(A_2\) of \(h(A_1) \cap B(c, \varepsilon - \delta)\) containing the point \(a' = h(a)\) separates \(B(c, \varepsilon - \delta)\) between \(p\) and \(q\).

Moreover, if \(\delta < \delta_0 / 4\), there are arcs \(pa'\) and \(a'q\) in \(B(c, \varepsilon - \delta - \delta_0 / 4)\) such that \((pa' \cup a'q) \cap A_2 = \{a'\}\).

\textbf{Proof.} By Proposition 5 of [P] the set \(h(A_1)\) separates the ball \(B(c, \varepsilon - \delta)\) between \(p\) and \(q\). Therefore a component \(A_2\) of \(h(A_1)\) so does (see Proposition 4 of [P]). Again by Proposition 5 of [P], noting that \(A_2\) is closed in \(B(c, \varepsilon - \delta)\), we see that the set \(h^{-1}(A_2) \subset A_1\) separates \(B(c, \varepsilon - 2\delta)\) between \(p\) and \(q\); and thus it intersects the set \(pa \cup aq\). By the assumption the only point of this intersection is \(a\), thus \(a' = h(a) \in A_2\).

Let \(\delta < \delta_0 / 4\). Put \(\delta_1 = \varepsilon - \delta - \delta_0 / 4\). Suppose there is no arc \(pa'\) in \(B(c, \delta_1)\) with \(pa' \cap A_2 = \{a'\}\). Let \(A_1\) be the component of \(B(c, \delta_1) \cap B(c, \varepsilon - \delta)\) containing \(a'\). Thus there is an arc \(Z \subset B(c, \delta_1)\) with end points \(p\) and \(a'\) such that \(Z \cap A_3 = \{a'\}\) (see Proposition 3 of [P] and Lemma 2). Let a point \(z \in Z \setminus \{a'\}\) be such that the arc \(za' \subset Z\) intersects \(A_3\) in the single point \(a'\). Thus, by the above assumption, \(A_2\) separates the ball \(B(c, \delta_1)\) between \(p\) and \(z\). Therefore some component \(A_4\) of \(B(c, \delta_1) \cap A_2\) separates \(B(c, \delta_1)\) between \(p\) and \(z\) (see Proposition 4 of [P]) and we have \(A_4 \neq A_3\). Hence \(A_4\) separates \(B(c, \delta_1)\) between \(p\) and \(a'\). By the proved part of the conclusion of this lemma the set \(A_3\) separates the ball \(B(c, \delta_1)\) between \(p\) and \(q\) (for \(h\) is also a \((\delta + \delta_0 / 4)\)-homeomorphism). Therefore the set \(A_4\) separates \(B(c, \delta_1)\) between \(p\) and \(q\). Thus, by Proposition 5 of [P], the set \(h^{-1}(A_4)\) separates \(B(c, \delta_1 - \delta)\) between \(p\) and \(q\). By the assumption on \(\delta\) we have \(pa \cup aq \subset B(c, \varepsilon - \delta_0) \subset B(c, \delta_1 - \delta)\), therefore the set \(h^{-1}(A_4) \subset A_1\) intersects the arc \(pa \cup aq\) in a point distinct from \(a\) (for \(a' = h(a) \notin A_4\)), a contradiction. The argument for the existence of an arc \(a'q\) runs similarly.

Now recall the theorem (the so-called \(\varepsilon\)-push property) which is a corollary to the well-known Effros theorem.

\textbf{4. Theorem} (Lemma 4 of [H, p. 37]). Let \(X\) be a homogeneous metric continuum. Then for every \(\varepsilon > 0\) there is \(\delta > 0\) (the so-called Effros number for the number \(\varepsilon\)) such that for two given points \(x, y \in X\) with \(d(x, y) < \delta\) there is an \(\varepsilon\)-homeomorphism \(h: X \to X\) sending \(x\) to \(y\).

\textbf{Proof of Theorem 1.} Let \(P\) be a homogeneous proper subcontinuum of a connected \((n + 1)\)-manifold \(M\). Suppose, on the contrary, \(P\) contains an \(n\)-dimensional umbrella. By the intrinsic invariance of open sets in the Euclidean spaces, since \(\partial M\) is either an \(n\)-manifold or the empty set, \(P\) cannot be contained in \(\partial M\). Because \(P\) is a boundary set in \(M\), there is a point \(a \in P \setminus \partial M\).
accessible from $M \setminus P$. Let $d$ be a metric on $M$ such that the ball $B(a, 1) \subset M$ is isometric to the appropriate ball of Euclidean $(n+1)$-space. By the homogeneity of $P$ there is an $n$-dimensional umbrella $T \subset B(a, 1) \cap P$ such that $T = A \cup ab$, where the set $A$ is homeomorphic to $I^n$, $ab$ is an arc with ends $a$ and $b$, and $A \cap ab = \{a\} \subset A \setminus \partial A$. Let a number $\varepsilon > 0$ be such that $\partial A \cap B(a, \varepsilon) = \emptyset$. Without loss of generality we may assume that $ab \subset B(a, \varepsilon)$.

Let $A_1$ be the component of $A \cap B(a, \varepsilon)$ containing $a$. Then $A_1$ separates the ball $B(a, \varepsilon)$ into exactly two components (see Proposition 3 of [P]): $V_b$ with $ab \setminus \{a\} \subset V_b$, and $V_c$ with some point $c \in V_c$. By the accessibility of $a$ there is an arc $ap \subset B(a, \varepsilon)$ with $ap \cap P = \{a\}$. By Lemma 2 there is an arc $ac \subset B(a, \varepsilon)$ with $ac \setminus \{a\} \subset V_c$.

(1) For any arc $ax$ with $ax \setminus \{a\} \subset V_c$ we have $P \cap (ax \setminus \{a\}) \neq \emptyset$.

In fact, let $cx \subset V_c$ be an arc (may be degenerate if $c = x$), and $\{z_m\}$ be a sequence of points of $ab \setminus \{a\}$ converging to $a$. By the $\varepsilon$-push property (Theorem 4) there are $\xi_m$-homeomorphisms $g_m: P \to P$ with $g_m(z_m) = a$ and $\lim_{m} \xi_m = 0$. By Proposition 5 of [P] the sets $g_m(A_1)$ separate the balls $B(a, \varepsilon - \xi_m)$ between $c$ and $b$, and do not intersect $cx$ and some fixed ball $B(b, \xi) \subset V_b$ for sufficiently great $m$. Since $g_m(b) \in B(b, \xi)$, the sets $B(b, \xi) \cup g_m(bz_m) \cup ax \cup cx$ are connected for almost all $m$. Noting $g_m(bz_m) \cap g_m(A_1) = \emptyset$, where $bz_m \subset ab$, we see that the set $g_m(A_1) \subset P$ intersects $ax \setminus \{a\}$ for large $m$.

By (1) we see that there is a sequence $\{a_m\}$ converging to $a$ with $a_m \in P \cap ac \setminus \{a\}$, and also

(2) $ap \setminus \{a\} \subset V_b$.

Let $pb \subset V_b$ be an arc, and let $\tau$ be a number such that

$$0 < \tau < \frac{1}{2} \min\{d(ap \cup pb \cup ab \cup ac, M \setminus B(a, \varepsilon)), d(pb \cup \{c\}, A_1)\}.$$

Further, let $\psi > 0$ be an Effros number for the number $\tau/4$ (see Theorem 4), and let $\phi > 0$ be an Effros number for the number $\psi$. Now, find $a_k$ with $d(a, a_k) < \phi$, and let $f: P \to P$ be a $\psi$-homeomorphism such that $f(a) = a_k$. Let $A_2$ denote the component of $B(a, \varepsilon - \tau/4) \cap f(A_1)$ containing $f(a)$. By Lemma 3 this component separates the ball $B(a, \varepsilon - \tau/4)$ between $c$ and $b$, and also between $c$ and $p$ (for obviously we have $\psi \leq \tau/4$). Moreover, the same lemma guarantees the existence of an arc $ca_k \subset B(a, \varepsilon - \tau/4 - \tau/2) = B(a, \varepsilon - 3\tau/4)$ with $ca_k \cap A_2 = \{a_k\}$. Let $r$ be the first point of the arc $f(ab)$ (in the ordering from $f(a)$ to $f(b)$) intersecting $A_1$. Since $d(a, a_k) < \psi$, we may find a point $q \in a_k, r \subset f(ab)$ with $a_k \neq q \neq r$ and $d(a, q) < \psi$.

Now consider the component $U$ of $B(a, \varepsilon - \tau/4) \setminus (A_1 \cup A_2)$ containing $q$. Observe that this component contains no point of the set $pb \cup \{c\}$. The point $a_k$ lies in the closures of $U$ and of the component $U_c$ of $B(a, \varepsilon - \tau/4) \setminus (A_2 \cup \text{Bd } U)$ containing $c$, for there exist the arcs $ca_k$ and $a_kq \subset f(ab)$. Since $a_k$ has a neighborhood in $A_2 \cup \text{Bd } U$ homeomorphic to $E^n$, there is, by Lemma 2, no
other component of $B(a, e - \tau/4) \setminus (A_2 \cup \text{Bd} U)$ containing $a_k$ in its closure, in particular, $a_k$ does not lie in the closure of the component $U_b$ containing $b$. Let $B$ be a connected open neighborhood of $a_k$ in $A_2$ with a positive distance from $A_1 \cup \text{Bd} U_b$. Then

the set $(A_2 \cup \text{Bd} U) \setminus B$ separates the ball $B(a, e - \tau/4)$ between $c$ and $b$.

For every $x \in q b' \cap \text{Bd} U \subset A_1 \setminus A_2$, where $b' = f(b)$ and $q b' \subset f(ab)$, find an open connected neighborhood $B_x$ of $x$ in $A_1 \setminus A_2$ such that

$\text{cl}(\bigcup \{B_x : x \in q b' \cap \text{Bd} U\}) \cap \text{cl} A_2 = \emptyset$.

Put $B_1 = \bigcup \{B_x : x \in q b' \cap \text{Bd} U\}$. Since $A_2 \subset (A_2 \cup \text{Bd} U) \setminus B_1$, we get

the set $(A_2 \cup \text{Bd} U) \setminus B_1$ separates the ball $B(a, e - \tau/4)$ between $c$ and $b$.

We also have

the set $(A_2 \cup \text{Bd} U) \setminus (B \cup B_1)$ does not separate the ball $B(a, e - \tau/4)$ between $c$ and $b$.

For, the segment between $b$ and $f(b)$ does not intersect $A_1 \cup A_2$, and, the arc $ca_k \cup f(ab)$ does not intersect $(A_2 \cup \text{Bd} U) \setminus (B \cup B_1)$.

Now, find a $(\tau/4)$-homeomorphism $h : P \to P$ such that $h(q) = a$. Then we obtain

the set $h(A_2 \cup \text{Bd} U) \setminus h(B \cup B_1)$ does not separate

the ball $B(a, e - 2\tau/4)$ between $c$ and $b$.

In fact, if not, then the set

$h^{-1}(h(A_2 \cup \text{Bd} U) \setminus h(B \cup B_1)) = (A_2 \cup \text{Bd} U) \setminus (B \cup B_1)$

would separate $B(a, e - 3\tau/4)$ between $c$ and $b$ (see Proposition 5 of [P]), an impossibility, for the segment between $b$ and $f(b)$, as well as the arc $ca_k \cup f(ab)$, lie in $B(a, e - 3\tau/4) \setminus ((A_2 \cup \text{Bd} U) \setminus (B \cup B_1))$.

By Proposition 5 of [P] and by (3) and (4) we have

each of sets $h(A_2 \cup \text{Bd} U) \setminus h(B)$ and $h(A_2 \cup \text{Bd} U) \setminus h(B_1)$

separates the ball $B(a, e - 2\tau/4)$ between $c$ and $b$.

The following statement contradicts the previous one, so it completes the proof of Theorem 1.

One of the sets $h(A_2 \cup \text{Bd} U) \setminus h(B)$ and

$h(A_2 \cup \text{Bd} U) \setminus h(B_1)$ fails to separate the ball $B(a, e - 2\tau/4)$ between $c$ and $b$.

Indeed, by (6) there is an arc $cb$ in $B(a, e - 2\tau/4) \setminus (h(A_2 \cup \text{Bd} U) \setminus h(B \cup B_1))$. By (7) this arc intersects the set $h(B \cup B_1)$. Since $d(B, B_1) > 0$, we have
This implies that going from $c$ to $b$ along the arc $cb$, we may find a point $y \in cb$ such that either $y \in h(B)$ and $cy \cap h(B_1) = \emptyset$ or $y \in h(B)$ and $cy \cap h(B) = \emptyset$, where $cy \subset cb$. If $y \in h(B)$, then let $ya' \subset h(B)$ be an arc (where $a' = h(a_k) = hf(a)$), and put $J(y, a) = ya' \cup h(a_kq)$, where $a_kq \subset f(ab)$. If $y \in h(B_1)$, then $y \in h(B_1)$ for some $x \in qb' \cap Bd U$, where $b' = f(b)$ and $qb' \subset f(ab)$. Let $yx' \subset h(B_x)$ be an arc, where $x' = h(x)$, and put $x'a = h(xq)$, where $xq \subset f(ab)$. Then put $J(y, a) = yx' \cup x'a$. Thus in the former case we get $J(y, a) \cap (h(A_2 \cup Bd U) \setminus h(B)) = \emptyset$, and, in the latter case we have $J(y, a) \cap (h(A_2 \cup Bd U) \setminus h(B_1)) = \emptyset$. But since both considered homeomorphisms are $(\tau/4)$-homeomorphisms, the set $h(A_2 \cup Bd U)$ does not intersect the arc $pa$. Therefore the connected set $cy \cup J(y, a) \cup pa \cup pb \subset B(a, e - 2\tau/4)$ does not intersect either $h(A_2 \cup Bd U) \setminus h(B)$ or $h(A_2 \cup Bd U) \setminus h(B_1)$. Thus we have (8).

The proof of Theorem 1 is complete.

A simple proof of the next fact is left to the reader.

5. Fact. A homogeneous locally connected continuum that topologically contains the cube $I^n$ and contains no $n$-dimensional umbrella, is an $n$-manifold.

Further, we get the following immediate consequence of Theorem 1.

6. Corollary. A proper homogeneous locally connected subcontinuum of a connected $(n+1)$-manifold, that topologically contains the cube $I^n$, is an $n$-manifold.

It was proved in [P] that each homogeneous subcontinuum of $E^{n+1}$, which topologically contains $I^n$ is locally connected. Thus we have the conclusion that forms the title of the paper.

7. Theorem. Each homogeneous continuum that lies in the Euclidean space $E^{n+1}$ and topologically contains the cube $I^n$ is an $n$-manifold.

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