ALGEBRAICALLY INVARIANT EXTENSIONS OF
σ-FINITE MEASURES ON EUCLIDEAN SPACE

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ABSTRACT. Let $G$ be a group of algebraic transformations of $\mathbb{R}^n$, i.e., the
group of functions generated by bijections of $\mathbb{R}^n$ of the form $(f_1, \ldots, f_n)$ where
each $f_j$ is a rational function with coefficients in $\mathbb{R}$ in $n$-variables. For a
function $\gamma: G \to (0, \infty)$ we say that a measure $\mu$ on $\mathbb{R}^n$ is $\gamma$-invariant when
$\mu(g[A]) = \gamma(g) \cdot \mu(A)$ for every $g \in G$ and every $\mu$-measurable set $A$. We
will examine the question: "Does there exist a proper $\gamma$-invariant extension of
$\mu$?" We prove that if $\mu$ is $\sigma$-finite then such an extension exists whenever
$G$ contains an uncountable subset of rational functions $H \subset (\mathbb{R}(X_1, \ldots, X_n))^n$
such that $\mu(\{x: h_1(x) = h_2(x)\}) = 0$ for all $h_1, h_2 \in H, h_1 \neq h_2$. In particular
if $G$ is any uncountable subgroup of affine transformations of $\mathbb{R}^n$, $\gamma(g)$ is the
absolute value of the Jacobian of $g \in G$ and $\mu$ is a $\gamma$-invariant extension of the
$n$-dimensional Lebesgue measure then $\mu$ has a proper $\gamma$-invariant extension.
The conclusion remains true for any $\sigma$-finite measure if $G$ is a transitive group
of isometries of $\mathbb{R}^n$. An easy strengthening of this last corollary gives also an
answer to a problem of Harazisvili.

0. INTRODUCTION: NOTATION AND HISTORY

Our terminology related to algebra, measure theory, set theory and model
theory follows [La, Ru, Je and CK] respectively.

Throughout the paper a measure on a set $X$ will stand for a nontrivial posi-
tive $\sigma$-additive measure, i.e., a function $\mu: \mathcal{M} \to [0, \infty]$ defined on a $\sigma$-algebra
$\mathcal{M}$ of subsets of $X$ containing all singletons such that

(i) $\mu(\bigcup_{i=0}^{\infty} A_i) = \sum_{i=0}^{\infty} \mu(A_i)$ for all pairwise disjoint sets $A_i$ from $\mathcal{M}$,
(ii) $\mu(\{x\}) = 0$ for all $x \in X$,
(iii) $0 < \mu(A) < \infty$ for some $A \in \mathcal{M}$.

If $\mu: \mathcal{M} \to [0, \infty]$ is a measure on $X$ and $A \subset X$ then the inner measure
of $A$ is defined in the standard way: $\mu_*(A) = \sup\{\mu(B): B \subset A & B \in \mathcal{M}\}$.

A measure on $X$ is said to be $\sigma$-finite if $X$ is a countable union of sets of
finite measure. A measure $\mu$ is complete if all subsets of every set of $\mu$
measure zero are $\mu$-measurable.

Received by the editors March 15, 1988 and, in revised form, June 22, 1988.
1980 Mathematics Subject Classification (1985 Revision). Primary 28C10; Secondary 14L35.
Key words and phrases. Invariant $\sigma$-finite measures, algebraic transformations of $\mathbb{R}^n$, isometries
of $\mathbb{R}^n$.

The results of this paper have been presented at MAA and AMS Joint Mathematics Meeting,
Phoenix, Arizona, January 1988 and at the Sixth Annual Auburn Miniconference on Real Analysis,

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0002-9947/90 $1.00 + .25 per page
If $G$ is a group of bijections of a set $X$ then a measure $\mu$ on $X$ is said to be $G$-invariant provided $\mu$ is $\gamma$-invariant where $\gamma(g) = 1$ for all $g \in G$.

For example, if $A_n$ is a group of affine transformations of $\mathbb{R}^n$ then every element of $A_n$ is uniquely represented as a superposition $T \circ L$ where $T$ is a translation and $L$ is a linear transformation of $\mathbb{R}^n$. Let $\gamma:A_n \to (0, \infty)$, where $\gamma(T \circ L)$ is defined as the absolute value of the Jacobian of $L$. Then $m$, the $n$-dimensional Lebesgue measure, is $\gamma$-invariant. Moreover, if $G_n$ is a group of isometries of $\mathbb{R}^n$ then $G_n \subset A_n$ and $m$ is $G_n$-invariant.

We say that a measure $\nu: \mathcal{N} \to [0, \infty]$ on a set $X$ is an extension of a measure $\mu: \mathcal{M} \to [0, \infty]$ defined on the same set $X$ if $\mathcal{M} \subset \mathcal{N}$ and $\nu(A) = \mu(A)$ for every $A \in \mathcal{M}$. Moreover, an extension is proper if $\mathcal{M} \neq \mathcal{N}$.

For a group $G$ of bijections of a set $X$ we say that a set $N \subset X$ is $G$-absolutely negligible if for every $G$-invariant $\sigma$-finite measure $\mu$ on $X$ and for every countable set $\{g_r: r = 0, 1, 2, \ldots \} \subset G$ we have $\mu_*\left(\bigcup_{r=0}^{\infty} g_r[N]\right) = 0$ (or, equivalently, if for every $G$-invariant $\sigma$-finite measure $\mu$ on $X$ there exists a $G$-invariant extension $\nu$ of $\mu$ such that $\nu(N) = 0$; compare Proposition 1.2(b)).

We say that a bijection $g$ of $\mathbb{R}^n$ is an algebraic transformation of $\mathbb{R}^n$ if $g$ is generated by bijections of $\mathbb{R}^n$ from the set $(\mathbb{R}(X_1, \ldots, X_n))_n$. For an algebraic transformation $g$ of $\mathbb{R}^n$ we say that $g$ is defined over the field $L \subset \mathbb{R}$ if $g$ is generated by some bijections of $\mathbb{R}^n$ from $(L(X_1, \ldots, X_n))_n$. For example, the functions

$$f(x, y) = (x^3 + 1, (y + 7)^5), \quad g(x, y) = \left(x, y + \frac{1}{x^2 + 1}\right)$$

and

$$(f^{-1} \circ g)(x, y) = \left((x - 1)^{1/3}, \left(y + \frac{1}{x^2 + 1}\right)^{1/5} - 7\right)$$

are algebraic transformations of $\mathbb{R}^2$ defined over $\mathbb{Q}$. Notice also that isometries and, more generally, nonsingular affine transformations of $\mathbb{R}^n$ are algebraic transformations of $\mathbb{R}^n$ that belong to the set $(\mathbb{R}(X_1, \ldots, X_n))_n$.

Now let $G$ be the group of all isometries of $\mathbb{R}^n$ and let $\mu$ be a $G$-invariant $\sigma$-finite measure on $\mathbb{R}^n$. Can we find a proper $G$-invariant extension of $\mu$?

This question has been discussed several times in the literature. In 1935 Szpilrajn proved that Lebesgue measure on $\mathbb{R}^n$ has a proper isometrically invariant extension (see [Sz]). In the same paper, he stated Sierpinski's question: "Does there exist a maximal isometrically invariant extension of Lebesgue measure on $\mathbb{R}^n$?" A negative answer to this question, i.e., the theorem "every isometrically invariant measure that extends Lebesgue measure on $\mathbb{R}^n$ has a proper isometrically invariant extension," was proved by several mathematicians. The first result of that kind was obtained independently by Phakadze (in 1958, see [Pk]) and Hulanicki (in 1962, see [Hu]) under the additional set-theoretical assumption that there does not exist a real measurable cardinal less
than or equal to continuum $2^\omega$, i.e., that there is no measure on $\mathbb{R}$ defined on all subsets of $\mathbb{R}$. In 1977, Harazisvili got the full result stated above without any set-theoretical assumptions for the one dimensional case, i.e., for $n = 1$ (see [Ha1]). Finally in 1983, Ciesielski and Pelc generalized Harazisvili’s result to all $n$-dimensional Euclidean spaces $\mathbb{R}^n$ (see [CP]; for more historical details of this issue see also [Ci]). In the same paper Ciesielski and Pelc stated the problem of characterizing those groups $G$ of isometries of $\mathbb{R}^n$ for which every $\sigma$-finite $G$-invariant measure has a proper $G$-invariant extension (see [CP, p. 6]). A more technical version of the same problem, i.e., the problem of characterizing those groups $G$ of isometries of $\mathbb{R}^n$ for which $\mathbb{R}^n$ is a union of countable many $G$-absolutely negligible sets, was also stated by Harazisvili in [Ha2].

In the present paper we will consider a generalization of this problem to the case of $\gamma$-invariant measure where $\gamma: G \rightarrow (0, \infty)$ and $G$ is a group of algebraic transformations of $\mathbb{R}^n$. In particular our main theorem (see Abstract, or Theorem 3.1) implies that

“If $G$ is a transitive group of isometries of $\mathbb{R}^n$ then $\mathbb{R}^n$ is a countable union of $G$-absolutely negligible sets.”

The above fact has been proved earlier by Harazisvili under the assumption of the continuum hypothesis (see [Ha2]). He also asked whether it is possible to remove this assumption from his theorem. Our results give an affirmative answer to this question.

The proof of our main theorem 3.1 uses a generalization of the technique of Ciesielski and Pelc [CP, Theorem 2.1, pp. 4–6]. The author wishes to thank Jan Mycielski for numerous important remarks about former versions of this paper. In particular it was Mycielski’s suggestion to replace in the proof of [CP, Theorem 2.1] the linear basis of $\mathbb{R}$ over $\mathbb{Q}$ by a transcendence basis of $\mathbb{R}$ over $\mathbb{Q}$ and to study in this way algebraic transformations of $\mathbb{R}^n$. Compare also the paper of Weglorz [We, Theorem 2.4] which was influenced by Mycielski in a similar way.

The author wishes also to thank Piotr Zakrzewski for calling his attention to the paper of Harazisvili [Ha2] and for other helpful remarks.

1. Measure theoretic preliminaries

In what follows we will need the following proposition essentially due to Szpilrajn (see [Sz, §2]).

**Proposition 1.1.** Let $\gamma: G \rightarrow (0, \infty)$ where $G$ is a group of bijections of a set $X$ and let $\mu: \mathcal{M} \rightarrow [0, \infty]$ be a $\gamma$-invariant measure on $X$. If a family $\mathcal{A}$ of subsets of $X$ is such that

1. $\mathcal{A}$ is closed under countable union,
2. if $A \in \mathcal{A}$ and $g \in G$ then $g[A] \in \mathcal{A}$,
3. every $A \in \mathcal{A}$ has $\mu$ inner measure zero,

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then \( \mu \) has a \( \gamma \)-invariant extension \( \nu: \mathcal{N} \to [0, \infty) \) such that \( \mathcal{A} \subset \mathcal{N} \) and \( \nu(A) = 0 \) for every \( A \in \mathcal{A} \).

The construction of such an extension is very simple. If \( \mathcal{I} \) is an ideal of subsets of \( X \) generated by the family \( \mathcal{A} \), and \( \mathcal{N} \) stands for a \( \sigma \)-algebra generated by \( \mathcal{I} \) then all elements of \( \mathcal{N} \) are of the form \( (M \cup I_1) \backslash I_2 \) where \( M \in \mathcal{M} \) and \( I_1, I_2 \in \mathcal{I} \). It is easy to see that \( \nu: \mathcal{N} \to [0, \infty] \) such that \( \nu((M \cup I_1) \backslash I_2) = \mu(M) \) is a well-defined \( \gamma \)-invariant measure on \( X \) extending \( \mu \).

In the proof of the next proposition, we use a method which goes back to Harazisvili’s paper [Ha1] (see also [CP, Proposition 1.9, p. 4]).

**Proposition 1.2.** Let \( G \) be a group of bijections of \( X \), \( \gamma: G \to (0, \infty) \) and let \( \mu \) be a \( \gamma \)-invariant \( \sigma \)-finite measure on \( X \).

(a) If \( N \subset X \) is such that there is an uncountable set \( H \subset G \) such that
\[
\mu_* (h_1[N] \cap h_2[N]) = 0 \quad \text{for distinct } h_1, h_2 \in H, \quad \text{then } \mu_* (N) = 0.
\]

(b) If \( N \subset X \) is such that for every countable set \( \{ g_r: r = 0, 1, 2, \ldots \} \subset G \) we have \( \mu_* (\bigcup_{r=0}^\infty g_r[N]) = 0 \) then there exists a \( \gamma \)-invariant extension \( \nu \) of \( \mu \) such that \( \nu(N) = 0 \).

(c) Moreover if \( X = \bigcup_{k=0}^\infty N_k \) where each \( N_k \) satisfies the assumption of (b) then \( \mu \) has a proper \( \gamma \)-invariant extension.

**Proof.** (a) If \( \mathcal{M} \subset \mathcal{M} \) is a subset of \( N \) then \( \mu(h_1[M] \cap h_2[M]) = 0 \) for every distinct \( h_1, h_2 \) from \( H \). But \( \mu(h[M]) = \gamma(h) \cdot \mu(M) \) and \( \gamma(h) \neq 0 \) for every \( h \) from \( H \). Hence, \( \sigma \)-finiteness of \( \mu \) implies that \( \mu(M) = 0 \) and so \( \mu_* (N) = 0 \).

(b) By Proposition 1.1 it is enough to notice that every element of the family \( \mathcal{A} = \{ \bigcup_{r=0}^\infty g_r[N]: g_r \in G \text{ for } r = 0, 1, 2, \ldots \} \) has \( \mu \) inner measure 0.

(c) By part (b), for each \( k = 0, 1, 2, \ldots \) there is a \( \gamma \)-invariant extension \( \nu_k \) of \( \mu \) such that \( \nu_k(N_k) = 0 \). But all \( N_k \)’s cannot have \( \mu \) measure zero. So some \( \nu_k \) must be a proper extension of \( \mu \).

In what follows, we will also use the following well-known fact. For the complex case the proof (using the Jensen’s Inequality) can be found in [GR, p. 9]. The direct proof follows also from Fubini’s theorem.

**Proposition 1.3.** If \( f: \mathbb{R}^n \to \mathbb{R} \) is a nonzero real analytic function then the set
\[ Z = \{ a \in \mathbb{R}^n: f(a) = 0 \} \]
has Lebesgue measure zero. In particular, if \( h, g \in (\mathbb{R}(X_1, \ldots, X_n))^n \) are different algebraic transformations of \( \mathbb{R}^n \) then the set
\[ \{ a \in \mathbb{R}^n: h(a) = g(a) \} \]
has Lebesgue measure zero.

2. **Algebraic Preliminaries**

A field \( L \subset \mathbb{R} \) is said to be algebraically closed in \( \mathbb{R} \) if \( L = M \cap \mathbb{R} \) where \( M \subset \mathbb{C} \) is an algebraic closure of \( L \). Notice, that an algebraically closed field in \( \mathbb{R} \) is real closed (i.e. satisfies the theory of real closed fields) in the sense defined in [CK or Ro]. The smallest field algebraically closed in \( \mathbb{R} \) containing \( L \subset \mathbb{R} \) is called a real closure of \( L \) and it will be denoted by \( \text{cl}_\mathbb{R}(L) \). The algebraic closure of a field \( K \) will be denoted by \( \text{cl}(K) \).
The next proposition will be used only in the case of algebraic transformation $g$ such that $g^{-1} \in (\mathbb{R}(X_1, \ldots, X_n))^n$. In this case this is a well-known fact and can be proved using standard algebraic technic. However we like to prove it in more general form (that possibly can be used to answer Problem 3 stated in the end of the paper). For this we will need the following model-theoretic definition (compare e.g. [CK]).

A model $\mathcal{L}$ is said to be an elementary submodel of a model $\mathcal{R}$ if $\mathcal{L} \subseteq \mathcal{R}$ and for every first order formula $\varphi(x_1, \ldots, x_m)$ and any parameters $a_1, \ldots, a_m$ from $\mathcal{L}$ the model $\mathcal{L}$ satisfies $\varphi(a_1, \ldots, a_m)$ if and only if $\mathcal{R}$ satisfies $\varphi(a_1, \ldots, a_m)$.

A theory $T$ is said to be model complete if and only if for all models $\mathcal{L}$ and $\mathcal{R}$ of $T$, if $\mathcal{L} \subseteq \mathcal{R}$ then $\mathcal{L}$ is an elementary submodel of $\mathcal{R}$.

We need the following important theorem of A. Robinson (see [CK, p. 110] or [Ro, §3.3]).

**Theorem 2.1.** The theory $T$ of real closed fields is model complete. In particular if $L \subseteq \mathbb{R}$ is a real closed field then $L$ is an elementary submodel of $\mathbb{R}$.

As a corollary of this fact we easily obtain

**Proposition 2.1.** If $g$ is an algebraic transformation of $\mathbb{R}^n$ defined over a real closed field $L \subseteq \mathbb{R}$ then

\[(2.1) \quad g[L^n] = L^n.\]

**Proof.** A first order formula $\varphi(x_1, \ldots, x_m, y_1, \ldots, y_n)$ defined by $g(x_1, \ldots, x_m) = (y_1, \ldots, y_n)$ has as its parameters only elements from $L$. If $a = (a_1, \ldots, a_n) \in L^n$ then $\mathbb{R}$ satisfies $\exists y_1 \cdots \exists y_n \varphi(a_1, \ldots, a_n, y_1, \ldots, y_n)$ and so does $L$ (by Theorem 2.1), i.e. $g(a_1, \ldots, a_n) \in L^n$. This proves $g[L^n] \subseteq L^n$. To show the converse inclusion it is enough to consider the formula $\exists x_1 \cdots \exists x_n \varphi(x_1, \ldots, x_n, a_1, \ldots, a_n)$.

### 3. The main theorem

From now on let $\mathcal{B}$ denote a transcendence base of $\mathbb{R}$ over $\mathbb{Q}$.

Now we are ready to prove our main lemma.

**Lemma 3.1.** Let $H \subseteq (\mathbb{R}(X_1, \ldots, X_n))^n$ be an uncountable set of algebraic transformations of $\mathbb{R}^n$. Then there exists an uncountable set $H' \subseteq H$, a finite set $A \subseteq \mathcal{B}$ and, for every $h \in H'$, a finite set $A_h \subseteq \mathcal{B} \setminus A$ with the following properties:

1. each $h \in H'$ (and so $h^{-1}$) is defined over the field $\text{cl}_{\mathbb{R}}(\mathbb{Q}(A \cup A_h))$;
2. $A_{h_1} \cap A_{h_2} = \emptyset$ for distinct $h_1, h_2 \in H'$;
3. for every $h_1, h_2 \in H'$ if $L = \text{cl}_{\mathbb{R}}(\mathbb{Q}(\mathcal{B} \setminus (A_{h_1} \cup A_{h_2})))$ then $a \in h_1^{-1}[L^n] \cap h_2^{-1}[L^n]$ implies $h_1(a) = h_2(a)$, i.e., $h_1^{-1}[L^n] \cap h_2^{-1}[L^n] \subseteq \{a : h_1(a) = h_2(a)\}$. 

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Proof. In the definition of each \( h \in H \) we use only finitely many parameters (i.e. coefficients) so for every \( h \in H \) there exists a finite set \( B_h \subset \mathcal{B} \) such that

\[
h = (h_1, \ldots, h_n) \in [\mathcal{cl}_R(\mathbb{Q}(B_h))(X_1, \ldots, X_n)]^n.
\]

Using for the family \( \{B_h; h \in H\} \) the \( \Delta \)-system argument (see e.g. [Je, Lemma 22.6, p. 226]) we can find an uncountable set \( H_0 \subset H \), a finite set \( A \subset \mathcal{B} \), a natural number \( m \) and, for every \( h \in H_0 \), a set \( A_h \) such that

(i) \( B_h = A \cup A_h \), and \( A \cap A_h = \emptyset \),

(ii) \( A_h \cap A_h' = \emptyset \) for distinct \( h_1, h_2 \in H_0 \),

(iii) \( A_h \) has exactly \( m \) elements.

Thus for the family \( H_0 \), the sets \( A, A_h \ (h \in H_0) \) already satisfy (1) and (2). Therefore it is enough to find an uncountable \( H' \subset H_0 \) which satisfies (3). We will do this in such a way that all elements of \( H' \) will have the same definitions with parameters from \( \mathcal{B} \).

Let \( Z = \{Z_1, \ldots, Z_m\} \) be a set of variables and, for \( h \in H_0 \), let \( \sigma'_h: A_h \to Z \) be a bijection. Then we can extend \( \sigma'_h \) to a field isomorphism \( \sigma''_h \) from \( \mathcal{cl}(\mathbb{Q}(\mathcal{B})) = \mathbb{C} \) to \( \mathcal{cl}(\mathbb{Q}(\mathcal{B}\setminus A_h)(Z)) \) in such a way that \( \sigma''_h(a) = a \) for every \( a \in \mathcal{cl}(\mathbb{Q}(\mathcal{B}\setminus A_h)) \). Let us extend \( \sigma''_h \) to \( \sigma_h: [\mathcal{cl}(\mathbb{Q}(\mathcal{B}))(X_1, \ldots, X_n)]^n \to [\mathcal{cl}(\mathbb{Q}(\mathcal{B}\setminus A_h)(Z))(X_1, \ldots, X_n)]^n \). But \( \sigma_h(h) \in [\mathcal{cl}(\mathbb{Q}(A \cup Z))(X_1, \ldots, X_n)]^n \) and the field \( \mathcal{cl}(\mathbb{Q}(A \cup Z)) \) is countable.

Define \( H' \subset H_0 \) as an uncountable set with the property

\[
(*) \quad \sigma_{h_1}(h_1) = \sigma_{h_2}(h_2) \quad \text{for every } h_1, h_2 \in H'.
\]

We prove that \( H' \) satisfies (3).

Let \( a \in h_1^{-1}[L^n] \cap h_2^{-1}[L^n] \), where \( L = \mathcal{cl}_R(\mathbb{Q}(\mathcal{B}\setminus (A_{h_1} \cup A_{h_2}))) \) and \( h_1, h_2 \in H' \). Notice that \( a \in L^n \) as, by Proposition 2.1, (1) and (2),

\[
a \in h_1^{-1}[L^n] \cap h_2^{-1}[L^n] \subset h_1^{-1}[\mathcal{cl}_R(\mathbb{Q}(\mathcal{B}\setminus A_{h_2})))^n] \cap h_2^{-1}[\mathcal{cl}_R(\mathbb{Q}(\mathcal{B}\setminus A_{h_1})))^n]
\]

\[
= \mathcal{cl}_R(\mathbb{Q}(\mathcal{B}\setminus A_{h_2})))^n \cap \mathcal{cl}_R(\mathbb{Q}(\mathcal{B}\setminus A_{h_1})))^n = L^n.
\]

Put \( h_1(a) = b_1 \) and \( h_2(a) = b_2 \). Thus \( b_1, b_2 \in L^n \). We have to prove that \( b_1 = b_2 \). But, by \((*)\) and the fact that \( \sigma_{h_1}(c) = c = \sigma_{h_2}(c) \) for every \( c \in L^n \),

\[
b_1 = \sigma_h(b_1) = \sigma_h(h_1(a)) = \sigma_{h_1}(h_1(a)) = \sigma_{h_1}(h_1(a)) = \sigma_{h_2}(h_2(a)) = \sigma_{h_2}(h_2(a)) = \sigma_{h_2}(b_2) = b_2.
\]

This finishes the proof of Lemma 3.1.

As a next step we will prove an essential part of the assumptions of Proposition 1.2.

Lemma 3.2. If \( G \) is a group of algebraic transformations of \( \mathbb{R}^n \) and \( H \subset (\mathbb{R}(X_1, \ldots, X_n))^n \) is an uncountable subset of \( G \) then there exists a countable
family of sets \{N_k: k = 0, 1, 2, \ldots\} such that \( \mathbb{R}^n = \bigcup_{k=0}^{\infty} N_k \) and that each \( N_k \) satisfies the condition:

for every countable set \( \{g_r: r = 0, 1, 2, \ldots\} \subset G \) there is an uncountable set \( H_0 \subset H \) such that for every distinct \( h_1, h_2 \in H_0 \)

\[
(3.1) \quad h_1^{-1} \left[ \bigcup_{r=0}^{\infty} g_r[N_k] \right] \cap h_2^{-1} \left[ \bigcup_{r=0}^{\infty} g_r[N_k] \right] \subset \{a \in \mathbb{R}^n: h_1(a) = h_2(a)\}.
\]

Proof. Let \( \mathscr{B} \) be a transcendence base of \( \mathbb{R} \) over \( Q \) and let \( H' \subset H \), \( A \) and \( A_h \) be as in Lemma 3.1. We choose an increasing sequence \( \mathscr{B}_0 \subset \mathscr{B}_1 \subset \mathscr{B}_2 \subset \cdots \)

of subsets of \( \mathscr{B} \) such that \( \mathscr{B} = \bigcup_{k=1}^{\infty} \mathscr{B}_k \) and for every \( k \) the set

\[
(*) \quad H^k = \{ h \in H': A_h \subset \mathscr{B}_{k+1} \setminus \mathscr{B}_k \}
\]

is uncountable.

Define \( N_k = [\text{cl}_R(Q(\mathscr{B}_k))]^n \). Then \( \bigcup_{k=0}^{\infty} N_k = \mathbb{R}^n \).

Let us fix \( \{g_r: r = 0, 1, 2, \ldots\} \subset G \) and a natural number \( k \). Choose also a countable set \( \mathscr{A} \subset \mathscr{B} \) such that \( A \subset \mathscr{A} \) and every \( g_r \) is defined over \( \text{cl}_R(Q(\mathscr{A})) \). Let \( H_0 = \{ h \in H^{k+1}: A_h \cap \mathscr{A} = \emptyset \} \).

By (*) the set \( H_0 \) is uncountable.

Let us fix arbitrary distinct \( h_1, h_2 \in H_0 \) and let \( L = \text{cl}_R(Q(\mathscr{B} \setminus (A_{h_1} \cup A_{h_2}))) \).

Then, by (*) and definitions of \( H_0 \) and \( N_k \), we can conclude that \( N_k \subset L^n \) and the \( g_r \)'s are defined over \( L \). Hence, by Proposition 2.1,

\[
h_1^{-1} \left[ \bigcup_{r=0}^{\infty} g_r[N_k] \right] \cap h_2^{-1} \left[ \bigcup_{r=0}^{\infty} g_r[N_k] \right] \subset h_1^{-1} \left[ \bigcup_{r=0}^{\infty} g_r[L^n] \right] \cap h_2^{-1} \left[ \bigcup_{r=0}^{\infty} g_r[L^n] \right] = h_1^{-1} [L^n] \cap h_2^{-1} [L^n]
\]

and, by (3) of Lemma 3.1, \( h_1^{-1}[L^n] \cap h_2^{-1}[L^n] \subset \{a: h_1(a) = h_2(a)\} \).

Therefore

\[
h_1^{-1} \left[ \bigcup_{r=0}^{\infty} g_r[N_k] \right] \cap h_2^{-1} \left[ \bigcup_{r=0}^{\infty} g_r[N_k] \right] \subset h_1^{-1}[L^n] \cap h_2^{-1}[L^n] \subset \{a: h_1(a) = h_2(a)\}.
\]

This finishes the proof of Lemma 3.2.

Theorem 3.1. Let \( G \) be a group of algebraic transformations of \( \mathbb{R}^n \), \( \gamma: G \to (0, \infty) \) and let \( \mu \) be a \( \gamma \)-invariant \( \sigma \)-finite measure on \( \mathbb{R}^n \). If \( G \) has an uncountable subset \( H \subset (\mathbb{R}(X_1, \ldots, X_n))^n \) with the property

\[
(3.2) \quad \mu_\gamma(\{a: h_1(a) = h_2(a)\}) = 0 \quad \text{for every } h_1, h_2 \in H, h_1 \neq h_2
\]

then \( \mu \) has a proper \( \gamma \)-invariant extension.

Proof. By (3.2) and Lemma 3.2 we have \( \mathbb{R}^n = \bigcup_{k=0}^{\infty} N_k \) where, by Proposition 1.2(a),

\[
\mu_\gamma(\bigcup_{r=0}^{\infty} g_r[N_k]) = 0 \quad \text{for every countable set } \{g_r: r = 0, 1, 2, \ldots\} \subset G
\]
and every $k = 0, 1, 2, \ldots$. Hence, by Proposition 1.2(c), $\mu$ has a proper $\gamma$-invariant extension.

**Corollary 3.1.** Let $G$ be a group of algebraic transformations of $\mathbb{R}^n$, $\gamma: G \to (0, \infty)$ and let $\mu$ be a $\gamma$-invariant $\sigma$-finite measure on $\mathbb{R}^n$. If at least one of the following conditions holds

1. $G$ contains uncountably many translations;
2. $\mu$ extends the $n$-dimensional Lebesgue measure and the set $G \cap (\mathbb{R}(X_1, \ldots, X_n))^n$ is uncountable;

then $\mu$ has a proper $\gamma$-invariant extension.

**Proof.** It is enough to show that both (C1) and (C2) imply (3.2).

If (C1) holds and $H$ is an uncountable set of translations then for every $h_1, h_2 \in H$, $h_1 \neq h_2$ the set $\{a: h_1(a) = h_2(a)\}$ is empty, so (3.2) is satisfied.

If (C2) holds then (3.2) is implied by Proposition 1.3.

To solve Harazisvili’s problem we will need the following lemma due to Harazisvili (see [Ha2, Remark 2, p. 507]).

**Lemma 3.3.** Let $G$ be a transitive group of isometries of $\mathbb{R}^n$, i.e., such that for every $a, b \in \mathbb{R}^n$ there exists $g \in G$ with the property $g(a) = b$. If $A \subset \mathbb{R}^n$ is a countable union of proper affine hyperplanes of $\mathbb{R}^n$ than $A$ is $G$-absolutely negligible.

**Proof.** For $k \leq n$ let $\mathcal{A}_k$ denote the family of countable unions of affine hyperplanes of $\mathbb{R}^n$ of dimension less than $k$. We prove by induction on $k \leq n$ that elements of $\mathcal{A}_k$ are $G$-absolutely negligible.

So let $k < n$ be such that the elements of $\mathcal{A}_k$ are $G$-absolutely negligible.

Let us fix an arbitrary $A \in \mathcal{A}_{k+1}$, a $G$-invariant $\sigma$-finite measure $\mu$ on $\mathbb{R}^n$ and a countable set \{\(g_r : r = 0, 1, 2, \ldots\)\} $\subset G$. By Proposition 1.2(a) it is enough to find a sequence \(\{h_\zeta: \zeta < \omega_1\}\) $\subset G$ such that for every $\zeta < \eta < \omega_1$

\[
\mu \left( h_\zeta \left[ \bigcup_{r=0}^\infty g_r[A] \right] \cap h_\eta \left[ \bigcup_{r=0}^\infty g_r[A] \right] \right) = 0.
\]

We will construct it by transfinite induction.

So let us assume that for some $\xi < \omega_1$ we have already constructed \(\{h_\zeta: \zeta < \xi\}\) $\subset G$ such that the condition (a) is satisfied for every $\zeta < \eta < \xi$. Let $A_i$ and $H_j$ ($i, j = 0, 1, 2, \ldots$) be affine hyperplanes of $\mathbb{R}^n$ of dimensions less than or equal to $k$ and such that

\[
\bigcup_{r=0}^\infty g_r[A] = \bigcup_{i=0}^\infty A_i \quad \text{and} \quad \bigcup_{\zeta < \xi} h_\zeta \left[ \bigcup_{r=0}^\infty g_r[A] \right] = \bigcup_{j=0}^\infty H_j.
\]

We have to find $h_\xi$ such that

\[
\mu \left( h_\xi \left[ \bigcup_{i=0}^\infty A_i \right] \cap \bigcup_{j=0}^\infty H_j \right) = 0.
\]
But if \( h_\xi[A_i] \neq H_j \), then \( h_\xi[A_i] \cap H_j \in \mathcal{A}_k \), i.e., by inductive hypothesis, it is enough to construct \( h_\xi \in G \) such that

\[
(\text{b}) \quad h_\xi[A_i] \neq H_j \quad \text{for every} \ i, j = 0, 1, 2, \ldots.
\]

Let \( w \in \mathbb{R}^n \) represents a vector in \( \mathbb{R}^n \) such that \( w \) is not parallel to any \( H_j \) \((j = 0, 1, 2, \ldots)\). Then for different reals \( a, b \) the distances

\[
\text{dist}(0, a \cdot w + H_j) \neq \text{dist}(0, b \cdot w + H_j) \quad \text{for every} \ j = 0, 1, 2, \ldots.
\]

So we can choose \( b \in \mathbb{R} \) such that

\[
(\text{c}) \quad \text{dist}(0, -b \cdot w + H_j) \neq \text{dist}(0, A_i) \quad \text{for every} \ i, j = 0, 1, 2, \ldots.
\]

Now let \( h_\xi \in G \) be such that \( h_\xi(0) = b \cdot w \). We prove that such \( h_\xi \) satisfies \( \text{(b)} \).

By way of contradiction let us assume that for some \( i \) and \( j \)

\[
(\text{d}) \quad h_\xi[A_i] = H_j.
\]

But \( h_\xi = T \circ L \), where \( L \) is an isometry of \( \mathbb{R}^n \) preserving origin and \( T \) is a translation such that \( T(x) = x + b \cdot w \) for every \( x \in \mathbb{R}^n \). Hence, by \( \text{(d)} \), \( L[A_i] = T^{-1}[H_j] = -b \cdot w + H_j \) and so

\[
\text{dist}(0, -b \cdot w + H_j) = \text{dist}(0, L[A_i]) = \text{dist}(0, A_i)
\]

contradicting \( \text{(c)} \).

Thus we proved that \( h_\xi \) satisfies \( \text{(b)} \). This finishes the proof of the lemma.

**Theorem 3.2.** If \( G \) is a transitive group of isometries of \( \mathbb{R}^n \) then \( \mathbb{R}^n \) is a countable union of \( G \)-absolutely negligible sets. In particular every \( \sigma \)-finite \( G \)-invariant measure on \( \mathbb{R}^n \) has a proper \( G \)-invariant extension.

**Proof.** Let \( \{N_k: k = 0, 1, 2, \ldots\} \) be the family given in Lemma 3.2 where \( H = G \). Then by Lemma 3.3 and Proposition 1.2(a) we have \( \mu_*(\bigcup_{r=0}^\infty g_r[N]) = 0 \) for every countable set \( \{g_r: r = 0, 1, 2, \ldots\} \subset G \) and every \( k = 0, 1, 2, \ldots \). Hence each \( N_k \) is \( G \)-absolutely negligible.

**Generalizations, examples and problems**

1. Let us remark first that although we have stated Theorem 3.1 only for measures on \( \mathbb{R}^n \) the theorem can be generalized for measures on \( K^n \) where \( K \) is either a real closed or algebraically closed field, since the theory of algebraic closed fields is also model complete (see [CK, p. 110]). Moreover, in the case of algebraically closed fields, the assumptions that \( H \subset (K(X_1, \ldots, X_n))^n \) may be dropped.

2. If \( X \subset K^n \) where \( K \) is as above and we define algebraic transformations on \( X \) in natural way, i.e., by functions generated by bijections of \( X \) from \((K(X_1, \ldots, X_n))^n\), then we can prove Theorem 3.1 for measures on \( X \). In particular we can conclude that it does not exist a maximal isometrically invariant extension of Lebesgue measure on \( n \)-dimensional sphere \( S^n \).
3. Theorem 3.1 and its generalizations as in 1 and 2 can be also proved for complex measures (see [Ru, Chapter 6]).

4. For the cardinal number \( \kappa \) we say that a measure \( \mu \) on a set \( X \) is \( \kappa \)-finite if \( X \) is a union of \( \kappa \) many sets of finite measure. Theorem 3.1 can be also generalized in the following way:

"Let \( \kappa \) be a cardinal number, \( G \) be a group of algebraic transformations of \( \mathbb{R}^n \), \( \gamma: G \to (0, \infty) \) and let \( \mu \) be a \( \gamma \)-invariant \( \kappa \)-finite measure on \( \mathbb{R}^n \). If \( G \) has a subset \( H \subset (\mathbb{R}(X_1, \ldots, X_n))^n \) of power greater than \( \kappa \) with the property

\[
\{a: h_1(a) = h_2(a)\} = \emptyset \quad \text{for every} \quad h_1, h_2 \in H, \quad h_1 \neq h_2,
\]

then \( \mu \) has a proper \( \gamma \)-invariant extension."

5. In 4 condition (*) can be replaced by the original condition (3.2) if we assume in addition that the measure \( \mu \) is \( \kappa^+ \)-additive.

6. We can also generalize the results from 4 and 5 in the way described in 1 and 2.

7. By 4, if in particular \( \kappa \) is less than continuum \( 2^\omega \), \( G \) is a group of all isometries of \( \mathbb{R}^n \) and \( \mu \) is a \( \kappa \)-finite \( G \)-invariant measure then there exists a proper \( G \)-invariant extension of \( \mu \). However for \( \kappa \) equal to continuum \( 2^\omega \) this cannot be proved as it was shown in [CP, Theorem 3.1].

8. An interesting example, suggested to the author by Jan Mycielski, can be obtained by considering a hyperbolic space \( \mathbb{H}^n \) for \( n \geq 2 \). If we identify \( \mathbb{H}^n \) with the model \( \{(a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}: a_{n+1} > 0\} \) then the group \( G \) of all isometries of \( \mathbb{H}^n \) is a group of algebraic transformations of \( \mathbb{R}^n \) and contains uncountably many translations. Moreover \( G \) is not a subgroup of a group of affine transformations of \( \mathbb{R}^n \) (see [MW or Be]). Let \( \nu \) be the hyperbolic invariant measure on \( \mathbb{H}^n \) induced by the Haar measure on \( G \). So \( \nu \) is a \( G \)-invariant \( \sigma \)-finite measure on \( \mathbb{H}^n \). Using the previous remarks and Corollary 3.1 we may conclude that the measure \( \nu \) does not have a maximal \( G \)-invariant extension.

9. Now we discuss the assumptions of Theorem 3.1, in particular condition (3.2).

First we prove that uncountability of \( H \subset G \) is essential (compare [Pe, Proposition 2.3, p. 14]).

Let \( G_0 \) be a group of all translations of \( \mathbb{R}^1 \) by rational numbers and let \( V \) be a Vitali set, i.e., \( V \cap H \) is a one element set for each orbit \( H \) of \( G_0 \). If we assume that there is a real measurable cardinal less than or equal to continuum (see [Je]) then there is a measure \( \nu_0: \mathcal{P}(V) \to [0, 1] \), where \( \mathcal{P}(V) \) is a family of all subsets of the set \( V \). Define a measure \( \mu: \mathcal{P}(\mathbb{R}^1) \to [0, \infty] \) by

\[
(4.1) \quad \mu(A) = \sum_{g \in G_0} \nu_0(g^{-1}[g[V] \cap A]).
\]

It is easy to see that \( \mu \) is \( G_0 \)-invariant and \( \sigma \)-finite. But \( \mu \) is defined on all subsets of \( \mathbb{R}^1 \) so it cannot have any proper extension.
10. It can be also proved that if there is a real measurable cardinal less than or equal to the continuum then for every countable group $G$ of bijections of $\mathbb{R}^1$ there exists a $G$-invariant measure defined on $\mathcal{P}(\mathbb{R}^1)$, however this needs a little more careful definition.

11. The group $G_0$ defined in 9 is related to an interesting open problem of Andrzej Pelc (see [Pe, p. 27]).

**Problem 1.** Let $\mu$ be a $G_0$-invariant extension of Lebesgue measure on $\mathbb{R}^1$. Does there exist a proper $G_0$-invariant extension of $\mu$?

12. The next example shows that we have to assume about $G$ something more than only uncountability.

**Example.** Let $G'$ be the group of all rotations of $\mathbb{R}^2$ about the origin and let $\nu: \mathcal{P}(\mathbb{R}^2) \to [0, \infty]$ be such that $\nu(A) = 1$ when $(0,0) \in A$ and $\nu(A) = 0$ otherwise. $\nu$ does not vanish at points, but still it is a $G'$-invariant measure. To correct this let $\mu$ and $G_0$ be as in Example 2 and let $\mu_1: \mathcal{P}(\mathbb{R}^3) \to [0, \infty]$ be a product measure of $\nu$ and $\mu$, i.e., $\mu_1(A) = \mu(\{x: (0,0,x) \in A\})$. Then $\mu$ is $\sigma$-finite and $G_1$-invariant, where the group $G_1 = \{(g', g''): g' \in G', g'' \in G\}$ is uncountable. It is also obvious that $\mu_1$ does not have any proper extension.

13. The reason that this example works is that $\mu_1$ is concentrated on a set $S = \{0\} \times \{0\} \times \mathbb{R}$ while $g[S] = S$ for every $g \in G_1$ and the group $\{g|_S: g \in G_1\}$ is countable. This suggests the following

**Definition.** Let $G$ be a group of bijections of a set $X$ and $\mu$ be a $G$-invariant measure on $X$. We say that $G$ is $\mu$-essentially countable if there is a set $S \subset X$ such that $\mu(X \setminus S) = 0$, $g[S] = S$ for all $g \in G$ and the group $\{g|_S: g \in G\}$ is countable.

**Problem 2.** Let $G$ be a group of algebraic transformations of $\mathbb{R}^n$ and $\mu$ be a $G$-invariant $\sigma$-finite measure of $\mathbb{R}^n$ such that $G$ is not $\mu$-essentially countable. Does $\mu$ have a proper $G$-invariant extension?

Recently the author has been informed that Piotr Zakrzewski proved the following result connected with the Problem 2: "If $G$ is a group of isometries of $\mathbb{R}^n$ and $\mu: \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ is $G$-invariant then the group $G$ is $\mu$-essentially countable."

14. In the next example we will construct a $\gamma$-invariant measure $\mu$ on $\mathbb{R}^1$ where $\gamma$ will not be given in a classical way by Jacobian.

**Example.** Let $G_0 = \{x^{3^n}: n \in \mathbb{Z}\}$ be a group of transformations of $\mathbb{R}^1$ and let $V \subset \mathbb{R}^1 \setminus \{0\}$ be such that $(V \cup \{0\}) \cap H$ contains exactly one element for every orbit $H$ of $G$. Let $\mu_0: \mathcal{P}(V) \to [0, 1]$ be a measure. For $n \in \mathbb{Z}$ let $g_n(x) = x^{3^n}$ and let $\mu_n: \mathcal{P}(g_n[V]) \to [0, 2^n]$ be defined by $\mu_n(g_n[A]) = 2^n \cdot \mu_0(A)$. Define
\[ \mu : \mathcal{P}(\mathbb{R}^1) \to [0, \infty) \] by
\[ \mu(A) = \sum_{n \in \mathbb{Z}} \mu_n([A_n]) = \sum_{n \in \mathbb{Z}} 2^n \cdot \mu_0(A_n) \]
where \( A_n \subset V \) are such that \( A \setminus \{0\} = \bigcup_{n \in \mathbb{Z}} g_n[A_n] \).

It is easy to see that \( \mu \) is a \( \sigma \)-finite measure. Moreover,
\[ \mu(g_m[A]) = \mu \left( \bigcup_{n \in \mathbb{Z}} (g_m \circ g_n)[A_n] \right) = \sum_{n \in \mathbb{Z}} 2^{m+n} \cdot \mu_0(A_n) = 2^m \cdot \mu(A), \]
i.e., \( \mu \) is \( \gamma_0 \)-invariant where \( \gamma_0 : G_0 \to (0, \infty) \) is defined by \( \gamma_0(g_n) = 2^n \). It is easy to see that \( \gamma_0 \) has little to do with a classical Jacobian.

Our group \( G_0 \) is countable. But if we consider a measure \( \nu \) being a product measure of \( \mu \) and a one-dimensional Lebesgue measure \( m \) then \( \nu \) is a \( \sigma \)-finite \( \gamma \)-invariant where \( \gamma : G \to (0, \infty), G = \{(g_n, i) : g_n \in G_0 \text{ and } i \text{ is an isometry of } \mathbb{R}^1\}, \) and \( \gamma(g_n, i) = 2^n \). It is also obvious that \( G \) is uncountable. Moreover about \( \nu \) we can prove that if \( f \) is a homeomorphism of \( \mathbb{R}^2 \) and the system \( \langle \mathbb{R}^2, \mu_f, G_f, \gamma_f \rangle \) is induced by \( f \) from the system \( \langle \mathbb{R}^2, \mu, G, \gamma \rangle \) then \( G \) is not a subgroup of affine transformations of \( \mathbb{R}^2 \).

15. **Problem 3.** Is the assumption \( H \subset (\mathbb{R}(X_1, \ldots, X_n))^n \) essential in Theorem 3.1?

**References**


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