ADDENDUM TO THE PAPER
"EXISTENCE OF WEAK SOLUTIONS
FOR THE NAVIER-STOKES EQUATIONS
WITH INITIAL DATA IN $L^p$"

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ABSTRACT. This paper considers the existence of global weak solutions for the Navier-Stokes equations in the infinite cylinder $\mathbb{R}^n \times \mathbb{R}_+$ with initial data in $L^r$, $n \geq 3$, $1 < r < \infty$. An imbedding theorem as well as related initial value problems are also studied, thus completing results in [2].

INTRODUCTION

This paper considers the initial value problem for the Navier-Stokes equations in the infinite cylinder $S_T = \mathbb{R}^n \times [0, T)$. Given $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, satisfying in the distributions sense $\text{div} \ f = 0$, $x \in \mathbb{R}^n$, we seek a solution vector $u(x, t) = (u_1(x, t), \ldots, u_n(x, t))$ and a pressure function $P(x, t)$ such that

\begin{equation}
D_t u_i - \sum_{j=1}^n D_{ij} u_i + \sum_{j=1}^n u_j D_j u_i + D_j P = 0, \quad (x, t) \in S_T,
\end{equation}

\begin{equation}
\sum_j D_j u_j = 0, \quad (x, t) \in S_T,
\end{equation}

\begin{equation}
\quad u(x, 0) = f(x).
\end{equation}

Here, $D_j$ and $D_t$ denote respectively, the distributional derivatives with respect to $x_j$ and $t$, $D_{ij}$ denotes the second order derivative with respect to $x_i, x_j$; likewise, $L(u)$ will denote the heat operator applied to $u$, and $\text{grad} u$, the square matrix $D_j u_i$. The first equation of (0.1) takes the form

\begin{equation}
L(u) + (\text{grad} u)(u) + \text{grad} P = 0.
\end{equation}

Following [4], I consider the functional spaces $L^p,q(S_T)$ consisting of the Lebesgue-measurable functions $u$, such that

\begin{equation}
\|u\|_{p,q}(T) < \infty.
\end{equation}
Let us define $U_p(t)$ as

$$
(0.4) \quad \left| \int_{\mathbb{R}^n} |u|^p \, dx \right|^{1/p},
$$

Then, the mixed norm $(0, 3)$ can be written as

$$
(0.5) \quad \|U_p\|_{q}(T) \quad \text{(usual $L^q$-norm over the interval $(0, T)$)}.
$$

The norms associated with the maximal operator $u^* = \sup_{t > 0} |u|$ (the supremum is taken over $t > 0$) are

$$
(0.6) \quad \|u^*\|_{p}(T) = \left( \int_0^T \left( \sup_{0 \leq t \leq T} |u|^p \right) \, dx \right)^{1/p}.
$$

In the particular case when we take $T = \infty$, we have

$$
(0.7) \quad \|u^*\|_{p}(\infty) = \left( \int_0^\infty \left( \sup_{t > 0} |u|^p \right) \, dx \right)^{1/p}.
$$

In the same fashion we introduce the norms $\|u\|_{p,q}^*$ as

$$
(0.8) \quad \|u\|_{p,q}^* = \left( \int_{\mathbb{R}^n} \left( \int_0^\infty |u(x,t)|^p \, dt \right)^{q/p} \, dx \right)^{1/q}.
$$

The aim of this paper is to complete the results in [2] concerning solutions of $(0, 1)$ for initial data in $L^n(\mathbb{R}^n)$. Likewise, in §IV below, a relation is established between the $L^{p,q}$ classes of existence and uniqueness introduced by Fabes, Jones, Riviere in [4] and the classes of solutions with initial data in $L^r(\mathbb{R})$.

### I. Main results

**Weak solutions** [4]. A function $u(x,t)$ is said to be a weak solution of the Navier-Stokes equations, with initial values $f$, $\text{div } f = 0$ (in the distributions sense), if for any $C^\infty$ rapidly decreasing vector function $v(x,t) = (v_1(x,t), \ldots, v_n(x,t))$, defined on $\mathbb{R}^{n+1}(x,t)$, such that $\text{div } v = 0$, $v(x,t) = 0$, $t > T$, we have

- (a) $u \in L^{p,q}(S_T)$ with $p, q \geq 2$,
- (b) $\int_0^T \int_{\mathbb{R}^n} (u, \text{L}^*(v) + (\text{grad } v)(u)) \, dx \, dt = -\int_{\mathbb{R}^n} (f(x), v(x,0)) \, dx$ where $\text{L}^*$ is the adjoint heat operator,
- (c) $\text{div } u(x,t) = 0$ (in the distributions sense) for a.e. $t$, such that $0 < t < T$.

**Theorem A.** Let $n$ be greater than or equal to 3 and $f(x)$, the initial data, a vector function, such that $f(x) \in L^r(\mathbb{R}^n)$, $1 < r < \infty$, and $\text{div } f = 0$ in the distributions sense. Let $F(x,t) = W \ast f$, where $W$ is the fundamental solution of the heat equation (see II below); the convolution is taken in the spatial coordinates. Then

$$
(0.1) \quad \|F\|_{p,q} < C_{p,q}\|f\|_r, \quad \frac{n}{r} = \frac{n}{p} + \frac{2}{q}, \quad q \geq p,
$$

where $C_{p,q}$ is a constant depending on $p, q$.
(ii) \( \|F\|_{p,q}^* < C_{p,q} \|f\|_r, \quad \frac{n}{r} = \frac{2}{p} + \frac{n}{q}, \quad p \geq q \),

(iii) If for some \( p, q \), \( 1 = \frac{n}{p} + \frac{2}{q}, \quad n < p \), we have

\[ \|F\|_{p,q} < \varepsilon_{p,q}(n), \]

where \( \varepsilon_{p,q}(n) \) is a fixed small quantity depending on \( p, q \) and \( n \) only. Then, there exists a unique solution \( u \) to the problem \((0, 1)\), that is global and satisfies the equations \((0, 1)\) and the initial data in the weak sense. The uniqueness holds in the class of functions \( u \) such that

\[ \|u\|_{p,q} < \infty. \]

Likewise, if for some \( (p, q) \); \( 1 = \frac{n}{q} + \frac{2}{p}, \quad n < q \), we have

\[ \|F\|_{p,q}^* < \varepsilon_{p,q}(n) \]

where \( \varepsilon_{p,q}(n) \) is a small quantity depending on \( p, q \) and \( n \) only; then, there exists a weak solution \( u \) of \((0, 1)\) for the initial data \( f \), that satisfies \((0, 1)\) for all time \( t > 0 \). The solution \( u \) is unique in the class of functions that satisfy \( \|u\|_{p,q}^* < \infty \).

**Corollary.** Let the initial data \( f \) belong to \( L^n(R^n) \). If for some \( L^{p,q} \)-norm we have

\[ \|F\|_{p,q} < \varepsilon_{p,q}(n), \quad 1 = \frac{n}{p} + \frac{2}{q}, \quad p < q. \]

Then, there exists a weak, global solution for the problem \((0, 1)\). The solution is unique in the corresponding class of functions \( L^{p,q}(S_\infty) \).

II. FUNDAMENTAL SOLUTIONS, THE BILINEAR FORM, MIXED POINTS

Fabes-Jones-Riviere, [4], extended to dimension \( n \) a formula found for the case \( n = 3 \) by Oseen [10]. The Oseen-Fabes-Jones-Riviere formula gives a divergence free matrix fundamental solution \( E_{ij}(x,t) \) for an \( n \)-dimensional heat equation. The matrix \( E_{ij}(x,t) \) is defined in the following way:

\[ E(x,t) = \delta_{ij} W(x,t) - R_i R_j W(x,t), \]

where \( W(x,t) = (4\pi t)^{-n/2} \exp -|x|^2/(4t) \), and \( R_j \) is the \( j \)th Riesz transform, namely,

\[ R_j(f) = \text{p.v.} \int (x_j - y_j)|x - y|^{-(n+1)} f(y) dy. \]

For details, see [4 and 13]. \( E_{ij}(x,t) \) is symmetric and divergence free, that is,

\[ \sum_j D_j E_{ij}(x,t) = 0, \quad t \geq 0. \]

In the above formula we take classical derivatives if \( t > 0 \), and distributional derivatives for the limit for \( t \) tending to 0. An other important property is the following one:

\[ \sum_j \int E_{ij}(x,t)f_j(y-x) dx \text{ tends to } f_j(y) \text{ in } L^p, \quad 1 < p < \infty, \]

as \( t \) tends to zero, provided that \( \text{div } f = 0 \).
A very important theorem in [4] asserts that $u(x, t)$ is a weak solution of the problem (0, 1) (see definition in §I) over $S_T$, with $g \in L^r$, $1 \leq r < \infty$, if and only if it is a solution of the following integral equation [4, Theorem 2.1, p. 226]:

\begin{equation}
(2.4) \quad u + B(u, u) = F(x, t).
\end{equation}

$F(x, t)$ stands for the convolution in the space variables of the initial data $f(x)$ with the fundamental solution $W(x, t)$. $B(u, v)$ is the bilinear form

\begin{equation}
(2.5) \quad \int_0^t \int_{\mathbb{R}^n} ((\text{grad} \, E(x - y, t - s))(v(y, s)), u(y, s)) \, dy \, ds.
\end{equation}

We have used here a notation consistent with (0, 2). In fact, $(\text{grad} \, E)(v)$ is the matrix $\sum_k D_k E_{ij} v_k$, whose $i$th row is dotted with $u$ to obtain the integrand of (2.5).

**Estimates for the bilinear operators and fixed-point properties.**

In what follows, we are going to consider Banach spaces of Lebesgue measurable functions defined on $S_T$ for which the operator $T(u, v) = B(u, v) + 1(u) + F(x, t)$, $(B(u, v)$ is bilinear and $1(u)$ is assumed to be linear) satisfies an inequality of the type

\begin{equation}
(2.6) \quad \|T(u, v)\| \leq C_1 \|u\| \|v\| + C_2 \|u\| + \|F\|.
\end{equation}

In (2.9) above, the norm is that of the Banach space in question.

**Lemma II.** The quadratic operator $T(u, u)$ maps the ball $\{\|u\| \leq s_1\}$ into itself if $s_1$ is the smallest root of the equation

\begin{equation}
(2.7) \quad C_1 s^2 + (C_2 - 1)s + \|F\| = 0
\end{equation}

provided that $C_1, C_2$ and $\|F\|$ satisfy

\begin{equation}
(2.8) \quad (1 - C_2)^2 > 4C_1 \|F\|, \quad C_1 > 0, \quad 0 \leq C_2 < 1.
\end{equation}

If $2s_1 C_1 + C_2 < 1$, $T(u, u)$ is a contraction mapping in the ball of radius $s_1$. In particular, $T(u, u)$ is a contraction mapping in the ball of radius $s_1$ if $C_1, C_2$ and $\|F\|$ satisfy

\begin{equation}
(2.9) \quad 2C_1 \|F\|((1 - C_2)^2 - 4C_1 \|F\|)^{-1/2} + C_2 < 1.
\end{equation}

For the proof see [2].

**III. ESTIMATES FOR THE BILINEAR FORM $B(u, u)$**

The bilinear form $B(u, v)$ admits the domination

\begin{equation}
(3.1) \quad |B(u, v)| \leq C \int_{\mathbb{R}^n} |x - y|^{-n+1} \int_0^t |x - y|^{-2}(1 + |x - y|^{-1} s^{1/2})^{-n-1} \times |u(y, t - s)||v(y, t - s)| \, ds \, dy.
\end{equation}
The above domination is a consequence of the estimate
\[ |D_k E_{ij}(x, t)| \leq C(|x| + t^{1/2})^{-n-1}. \]

Calling \( M(u) \) the maximal function of Hardy-Littlewood of \(|u|\) on the space variables and \( u^* \) the sup on \( t > 0 \) of \(|u|\), we have
\[ |B(u, u)| \leq C_0 \int_{\mathbb{R}^n} |x - y|^{-n+1} (M(u^*))^2 \, dy. \]

The constant \( C_0 \) does not exceed
\[ C \int_0^\infty (1 + t^{1/2})^{-n-1} dt. \]

The right-hand side of (3.14) does not depend on \( t \), hence
\[ |B(u, u)|^* \leq C_0 \int_{\mathbb{R}^n} |x - y|^{-n+1} (M(u^*))^2 \, dy. \]

We now apply Hardy-Littlewood-Sobolev potential inequality to (3.5) with exponents \( \frac{1}{q} = \frac{2}{n} - \frac{1}{n} \) (see [13, pp. 119, 120]) and obtain
\[ \|B(u, u)^*\|_n(\infty) \leq C \{\|M(u^*)\|_n(\infty)\}^2 \leq C' \{\|u^*\|_n(\infty)\}^2. \]

We now need estimates on \( D_k E_{ij} \) of the type introduced by Benedek-Panzone in [1] (see p. 321, Theorem 1), namely,
\[ |D_k E_{ij}| \leq \frac{C}{|x|^{n-\theta}(t^{1/2})^{1+\theta}}. \]

Here, \( C \) is an independent constant, \( 0 < \theta < 1 \), \( t > 0 \), \( x \in \mathbb{R}^n \). A simple adaptation of Theorem 1, p. 321 in [1] (see also [4, Theorem (3.1)]) gives for the operator
\[ T(f) = \int_0^t \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-\theta}(t - \tau)^{1+\theta/2}} f(y, \tau) \, dy \, d\tau \]
the estimates
\[ \|T(f)\|_{p^*, q^*} \leq C_{p, q} \|f\|_{p, q} \]
where
\[ \frac{1}{p^*} = \frac{1}{p} - \frac{\theta}{n} = \frac{1}{q} - \frac{1 - \theta}{2}. \]

Similar results hold for the norms \( \|\|_{p, q}^* \), namely,
\[ \|T(f)\|_{p^*, q^*} \leq C_{p, q} \|f\|_{p, q}^*. \]

Here,
\[ \frac{1}{p^*} = \frac{1}{p} - \frac{1 - \theta}{2} = \frac{1}{q} - \frac{\theta}{n}. \]

The estimates (3.5), (3.7), (3.8) and (3.6) lead to the following inequalities for \( B(u, u) \):
\[ \|B(u, u)\|_{\frac{q}{\theta}, \frac{2}{1-\theta}} \leq C_{\theta}(\|u\|_{\frac{q}{\theta}, \frac{2}{1-\theta}})^2, \]
0 < \theta < 1$, which is a consequence of
\begin{equation}
|B(u, u)| \leq CT(|u|^2) .
\end{equation}
$T$ as defined in (3.6) above. Likewise we get for the $\| \cdot \|^*_{p,q}$ norms the estimate
\begin{equation}
\|B(u, u)\|^*_{\frac{2}{1-\theta}, \frac{n}{\theta}} \leq C_\theta (\|u\|^*_{\frac{2}{1-\theta}, \frac{n}{\theta}})^2 ,
\end{equation}
$0 < \theta < 1$. In the above expressions we may replace $\frac{n}{\theta}$ and $\frac{2}{1-\theta}$ by $p$ and $q$ respectively, satisfying
\[ 1 = \frac{n}{p} + \frac{2}{q} ; \quad p > n . \]

IV. IMBEDDING THE INITIAL DATA

As we have seen in (2.4), $F(x, t)$ is the term arising from the initial data:
\begin{equation}
F(x, t) = W * f .
\end{equation}
The above convolution is on the spatial variables only. The “a priori estimate” (3.86) [2] and Lemma A3 in [2] give the following result as a trivial consequence:

Lemma IV. Let $1 \leq n < \infty$, $f \in L^p(\mathbb{R}^n)$, $1 < p \leq \infty$. Then, the function $F(x, t)$ as defined above, satisfies
\begin{enumerate}
  \item $\|F\|_{p,q} \leq C_p(n)\|f\|_p$,
  \item $\|F\|^*_{p,q} \leq C_p(n)\|f\|^*_{p,q}$.
\end{enumerate}

$C_p(n)$ depends only on $p$ and $n$.

Theorem B. The function $F(x, t)$ defined above satisfies
\begin{enumerate}
  \item $\|F\|_{p,v} \leq C_r\|f\|_r$,
  \item $\|F\|^*_{v,s} \leq C_r\|f\|^*_{r,s}$, $1 < r < \infty$, $v > s$, $\frac{1}{r} = \frac{1}{s} + \frac{2}{nv}$.
\end{enumerate}
$C_r$ depends only on $r, v, s$ and $n$.

Proof. We shall consider the norms $\| \cdot \|_{p,q}$ only, since $\| \cdot \|^*_{p,q}$ can be dealt with in a similar manner ($\tilde{u}(t, x) = u(x, t)$, thus $\|\tilde{u}\|_{p,q} = \|u\|^*_{p,q}$). On one hand we have
\[ \|F\|_{p,\infty} < C_p\|f\|_p , \quad 1 < p \leq \infty , \]
and from Lemma IV
\[ \|F\|_{n+2_q n^{-2}_q} < C_q\|f\|_q , \quad 1 < q \leq \infty . \]
The Benedek-Panzone interpolation theorem for mixed norms (see [1, Theorems 1, 2]); gives the desired result for
\begin{equation}
\begin{align*}
\frac{1}{r} &= \frac{t}{p} + \frac{1-t}{q} , \\
\frac{1}{s} &= \frac{t}{p} + \frac{1-t}{q} \frac{n}{n+2} , \\
\frac{1}{v} &= \frac{1-t}{q} \frac{n}{n+2} , \quad 0 < t < 1 ,
\end{align*}
\end{equation}
or
\[ \frac{1}{r} = \frac{1}{s} - \frac{1}{v} + \frac{n+2}{n} \frac{1}{v}, \quad v > s, \]
hence,
\[ (4.3) \quad \frac{1}{r} = \frac{1}{s} + \frac{2}{nv}. \]
Concerning the norms \( \| \cdot \|_{p,q}^* \), one should notice that
\[ \| F \|_{\infty, p}^* < C_p \| f \|_p, \quad 1 < p \leq \infty, \]
is a consequence of the maximal theorem associated with the Weierstrass kernel. This finishes the proof.

V. PROOF OF THEOREM A

Parts (i) and (ii) follow from the imbedding results of §IV. Part (iii) follows using the estimates (3.9) and (3.11) and Lemma II. This concludes the proof.

REFERENCES


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