

METRIZABLE SPACES WHERE THE INDUCTIVE DIMENSIONS DISAGREE

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ABSTRACT. A method for constructing zero-dimensional metrizable spaces is given. Using generalizations of Roy's technique, these spaces can often be shown to have positive large inductive dimension. Examples of \mathbf{N} -compact, complete metrizable spaces with $\text{ind} = 0$ and $\text{Ind} = 1$ are provided, answering questions of Mrowka and Roy. An example with weight \mathfrak{c} and positive Ind such that subspaces with smaller weight have $\text{Ind} = 0$ is produced in ZFC. Assuming an additional axiom, for each cardinal λ a space of positive Ind with all subspaces with weight less than λ strongly zero-dimensional is constructed.

I. INTRODUCTION

There are three classical notions of dimension for a normal topological space X . The small inductive dimension, denoted $\text{ind}(X)$, is defined by $\text{ind}(X) = -1$ if X is empty, $\text{ind}(X) \leq n$ if each $p \in X$ has a neighborhood base of open sets whose boundaries have $\text{ind} \leq n - 1$, and $\text{ind}(X) = n$ if $\text{ind}(X) \leq n$ and n is as small as possible. The large inductive dimension, denoted $\text{Ind}(X)$, is defined by $\text{Ind}(X) = -1$ if X is empty, $\text{Ind}(X) \leq n$ if each closed $C \subseteq X$ has a neighborhood base of open sets whose boundaries have $\text{Ind} \leq n - 1$, and $\text{Ind}(X) = n$ if $\text{Ind}(X) \leq n$ and n is as small as possible. The covering dimension, denoted by $\text{dim}(X)$, is defined by $\text{dim}(X) = -1$ if X is empty, $\text{dim}(X) \leq n$ if for each finite open cover of X there is a finite open refining cover such that no point is in more than $n + 1$ of the sets in the refining cover and, $\text{dim}(X) = n$ if $\text{dim}(X) \leq n$ and n is as small as possible. It is well known that for separable metric spaces, all three dimensions coincide and that for arbitrary metric spaces $\text{Ind} = \text{dim}$; however, it is possible to have $\text{ind} < \text{Ind}$, and we will call a metric space with this property a gap metric space. Very few examples of gap metric spaces are in the literature. The original example is Roy's space Δ , described in 1962 (see [R1, R2]). Nyikos showed that the Engelking-Mrowka conjecture is false in [NY] by showing that Δ is not \mathbf{N} -compact, that is, that Δ is not homeomorphic to a closed subspace of \mathbf{N}^α for any cardinal α . Later Mrowka produced an \mathbf{N} -compact gap metric space, which he called μ_0 , in [M]. Most recently, examples with weight equal to ω_1

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were produced independently by Ostaszewski in [O] and by the author in [K1].

All of the examples mentioned above as well as those constructed here satisfy $\text{ind} = 0$, that is they have bases of clopen sets, sets which are simultaneously closed and open; such spaces are called zero-dimensional. We are concerned with metrizable spaces where $\text{ind} = 0$ and one (hence both) of the other dimensions are not. (They will be 1.) It is apparent from the definition that $\text{Ind}(X) = 0$ if and only if whenever F_1 and F_2 are disjoint closed subsets of X , there is a clopen set containing F_1 which is disjoint from F_2 . Thus our approach to showing $\text{Ind} > 0$ will be to exhibit a pair of disjoint closed sets for which no such clopen set exists. We will call a metric space for which $\text{Ind} = 0$ strongly zero-dimensional, although this term refers to another dimension (Dim , not discussed here) being zero; there is no confusion because $\text{Dim} = \text{Ind}$ for metric spaces.

In this paper a rather general method for producing gap metric spaces is described; the method involves using well-chosen subspaces of countably infinite products of well-chosen factor spaces. The advantages of this technique are that the examples are, in general, much simpler than existing examples, and the properties can be varied to a large extent. Furthermore, these spaces are fairly common. For instance, the example in [K1] (the first application of this method) is a subspace of ω_1^ω , and one of the examples in this paper is a subspace of M^ω (M denotes the Michael line, the space whose point set is the real numbers and whose topology is obtained by taking sets of the form $O \cup P$ as open, where O is a usual open set and P is a subset of the irrationals). Finally, the product space gives a reasonable setting in which to look for spaces where the discrepancy in dimension is greater than 1.

In §II general notations and key background lemmas are given. §III gives the method for producing zero-dimensional metrizable spaces; theorems are proved which show the spaces are in fact zero-dimensional, and a further theorem gives conditions which guarantee completeness.

In §IV we apply the method of §III to produce two \mathbf{N} -compact examples of gap metric spaces. Both are complete, and thus affirmatively answer the question of Mrowka, "Do there exist complete \mathbf{N} -compact metric spaces with $\text{dim} > 0$?", posed in [A]. The first space is simple to describe. The second, additionally, answers a question of Roy's. He had asked if non- \mathbf{N} -compactness was a property of all subspaces of Δ with positive Ind . As the second example embeds in Δ (the technical proof of which is in [K2]), it gives a negative answer to Roy's question. The second example is best understood as a modification of the first.

Since the discovery of gap metric spaces of weight ω_1 (the weight of a topological space is the minimal cardinality of a base for the topology), an obvious question becomes "Does every gap metric space contain a gap metric subspace of weight ω_1 ?" or, more generally, "Is there a cardinal λ such that every gap metric space contains a gap metric subspace of weight $\leq \lambda$?" It is not too hard,

using the ideas in [O], to show Δ contains a weight ω_1 gap metric subspace; for μ_0 the answer is not clear. In §V, a gap metric space of weight \mathfrak{c} is described with the property that every subspace of weight less than \mathfrak{c} is strongly zero-dimensional; thus the first question has a negative answer assuming the negation of CH. In §VI, assuming a set theoretic axiom E^* , it is shown that, for any cardinal λ , there is a gap metric space such that any subspace of cardinality less than λ has $\text{Ind} = 0$, giving a negative answer to the second question. The examples of this section seem to be the first examples of nonstrongly zero-dimensional metrizable spaces with all small (where small is as large as desired) subspaces strongly zero-dimensional. Furthermore, in the examples, closed sets of cardinality less than λ have neighborhood bases of clopen sets; the closed sets which exhibit $\text{Ind} > 0$ have cardinality at least λ .

With regard to the methods of this paper, they should be viewed as simplifications and generalizations of methods employed by Roy and later by Mrowka.

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II. PRELIMINARIES

For a set I and a collection $\{X_i : i \in I\}$ of sets, $\prod_{i \in I} X_i$ denotes $\{f : f \text{ is a function, } \text{domain}(f) = I, \text{ and } \forall i \in I, f(i) \in X_i\}$; correspondingly, X^I will denote $\{\sigma : \text{domain}(\sigma) = I, \text{ and } \text{image}(\sigma) \subseteq X\}$. If $I \in \omega + 1$ and $\text{domain}(\sigma) = I$, then σ is called a sequence. If σ is a sequence and D is a collection of sequences, then $|\sigma|$ denotes $\text{domain}(\sigma)$ and

$$|D| = \sup\{|\tau| : \tau \in D\};$$

also $\sigma|_n$ and $D|_n$ denote the appropriate restrictions to n . We do not distinguish between σ and the ordered $|\sigma|$ -tuple $(\sigma(0), \sigma(1), \sigma(2), \dots)$.

We adopt the following notation which is useful in manipulating sequences. Let $\{X_i : i \in \omega\}$ be a given collection of sets, let σ, τ be sequences, and let D, E be collections of sequences such that $\gamma(i) \in X_i$ for $\gamma \in \{\sigma, \tau\} \cup D \cup E$ and $i < |\gamma|$, and such that $|\sigma| < \omega$ and $|D| < \omega$. We define

$$\langle \sigma, \tau \rangle = (\sigma(0), \sigma(1), \dots, \sigma(|\sigma|), \tau(0), \tau(1), \dots),$$

that is $\langle \sigma; \tau \rangle$ is the concatenation of σ and τ .

$$\langle \sigma; E \rangle = \{\langle \sigma, \varepsilon \rangle : \varepsilon \in E\}, \quad \langle D; E \rangle = \{\langle \delta, \varepsilon \rangle : \delta \in D, \varepsilon \in E\},$$

and

$$\overline{\langle \sigma; D \rangle} = \left\{ \psi \in \prod X_i : \text{there is } n \in \omega \text{ such that } \psi|_n \in \langle \sigma; D \rangle \right\}.$$

Later on, when we are discussing a special subset R of $\prod X_i$, $\overline{\langle \sigma; D \rangle}$ will instead be used (ambiguously) to mean $\overline{\langle \sigma; D \rangle} \cap R$. If P is a property of subsets of X , then P is countably coadditive on X means that, if $X = \bigcup_{i \in \omega} X_i$, then one of the X_i 's has property P . Most notably for our purposes, if X is an uncountable set then the property U of being uncountable is countably

coadditive on X ; also if X is a second category in its own topological space, then the property S of being second category in X is countably coadditive.

The following definition and lemmas are used to show that the examples constructed have positive Ind. Special cases of Lemmas 3 and 4 were known to Roy [R2]; these lemmas were discovered independently by Fleissner [F]; Lemmas 3, 4 and 5 were used by Mrowka in [M] in the language of ramified products.

1. Definition. Let $\{X_i: i \in I\}$, where $I \in \omega + 1$, be a collection of sets such that property P is countably coadditive on each X_i . Then $Y \subseteq \prod X_i$ is a P -full subset of $\prod X_i$ iff for each $\sigma \in Y$ and $k \in I$, the subset $\{\tau(k): \tau|_{k-1} = \sigma|_{k-1}\}$ of X_k has property P on X_k .

As remarked above, we will be especially concerned with U -full and S -full subsets.

2. Definition. Let $\{X_i: i \in I\}$, where $I \in \omega + 1$, be a collection of second countable spaces. Then $Y \subseteq \prod X_i$ is a *super S -full* subset of $\prod X_i$ iff for each $\sigma \in Y$ and $k \in I$, $\{\tau(k): \tau|_{k-1} = \sigma|_{k-1}\}$ is second category everywhere on X_k .

3. Lemma. Let $\{X_i: i < n\}$ where $n \in \omega$, be a collection of sets with property P countably coadditive on each X_i , and let $Y \subseteq \prod X_i$ be a P -full subset of $\prod X_i$. Then, if $Y = \bigcup_{i \in \omega} Y_i$, there is a $j \in \omega$ such that Y_j contains a P -full subset of $\prod X_i$.

4. Lemma. Let $\{X_i: i \in \omega\}$ be a collection of sets with property P countably coadditive on each X_i ; let $X = \prod X_i$ and let $\{C_i: i \in \omega\}$ be a collection of subsets of X satisfying:

- (i) $C_i \subseteq C_{i+1} \quad \forall i \in \omega$,
- (ii) $X = \bigcup C_i$, and
- (iii) if $\sigma \in C_n$, then $\{\tau \in X: \tau|_n = \sigma|_n\} \subseteq C_n$.

Then there is a k such that $C_k|_k$ contains a P -full subset of $\prod_{i < k} X_i$.

5. Lemma. Let $\{X_i: i < n\}$ be a collection of second countable spaces, and let K be an S -full subset of $\prod X_i$; there is a $K' \subseteq K$ and an $O_i \subseteq X_i$ for each $i < n$, where O_i is a nonempty open subset of X_i such that K' is a *super S -full* subset of $\prod O_i$.

If X is a topological space, then $Y \subseteq X$ is a Bernstein subset of X provided $Y \cap K \neq \emptyset$ and $(X \setminus Y) \cap K \neq \emptyset$ if K is an uncountable closed subset of X . If X is a complete, separable metric space with no isolated points then X contains a Bernstein set, and each Bernstein subset of X is second category everywhere on X . Furthermore, for such X , if O is an open cover in X of a Bernstein subset of X , then $X \setminus (\bigcup O)$ is countable.

III. A METHOD FOR CONSTRUCTING ZERO DIMENSIONAL
METRIZABLE SPACES

1. The spaces will be subspaces specified in Definition 1.3 where the factor spaces are always of the form given in Definition 1.1. Theorem 1.4 gives the main result, and Theorem 1.6 also gives conditions which guarantee completeness of a space satisfying the hypotheses of 1.4. The rest of the section is devoted to proving Theorem 1.4 and 1.6.

1.1. **Definition.** A triple (X, A, B) is a *tame triple* provided X is a first countable, T_1 topological space, $X = A \cup B$ where $A \cap B = \emptyset$, and the elements of A are isolated in X .

1.2. **Examples of the tame triples.** (i) (E_λ, S, L) , where λ is an ordinal with the order topology, S is the set of successor ordinals in λ , L is the set of limit ordinals in λ with countable cofinality, and $E_\lambda = S \cup L$.

(ii) (M, I, Q) , where M denotes the Michael line, I the irrational points and Q the rational points.

(iii) $(A(Y), N, O)$, where $A(Y)$ denotes the Alexandroff duplicate of the (first countable) space Y , O represents the "old" points, and N the "new" points.

1.3. **Definition.** Let $\{(X_i, A_i, B_i) : i \in \omega\}$ be a collection of tame triples. A set $X \subseteq \prod X_i$ respects $\{(X_i, A_i, B_i) : i \in \omega\}$ provided X satisfies the following:

(i) $X = P_1 \cup P_2^0 \cup P_2^1 \cup P_2^2 \cup \dots$, where $P_1 = X \cap (\prod_{i \in \omega} A_i)$, and for $k \in \omega$,

$$P_2^k = X \cap \left(\left(\prod_{i < k} A_i \right) \times B_k \times \left(\prod_{i > k} A_i \right) \right).$$

(ii) There is a function $\phi_1 : \omega \rightarrow \omega$ satisfying, for each $k \in \omega$:

(a) if $b_1, b_2 \in B_k$, and $b_1 \neq b_2$, then

$$\pi_{\phi_1(k)}(X \cap \pi_k^{-1}(b_1)) \cap \pi_{\phi_1(k)}(X \cap \pi_k^{-1}(b_2)) = \emptyset,$$

and

(b) $\phi_1(k) > k$.

(iii) There is a function $\phi_2 : \omega^2 \rightarrow \omega$ satisfying for each $k_1, k_2 \in \omega$, where $k_1 \neq k_2$:

(a) $\pi_{\phi_2(k_1, k_2)}(P_2^{k_1}) \cap \pi_{\phi_2(k_1, k_2)}(P_2^{k_2}) = \emptyset$, and

(b) $\phi_2(k_1, k_2) > \max\{k_1, k_2\}$.

The following is the main theorem for verifying that examples are metrizable.

1.4. **Theorem.** Let $\{(X_i, A_i, B_i) : i \in \omega\}$ be a collection of tame triples and let $X \subseteq \prod X_i$ be a subspace which respects $\{(X_i, A_i, B_i) : i \in \omega\}$. Then:

- (i) X is zero-dimensional,
- (ii) X is metrizable, and
- (iii) $\text{Ind}(X) \leq 1$.

We remark that it is not necessary that the factor spaces themselves be zero-dimensional. The proof of this theorem is postponed until subsection 3. We continue giving conditions which guarantee completeness.

1.5. Definition. Let $\{X_i: i \in \omega\}$ be a collection of sets. A set $X \subseteq \prod X_i$ is an *approximately complete* subset of $\prod X_i$ provided: if $\sigma \in \prod X_i$ and $\{\tau \in X: \tau|_k = \sigma|_k\} \neq \emptyset$ for each $k \in \omega$, then $\sigma \in X$.

We remark that the approximately complete subsets are precisely those which are closed in the product $\prod(X_i, D_i)$ where, for each i , D_i is the discrete topology on X_i .

1.6. Theorem. If $\{(X_i, A_i, B_i): i \in \omega\}$ is a collection of tame triples, $X \subseteq \prod X_i$ respects $\{(X_i, A_i, B_i): i \in \omega\}$ and is an approximately complete subset of $\prod X_i$, then X is completely metrizable.

2. Theorem 1.6 will be proved in subsection 4.2. We now introduce some notation, which will be standard, and prove lemmas which will be necessary for proving 1.4 and 1.6. Throughout we will assume a fixed collection $\{(X_i, A_i, B_i): i \in \omega\}$ of tame triples and a fixed $X \subseteq \prod X_i$ which respects $\{(X_i, A_i, B_i): i \in \omega\}$.

2.1. For each $i \in \omega$, and each $p \in B_i$, let $\{N_n^i(p): n \in \omega\}$ be a nested local base at p in X_i .

2.2. For each $\sigma \in X$, we define a nested local base $\{V_i(\sigma): i \in \omega\}$ as follows:

- (i) if $\sigma \in P_1$, then $V_i(\sigma) = \{\tau \in X: \tau|_i = \sigma|_i\}$ for $i \in \omega$,
- (ii) if $\sigma \in P_2^k$ and $i > k$, then $V_i(\sigma) = \{\tau \in X: \tau|_k = \sigma|_k, \tau(k) \in N_i^k(\sigma(k)), \text{ and for } k < j < i, \tau(j) = \sigma(j)\}$. For $i \leq k$ let $V_i(\sigma) = V_{k+1}(\sigma)$.

It is easy to check that for $\sigma \in X$, $\{V_i(\sigma): i \in \omega\}$ is a nested local base at σ in the subspace topology on X (as a subset of $\prod X_i$).

2.3. For $i \in \omega$, let $G_i = \{V_i(\sigma): \sigma \in X\}$. It is clear that G_i covers X , G_{i+1} refines G_i , and $G = \bigcup\{G_i: i \in \omega\}$ is a base for X .

We now prove lemmas which establish relationship among the various basis elements we have just described.

2.4. Lemma. For each $k \in \omega$, $n > \max\{k + 1, \phi_1(k)\}$, and $\sigma, \tau \in P_2^k$, either $V_n(\sigma) = V_n(\tau)$ or $V_n(\sigma) \cap V_n(\tau) = \emptyset$.

Proof. Either $\sigma|_n = \tau|_n$ in which case $V_n(\sigma) = V_n(\tau)$, or else there is a $j < n$ such that $\sigma(j) \neq \tau(j)$. If $j \neq k$ then obviously $V_n(\sigma) \cap V_n(\tau) = \emptyset$. If $j = k$, then $\sigma(\phi_1(k)) \neq \tau(\phi_1(k))$ and $V_n(\sigma) \cap V_n(\tau) = \emptyset$. \square

2.5. Lemma. If $\sigma \in P_2^{k_1}$ and $\tau \in P_2^{k_2}$, where $k_1 \neq k_2$, then $V_n(\sigma) \cap V_n(\tau) = \emptyset$ for $n > \phi_2(k_1, k_2)$.

Proof. $V_n(\sigma) \subseteq \{\gamma \in X: \gamma(\phi_2(k_1, k_2)) = \sigma(\phi_2(k_1, k_2))\}$ and $V_n(\tau) \subseteq \{\gamma \in X: \gamma(\phi_2(k_1, k_2)) = \tau(\phi_2(k_1, k_2))\}$. By assumption the two sets do not intersect. \square

2.6. Lemma. *If $\sigma \in P_1$ and $k \in \omega$, then there is $t \in \omega$ such that $V_t(\sigma) \cap V_t(\tau) = \emptyset$ whenever $\tau \in P_2^k$.*

Proof. Let $t_1 = \phi(k) + 1$. If $\tau \in P_2^k$ and $V_{t_1}(\sigma) \cap V_{t_1}(\tau) \neq \emptyset$, then, since $\tau(k)$ is determined by $\sigma(\phi_1(k))$, there is a neighborhood $N_m^k(\tau(k))$ in X_k which does not contain $\sigma(k)$, since $\sigma(k) \in A_k$ and $\tau(k) \in B_k$. But then $t = \max\{t_1, m\}$ suffices for the lemma. \square

2.7. Lemma. *If $\sigma \in P_1$ and $m \in \omega$, then there is $t \in \omega$ such that $V_t(\sigma) \cap V_t(\tau) = \emptyset$ for all $\tau \in P_2^i$ where $i < m$.*

Proof. Apply Lemma 2.6 m times. \square

2.8. Lemma. *If $m \in \omega$, $\sigma \in P_1 \cup \bigcup_{j < m} P_2^j$ and $\tau \notin P_2^i$ for $i \leq m$, then either $V_t(\tau) \subseteq V_m(\sigma) \forall t > m$, or else $V_t(\tau) \cap V_m(\sigma) = \emptyset \forall t > m$.*

Proof. If $\tau \in V_m(\sigma)$, then $\{\gamma: \gamma|_{m+1} = \tau|_{m+1}\} \subseteq V_m(\sigma)$; hence $V_t(\tau) \subseteq V_m(\sigma)$ for all $t > m$, as $V_m(\tau) \subseteq V_m(\sigma)$. On the other hand, $\tau \notin V_m(\sigma)$ implies $V_m(\tau) \cap V_m(\sigma) = \emptyset$, thus $V_t(\tau) \cap V_m(\sigma) = \emptyset$ for $t > m$. \square

3. Proof of Theorem 1.4. First we show

3.1. X is zero-dimensional.

This will follow immediately from 3.1.1 and 3.1.2.

3.1.1. If $\sigma \in P_1$ and $i \in \omega$, then $V_i(\sigma)$ is a clopen set in X .

Proof. $\tau \in V_i(\sigma)$ iff $\tau|_i = \sigma|_i$. If $\tau(j) \neq \sigma(j)$ for some $j < i$, then either $\tau(j) \in A_j$, in which case $V_i(\tau) \cap V_i(\sigma) = \emptyset$, or else $\tau(j) \in B_j$, in which case there is $N_m^j(\tau(j))$ which does not contain $\sigma(j)$. Thus for $r \geq \max\{i, m\}$, $V_r(\tau) \cap V_i(\sigma) = \emptyset$. Hence $V_i(\sigma)$ is clopen. \square

3.1.2. If $\sigma \in P_2^k$ and $i > \max\{k + 1, \phi_1(k)\}$, then $V_i(\sigma)$ is clopen.

Proof. Suppose τ is in the closure of $V_i(\sigma)$. If $\tau \in P_2^k$, then by Lemma 2.4, $\tau \in V_i(\sigma)$. Clearly if $\tau \in P_1$, $\tau \in V_i(\sigma)$. If $\tau \in P_2^j$, where $i < j$, then Lemma 2.8 gives that $\tau \in V_i(\sigma)$. Thus we only need to consider $\tau \in P_2^j$, where $j \leq i$ and $j \neq k$. Here an argument similar to that in 3.1.1 works. \square

3.2. X is metrizable. We make use of the following theorem due to Frink (see [N, p. 33]).

3.2.1. Theorem. *A T_2 space X is metrizable if there exists a countable open neighborhood base $\{V_i(\sigma): i \in \omega\}$ for each $\sigma \in X$ satisfying: For each $\sigma \in X$ and $i \in \omega$ there is a $t \in \omega$ such that $V_i(\sigma) \cap V_t(\tau) \neq \emptyset$ implies $V_t(\tau) \subseteq V_i(\sigma)$.*

We will show that the sets $\{V_i(\sigma): i \in \omega\}$ for $\sigma \in X$ satisfy the condition of Theorem 3.2.1. Assume $i \in \omega$ and $\sigma \in X$ are given.

3.2.2. The condition of Theorem 3.2.1 is met if $\sigma \in P_1$.

Proof. If $\tau \in P_2^j$ where $j > i$, then Lemma 2.8 guarantees that for any $s > i$ either $V_s(\tau) \subseteq V_s(\sigma)$ or $V_s(\tau) \cap V_i(\sigma) = \emptyset$. In case $\tau \in P_1$ this is also obviously true. By Lemma 2.7 we can find $u \in \omega$ such that $V_u(\sigma) \cap V_u(\tau) = \emptyset$ for any $\tau \in P_2^j$ where $j \leq i$. Thus we may choose $t = \max\{u, i + 1\}$. \square

3.2.3. The condition of Theorem 3.2.1 is met for $\sigma \in P_2^k$.

Proof. For $\tau \in P_1 \cup (\bigcup_{j < i} P_2^j)$, as in 3.2.2, for any $s > i$, $V_s(\tau) \subseteq V_i(\sigma)$ or $V_s(\tau) \cap V_i(\sigma) = \emptyset$. Choose $u = \max\{\phi_2(k, j) : j \leq i\}$. Then $V_u(\tau) \cap V_u(\sigma) = \emptyset$ for $\tau \in P_2^i, i \leq m$. Thus let $t = \max\{u, i + 1\}$. \square

From 3.2.2 and 3.2.3 we conclude, via Frink's Theorem, that X is metrizable. It remains to show:

3.3. $\text{Ind}(X) \leq 1$. We will show for each $k \in \omega$, that $\text{Ind}(P_2^k) = 0$, that P_2^k is closed in X , and that $\text{Ind}(P_1) = 0$. It follows from [NA, Theorem 9–10] that $\text{Ind}(\bigcup_{k \in \omega} P_2^k) = 0$, and so X is the metrizable union of two subspaces P_1 and $\bigcup_{k \in \omega} P_2^k$ which have $\text{Ind} = 0$. By [NA, Theorem 12–6], $\text{Ind}(X) \leq 1$.

3.3.1. For $k \in \omega, \text{Ind}(P_2^k) = 0$.

Proof. Lemma 2.4 gives that, for $n \geq \phi_1(k)$, the collection $O_n^k = \{V_n(\sigma) : \sigma \in P_2^k\}$ is an order one cover of P_2^k . But O_{n+1}^k refines O_n^k , and $\bigcup_{j \in \omega} O_j^k$ is a base for P_2^k . Hence by [NA, Theorem 12-6], $\text{Ind}(P_2^k) = 0$. \square

3.3.2. P_2^k is a closed set in X .

Proof. This follows immediately from Lemmas 2.5 and 2.6. \square

3.3.3. $\text{Ind}(P_1) = 0$.

Proof. Let $O_n = \{V_n(\sigma) : \sigma \in P_1\}$ and proceed as in 3.3.1. \square

Thus we have shown $\text{Ind}(X) \leq 1$, and the proof of Theorem 1.4 is complete.

4. We prove Theorem 1.6, so we assume X is additionally approximately complete. We only need to show (e.g., see [N, 3.5–3.8]):

4.1. $\{G_i : i \in \omega\}$ is a complete sequence of open covers for X , that is, if $\{g_i : i \in \omega\}$ is a collection of sets in G such that $g_i \in G_i$ for all $i \in \omega$ and $\{g_i : i \in \omega\}$ has the finite intersection property, then $\bigcap_{i \in \omega} g_i \neq \emptyset$.

We will use, without proof, the following easy reduction of 4.1 (use e.g. Theorem 3.2.1).

4.2. **Lemma.** *If for $\{g_i\}$ as in 4.1 there is a subsequence $\{\eta_i : i \in \omega\}$ of ω such that $\bigcap_{i \in \omega} g_{\eta_i} \neq \emptyset$, then $\bigcap_{i \in \omega} g_i \neq \emptyset$.*

Theorem 1.6 is proved by considering three cases. Assume $\{g_i\}$ as in 4.1; for each $i \in \omega$ choose $\sigma_i \in X$ such that $g_i = V_i(\sigma_i)$.

Case 1. There is an increasing sequence $\{\eta_i: i \in \omega\}$ with $\sigma_{\eta_i} \in P_1$. Since no finite intersection of the g_i 's is empty, then for $i, j > t$, $\sigma_{\eta_i}|_t = \sigma_{\eta_j}|_t$. Setting $\tau = (\sigma_{\eta_1}(0), \sigma_{\eta_2}(1), \sigma_{\eta_3}(2), \dots)$, approximate completeness guarantees that $\tau \in X$; furthermore it is clear that $\tau \in \bigcap_{i \in \omega} g_{\eta_i}$, hence by Lemma 4.2, $\bigcap_{i \in \omega} g_i \neq \emptyset$.

Case 2. There is a fixed $k \in \omega$ and an increasing sequence $\{\eta_i: i \in \omega\}$ with $\sigma_{\eta_i} \in P_2^k$. We may assume $\eta_0 > k$; as in Case 1, $\tau = (\sigma_{\eta_1}(0), \sigma_{\eta_2}(1), \dots) \in X$ by approximate completeness, and $\tau \in \bigcap_{i \in \omega} g_i$.

Case 3. There is an increasing sequence $\{\eta_i: i \in \omega\}$ with $\sigma_{\eta_i} \in P_2^{k_{\eta_i}}$ where $\{k_{\eta_i}: i \in \omega\}$ is also increasing with i . Thus for $\eta_i, \eta_j > t$, $\sigma_{\eta_i}|_t = \sigma_{\eta_j}|_t = \tau|_t$ where $\tau = (\sigma_{\eta_1}(0), \sigma_{\eta_2}(1), \dots)$. Again we get $\tau \in \bigcap_{i \in \omega} g_i$.

Since one of the three cases must occur, the conclusion of 4.1 must follow. \square

IV. APPLICATIONS TO N-COMPACT SPACES

The letter C will denote both the point set $\{0, 1\}^\omega$ and the set with the usual Cantor set topology. For $x \in C$, $J_n(x)$ denotes the usual basic neighborhood of x in C given by $\{y \in C: y|_n = x|_n\}$. End denotes the set of end points of C (those which take on either 0 or 1 only finitely many times), and Int denotes the set of interior points ($C \setminus \text{End}$). By M_C , we mean the topological space with point set C and a base given by sets $\{\{x\}: x \in \text{Int}\} \cup \{J_n(x): x \in \text{End}\}$. Thus M_C is the "Michael Cantor set"; we use M_C instead of M purely for technical reasons.

1. **Example E1.** Let $S = \{\text{End}\} \cup \{A_{i,j}: i, j \in \omega\}$ be a partition of C such that the $A_{i,j}$ are second category everywhere subsets of C (Bernstein subsets will do), $\forall i, j \in \omega$. Let $\{e_i: i \in \omega\}$ be an enumeration of End. Then $E1 = P_1 \cup (\bigcup_{i \in \omega} P_2^i)$, where:

- (i) $P_1 = (\text{Int})^\omega \subseteq (M_C)^\omega$, and
- (ii) $P_2^k = \{\sigma \in (M_C)^\omega: \sigma(i) \in \text{Int} \text{ for } i < k, \sigma(k) \in \text{End}, \text{ and for } i > k, \sigma(i) \in A_{j,k} \text{ where } \sigma(k) = e_j\}$ for each $k \in \omega$.

We check that E1 satisfies the hypotheses of Theorem 1.4 of §III. It is obvious that $(M_C, \text{Int}, \text{End})$ is a tame triple, so we only need to check that E1 respects $\{(M_C, \text{Int}, \text{End})_i: i \in \omega\}$. From the definition of E1, condition (i) is satisfied. Letting $\phi_1: \omega \rightarrow \omega$ be defined by $\phi_1(i) = i + 1$, and $\phi_2: \omega^2 \rightarrow \omega$ be defined by $\phi_2(i, j) = \max\{i, j\} + 1$, conditions (ii) and (iii) are met. Hence E1 is a zero-dimensional metrizable space, with $\text{Ind}(E1) \leq 1$. In order to show E1 is complete, by Theorem 1.6 of §I we only need to check that it is approximately complete. To this end, suppose $\sigma \in (M_C)^\omega$ such that for all $i \in \omega$ there is $\sigma_i \in E1$ with $\sigma_i|_i = \sigma|_i$. If no $\sigma(j) \in \text{End}$, then $\sigma \in P_1$, so suppose $\sigma(t) \in \text{End}$. Say $\sigma(t) = e_r$. Then for each $i > t$, σ_i must be in P_2^t , so

$\sigma(j) \in \text{Int}$ for $j < t$, $\sigma(t) = \sigma_i(t) = e_r$, and $\sigma_i(j) \in A_{r,t}$ for $t < j < i$. It follows that $\sigma \in P_2'$. Thus $E1$ is completely metrizable. That $\text{Ind}(E1) > 0$ will be shown in subsection 3 of this section; and \mathbf{N} -compactness is shown in subsection 4.

The following simpler modification of $E1$ can also be shown to have all of the important properties of $E1$. Let M denote the Michael line, \mathbf{Q} the rational points, and \mathbf{I} the irrational points. Let $\{A_i : i \in \mathbf{Q}\}$ partition \mathbf{I} into second category everywhere sets.

Then $E_1' = R_1 \cup (\bigcup_{i \in \omega} R_2^i)$, where

- (i) $R_1 = \mathbf{I}^\omega \subseteq M^\omega$, and
- (ii) $R_2^k = \{\sigma \in M^\omega : \sigma(i) \in I \text{ for } i < k, \sigma(k) \in \mathbf{Q}, \text{ and for } i > k, \sigma(i) \in A_{\sigma(k)}\}$ for $k \in \omega$.

We remark that $E1$ and $E1'$ have first countable zero-dimensional compactifications. To see this observe that M_C and M embed, in a natural way, in the Alexandroff duplicate, $A(C)$, of C . Thus $E1$ and $E1'$ embed in $A(C)^\omega$.

2. Example E2. In order to get a subspace of Δ , $E1$ must be changed so that factors are locally compact and locally countable, yet closures of subsets of A are still "sufficiently large" in each factor.

Let $P = \{B\} \cup \{A_{b,k} : b \in B, K \in \omega\}$ be a partition of C into Bernstein sets. Let $A = C \setminus B$, and let $S(B) = \{(s_1, s_2) : s_1, s_2 \text{ are countably infinite subsets of } A \text{ with } |\text{cl}(s_1) \cap \text{cl}(s_2)| = c\}$. A simple transfinite induction argument yields:

2.1. Lemma. *There is a bijection $f : B \rightarrow S(B)$ satisfying: if $f(b) = (s_1, s_2)$ then $|J_n(b) \cap s_1| = |J_n(b) \cap s_2| = \omega$ for all $n \in \omega$.*

Use this lemma to get f ; let $f(b)$ be denoted by $(f_1(b), f_2(b))$. A sequence τ_b can be chosen so that, for $i \in \omega$:

- (i) $\tau_b(2i + 1) \in f_1(b)$,
- (ii) $\tau_b(2i) \in f_2(b)$, and
- (iii) $\tau_b(i) \in J_i(b)$.

Let $\tau_b^n = \{\tau_b(m) : m \geq n\}$, and let $A_k = \bigcup_{b \in B} A_{b,k}$. Finally let C' denote the topological space with point set C , and a basis

$$\{\{a\} : a \in A\} \cup \{\{b\} \cup \tau_b^n : b \in B, n \in \omega\}.$$

It is clear that C' is a locally countable, locally compact, first countable refinement of C and that (C', A, B) is a tame triple, in the van Douwen line tradition.

Example $E2 = P_1 \cup (\bigcup_{i \in \omega} P_2^i)$, where:

- (i) $P_1 = A^\omega \subseteq (C')^\omega$, and
- (ii) $P_2^k = \{\sigma \in (C')^\omega : \sigma(i) \in A \text{ for } i < k, \sigma(k) \in B, \text{ and for } i > k, \sigma(i) \in A_{b,k} \text{ where } \sigma(k) = b\}$ for $k \in \omega$.

Showing E2 satisfies the hypotheses of Theorems 1.4 and 1.6 of §III is much like the proof for E1; we conclude that E2 is a zero-dimensional, completely metrizable space and that $\text{Ind}(E2) \leq 1$.

3. In this subsection we show that $\text{Ind}(E1)$ and $\text{Ind}(E2)$ are positive. The technique here is based on Mrowka's modification of Roy's method. We prove a theorem which is general enough to show that a class of spaces containing E1 and E2 have positive Ind. There is an analogy here with §III; there we wanted the subspace of the product to be small enough to be metrizable; here we want it large enough to have positive dimension.

3.1. Definition. A topology T for C is an *acceptable refinement* of C provided there are subsets A and B of C satisfying:

- (i) $((C, T), A, B)$ is a tame triple,
- (ii) A is second category everywhere on C ,
- (iii) (C, T) is a refinement of C , and
- (iv) if O is open in C , and S_1, S_2 are subsets of A which are both second category everywhere on O , then there is $b \in B$ with $b \in \text{cl}_{(C, T)}(S_1) \cap \text{cl}_{(C, T)}(S_2)$.

3.2. Definition. Let $\{T_i: i \in \omega\}$ be a collection of acceptable refinements of C , with sets A_i, B_i corresponding to sets A and B in Definition 3.1. A subspace K of $\prod_{i \in \omega} (C, T_i)$ is a *full enough* subspace of $\prod_{i \in \omega} (C, T_i)$ provided:

- (i) $\prod_{i \in \omega} A_i \subseteq K$ and,
- (ii) if $a_i \in A_i$ for $i < n$ and $b_n \in B_n$, then there are sets $A'_j \subseteq A_j$ for each $j > n$ such that A'_j is second category everywhere on C and $\langle (a_0, a_1, \dots, a_{n-1}, b_n); D(a_0, a_1, \dots, a_{n-1}, b_n) \rangle \subseteq K$ where $D(a_0, a_1, \dots, a_{n-1}, b_n) = \prod_{n < j < \omega} A'_j$.

We remark that both E1 and E2 are full enough subspaces of products of acceptable refinements of C . Thus the following theorem gives that $\text{Ind}(E1) = \text{Ind}(E2) = 1$.

3.3. Theorem. Let $\{T_i: i \in \omega\}$ be a collection of acceptable refinements of C , with $((C, T_i), A_i, B_i)$ denoting the appropriate tame triples, and let K be a full enough subspace of $\prod_{i \in \omega} (C, T_i)$ which respects $\{((C, T_i), A_i, B_i): i \in \omega\}$. Then

- (i) K is a zero-dimensional metrizable space, and
- (ii) $\text{Ind}(K) = 1$.

Since K is respectful we need only show that $\text{Ind}(K) > 0$, since Theorem 1.4 of §III applies.

3.4. Proof of Theorem 3.3. We assume for all $i \in \omega$ and $b \in B_i$, a nested local base $\{N_n^i(b): n \in \omega\}$ in (C, T_i) (as in 2.1 of §III), with the further restriction that $N_n^i(b) \subseteq J_n(b)$; we also assume neighborhoods $\{V_i(\sigma): i \in \omega, \sigma \in k\}$ as

in Definition 2.2 of §III, and that $K = P_1 \cup P_2^0 \cup P_2^1 \cup \dots$ as in Definition 1.3 of §III.

We now prove several lemmas which culminate in producing two closed subsets of K with no partition of K between them.

3.4.1. Lemma. *Suppose the following are given.*

- (a) $k, s \in \omega, s \geq 1,$
- (b) $a_i \in A_i$ for $i < k,$
- (c) $b_k \in B_k,$
- (d) open sets O_1, O_2, \dots, O_s in C with $b_k \in O_1,$ and
- (e) $\{U_1, U_2\}$ an open cover of $K.$

Then there are

- (a') $i \in \{1, 2\},$
- (b') $s' \geq s,$
- (c') open sets $O'_1, O'_2, \dots, O'_{s'}$ in $C,$ and
- (d') a subset D of $D(a_0, a_1, \dots, a_{k-1}, b_k)|_{s'}$

which is super S -full on $\prod_{j=1}^{s'} (O'_j \cap A_{k+j})$ and which satisfies:

- (i) $O'_j \subseteq O_{j+1}$ for $1 \leq j < s,$ and
- (ii) if $\sigma \in \overline{\langle (a_0, \dots, a_{k-1}, b_k); \bar{D} \rangle},$ then $V_{k+s'}(\sigma) \subseteq U_i.$

Proof. Let \hat{D} denote the set $\{d \in D(a_0, a_1, \dots, a_{k-1}, b_k) : d(i) \in O_{i+2}$ for $i \leq s - 2\}.$ (We view \hat{D} as a collection of sequences.) For each $d \in \hat{D},$ let $\sigma_d = \langle (a_0, \dots, a_{k-1}, b_k); d \rangle \in K.$ For $m \in \omega,$ let $S_m = \{d \in \hat{D} : V_{k+m}(\sigma_d) \subseteq U_1 \text{ or } U_2\}.$ Then it is easy to check that $\{S_m : m \in \omega\}$ satisfies the conditions of Lemma 4 of §II, so there is s' such that $S_{s'}$ contains $\hat{S}_{s'}$ where $\hat{S}_{s'}|_{s'}$ is an S -full subset of $\prod_{k < j \leq k+s'} (C, T_j).$ Furthermore, if $e \in \hat{S}_{s'}|_{s'}$ and $\sigma \in \overline{\langle (a_0, a_1, \dots, a_{k-1}, b_k); e \rangle},$ then $V_{k+s'}(\sigma) \subseteq U_1 \text{ or } U_2.$ We may assume $s' \geq s.$

By the conditions on $\hat{D}, \hat{S}_{s'}|_{s'}$ is S -full on $\prod_{1 < j \leq s} O_j \times \prod_{k+s \leq j \leq k+s'} (C, T_j).$ Let $H_i = \{e \in \hat{S}_{s'} : V_{k+s'}(\sigma_e) \subseteq U_i\}$ for $i = 1, 2.$ Then $H_1 \cup H_2 = \hat{S}_{s'},$ and by Lemma 3 of §II, there is $i \in \{1, 2\}$ such that $H_i|_{s'}$ contains a subset D' of $\hat{S}_{s'}|_{s'}$ which is S -full on $\prod_{1 < j \leq s} O_j \times \prod_{k+s \leq j \leq k+s'} (C, T_j).$ By applying Lemma 5 of §II we get $O'_1, O'_2, \dots, O'_{s'},$ and $D \subset D'.$ \square

3.4.2. Lemma. *Suppose the following are given:*

- (a) $k, s \in \omega, s \geq 1,$
- (b) $a_j \in A_j$ for $j < k,$
- (c) O_1, O_2, \dots, O_s open sets in $C,$
- (d) $\{U_1, U_2\}$ an open cover of $K,$ and
- (e) D_1, D_2 super S -full subsets of $\prod_{1 \leq j \leq s} (O_j \cap A_{k+j-1})$ satisfying, for $i \in \{1, 2\}, \overline{\langle (a_0, \dots, a_{k-1}); D_i \rangle} \subseteq U_i.$

Then there are

- (a') $s' \geq s$,
- (b') $a_k \in A_k$,
- (c') O'_1, \dots, O'_s , open sets in C , and
- (d') D'_1, D'_2 super S -full subsets of $\prod_{1 \leq j \leq s'} (O'_j \cap A_{k+j})$ satisfying, for $i \in \{1, 2\}$, $\langle (a_0, \dots, a_{k-1}, a_k); D'_i \rangle \subset U_i$.

Proof. For $j \in \{1, 2\}$, let $F_j = \{d(0) : d \in D_j\}$. Then F_1, F_2 are both second category everywhere on O_1 , thus, from the fact that T_k is an acceptable refinement of C , there is $b_k \in B_k$ with $b_k \in \text{cl}_{(C, T_k)}(F_1) \cap \text{cl}_{(C, T_k)}(F_2)$. Clearly b_k can be chosen to be in O_1 . Using k, s , the O_i 's, b_k , and $\{U_1, U_2\}$, we can apply the previous lemma to get

- (i) $i \in \{1, 2\}$,
- (ii) $s' \geq s$,
- (iii) open sets O'_1, O'_2, \dots, O'_s , in C , and
- (iv) a subset D'_i of $D(a_0, \dots, a_{k-1}, b_k)|_{s'}$ which is super S -full on $\prod_{1 \leq j \leq s'} (O'_j \cap A_{k+j})$, and such that $O'_j \subseteq O_{j+1}$ for $1 \leq j \leq s-1$ and for $\sigma \in \langle (a_0, \dots, a_{k-1}, b_k); D'_i \rangle$, $V_{k+s'}(\sigma) \subseteq U_i$.

Now, in (C, T_k) , if $a \in A_k \cap N_{k+s'}^k(b_k)$, then $\langle (a_0, a_1, \dots, a_{k-1}, a); D'_i \rangle \subseteq U_i$. We may choose $a_k \in A_k \cap N_{k+s'}^k(b_k)$ to be in F_2 if $i = 1$ or F_1 if $i = 2$. Suppose $i = 1, a_k \in F_2$. Then D'_1 is as required (using the O_i 's).

To find D'_2 , first let $\hat{D}_2 = \{\sigma \in \prod_{k \leq j < \omega} A_j : \sigma|_s \in D_2, \sigma(0) = a_k, \text{ and } \sigma(i) \in O'_i \text{ for } 1 \leq i \leq s'\}$; let $\hat{\hat{D}}_2 = \{(\sigma(1), \sigma(2), \dots) : \sigma \in \hat{D}_2\}$ and finally let $D'_2 = \hat{\hat{D}}_2|_{s'}$. The super S -fullness on D'_2 follows from the super S -fullness of D_2 and the fact that all A_n 's are second category everywhere. \square

3.4.3. Lemma. Let $n \in \omega$ and let F_1, F_2 be subsets of $\prod_{i < n} A_i$ such that there are two distinct integers n_1 and n_2 with $\pi_{n_1}(F_1) \cap \pi_{n_1}(F_2) = \emptyset$ for $i \in \{1, 2\}$. Then $\text{cl}_K(\overline{\langle F_1 \rangle}) \cap \text{cl}_K(\overline{\langle F_2 \rangle}) = \emptyset$.

Proof. If $\sigma \in P_1$, then either $\sigma(n_1) \notin \pi_{n_1}(F_1)$ or $\sigma(n_1) \notin \pi_{n_1}(F_2)$. Thus $V_{n_1+1}(\sigma)$ will miss one of F_1 and F_2 . For $\sigma \in P_2^k$, then one of $n_1, n_2 \neq k$, say $n_1 \neq k$. It follows that $V_{n_1+1}(\sigma)$ misses either F_1 or F_2 . \square

We now exhibit two closed sets in K which permit no clopen partition of K between them. For $i \in \{1, 2\}$, partition A_i into two sets A_i^1 and A_i^2 such that each of the partition elements is second category everywhere on C . Let $D_i = A_i^1 \times A_i^2$ for $i \in \{1, 2\}$. Choose $a_0 \in A_0$ arbitrarily.

Let $F_i = \text{cl}_K(\overline{\langle (a_0); D_i \rangle})$. It is clear from the above lemma that F_1 and F_2 are nonintersecting closed sets in K .

3.4.4. There is no clopen partition of K between F_1 and F_2 , hence $\text{Ind}(K) > 0$.

Proof. Suppose $\{U_1, U_2\}$ is an open cover of K with $F_1 \subseteq U_1$, $F_2 \subseteq U_2$. Using $k = 1$, $s = 2$, $O_1 = O_2 = C$, a_0 , and D_1, D_2 as above, we can apply Lemma 3.4.3 to get $s' \geq s$, $a_1 \in A_1$, $O'_1, \dots, O'_{s'}$ open in C and D'_1, D'_2 super S -full sets satisfying the conditions of 3.4.3. Lemma 3.4.3 can be applied repeatedly in this fashion to produce a_2, a_3, a_4, \dots such that letting $\sigma = (a_0, a_1, a_2, \dots)$, we have $\sigma \in P_1$, and each neighborhood of σ intersects U_1 and U_2 . Since $\{U_1, U_2\}$ covers K , $U_1 \cap U_2 \neq \emptyset$. \square

4. The purpose of this section is to show that spaces E1 and E2 are both N-compact. The following result is easy and well known.

4.1. **Lemma.** *If (C', T) is zero-dimensional, where $C' \subseteq C$ and T gives a topology finer than the usual topology on C' as a subspace of C , then (C', T) is hereditarily N-compact.*

The following corollary gives that both E1 and E2 are N-compact.

4.2. **Corollary.** *Let $\{(C_i, T_i) : i \in \omega\}$ be a collection of refinements of C ; then $\prod_{i \in \omega} (C_i, T_i)$ is hereditarily N-compact on zero-dimensional subspaces.*

Proof. $\prod_{i \in \omega} (C_i, T_i)$ is a refinement of C^ω which is homeomorphic to C , thus Lemma 4.1 can be applied. \square

V. A GAP METRIC SPACE OF WEIGHT ϵ WHOSE SUBSPACES OF WEIGHT SMALLER THAN ϵ ARE ALL STRONGLY ZERO-DIMENSIONAL

A gap metric space is now described which contains no gap metric subspace of weight less than ϵ . The factor spaces are refinements of the usual plane; in fact, any Euclidean space of dimension greater than or equal to two could be used in place of the plane, because we only make use of certain of its connectedness properties. It is interesting to compare this example with those in §II; here highly connected factor spaces allow the sets $A_{b,k}$ to be spread out in relation to each other, whereas in §II, in order to get N-compactness, the factor spaces were highly disconnected and the sets $A_{b,k}$ then had to be extremely dense.

1.1. The factor space is a refinement of R^2 (the plane with the usual topology). Let B be a Bernstein subset of R^2 , and let $A = R^2 \setminus B$. Then R^{2*} denotes the points set R^2 with topology generated by $B = \{\{\sigma\} : \sigma \in A\} \cup \{N_n(b) : b \in B\}$ where $N_n(b)$ is the set of points whose distance to b is less than $1/n$. It is easy to verify that (R^2, A, B) is a tame triple.

1.2. **Example E3.** Let $\{A_i : i \in \omega\}$ be a collection of subsets of A such that:

(i) if $a_i \in A_i$, $a_j \in A_j$ and $i \neq j$, then the Euclidean distance between a_i and a_j is at least 1, and

(ii) $|A_i| = \epsilon$ for all $i \in \omega$.

By (ii), each A_i can be partitioned into $\{A_{b,i} : b \in B\}$ where each $A_{b,i}$ is uncountable.

Example E3 is the subspace of $(R^{2*})^\omega$ given by $E3 = P_1 \cup (\bigcup_{i \in \omega} P_2^i)$ where $P_1 = A^\omega$, and for $k \in \omega$, $P_2^k = \{\sigma \in (R^{2*})^\omega : \sigma(i) \in A \text{ for } i < k, \sigma(k) \in B, \text{ and for } i > k, \sigma(i) \in A_{b,k} \text{ where } \sigma(k) = b\}$.

Clearly E3 satisfies the hypotheses of Theorem 1.4 of §III, and so is metrizable, zero-dimensional, and $\text{Ind}(E3) \leq 1$. It is also easy to check that E3 is complete using Theorem 1.6 of §III.

1.3. $\text{Ind}(E3) \geq 1$. The proof of this follows the same general pattern as the proof in §IV for examples E1 and E2; in fact, in many ways it is easier. The following lemma is analogous to Lemma 3.4.1 of §IV, and the proof is omitted.

1.3.1. **Lemma.** *Suppose $\{U_1, U_2\}$ is an open cover of E3, $k \in \omega$, for $j < k$, $a_j \in A$, and $b_k \in B$ are all given. Then there is an $m > 0$, an $i \in \{1, 2\}$ and a U -full subset C of $(A_{b,k})^m$ satisfying: If $\sigma \in \overline{\langle\langle a_0, \dots, a_{k-1}, b_k \rangle\rangle; C}$, then $V_{k+m}(\sigma) \subseteq U_i$.*

(Here $V_t(\sigma)$ is defined as in §III, 2.2, so that for $\sigma \in P_1$, $V_t(\sigma) = \{\tau \in E3 : \tau|_t = \sigma|_t\}$ and for $\sigma \in P_2^k$ and $t > k$, $V_t(\sigma) = \{\tau \in E3 : \tau|_k = \sigma|_k, \tau(k) \in N_t(\sigma(k))\}$, and for $k < j < t$, $\tau(j) = \sigma(j)\}$.)

The next lemma is analogous to Lemma 3.4.2 of §IV, but displays the use of the connectedness of the plane.

1.3.2. **Lemma.** *Suppose $\{U_1, U_2\}$ is an open cover of E3, $k \in \omega$, and for $i < k$, $a_i \in A$ are given. Further suppose that for each $j \in \{1, 2\}$ there is $m_j > 0$ and C_j a U -full subset of A^{m_j} such that $\overline{\langle\langle a_0, \dots, a_{k-1} \rangle\rangle; C_j} \subseteq U_j$. Then there is $a_k \in A$, and for $j \in \{1, 2\}$ an $n_j > 0$ and U -full sets $D_j \subseteq A^{n_j}$ such that*

$$\overline{\langle\langle a_0, \dots, a_{k-1}, a_k \rangle\rangle; D_j} \subseteq U_j.$$

Proof. For each $b \in B$, apply Lemma 1.3.1 to get i_b, m_b and C_b with C_b a U -full set in A^{m_b} satisfying: if $\sigma \in \overline{\langle\langle a_0, \dots, a_{k-1}, b \rangle\rangle; C_b}$, then $V_{k+m_b}(\sigma) \subseteq U_{i_b}$. Thus for $a \in A \cap N_{k+m_b}(b)$, $\langle\langle a_0, \dots, a_{k-1}, a \rangle\rangle; C_b \subset U_{i_b}$.

The collection $\{N_{k+m_b}(b) : b \in B\}$ is an open cover of B (in R^2), hence of all of R^2 except possibly a countable set. For $j \in \{1, 2\}$, let $V_j = \{N_{k+m_b}(b) : i_b = j\}$, and consider two cases.

Case 1. One of the V_j 's is empty, say $V_1 = \emptyset$. Because C_1 is U -full, $C'_1 = \{c(0) : c \in C_1\}$ is uncountable, and since B is Bernstein, there is $b \in B$ with $N_{k+m_b}(b)$ containing $a \in C'_1$. Let $a = a_k$ and let $n_1 = 1$. Then let $D_1 = \{c(1), \dots, c(m_1 - 1) : c \in C_1, \text{ and } c(0) = a_k\}$. (If $m_1 = 1$, let $n_1 = 1$ and let D_1 be any uncountable subset of A .) Let $n_2 = m_b$ and $D_2 = C_b$.

Case 2. Both V_1 and V_2 are nonempty. Then there are for $j \in \{1, 2\}$, $b_j \in B$ with $N_{k+m_{b_j}}(b_j) \in V_j$ and $N_{k+m_{b_1}}(b_1) \cap N_{k+m_{b_2}}(b_2) \neq \emptyset$. This is because no countable set separates the plane. Then $a_k \in N_{k+m_{b_1}}(b_1) \cap N_{k+m_{b_2}}(b_2) \cap A$ can

be chosen, and setting $n_j = m_{b_j}$ and $D_j = C_{b_j}$, the conditions of the lemma are satisfied. \square

1.3.3. We describe the closed sets which exhibit $\text{Ind} > 0$. Choose $a \in A$ and let S_1, S_2 be a pair of nonintersecting uncountable closed subsets of R^2 (so $|S_i \cap A| = \mathfrak{c}$ for $i \in \{1, 2\}$). Then, setting $F_i = \overline{\langle (a); S_i \rangle}$ and repeatedly applying Lemma 1.3.2, it is shown that there is no clopen set containing F_1 and which is disjoint from F_2 .

2. We now show that if $E3' \subset E3$ and $\text{weight}(E3') < \mathfrak{c}$, then $\text{Ind}(E3') = 0$.

2.1. **Lemma.** *If $E3' \subset E3$, then for each $i \in \omega$, $|\pi_i(E3')| \leq \text{weight}(E3)$. Thus if $\text{weight}(E3') < \mathfrak{c}$, $\pi_i(E3')$ is strongly zero-dimensional when viewed as a subset of R^2 .*

Proof. Let $\alpha = |\pi_i(E3')|$. We show $\alpha \leq \text{weight}(E3')$ by showing that $E3'$ contains α many pairwise disjoint distinct open sets. Suppose $|\pi_i(E3') \cap A| = \alpha$. Then for each $a \in (\pi_i(E3') \cap A)$ let σ_a be a point of $P_1 \cap E3'$ such that $\sigma_a(i) = a$. It follows that $\{V_{i+1}(\sigma_a) \cap E3'\}$ is a pairwise disjoint collection of nonempty open sets in $E3'$ of cardinality α ; hence $\alpha \leq \text{weight}(E3')$. Similarly, if $|\pi_i(E3') \cap B| = \alpha$, we can find $\tau_b \in P_2 \cap E3'$ for each $b \in (\pi_i(E3') \cap B)$, with $\tau_b(i) = b$. Then $\{V_{i+2}(\tau_b) \cap E3'\}$ is the desired collection. \square

By this result we will be done when we show

2.2. **Lemma.** *If $E3' \subset E3$, and for each $i \in \omega$, $\pi_i(E3')$ is a zero-dimensional subset of R^2 , then $\text{Ind}(E3') = 0$.*

Proof. The goal is to construct a sequence $\{H_i : i \in \omega\}$ of covers of $E3'$ such that for all $i \in \omega$, H_{i+1} refines H_i , $\text{order}(H_i) = 1$ and $\bigcup_{i \in \omega} H_i$ forms a base for $E3'$. It will follow by [N, Theorem 12-6] that $\text{Ind}(E3') = 0$.

First we get such covers in the factors. For each $i \in \omega$, let $\{G_n^i : n \in \omega\}$ be a collection of covers of $\pi_i(E3')$ (open in the R^2 topology) such that for each $n \in \omega$, $\text{order}(G_n^i) = 1$, G_{n+1}^i refines G_n^i , and if $g \in G_n^i$ then the (R^2) diameter of g is less than $1/n + 1$. The G_n^i 's can be obtained because $\pi_i(E3')$ is a strongly zero-dimensional subset of R^2 . Note that if $g \in \bigcup_{n \in \omega} G_n^i$ then g can intersect no more than one element of $\{A_k : k \in \omega\}$, by our choice of A_i 's. For $p \in \pi_k(E3')$, let $g_n^k(p)$ denote the element of G_n^k which contains p . We may assume that if $\sigma \in P_1$, and $\sigma|_n = \tau|_n$ for some $\tau \in E3'$, then there is $\gamma \in E3' \cap P_1$ with $\gamma|_n = \sigma|_n$. Simply add the sequence δ with $\delta(i) = \sigma(i)$ for $i < n$, and $\delta(i) = 0$ for $i > n$. (Make sure 0 is not in B .) We can also assume that if $\tau \in E3' \cap P_1$, then $\{\gamma(n) : \gamma|_n = \tau|_n\}$ is dense in R^2 , by choosing a countable dense subset of $A \cap R^2$, and letting elements of that set "follow" $\tau|_n$.

We now define a new base on $E3'$.

- (i) For $\sigma \in E3' \cap P_1$, and $m \in \omega$, $U_m(\sigma) = \{\tau \in E3' : \tau|_m = \sigma|_m \text{ and } \tau(m) \in g_m^m(\sigma)\}$;

- (ii) For $k \in \omega$, $\sigma \in E3' \cap P_2^k$, and $m > k$, let $U_m(\sigma) = \{\tau \in E3': \tau|_k = \sigma|_k, \tau(k) \in g_m^k(\sigma(k)), \text{ for } k < i < m, \tau(i) = \sigma(i), \text{ and } \tau(m) \in g_m^m(\sigma(m))\}$.

It is easy to verify that $\{U_m(\sigma): \sigma \in E3', m \in \omega (m > k \text{ if } \sigma \in P_2^k)\}$ forms a base for $E3'$.

Setting $H_i^1 = \{U_i(\sigma): \sigma \in P_1\}$ and $H_i^2 = \{U_i(\sigma): \sigma \in P_2^0 \cup \dots \cup P_2^{i-1}\}$, $H_i^1 \cup H_i^2$ covers $E3'$, and each is order one. If $h \in H_i^1$ and $k \in H_i^2$ intersect, then $h \subseteq k$. Let $H_i = H_i^2 \cup \{h \in H_i^1: h \text{ is not a subset of } \bigcup H_i^2\}$. Then $\{H_i: i \in \omega\}$ satisfies the conditions laid out at the beginning of the proof. \square

2.3. *Remark.* By carefully choosing the sets $\{A_{b,i}: b \in B, i \in \omega\}$ and a subspace of the so-defined version of E3 an example which is both \mathbf{N} -compact and contains no subspace with weight less than \mathfrak{c} of positive dimension can be found.

VI. CONSISTENT EXAMPLES OF METRIZABLE SPACES OF POSITIVE Ind FOR WHICH SMALL SUBSPACES HAVE Ind = 0

For λ an ordinal, we consider λ a topological space with the order topology; a subset of λ is assumed to have the subspace topology. We let $\text{Suc}(\lambda)$ and $\text{Lim}(\lambda)$ respectively denote the sets of successors and limits in λ .

For a cardinal κ , $E(\kappa)$ is the statement: There is a set E of ordinals of cofinality ω which is stationary in κ , but for all $\beta < \kappa$, $E \cap \beta$ is not stationary in β .

The following statement, which we call E^* , is consistent with the usual axioms of set theory.

E^* : For all singular strong limit cardinals κ , $2^\kappa = \kappa^+$ and $E(\kappa^+)$.

For a discussion of E^* , the reader should consult [F]. The author would like to thank Bill Fleissner for pointing out E^* and also for the following lemma.

1. **Lemma.** Let κ be a cardinal for which $E(\kappa)$ holds, and let E_κ denote the set guaranteed by $E(\kappa)$. Let $\delta < \kappa$ be fixed, and let $M_\delta = \delta \cap (\text{Suc}(\kappa) \cup E_\kappa)$. Then M_δ is metrizable and $\text{Ind}(M_\delta) = 0$.

Proof. We first get an open cover H of M_δ which is locally finite such that $H = \{\{\beta\}: \beta \in \text{Suc}(\delta)\} \cup \{(\beta', \beta]: \beta \in E_\kappa \cap M_\delta \text{ and } \beta' \text{ depends on } \beta\}$. To get this cover, proceed by transfinite induction; assume for all $\alpha < \delta$ the cover exists, and $\delta < \kappa$. Then there is a C which is closed and unbounded in δ and disjoint from E_κ . Let $D_c = \{c_\beta: \beta < \text{cof}(\delta)\}$ be a relatively discrete, cofinal in δ subset of C . For $c_\beta \in D_c$, let $s_\beta = \sup\{c_\gamma: \gamma < \beta\}$. By the inductive hypothesis there is a cover H'_β of $[0, c_\beta] \cap M_\delta$ as specified. Let $H_\beta = \{h \cap (s_\beta, c_\beta] \cap M_\delta: h \in H'_\beta\}$, and let $H = \bigcup_{\beta < \text{cof}(\delta)} H_\beta$. Now for $\beta \in E_\kappa \cap M_\delta$ let $\{\beta_i: i \in \omega\}$ be an increasing sequence in δ converging to β such

that $(\beta_0, \beta] \subseteq (\beta', \beta]$ where $M_\delta \cap (\beta', \beta] \in H$. For $i \in \omega$, let

$$H_i = \{\{\beta\} : \beta \in \text{Suc}(\delta)\} \cup \{(\beta_i, \beta] \cap M_\delta : \beta \in E_\kappa \cap \delta\}.$$

One easily checks that H_{i+1} refines H_i , $\bigcup_{i \in \omega} H_i$ is a base for M_δ , and H_i is a locally finite collection of clopen sets. Thus M_δ is a metrizable space and $\text{Ind}(M_\delta) = 0$. \square

Immediately from this lemma we get

2. Lemma. *If κ is a regular cardinal and $E(\kappa)$ holds, with E_κ the set guaranteed by $E(\kappa)$, and $\delta < \kappa$, then if $X \subseteq (\text{Suc}(\kappa) \cup E_\kappa)^\omega$ and $\text{weight}(X) \leq \delta$, then X is metrizable and $\text{Ind}(X) = 0$.*

Proof. Let $\alpha = \sup\{\pi_k(X) : k \in \omega\}$. Then $\alpha < \kappa$, and

$$X \subseteq ((\text{Suc}(\kappa) \cup E_\kappa) \cap \alpha)^\omega,$$

and by the previous lemma $((\text{Suc}(\kappa) \cup E_\kappa) \cap \alpha)$ is metrizable and has $\text{Ind} = 0$. Thus $((\text{Suc}(\kappa) \cup E_\kappa) \cap \alpha)^\omega$ is metrizable with $\text{Ind} = 0$, and hence X is metrizable with $\text{Ind} = 0$. \square

3. Theorem (Assume E^*). *For each cardinal λ there is a metrizable space $X(\lambda)$ with $\text{Ind}(X(\lambda)) = 1$ and for all $Y \subseteq X(\lambda)$, with $\text{weight}(Y) \leq \lambda$, $\text{Ind}(Y) = 0$.*

Proof. Our proof is patterned after that in [K1]; we intend to replace the set ω_1 with the set $\text{Suc}(\kappa) \cup E(\kappa)$ for a sufficiently large κ .

Let κ be a strong limit cardinal greater than λ ; then κ^+ is a regular cardinal, and E^* gives that $E(\kappa^+)$ holds. Let E_{κ^+} denote the set it gives.

Our example is a subspace of W^ω where $W = \text{Suc}(\kappa^+) \cup E(\kappa^+)$. Observe that by the previous lemma, any subspace of W^ω of weight $\leq \lambda$ is necessarily metrizable with $\text{Ind} = 0$. So we need only produce a positive dimensional subspace of W^ω . Since $(W, \text{Suc}(\kappa^+), E(\kappa^+))$ is a tame triple we want to satisfy the other hypothesis of Theorem 1.4 of §III. We define $X(\lambda) \subseteq W^\omega$ by $X(\lambda) = P_1 \cup (\bigcup_{i \in \omega} P_2^i)$ where:

- (i) $P_1 = (\text{Suc}(\kappa^+))^\omega$ and for $k \in \omega$,
- (ii) $P_2^k = \bigcup_{\alpha \in E_\kappa} \{\sigma \in W^\omega : \sigma(i) \in \text{Suc}(\kappa^+) \text{ for } i < k, \sigma(k) = \alpha, \sigma(k+1) = \alpha + k, \text{ and for } i > k + 1, \sigma(i) \in \{e + k : e \in \text{Lim}(\kappa^+)\}\}$.

One can check, using $\phi_1(i) = i + 1$ and $\phi_2(i, j) = \max\{i, j\} + 1$, that the hypotheses of Theorem 1.4 of §III are satisfied; also one can show $X(\lambda)$ is complete using Theorem 1.6 of §III. Proving that $\text{Ind}(X(\lambda)) > 0$ is very similar to the proof that $\text{Ind}(Z) > 0$ in [K1], the main difference is in substituting $U(\kappa^+)$ -full for U -full, where a set S has property $U(\kappa^+)$ iff $|S| = \kappa^+$. Also, it can be shown that if F_1, F_2 are disjoint closed subsets of $X(\lambda)$ and the weight of F_1 is less than or equal to λ , then there is a clopen set containing F_1 and missing F_2 . This can be done using the methods in §5 of [K1]. \square

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