TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 318, Number 2, April 1990

SIMPLE LIE ALGEBRAS OF CHARACTERISTIC p WITH DEPENDENT ROOTS

GEORGIA BENKART AND J. MARSHALL OSBORN

ABSTRACT. We investigate finite dimensional simple Lie algebras over an algebraically closed field $\mathbf{F}$ of characteristic $p \geq 7$ having a Cartan subalgebra H whose roots are dependent over $\mathbf{F}$. We show that H must be one-dimensional or for some root α relative to H there is a 1-section $L^{(\alpha)}$ such that the core of $L^{(\alpha)}$ is a simple Lie algebra of Cartan type $H(2 : \underline{m} : \Phi)^{(2)}$ or $W(1 : \underline{n})$ for some $n > 1$. The results we obtain have applications to studying the local behavior of simple Lie algebras and to classifying simple Lie algebras which have a Cartan subalgebra of dimension less than $p - 2$.

1. INTRODUCTION

Let L be a finite-dimensional Lie algebra over an algebraically closed field $\mathbf{F}$ of characteristic $p > 0$ with Cartan subalgebra H. We say that H has *toral rank* m if $\dim_P P\Delta = m$, where Δ is the set of roots of L with respect to H, P is the prime field, and $P\Delta$ is the P-vector space spanned by Δ. We call the roots *dependent* if any two of them are $\mathbf{F}$-linearly dependent. Every one-dimensional Cartan subalgebra has dependent roots, but it may have arbitrarily large toral rank. Any Cartan subalgebra of toral rank one also has dependent roots.

For each $\alpha \in \Delta$, let $L_\alpha = \{x \in L \mid (\mathrm{ad}_h - \alpha(h)I)^n x = 0$ for all $h \in H$, some $n\}$, and let $L^{(\alpha)} = \sum_{i \in P} L_{i\alpha}$. We say that $L^{(\alpha)}$ is the 1-*section* of L determined by α, and when we speak of a 1-section of L we mean $L^{(\alpha)}$ for some $\alpha \in \Delta$. A 1-section is then, in particular, a subalgebra of L which contains H and has toral rank one relative to H. We shall also make use of the subalgebras $L^{(\alpha, \beta)} = \sum_{i, j \in P} L_{i\alpha + j\beta}$ for $\beta \notin P\alpha$, which we refer to as 2-*sections*.

In the case that L is simple, the action of ad_H on each root space can be simultaneously upper triangularized with each ad_h having a unique eigenvalue down the diagonal (see [BO3, Introduction]). With that in mind, we say in the more general case that a Cartan subalgebra H is *triangulable* if for each root space there is a basis relative to which ad_H is upper triangular. If H is triangulable, then in particular, $\alpha([H, H]) = 0$ for each root α.

Received by the editors June 25, 1987 and, in revised form, August 8, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 17B50, 17B20.
The authors gratefully acknowledge the support of N.S.F. grant #DMS 85-01514.

Any 1-section $L^{(\alpha)}$ contains a maximal solvable ideal which we denote by $\operatorname{Rad} L^{(\alpha)}$. Winter [Win] has shown that $L^{(\alpha)} = \operatorname{Rad} L^{(\alpha)}$ if and only if

$$\alpha([L_{i\alpha}, L_{-i\alpha}]) = 0$$

for $1 \leq i \leq p-1$. When H is a triangulable Cartan subalgebra and when $\operatorname{Rad} L^{(\alpha)} \neq L^{(\alpha)}$, then it has been shown in [BO3] that $L^{(\alpha)} / \operatorname{Rad} L^{(\alpha)}$ consists of a simple algebra called $\operatorname{Core} L^{(\alpha)}$ together with certain derivations of $\operatorname{Core} L^{(\alpha)}$ which are contained in $(H + \operatorname{Rad} L^{(\alpha)}) / \operatorname{Rad} L^{(\alpha)}$. When $\operatorname{Core} L^{(\alpha)} \neq (0)$, the Cartan subalgebra $\overline{H}$ of $\operatorname{Core} L^{(\alpha)}$ induced by H has toral rank one. We can now state our main theorem.

Theorem 1.1′. *Let L be a finite-dimensional simple Lie algebra over an algebraically closed field $\mathbf{F}$ of characteristic $p \geq 7$. Assume H is a Cartan subalgebra of L such that the roots of L with respect to H are dependent. Then one of the following holds:*

(i) $\dim_{\mathbf{F}} H = 1$, *or*

(ii) *for some root α of H on L,* $\operatorname{Core} L^{(\alpha)}$ *is a simple Lie algebra of Cartan type $H(2 : \underline{m} : \Phi)^{(2)}$ or $W(1 : \underline{n})$ for $n > 1$.*

In particular, if $\dim_{\mathbf{F}} H < p - 2$, *then* (i) *must hold.*

The simple algebras having a one-dimensional Cartan subalgebra have been classified in [BO2]. They are the Albert-Zassenhaus algebras (which are isomorphic to the algebras $W(1 : \underline{n})$ or to certain of the algebras $H(2 : \underline{m} : \Phi)^{(2)}$) or they are $\mathrm{sl}(2)$. Wilson [Wil2] has shown that a simple Lie algebra having a Cartan subalgebra of toral rank one must be $\mathrm{sl}(2)$, or an algebra of Cartan type $H(2 : \underline{m} : \Phi)^{(2)}$ or $W(1 : \underline{n})$ for $n \geq 1$. Since $L = \operatorname{Core} L^{(\alpha)}$ for some root α whenever L is a simple algebra having a Cartan subalgebra of toral rank one, each toral rank one simple algebra either is in (i) if it is $\mathrm{sl}(2)$ or $W(1 : \underline{1})$ or it occurs as a special case of (ii).

It follows from Wilson's result that the core of every 1-section relative to a triangulable Cartan subalgebra H is (0), $\mathrm{sl}(2)$, $H(2 : \underline{m} : \Phi)^{(2)}$ or $W(1 : \underline{n})$. If H has dimension $< p - 2$, then the core of each 1-section relative to H must be (0), $\mathrm{sl}(2)$, or $W(1 : \underline{1})$ by [BO3]. In light of this, our strategy for proving Theorem 1.1 is to prove the following theorem which is equivalent to Theorem 1.1.

Theorem 1.1′. *Let L be a finite-dimensional simple Lie algebra over an algebraically closed field $\mathbf{F}$ of characteristic $p \geq 7$. Assume H is a Cartan subalgebra of L such that the roots of L with respect to H are dependent and the core of each 1-section relative to H is (0), $\mathrm{sl}(2)$, or the Witt algebra $W(1 : \underline{1})$. Then* $\dim_{\mathbf{F}} H = 1$.

The proof of Theorem 1.1′ is given in §3. Contained in §2 are some definitions and results not requiring the more specialized hypotheses of this theorem.

Perhaps it should be remarked that the known simple Lie algebras over algebraically closed fields of characteristic $p \geq 7$ are either classical (analogues of the complex simple algebras) or of Cartan type. In each classical algebra the Cartan subalgebras are conjugate under the group of automorphisms, and each 1-section is isomorphic to $\mathrm{sl}(2)$. In a Cartan type Lie algebra there may exist infinitely many nonconjugate Cartan subalgebras, and some may have cores different from (0), $\mathrm{sl}(2)$ or $W(1:\underline{1})$. (This phenomenon occurs, for example, in the algebra $H(4:\underline{1})^{(2)}$. The proof of Lemma 2.12 below shows that it can happen even when the roots are dependent.) Since the core of each 1-section must be (0), $\mathrm{sl}(2)$, or $W(1:\underline{1})$ if the Cartan subalgebra has dimension less than $p-2$, Theorem 1.1 provides information helpful in classifying simple Lie algebras having Cartan subalgebras of small dimension.

2. Some general lemmas

Throughout the remainder of the paper we assume that L is a finite-dimensional Lie algebra over an algebraically closed field $\mathbf{F}$ of characteristic $p \geq 7$ and H is a Cartan subalgebra of L. For any two roots α and γ of L, let $N_\gamma^\alpha = \{x \in L_\gamma \mid \alpha([x, L_{-\gamma}]) = 0\}$, and let $N^\alpha = \sum_{\gamma\in\Delta} N_\gamma^\alpha + \sum_{\gamma\in\Delta}[N_\gamma^\alpha, N_{-\gamma}^\alpha]$.

Lemma 2.1. *If H is triangulable, then N^α and $H + N^\alpha$ are subalgebras of L, and N^α is an ideal of $H + N^\alpha$.*

Proof. From $\alpha([H, H]) = 0$ we have

$$\alpha([[N_\gamma^\alpha, H], L_{-\gamma}]) \subseteq \alpha([[N_\gamma^\alpha, L_{-\gamma}], H]) + \alpha([N_\gamma^\alpha, [L_{-\gamma}, H]]) = 0$$

to show $[N_\gamma^\alpha, H] \subseteq N_\gamma^\alpha$. Also,

$$[[N_\gamma^\alpha, N_{-\gamma}^\alpha], H] \subseteq [[N_\gamma^\alpha, H], N_{-\gamma}^\alpha] + [N_\gamma^\alpha, [N_{-\gamma}^\alpha, H]] \subseteq [N_\gamma^\alpha, N_{-\gamma}^\alpha],$$

which will prove that N^α is an ideal in $H+N^\alpha$ and that $H+N^\alpha$ is a subalgebra once we know that N^α is a subalgebra.

Now, for $\gamma, \beta \in \Delta$ with $\gamma \neq -\beta$,

$$\alpha([[N_\gamma^\alpha, N_\beta^\alpha], L_{-\gamma-\beta}]) \subseteq \alpha([[N_\gamma^\alpha, L_{-\gamma-\beta}], N_\beta^\alpha]) + \alpha([N_\gamma^\alpha, [N_\beta^\alpha, L_{-\gamma-\beta}]]) = 0,$$

to give $[N_\gamma^\alpha, N_\beta^\alpha] \subseteq N^\alpha$. Also, for $\gamma, \beta \in \Delta$,

$$[[N_\beta^\alpha, N_{-\beta}^\alpha], N_\gamma^\alpha] \subseteq [H, N_\gamma^\alpha] \subseteq N_\gamma^\alpha,$$
$$[[N_\beta^\alpha, N_{-\beta}^\alpha], [N_\gamma^\alpha, N_{-\gamma}^\alpha]] \subseteq [[N_\beta^\alpha, N_{-\beta}^\alpha], H] \subseteq N^\alpha,$$

which shows that N^α is a subalgebra. □

If the roots of L are dependent, then $N^\alpha = N^\beta$ for any two roots α and β. Thus, in the dependent case, we shall just write N for N^α. Also, the set $H_0 = \{h \in H | \alpha(h) = 0\}$ is independent of the root used to define it in the dependent case. We can then express the dependent case of Lemma 2.1 as

Corollary 2.2. *Assume that* H *is a triangulable Cartan subalgebra with dependent roots, and let* $N_\gamma = \{x \in L_\gamma | [x, L_{-\gamma}] \subseteq H_0\}$ *and* $N = \sum_{\gamma\in\Delta} N_\gamma + \sum_{\gamma\in\Delta}[N_\gamma, N_{-\gamma}]$. *Then* N *and* $H+N$ *are both subalgebras, and* N *is an ideal of* $H+N$.

If α and β are two P-independent roots, then the set of roots $\{i\alpha+\beta\}_{i\in P}$ is called the *α-string of roots through β*. The corresponding sum of root spaces, $S(\alpha, \beta) = \sum_{i\in P} L_{i\alpha+\beta}$ is a module for the 1-section $L^{(\alpha)}$. The irreducible factor modules into which $S(\alpha, \beta)$ decomposes, either regarding $S(\alpha, \beta)$ as an $L^{(\alpha)}$-module or regarding $S(\alpha, \beta)$ as a module for some subalgebra of $L^{(\alpha)}$, are frequently modules of dimension p with dimension one in each root space. Such a factor module will sometimes be described as an *α-string of elements through β*.

We recall next the definition of a long filtration in a Lie algebra L. Let S_0 be any maximal subalgebra of L, and let S_{-1} be any subspace of L containing S_0 such that S_{-1}/S_0 is an irreducible S_0-submodule of L/S_0. For $j \geq 0$, define S_{j+1} inductively by $S_{j+1} = \{x \in S_j | [x, S_{-1}] \subseteq S_j\}$, and for $j \geq 1$ define S_{-j-1} inductively by $S_{-j-1} = [S_{-j}, S_{-j}] + S_{-j}$. In particular, S_1 is the kernel of the action of S_0 on S_{-1}/S_0. It is well known that $[S_i, S_j] \subseteq S_{i+j}$, and that

$$L = S_{-k} \supset \cdots \supset S_{-1} \supset S_0 \supset S_1 \supset \cdots \supset S_l = S_{l+1} = \cdots, \tag{2.3}$$

for some integers k and l. The space S_l is an ideal of L which must be (0) if L is simple. We call (2.3) the filtration of L induced by S_0 and S_{-1}. Corresponding to this filtration is the graded Lie algebra $G = \bigoplus_j G_j$, where $G_j = S_j/S_{j+1}$ for $-k \leq j \leq l$. In practice, once a maximal subalgebra S_0 has been chosen, we talk about S_1 and the other spaces in the filtration (2.3) as well as about G_0 and the other components of G with the implicit understanding that some S_{-1} has been chosen to determine completely the filtration and the resulting graded algebra. We also adopt the convention that if $v \in S_j - S_{j+1}$, then $\overline{v} = v + S_{j+1} \in G_j$. We will need the following result in §3.

Proposition 2.4. *Let* S_0 *be a maximal subalgebra of* L *containing the Cartan subalgebra* H *of* L. *Then either* (i) G_0 *is simple,* (ii) G_0 *contains a proper ideal, or* (iii) G_0 *is one-dimensional and* S_0 *contains an ideal* I *of* L *such that* S_0/I *is solvable and* L/I *has toral rank one relative to* $(H+I)/I$. *If* $\overline{J}$ *is a proper ideal of* G_0, *then* $\overline{J}$ *contains an* $\overline{x} \in \overline{J}_\zeta$ *for some* $\zeta \in \Delta \cup \{0\}$ *such that* $\mathrm{ad}_{\overline{x}}$ *acts nonnilpotently on* G_{-1}. *If* $L^{(\alpha)} \subseteq S_0$ *for some* $\alpha \in \Delta$ *and if* L *has toral rank two relative to* H, *then* $x \in L_{i\alpha}$ *for some* $i \in P$.

Proof. Clearly, there are three possibilities: (i) G_0 is simple, (ii) G_0 contains a proper ideal, or (iii) G_0 is one-dimensional. In the third case, S_0 is spanned modulo S_1 by an element of H. Any weight vector of G_{-1} spans a G_0-submodule of G_{-1} so that G_{-1} must be one-dimensional. Hence $S_{-1} = \mathbf{F}v + S_0$, and S_{-1} is a subalgebra, which is necessarily L by the maximality of S_0. If v corresponds to the root β, then by an inductive argument it follows that on S_j/S_{j+1} the Cartan subalgebra H has only the root $-j\beta$. Let l be the least integer such that $S_l = S_{l+1} = \cdots$. Then S_l is an ideal of L contained in S_1 and S_0/S_l is solvable. Modulo S_l the Cartan subalgebra $H + S_l$ has toral rank one. This establishes the first assertion of the proposition.

Suppose now that $\overline{J}$ is a proper ideal of G_0, and that $\overline{J}$ does not contain a root vector or an element of $H + S_1/S_1$ which acts nonnilpotently on G_{-1}. Then $\bigcup_{\zeta\in\Delta\cup\{0\}} \operatorname{ad}\overline{J}_\zeta$ is a weakly closed set of nilpotent transformations on G_{-1} so that by [J, Theorem 3.1′] or [BW, Theorem 1.10.1] there exists a nonzero vector in G_{-1} annihilated by $\operatorname{ad}_{\overline{J}}$. The space of all vectors annihilated by $\operatorname{ad}_{\overline{J}}$ is a G_0-submodule of G_{-1}, and so is all of G_{-1}. This contradicts the fact that G_0 must act faithfully on G_{-1} by its definition, and shows that there exists an $\overline{x} \in \overline{J}_\zeta$ for some $\zeta \in \Delta \cup \{0\}$ acting nonnilpotently on G_{-1}.

Assume now that $L^{(\alpha)} \subseteq S_0$ for some α and that L has toral rank two relative to H. An element $\overline{x} \in \overline{J}_\zeta$ cannot act nonnilpotently on G_{-1} for $\zeta \notin P\alpha$, since $\operatorname{ad}_{\overline{x}}$ will take each root vector of G_{-1} into $\sum_j G_{-1,j\alpha} = (0)$ after a finite number of steps. Thus, for any $\overline{x} \in \overline{J}_\zeta$ acting nonnilpotently on G_{-1}, $\zeta \in P\alpha$. □

Consider now the associative commutative divided power algebra $\mathbf{O}_n$ over $\mathbf{F}$ having a basis of formal powers, $\{x^{(j)} | j = 0, \dots, p^n - 1\}$ and having multiplication given by $x^{(i)}x^{(j)} = \binom{i+j}{j}x^{(i+j)}$. The derivation algebra of $\mathbf{O}_n$ is denoted by $W(1 : \underline{n})$ in Wilson's notation [Wil1], and it is a simple Lie algebra of dimension p^n whenever $p^n \geq 3$. Let ∂ be the derivation of $\mathbf{O}_n$ given by $\partial(x^{(j)}) = x^{(j-1)}$. Then $W(1 : \underline{n})$ has as basis the derivations $y_j = x^{(j+1)}\partial$ where $j = -1, 0, \dots, p^n - 2$, with multiplication given by

$$[y_i, y_j] = \left(\binom{i+j+1}{j} - \binom{i+j+1}{i}\right) y_{i+j}. \tag{2.5}$$

The elements y_j for $j = 0, 1, \dots, p^n - 2$ span a maximal subalgebra which is solvable.

Suppose now that we are in the special case where $n = 1$, and let $t^i = i!x^{(i)}$. Then $t^i t^j = t^{i+j}$ whenever $0 \leq i + j \leq p - 1$, and otherwise the product is 0. Thus, $\mathbf{O}_1$ is isomorphic to the truncated polynomial algebra in one variable. The derivation algebra $W = W(1 : \underline{1})$ is the *Witt algebra*, and it has as a basis the elements $e_i = t^{i+1}\partial/\partial t$ for $i = -1, 0, \dots, p-2$, which have multiplication given by

$$[e_i, e_j] = (j - i)e_{i+j}, \tag{2.6}$$

where $e_{i+j} = 0$ if $i + j = -2$ or if $i + j > p - 2$. An algebra W with such a basis and multiplication will be referred to as a *Witt algebra with proper basis.* On the other hand, if the elements $e_i = (1+t)^{i+1}\partial/\partial t$ are used instead as a basis for W, then the multiplication is given by (2.6) with the rule that the subscript of e_{i+j} is to be interpreted modulo p, and W is called a *Witt algebra with* *group basis*. Any Cartan subalgebra H of W is one-dimensional with $H = \mathbf{F}e_0$ where e_0 is part of a proper or group basis. We will need the following result of Weisfeiler.

Proposition 2.7 [We]. *Let L be a simple Lie algebra over an algebraically closed field of characteristic $p \geq 7$, and let S_0 be a maximal subalgebra of L. If S_0 is solvable, then L is isomorphic to either* sl(2) *or to $W(1:\underline{n})$ for some $n \geq 1$.*

A refinement of Weisfeiler's result that will be most helpful to us is

Proposition 2.8. *Let L be a simple Lie algebra with a Cartan subalgebra H, and let S_0 be a maximal subalgebra of L containing H. If S_0 is solvable, then L has toral rank one relative to H, and H has dimension p^{n-1} for some $n \geq 1$.*

Proof. By Proposition 2.7, L is either sl(2) or $W(1:\underline{n})$. When L is sl(2) then every Cartan subalgebra has both rank one and toral rank one so the result holds in this case. Brown [Br] has shown that a Cartan subalgebra of $W(1:\underline{n})$ is one of the following: (i) a Cartan subalgebra of dimension p^{n-1} contained in a subalgebra of codimension one, or (ii) a one-dimensional Cartan subalgebra. In the second case there is a basis $\{u_\sigma | \sigma \in \Omega\}$ for some additive subgroup Ω of $\mathbf{F}$ with $H = \mathbf{F}u_0$, and with multiplication given by $[u_\sigma, u_\tau] = (\tau - \sigma)u_{\sigma+\tau}$ (see [Br, Theorem 4]). By Corollary 3.4 of [BIO], the subalgebra of $W(1:\underline{n})$ of codimension one is unique, and hence it is the solvable maximal subalgebra M_0 spanned by the elements y_j for $j \geq 0$ as in (2.5). Therefore in the first possibility, the Cartan subalgebra is contained in M_0. Since L/M_0 is one-dimensional in this situation, it follows as in case (iii) of Proposition 2.4 or from [Be, Theorem 3.24], that H has toral rank one. In the remaining case to be considered there is a basis $\{u_\sigma | \sigma \in \Omega\}$ with $H = \mathbf{F}u_0$ and $[u_\sigma, u_\tau] = (\tau - \sigma)u_{\sigma+\tau}$ for all $\sigma, \tau \in \Omega$. If S_0 is a solvable maximal subalgebra containing such a Cartan subalgebra H, then S_0 must contain at least two root vectors u_σ, u_τ for $\sigma \neq \tau$, or it would not be maximal. If σ and τ are P-dependent, the subalgebra generated by u_σ and u_τ contains a copy of sl(2). If σ and τ are P-independent, then repeated applications of ad_{u_σ} and ad_{u_τ} show that both $u_{\sigma+\tau}$ and $u_{(p-1)\sigma+(p-1)\tau}$ lie in S_0. However, $u_{\sigma+\tau}$, u_0, $u_{(p-1)\sigma+(p-1)\tau}$ span a copy of sl(2). Thus in either event, S_0 contains a copy of sl(2), which contradicts the solvability of S_0. Hence this final case cannot occur, and every Cartan subalgebra contained in a solvable maximal subalgebra has toral rank one and dimension p^{n-1} for some $n \geq 1$. □

The p-dimensional irreducible modules for the Witt algebra will play a role in our proof of Theorem 1.1$'$. In particular, we will need the following result of Block describing them.

Lemma 2.9 [Bl1, Theorem 6.1]. *Let W be a Witt algebra over $\mathbf{F}$ with group basis $\{f_i | i = -1, \ldots, p-2\}$, and let λ and μ be any two elements of $\mathbf{F}$.*

Then there is a p-dimensional W-module $V(\lambda, \mu)$ *with basis* $v_0, v_1, \dots, v_{p-1}$, *such that*

$$f_i v_k = (\lambda + i\mu + k)v_{i+k}, \tag{2.10}$$

where the subscripts are to be read modulo p. *The module* $V(\lambda, \mu)$ *is irreducible unless* $\lambda \in P$ *and* $\mu = 0$ *or* 1. *Every irreducible* W*-module of dimension* p *is of the form* $V(\lambda, \mu)$ *for some* $\lambda, \mu \in \mathbf{F}$.

Lemma 2.11. *Let* W *be a Witt algebra with* $f_{-1}, f_0, \dots, f_{p-2}$ *as a group basis, let* U *and* V *be two irreducible* W*-modules of dimension* p *on which* f_0 *acts with nonintegral weights, and assume there is a nonzero* W*-module homomorphism* $V \otimes U \to W$. *Then*

(i) f_j *acts nilpotently on* V *if and only if it acts nilpotently on* U, *and at most one such nilpotent* f_j *can occur*;

(ii) *for each* $j \in P$ *at least* $p-2$ *of the products of basis elements of* V *with basis elements of* U *give nonzero multiples of* f_j; *and*

(iii) *for all but at most one* $j \in P$, *at least* $p-1$ *of those products are nonzero.*

Proof. From Lemma 2.9 we may assume that $V = V(\lambda, \mu)$ and $U = V(\lambda', \mu')$ for some constants $\lambda, \lambda', \mu, \mu' \in \mathbf{F}$. We may further suppose that $v_0, v_1, \dots, v_{p-1}$ is a basis of V and $u_0, u_1, \dots, u_{p-1}$ is a basis of U with $F_i v_k = (\lambda + i\mu + k)v_{i+k}$ and $F_i u_k = (\lambda' + i\mu' + k)u_{i+k}$ for all $i, k \in P$. (Here we are using F_i to denote the action of f_i on U, V, or W.) The nonzero W-homomorphism $V \otimes U \to W$ necessarily is onto because W is simple. Thus for some l and s, $v_l \otimes u_{-l-s}$ maps to a nonzero multiple of f_0, and so must have weight 0 relative to F_0. This implies that $\lambda' = -\lambda + s$. By setting $u'_k = u_{k-s}$ for all k, we see that $F_i u'_k = (-\lambda + i\mu' + k)u'_{k+i}$. Now F_{-1}^p is a scalar multiple of the identity map on each irreducible module, and for V that scalar is $(\lambda - \mu)^p - (\lambda - \mu)$; for U, $(-\lambda - \mu')^p - (-\lambda - \mu')$; and for W, zero. We conclude from applying F_{-1}^p to $V \otimes U$ that $(\lambda - \mu)^p - (\lambda - \mu) + (-\lambda - \mu')^p - (-\lambda - \mu') = 0$, and hence that $\mu + \mu' = m \in P$. Thus, we may assume that $u_0, u_1, \dots, u_{p-1}$ have been chosen so that the action of W on U is given by $F_i u_k = (-\lambda - i\mu + im + k)u_{i+k}$ for some $m \in P$. Suppose that $v_j \otimes u_k \to [v_j, u_k] \in W$. Then by considering weights of F_0, we see that $[v_j, u_{k-j}]$ is a multiple of f_k.

Now $F_i v_k = 0$ for some $i, k \in P$ if and only if $\lambda + i\mu + k = 0$. If $\mu \in P$, then $\lambda + i\mu + k \neq 0$ for any i or k since λ is nonintegral, and hence it must be that no F_i acts nilpotently in that case. We may assume then that $\mu \notin P$, and that $F_i v_k = 0 = F_j v_n$ for some $i, j, k, n \in P$. It follows that $\lambda + i\mu + k = 0 = \lambda + j\mu + n$, and hence that $(k - n) + (i - j)\mu = 0$. Therefore $i = j$ and $k = n$, and at most one f_i acts nilpotently on V in this case. Since each f_i acts nilpotently on W, a given f_i will act nilpotently on U if and only if it acts nilpotently on V. This establishes part (i).

Suppose now that $[v_{i-k}, u_{j-i+k}] = 0 = [v_i, u_{j-i}]$ for some $i, j, k \in P$ with

$k \neq 0$. Since the image of $V \otimes U$ in W is a nonzero ideal, it must be that each f_j is in the image, and hence for some l, $[v_{i+lk}, u_{j-i-lk}] \neq 0$. Therefore by replacing i by $i+k$ one or more times as necessary, we may assume that in addition to the two relations above we have $[v_{i+k}, u_{j-i-k}] \neq 0$. Then

$$\begin{aligned} 0 &= F_k F_{-k}[v_i, u_{j-i}] \\ &= [F_k F_{-k} v_i, u_{j-i}] + [F_k v_i, F_{-k} u_{j-i}] + [F_{-k} v_i, F_k u_{j-i}] + [v_i, F_k F_{-k} u_{j-i}] \\ &= [F_k v_i, F_{-k} u_{j-i}], \end{aligned}$$

so that either $F_k v_i = 0$ or $F_{-k} u_{j-i} = 0$. Suppose that $F_k v_i = 0$, so that F_k is the only F acting nilpotently on U or V, and v_i is the only root vector of V that it annihilates. If $[v_{i+q}, u_{j-i-q}] = 0$ for some $q \in P$ with $q \neq \pm k, 0$, then applying $F_q F_{-q}$ to $[v_{i+q}, u_{j-i-q}] = 0$ gives $[F_q v_{i+q}, F_{-q} u_{j-i-q}] = 0$. Since F_q and F_{-q} act nonnilpotently on U and V, this implies that $[v_{i+2q}, u_{j-i-2q}] = 0$. Proceeding by induction, we can establish that $[v_{i+nq}, u_{j-i-nq}] = 0$ for each positive integer n by applying $F_q F_{-q}$ to $[v_{i+(n-1)q}, u_{j-i-(n-1)q}] = 0$. But then f_j fails to lie in the image of $V \otimes U$, a contradiction. Thus, $[v_{i+q}, u_{j-i-q}] = 0$ only when $q = 0, -k$. We have proved that there are at most two cases when the product into f_j is zero under the assumption that $F_k v_i = 0$, and the case when $F_{-k} u_{j-i} = 0$ follows by symmetry. This proves part (ii).

Assume as above that $[v_{i-k}, u_{j-i+k}] = 0 = [v_i, u_{j-i}]$, $[v_{i+k}, u_{j-i-k}] \neq 0$, and $F_k v_i = 0$. (The case when $F_{-k} u_{j-i} = 0$ is symmetric and so will be omitted.) Then applying $F_k F_{-k}$ to $[v_{i-k}, u_{j-i+k}] = 0$ gives $[F_{-k} v_{i-k}, F_k u_{j-i+k}] = 0$, and so $F_k u_{j-i+k} = 0$, since F_{-k} acts nonnilpotently on U and V. Suppose now that for $j' \in P$ there are two products into $f_{j'}$ which are zero, say $[v_{i'}, u_{j'-i'}] = 0 = [v_{i'-k'}, u_{j'-i'+k'}]$. Then the preceding calculations with i, j, k replaced by i', j', k' give that $F_{k'} v_{i'} = 0 = F_{k'} u_{j'-i'+k}$. But k is the only element of P for which F_k is nilpotent, so that $k' = k$. Since v_i is the only root vector of V annihilated by F_k, we also have $i' = i$. Finally, u_{j-i+k} is the only root vector of U annihilated by F_k, so that $j' = j$, to establish part (iii). □

We close this section with a result which will be needed in §3 to complete the proof of Theorem 1.1$'$.

Lemma 2.12. *Let L be a Lie algebra with triangulable Cartan subalgebra H such that the core of each 1-section relative to H is either* sl(2), *$W(1:\underline{1})$ or (0). Let S_0 be a maximal subalgebra containing H and assume*: (1) *S_0 contains no nonzero ideals of L*; (2) *L/S_0 is an irreducible S_0-module*; *and* (3) *the associated graded algebra $G = \bigoplus \sum_i G_i$ has $G_0 \cong W(1:\underline{1})$. Then L has toral rank one relative to H or $\dim_{\mathbf{F}} H = 1$.*

Proof. The assumptions imply that G is an irreducible graded Lie algebra with component $G_0 \cong W(1:\underline{1})$ in Kostrikin's terminology [K]. Kostrikin shows that such algebras can be obtained from two series $\mathbf{A}_n$ and $\mathbf{B}_n$ of algebras which can be described as follows: Let $\mathbf{O}$ be the truncated polynomial algebra $F[t]/\langle t^p\rangle$. Let $G = \bigoplus\sum_i G_i$ where $i = -1, 0, \dots, p^{n-1}-2$ and suppose that G_i has as basis $\{A(i, t^k) | k = 0, 1, \dots, p-1\}$ where if $i = -1$ or $p^{n-1}-2$ then $k = p-1$ is omitted. Here $aA(i, f) + bA(i, g) = A(i, af + bg)$ for all $f, g \in \mathbf{O}$ and $a, b \in \mathbf{F}$, and multiplication is given by

$$[A(-1, f), A(0, g)] = A(-1, (fg)'), \qquad [A(-1, f), A(j, g)] = A(j-1, fg), \quad j \geq 1,$$

$$[A(i, f), A(j, g)] = A(i+j, h), \quad \text{where } h = \binom{i+j+1}{j} f'g - \binom{i+j+1}{i} fg', \qquad i, j \geq 0;$$

here f' is the usual derivative of f. We denote this algebra by $\mathbf{A}_n$, and we let $\mathbf{A}_n^\sharp = \mathbf{A}_n + V$, where V is any subspace of the two-dimensional vector space $\langle A(r, t^{p-1}), A(r+1, 1)\rangle$ and $r = p^{n-1}-2$. Products in $\mathbf{A}_n^\sharp$ of elements in V with elements of $\mathbf{A}_n$ are as given in the equations above, and $[V, V] = 0$. It follows that $\mathbf{A}_n$ is an ideal of the Lie algebra $\mathbf{A}_n^\sharp$.

The second series consists of the algebras $G = \mathbf{B}_n$ in which the graded component G_i has as basis $\{B(i, t^k) | k = 0, 1, \dots, p-1\}$ for $i = -1, 0, \dots, p^{n-1} - 2$, and $aB(i, f) + bB(i, g) = B(i, af + bg)$. The multiplication here is specified by

$$(2.13) \quad [B(i, f), B(j, g)] = B(i+j, h), \quad \text{where } h = \binom{i+j+1}{j} \Delta fg - \binom{i+j+1}{i} f\Delta g;$$

where $\Delta f = f' + ft^{p-1}$. Kostrikin's result [K] states that every irreducible graded Lie algebra G with G_0 isomorphic to a Witt algebra is one of the algebras $\mathbf{A}_n$, $\mathbf{A}_n^\#$, or $\mathbf{B}_n$.

Suppose now that L satisfies the hypotheses of the lemma. Then $\overline{H} = H + S_1/S_1$ is a Cartan subalgebra of the Witt algebra G_0. If G is of type $\mathbf{A}_n$ or $\mathbf{A}_n^\#$, then each G_i is a restricted G_0-module. Hence, each Cartan subalgebra of G_0 acts with integral eigenvalues on G_i. Thus, since the roots of H are determined by the roots of $\overline{H}$ on G, we see that L has toral rank one relative to H in this case.

Suppose now that G is of type $\mathbf{B}_n$, and let $\overline{v}_0, \dots, \overline{v}_{p-1}$ be a basis for G_{-1} chosen so that $v_0, \dots, v_{p-1}$ are root vectors relative to H. If H has integral eigenvalues on G_{-1}, then as above H has toral rank one. Thus, we may assume that the vector $\overline{v}_i$ corresponds to the root $\beta + i$ for some $\beta \notin P$. Since the homogeneous component G_j is an irreducible G_0-module for each j, it follows that the roots of $\overline{H}$ on G_j are of the form $-j\beta + i$. Now $\overline{v}_i = B(-1, f_i)$ for some polynomial $f_i \in \mathbf{O}$. If $f_i = \sum_j a_{i,j} t^j$, then it must be that $a_{k,1} \neq 0$ for some value k. Therefore

$$[B(-1, f_k), B(j, g)] = B(j-1, f'_k g + f_k g t^{p-1}) \neq 0 \tag{2.14}$$

for each $g \neq 0$ and $j \neq -1$ by (2.13). Consider a basis $u_{-1}, \dots, u_r$ of the 1-section of L containing v_k. Suppose that $u_{-1} = v_k$ and that $B(j, g_j) = \overline{u}_j$. Then the span $\widetilde{G}$ of the elements $B(j, g_j)$ is a Lie algebra with self-centralizing ad-nilpotent element $B(-1, g_{-1}) = B(-1, f_k)$ by (2.14). By results of [BIO], $\widetilde{G}$ is a simple Lie algebra which is an Albert-Zassenhaus algebra. Since the elements $B(j, g_j)$ with $j \geq 0$ form a subalgebra of codimension one in $\widetilde{G}$, it follows from [BIO, Theorem 3.9] that in fact $\widetilde{G}$ is the algebra $W(1 : \underline{n-1})$. But since each 1-section is assumed to have core equal to sl(2), $W(1 : \underline{1})$ or 0, it must be that $n = 2$. Thus G, and hence L, is p^2-dimensional, and H is a Cartan subalgebra of dimension one. □

Remark. The proof above shows that the algebras $\mathbf{B}_n$, which are simple and have dependent roots relative to the Cartan subalgebra which is the centralizer of $B(0, t)$, have a core different from (0), sl(2), or $W(1 : \underline{1})$ when $n > 2$.

3. Proof of the main result

We assume throughout this section that K is a Lie algebra having toral rank two relative to a Cartan subalgebra H which is triangulable and has dependent roots. As before we denote the set of roots relative to H by Δ. Further we suppose that each 1-section relative to H has as its core either sl(2), a Witt algebra, or (0). When $\operatorname{Core} K^{(\gamma)} \neq (0)$ for some $\gamma \in \Delta$, then $\operatorname{Rad} K^{(\gamma)} \cap H$ is contained in the kernel of γ, which is H_0 because roots are dependent. Therefore, if $y \in \operatorname{Rad} K^{(\gamma)} \cap K_\gamma$, then $y \in N_\gamma$. Since $\operatorname{Core} K^{(\gamma)} = \mathrm{sl}(2)$, or a Witt algebra, this implies that the space N_γ has codimension 0 or 1 in K_γ. When $K^{(\gamma)}$ is solvable, then $N_\gamma = K_\gamma$ by Winter's result [Win]. Therefore, N_γ has codimension 0 or 1 in K_γ for each $\gamma \in \Delta$.

Lemma 3.1. *Let S_0 be a maximal subalgebra of K containing $H + N$, and assume that G_0 contains a proper ideal $\overline{J}$. Then there is a vector $\overline{x} \in \overline{J}_\zeta$ for some $\zeta \in \Delta$, such that $\mathrm{ad}_{\overline{x}}$ acts nonnilpotently on G_{-1}. If $S_0 \supseteq K^{(\alpha)}$, then $\overline{x} \in \overline{J}_{q\alpha}$ for some $q \neq 0$ in P.*

Proof. The existence of such a nonnilpotent vector $\overline{x}$ is guaranteed by Proposition 2.4; however, it remains to be shown that the case $\overline{x} \in \overline{J}_0$ cannot occur. Assume that $\overline{x} \in \overline{J}_0$. Then $x \in H - H_0$ by the nonnilpotence of $\mathrm{ad}_{\overline{x}}$. Applying $\mathrm{ad}_{\overline{x}}$ to the elements of each root space $(G_0)_\beta$ we see that $\sum_{\beta \in \Delta} (G_0)_\beta \subset \overline{J}$. Thus, G_0 is spanned modulo $\overline{J}$ by elements of $H + S_1/S_1$. Let $h + S_1/S_1$ be such an element. Then some linear combination of x and h is in $H_0 \cap (S_0 - S_1)$. But this gives a contradiction, for the elements of H_0 lie in S_1 since they act nilpotently on K and the root spaces of $K/H + N$, hence of K/S_0, are one-dimensional. Thus, it must be that such a nonnilpotent $\overline{x}$ lies in a space corresponding to a nonzero root. □

For a proof of the next lemma we will use the well-known fact that if A is a Lie algebra with an irreducible module M, and if D is an ideal of A acting nilpotently on M, then D annihilates M.

Lemma 3.2. *Suppose $\alpha \in \Delta$, and for each root $\gamma \notin P\alpha$ assume that the core of $K^{(\gamma)}$ is* sl(2) *or a Witt algebra. For each $i \neq 0$, let $T(\gamma, i\alpha) = \{x \in S(\gamma, i\alpha) | [x, S(\gamma, -i\alpha)] \subseteq \operatorname{Rad} K^{(\gamma)}\}$. Then,*

(i) *$T(\gamma, i\alpha)$ is the largest $K^{(\gamma)}$-submodule of $S(\gamma, i\alpha)$ contained in N;*

(ii) *for each $\gamma \notin P\alpha$, either $\operatorname{Rad} K^{(\gamma)}$ contains an element which acts nonnilpotently on K/N, or else $S(\gamma, i\alpha)/T(\gamma, i\alpha)$ has a composition series whose factors are trivial $\operatorname{Rad} K^{(\gamma)}$-modules, hence are modules for $K^{(\gamma)}/\operatorname{Rad} K^{(\gamma)}$;*

(iii) *if for $\gamma \notin P\alpha$ the core of $K^{(\gamma)}$ is a Witt algebra with a group basis, then either $\operatorname{Rad} K^{(\gamma)}$ contains an element which acts nonnilpotently on K/N, or else $[\operatorname{Rad} K^{(\gamma)}, S(\gamma, i\alpha)] \subseteq N$ for each i.*

Proof. For ease of notation in the remainder of the proof, let $S = S(\gamma, i\alpha)$, $T = T(\gamma, i\alpha)$, $S' = S(\gamma, -i\alpha)$, and $T' = T(\gamma, -i\alpha)$. Then T is a $K^{(\gamma)}$-submodule of S contained in N, and we claim that it is the largest $K^{(\gamma)}$-submodule of S contained in N. Indeed, suppose that $X \subseteq S \cap N$ is a $K^{(\gamma)}$-submodule with $T \subsetneq X$. Then there is a nonzero $K^{(\gamma)}$-module homomorphism $X \otimes S' \to K^{(\gamma)}/\operatorname{Rad} K^{(\gamma)}$ induced by the product. Since $K^{(\gamma)}/\operatorname{Rad} K^{(\gamma)}$ is simple, this map must be onto. But $[X, S'] \cap H \subseteq H_0$ to give a contradiction.

Let $S = U_r \supset \cdots \supset U_1 \supset U_0 = T$ be a composition series of $K^{(\gamma)}$-modules. Consider a factor module $M = U_i/U_{i-1}$ with associated representation $\pi\colon K^{(\gamma)} \to \mathrm{gl}(M)$. We suppose that the ideal $\ker \pi \cap \operatorname{Rad} K^{(\gamma)}$ has been factored out, and let D be the last term of the derived series of $\operatorname{Rad} K^{(\gamma)}$ modulo $\ker \pi \cap \operatorname{Rad} K^{(\gamma)}$. Since $\operatorname{ad}_x^p = 0$ on $K^{(\gamma)}$ for each $x \in D$, $\pi(x)^p = \lambda_x I$ on M for some scalar λ_x. If $\lambda_x = 0$ for all x, then D must annihilate M. In this case M is a trivial module for $\operatorname{Rad} K^{(\gamma)}$. Consequently, either S/T has a composition series whose factors are all trivial $\operatorname{Rad} K^{(\gamma)}$-modules, or else for some factor module $M = U_i/U_{i-1}$ and some $x \in \operatorname{Rad} K^{(\gamma)}$, $\pi(x)^p = \lambda_x I \neq 0$ on M. Now there is a $y \in U_i - U_{i-1}$ with $y \notin N$, and $\pi(x)^p y \equiv \lambda_x y$ modulo U_{i-1}. Thus, x acts nonnilpotently on K/N, and (ii) is seen to hold.

Suppose now that the core of $K^{(\gamma)}$ is a Witt algebra with a group basis, and assume that no element of $\operatorname{Rad} K^{(\gamma)}$ acts nonnilpotently on K/N. Then by part (ii), S/T has a composition series with factors which are irreducible $(K^{(\gamma)}/\operatorname{Rad} K^{(\gamma)})$-modules. If S/T is irreducible, it follows that $[\operatorname{Rad} K^{(\gamma)}, S] \subseteq T \subseteq N$, so that part (iii) holds in this case. Hence, we may assume that there exists a series $S = U_r \supset \cdots \supset U_1 \supset U_0 = T$ whose factors are irreducible modules for $K^{(\gamma)}/\operatorname{Rad} K^{(\gamma)} \cong W(1:\underline{1})$, and that $[\operatorname{Rad} K^{(\gamma)}, S] = U_q$ for some $q > 0$. Since $T \cap K_{i\alpha+j\gamma} \subseteq \bigcap_{k\neq 0}\{v \in K_{i\alpha+j\gamma} | [v, K_{-i\alpha+k\gamma}] \subseteq \operatorname{Rad} K^{(\gamma)}\}$, we see that $T \cap K_{i\alpha+j\gamma}$ has codimension $\leq p$ in S, and hence that $\dim_{\mathbf{F}} S/T \leq p^2$. Thus, we may further suppose that each U_j/U_{j-1} is a p-dimensional module for $W(1:\underline{1})$. There is an analogous series for S', say $S' = V_s \supset \cdots \supset V_1 \supset V_0 = T'$. Suppose that $[V_{j-1}, U_1] \subseteq \operatorname{Rad} K^{(\gamma)}$, but $[V_j, U_1] \not\subseteq \operatorname{Rad} K^{(\gamma)}$. Then there is a nonzero homomorphism $V_j/V_{j-1} \otimes U_1/T \to K^{(\gamma)}/\operatorname{Rad} K^{(\gamma)}$. We conclude from Lemma 2.11 that $U_1 \cap N$ has codimension $\geq p-2$ in U_1. Now if $j > 1$, then the map $V_1/T' \otimes U_1/T \to K^{(\gamma)}/\operatorname{Rad} K^{(\gamma)}$ induced from the product is zero. We see that $V_1 \cap N$ has codimension ≤ 2 in V_1. However by the definition of T', there exists a least integer k such that $V_1/T' \otimes U_k/U_{k-1} \to K^{(\gamma)}/\operatorname{Rad} K^{(\gamma)}$ is nonzero. Thus it must be that $V_1 \cap N$ has codimension $\geq p-2$ in V_1. This contradiction shows that $j = 1$, and $V_1/T' \otimes U_1/T \to K^{(\gamma)}/\operatorname{Rad} K^{(\gamma)}$ is nonzero. Since $U_1 \subseteq [\operatorname{Rad} K^{(\gamma)}, S]$, we may assume there is an $x \in \operatorname{Rad} K^{(\gamma)}$, $w \in S$, and $v \in V_1$ such that $[[x, w], v] = h \in H - H_0$. But then

$$\begin{aligned} h = [[x, w], v] &= [[x, v], w] + [x, [w, v]] \\ &\in [T', S] + [\operatorname{Rad} K^{(\gamma)}, K^{(\gamma)}] \subseteq \operatorname{Rad} K^{(\gamma)}. \end{aligned}$$

This contradiction shows that this final case cannot occur and finishes the proof. □

Lemma 3.3. *Let S_0 be a maximal subalgebra of K such that $H + N \subseteq S_0$. Let $\overline{J}$ be a proper ideal of G_0, and assume that for $\alpha \in \Delta$ and some $x \in (S_0 - S_1)_\alpha$, $\overline{x} = x + S_1/S_1 \in \overline{J}_\alpha$ acts nonnilpotently on $G_{-1} = S_{-1}/S_0$. Then for some root $\gamma \notin P\alpha$ there is an α-string of elements through γ in G_{-1} on which $\operatorname{ad}_{\overline{x}}$ acts nonnilpotently, and the following hold*:

(i) ad_x *acts nonnilpotently on the elements of $K_{j\alpha+k\gamma}$ not in $N_{j\alpha+k\gamma}$ for each $k \neq 0$. For $i = 0$ or -1, if $(G_i)_{j\alpha+k\gamma} \neq (0)$ for some $k \neq 0$ and some j, then $(G_i)_{j\alpha+k\gamma} \neq (0)$ for all j. Moreover, $\overline{J} \supset (G_0)_{j\alpha+k\gamma}$ for all j and all $k \neq 0$, so that G_0 is spanned modulo $\overline{J}$ by elements of $G_0^{(\alpha)}$.*

(ii) *If some section $K^{(r\alpha+\gamma)}$ has as its core a Witt algebra with a group basis, then every section, except possibly $K^{(\alpha)}$, has as its core a Witt algebra with a group basis.*

(iii) $(G_0)^{(\alpha)}$ *is solvable, but G_0 is nonsolvable.*

(iv) *Either* (a) *$K/H+N$ consists of α-strings through γ and $-\gamma$ plus elements in $(K/H+N)_{i\alpha}$ for each $i \neq 0$, the core of $K^{(\alpha)}$ is a Witt algebra with a group basis, and S_0 contains at most one root space of the form $K_{i\alpha}$*; *or* (b) *$S_0 = H + N$, G_{-1} has dimension $p^2 - 1$, for some $\beta \notin P\alpha$ there is a $\overline{y} \in (G_0)_\beta$ which acts nonnilpotently on G_{-1}, and every section has as its core a Witt algebra with a group basis.*

(v) *G_0 is semisimple.*

Proof. The nonnilpotence of $\mathrm{ad}_{\overline{x}}$ on G_{-1} implies that for some root γ there exists an entire α-string of elements through γ in G_{-1} on which $\mathrm{ad}_{\overline{x}}$ acts nonnilpotently. Note first that $\gamma \notin P\alpha$ since $H \subseteq S_0$. Thus, there must also exist an α-string of elements through $-\gamma$ which are not in N. Observe next that $(\mathrm{ad}_{\overline{x}})^p = \nu I$ on $\sum_{i\in P}(G_{-1})_{i\alpha+\gamma}$ for some $\nu \neq 0$ in $\mathbf{F}$ since root spaces of G_{-1} are one-dimensional.

If $x \in (S_0 - S_1)_\alpha$ is such that $x + S_1 = \overline{x}$, then $(\mathrm{ad}_x)^p$ decomposes each root space $K_{i\alpha+j\alpha}$ into generalized eigenspaces, and all but at most one of these eigenspaces lies in N. In particular for the root spaces of form $K_{i\alpha+\gamma}$, the eigenspaces corresponding to eigenvalues different from ν lie in N. We claim that for the root spaces of the form $K_{j\alpha-\gamma}$ the eigenspaces corresponding to the eigenvalues different from $-\nu$ lie in N. To see this, consider first the case that $K^{(\alpha)}$ is solvable. In this case $\sum_{i\in P}[K_{i\alpha}, K_{-i\alpha}] \subseteq H_0$, so that there is an ideal $R \supseteq [K^{(\alpha)}, K^{(\alpha)}]$ and an $h \in H - H_0$ with $K^{(\alpha)} = \mathbf{F}h + R$. For each i, the product induces a nonzero mapping of $[K_{i\alpha+\gamma}, K_{-i\alpha-\gamma}]$ into $\mathbf{F}h+R/R$. Since ad_x acts nilpotently on $\mathbf{F}h+R/R$, the generalized eigenspaces of $(\mathrm{ad}_x)^p$ on $K_{-i\alpha-\gamma}$ corresponding to eigenvalues not equal to $-\nu$ lie in N, while the space corresponding to $-\nu$ has a nonzero product with $K_{i\alpha+\gamma}$ into $\mathbf{F}h+R/R$. In the other case $K^{(\alpha)}$ is nonsolvable, and there is a nonzero product between these two strings into $K^{(\alpha)}/\operatorname{Rad} K^{(\alpha)}$. Then since ad_x acts nilpotently on $K^{(\alpha)}/\operatorname{Rad} K^{(\alpha)}$, we again have that the generalized eigenspace of $(\mathrm{ad}_x)^p$ on $K_{-i\alpha-\gamma}$ corresponding to eigenvalues unequal to $-\nu$ lie in N, while the space with value $-\nu$ has a nonzero product with $K_{i\alpha+\gamma}$.

We argue next that a similar result holds for each space $K_{j\alpha+k\gamma}$ with $k \neq 0, \pm 1$, whenever $K_{j\alpha+k\gamma} \neq N_{j\alpha+k\gamma}$. Indeed, if $K_{j\alpha+k\gamma} \neq N_{j\alpha+k\gamma}$ for some $k \neq 0, \pm 1$ and some j, then some 1-section $K^{(r\alpha+\gamma)}$ has as its core a Witt algebra with a group basis. By replacing $r\alpha+\gamma$ by γ to simplify the notation, we may assume that the section $K^{(\gamma)}$ has that property. Then $N_{j\gamma} = (\operatorname{Rad} K^{(\gamma)})_{j\gamma}$ for all $j \in P$, $j \neq 0$, and there exists a nonzero mapping $[K_{-\gamma}, K_{2\gamma}] \to K_\gamma/N_\gamma$ induced by the product. By looking at generalized eigenspaces relative to $(\mathrm{ad}_x)^p$, we see that the spaces in $K_{2\gamma}$ corresponding to eigenvalues not equal to 2ν lie in N, while the space corresponding to 2ν does not. It also follows that there is a whole α-string of elements through 2γ on which $(\mathrm{ad}_x)^p$ acts with eigenvalue 2ν, and the elements of that string map to $K_{2\gamma} - N_{2\gamma}$ after an appropriate number of applications of ad_x.

Proceeding by induction, we suppose that there is an α-string through $(k-1)\gamma$ on which $(\mathrm{ad}_x)^p$ acts with eigenvalue $(k-1)\nu$, and which map to $K_{(k-1)\gamma} - N_{(k-1)\gamma}$ after applying ad_x a suitable number of times. By considering the product from $[K_{-\gamma}, K_{k\gamma}]$ into $K_{(k-1)\gamma}/N_{(k-1)\gamma}$, we obtain the corresponding result for $k\gamma$. Hence, we see from this inductive argument that for each nonzero $k \in P$, the generalized eigenspaces of $K_{k\gamma}$ relative to $(\mathrm{ad}_x)^p$ corresponding to eigenvalues unequal to $k\nu$ lie in N, while that corresponding to $k\nu$ does not.

Also, there is a whole α-string of elements through $k\gamma$ which map to $K_{k\gamma} - N_{k\gamma}$ after applying ad_x a number of times. We claim that in this case the core of each section, except possibly $K^{(\alpha)}$, is a Witt algebra with a group basis. Indeed, if $\mathrm{Core}\, K^{(\zeta)}$ for $\zeta = s\alpha + \gamma$ is not a Witt algebra with a group basis, then $K_{2\zeta} \oplus \cdots \oplus K_{(p-2)\zeta} \subset N \subset S_0$. Applying ad_x, we see that $K_{2\gamma} \oplus \cdots \oplus K_{(p-2)\gamma} \subset N \subset S_0$. But this implies $K_\gamma \subset S_0$ since $K^{(\gamma)}$ has as its core a Witt algebra with a group basis. We have contradicted the fact that γ is a root of G_{-1}. Thus, it must be that every section $K^{(s\alpha+\gamma)}$ has as its core a Witt algebra with a group basis. Moreover, replacing γ with each of the values $s\alpha+\gamma$ in the above argument shows for each $k \neq 0$ and each j that the generalized eigenspaces of $(\mathrm{ad}_x)^p$ on $K_{j\alpha+k\gamma}$ corresponding to eigenvalues different from $k\nu$ lie in N while the space corresponding to $k\nu$ does not. We conclude from this argument that if $(G_{-1})_{j\alpha+k\gamma} \neq (0)$ then $(\mathrm{ad}_{\overline{x}})^p = k\nu I$ on that space, and if $k \neq 0$, there is an α-string of elements through $k\gamma$ in G_{-1}.

Now if $\overline{u} \in (G_0)_{j\alpha+k\gamma}$ for $k \neq 0$, then $\overline{u}$ moves at least one element of some space $(G_{-1})_{l\alpha+m\gamma}$ into a nonzero element of $(G_{-1})_{(j+l)\alpha+(k+m)\gamma}$ since G_{-1} is a faithful G_0-module. Thus if $\overline{u}$ lies in a generalized eigenspace relative to $(\mathrm{ad}_{\overline{x}})^p$, it follows that $(\mathrm{ad}_{\overline{x}})^p$ has the eigenvalue $k\nu$ on $\overline{u}$. Therefore, we see that $(G_0)_{j\alpha+k\gamma}$ has only the eigenvalue $k\nu$ under $(\mathrm{ad}_{\overline{x}})^p$. But then $(G_0)_{j\alpha+k\gamma} = (\mathrm{ad}_{\overline{x}})^p((G_0)_{j\alpha+k\gamma}) \subseteq \overline{J}$, and G_0 is spanned modulo $\overline{J}$ by elements in $(G_0)^{(\alpha)}$. Hence we have shown that parts (i) and (ii) hold.

We claim next that $(G_0)^{(\alpha)}$ is solvable. Indeed since $H_0 \subseteq S_1$, $H + S_1/S_1$ is a one-dimensional Cartan subalgebra of G_0. Thus if $(G_0)^{(\alpha)}$ is nonsolvable, it is a Yermolaev algebra in the terminology of [BO2]. Its radical contains $(G_0)^{(\alpha)} \cap \overline{J}$, which is nonzero. Now it has been argued in [BO1, Theorems 2.12 and 2.16] that any irreducible representation of a Yermolaev algebra on which the radical acts nonnilpotently must have dimension at least p^2. Since $\dim_{\mathbf{F}} K/S_0 \leq p^2 - 1$, it follows that $(G_0)^{(\alpha)}$ must be solvable, to give the first assertion of (iii). If G_0 is solvable, then since S_1 is solvable, we see that S_0 is solvable. Let K^∞ denote the term of the lower central series of K where the series stabilizes. Then K is spanned modulo K^∞ by elements of H_0, and $K^\infty = [K^\infty, K^\infty]$. Let M be a maximal ideal of K^∞. If K^∞/M is one-dimensional, then $K^\infty = [K^\infty, K^\infty] \subseteq M$, a contradiction. Thus it must be that K^∞/M is a simple algebra. Moreover, $S_0 \cap K^\infty + M/M$ is a solvable maximal subalgebra in K^∞/M. Since $K_{i\alpha+j\gamma} \not\subseteq M$ for $j = \pm 1$ and any i, K^∞/M has toral rank two relative to $H \cap K^\infty + M/M$. But this contradicts Proposition 2.8. Thus G_0 is nonsolvable as claimed in (iii).

Suppose now that $K/H+N$ is composed of more than the α-strings through γ and $-\gamma$ and possibly elements from $K^{(\alpha)}$. Then some section $K^{(r\alpha+\gamma)}$ has as its core a Witt algebra with a group basis. By part (ii), every section, except possibly $K^{(\alpha)}$, has as its core a Witt algebra with a group basis. Assume that is the case. Then by Lemma 3.2(iii), either for some $\beta \notin P\alpha$, $\operatorname{Rad} K^{(\beta)}$ has an element y which acts nonnilpotently on K/N, or for all $\beta \notin P\alpha$, we have $[\operatorname{Rad} K^{(\beta)}, S(\beta, i\alpha)] \subseteq N \subseteq S_0$ for all $i \neq 0$. If $y \in \operatorname{Rad} K^{(\beta)}$ acts nonnilpotently, then $y \in S_0 - S_1$. If $\operatorname{ad}_{\bar{y}}$ is nilpotent on G_{-1}, then an inductive argument shows that $\operatorname{ad}_{\bar{y}}$ is nilpotent on all of G. Hence, we may suppose that there is a root vector in $(G_{-1})_{s\alpha+\gamma}$ for some s on which $\operatorname{ad}_{\bar{y}}$ acts nonnilpotently. Since $\beta \notin P\alpha$, this implies that G_{-1} has elements in the α-string through $k\gamma$ for each k together with elements in the α-section. By part (i), $(G_{-1})_{l\alpha+k\gamma} \neq (0)$ for all l, and all $k \neq 0$. Applying part (i) with the root β in place of α shows that $(G_{-1})_{l\alpha} \neq (0)$ for all $l \neq 0$. Since each of these root spaces is at most one-dimensional, we obtain $\dim_{\mathbf{F}} G_{-1} = p^2 - 1$. The conclusions in (iv)(b) follow.

Assume then for each $\beta \notin P\alpha$ that the core of $K^{(\beta)}$ is a Witt algebra with a group basis, and that $[\operatorname{Rad} K^{(\beta)}, S(\beta, i\alpha)] \subseteq N \subseteq S_0$ for each $i \neq 0$. Since $\operatorname{Rad} K^{(\beta)} \subseteq N \subseteq S_0$, it must be that $\operatorname{Rad} K^{(\beta)} \subseteq S_1$ in this case. Thus, $\dim_{\mathbf{F}}(G_0)_\beta \leq 1$ for each β. Suppose now that $(G_0)_{r\alpha+k\gamma} \neq 0$ and $(G_0)_{s\alpha+l\gamma} \neq (0)$ for some nonzero k, l with $k \neq l$. Then by part (i), $(G_0)_{k\gamma} \neq (0)$ and $(G_0)_{l\gamma} \neq (0)$. If $l \neq -k$, then since $K^{(\gamma)}$ is a Witt algebra with a group basis, we obtain $K^{(\gamma)} \subseteq S_0$. This contradiction shows that if such k and l exist, then $l = -k$. Moreover, at most one such pair is possible. Given such a pair, consider the $k\gamma$-string through α in G_0. This is a module for $(G_0)_{k\gamma} + (G_0)_{-k\gamma} + (G_0)_0 \cong \mathrm{sl}(2)$, and as such it contains an irreducible $\mathrm{sl}(2)$-submodule. But since $\gamma \notin P\alpha$, the irreducible submodule must be p-dimensional and G_0 must contain the roots $\alpha + m\gamma$ for all m. This contradiction shows that it is impossible to have such a pair in G_0. The only alternative is that G_0 contains at most one α-string in addition to elements in $(G_0)^{(\alpha)}$. In this situation G_0 is solvable which contradicts (iii). Thus, this case cannot occur. We conclude that whenever every section, except possibly $K^{(\alpha)}$, has as its core a Witt algebra with a group basis, then (iv)(b) must hold.

We suppose that $K/H + N$ consists of α-strings through γ and $-\gamma$ and perhaps elements from $K^{(\alpha)}$. Then for each root $\beta \neq P\alpha$, it must be that $\operatorname{Rad} K^{(\beta)}$ acts nilpotently on K/N. Hence, we may assume by Lemma 3.2 that for each $i \neq 0$, $S/T = S(\gamma, i\alpha)/T(\gamma, i\alpha)$ has a composition series with factors which are irreducible modules for $K^{(\gamma)}/\operatorname{Rad} K^{(\gamma)}$ which is $\mathrm{sl}(2)$ or a Witt algebra with a proper basis. By refining the series if necessary, we may suppose that the factors are irreducible modules for the copy of $\mathrm{sl}(2)$ in $K^{(\gamma)}/\operatorname{Rad} K^{(\gamma)}$ which is generated by the γ and $-\gamma$ root spaces. Thus, suppose that

$$S = U_r \supset \cdots \supset U_1 \supset U_0 = T,$$

where U_{j+1}/U_j is an irreducible $\mathrm{sl}(2)$-module. If $S' = S(\gamma, -i\alpha)$ and $T' = T(\gamma, -i\alpha)$, then there is an analogous series

$$S' = V_s \supset \cdots \supset V_1 \supset V_0 = T'.$$

Let j be the smallest integer such that the product induces a nonzero sl(2)-module homomorphism $U_1/T\otimes V_j/V_{j-1} \to K^{(\gamma)}/\operatorname{Rad} K^{(\gamma)}$. Now $K^{(\gamma)}/\operatorname{Rad} K^{(\gamma)}$ is either isomorphic to sl(2), or as an sl(2)-module it consists of a $p-3$-dimensional module on top of a copy of sl(2). Since $K/H+N$ has only two strings and possibly elements in $K^{(\alpha)}$, it follows that the $\pm 2\gamma + i\alpha$ and $\pm 3\gamma + i\alpha$ spaces of U_1/T tensored with the $\mp 2\gamma - i\alpha$ and $\mp 3\gamma - i\alpha$ spaces of V_j/V_{j-1}, respectively, are mapped to zero. However, because the map is onto, it must be that there is a nonzero pairing of the $\gamma + i\alpha$, $-\gamma + i\alpha$, or $i\alpha$ space of U_1/T with the $-\gamma - i\alpha$, $\gamma - i\alpha$, or $-i\alpha$ space of V_j/V_{j-1}. Regardless of which pair has a nontrivial image, it follows from [BO1, Lemma 2.9], that $(U_1/T)_{i\alpha}\otimes(V_j/V_{j-1})_{-i\alpha}$ is not sent to zero. Thus, it must be for all $i \neq 0$, that $K_{i\alpha} \neq N_{i\alpha}$. Thus, $K^{(\alpha)}$ has as its core a Witt algebra with a group basis. If $K_{i\alpha} \subseteq (S_0)_{i\alpha}$ and $K_{j\alpha} \subseteq (S_0)_{j\alpha}$ for $i \neq j$, then $(G_0)^{(\alpha)}$ is not solvable. Hence it must be that at most one of the root spaces $K_{i\alpha}$ is contained in S_0. This concludes the proof of (iv).

Assume that $\overline{A}$ is an abelian ideal of G_0. Then $\overline{A}$ is proper by (iii), and so there must exist an $\overline{a} \in \overline{A}_\zeta$ for some $\zeta \in \Delta$, which acts nonnilpotently on G_{-1}. By (i) G_0 is spanned modulo $\overline{A}$ by elements in $G_0^{(\zeta)}$, which by (iii) is solvable. But then G_0 is solvable to contradict (iii). Thus, G_0 has no abelian ideals, hence must be semisimple. This gives (v) and finishes the proof of the lemma. □

Lemma 3.4. *Let S_0 be a maximal subalgebra of K such that $H+N \subseteq S_0$. Assume that $H + S_1/S_1 \not\subseteq [G_0, G_0]$, and assume that for $\beta \in \Delta$ and some $x \in (S_0 - S_1)_\beta$, $\overline{x} = x + S_1/S_1$ acts nonnilpotently on $G_{-1} = S_{-1}/S_0$. Then* $\dim_{\mathbf{F}} \operatorname{Rad} K^{(\beta)} + S_1/S_1 \leq p-1$, *each root space of* $\operatorname{Rad} K^{(\beta)} + S_1/S_1$ *is at most one-dimensional, every nonzero root vector of* $\operatorname{Rad} K^{(\beta)} + S_1/S_1$ *acts nonnilpotently on G_{-1}, and* $\operatorname{Rad} K^{(\beta)} + S_1/S_1$ *is abelian.*

Proof. By assumption $H + S_1/S_1 \not\subseteq [G_0, G_0]$, so that $[G_0, G_0]$ is a proper ideal of G_0, and $[G_0, G_0]$ is the sum of all the root spaces $(G_0)_\tau$ where $\tau \neq 0$. Therefore, since $\overline{x} \in (G_0)_\beta$ acts nonnilpotently on G_{-1}, we have by (iii) of Lemma 3.3 that $G_0^{(\beta)}$ is solvable. Moreover by (i) of Lemma 3.3, if $\overline{x}$ acts nonnilpotently on the β-string through δ in G_{-1}, then $\overline{x}$ acts nonnilpotently on the β-string through $l\delta$ in G_{-1} for all $l \neq 0$. Let $C = \sum_{i\neq 0}(G_0)_{i\beta}$. Since $\overline{y} \in (G_0)_{i\beta}$ implies $\operatorname{ad}_{\overline{y}}$ acts nilpotently on C, it follows from Jacobson's result on weakly closed sets that C is nilpotent. In fact, C is a Cartan subalgebra of $[G_0, G_0]$ because $\overline{x}$ acts nonnilpotently on the part of $[G_0, G_0]$ outside of C. Let $C' = \operatorname{Rad} K^{(\beta)} + S_1/S_1$, and observe that C' is an ideal of C. If $C' = (0)$, all the conclusions hold trivially. Hence we may assume that $C' \neq (0)$. Every nonzero ideal of a nilpotent algebra intersects the center nontrivially (see [H, Lemma 3.3]). Therefore, $C' \cap Z(C) \neq (0)$, and since both of these spaces are invariant under $H + S_1/S_1$ we may assume that $\bar{z}$ is a nonzero element of $(C' \cap Z(C))_{j\beta}$ for some j. Since $\bar{x}$ and $\bar{z}$ commute, they possess a common eigenvector on each string in G_{-1}—namely the sum of the root vectors after suitable normalization. If $\bar{z}$ acts nonnilpotently on one string, it acts nonnilpotently on the β-string through $l\delta$ for all $l \neq 0$ by (i) of Lemma 3.3. Therefore, if $\bar{z}$ acts nilpotently on one such string, it acts nilpotently on all of them and annihilates the eigenvectors. But then it annihilates all the root vectors in those strings. Since $\bar{z} \in \operatorname{Rad} K^{(\beta)} + S_1/S_1$ it annihilates $\sum_{i\neq 0}(G_{-1})_{i\beta}$ as well. However then $\operatorname{ad}_{\bar{z}}$ would annihilate all of G_{-1}, which would imply it is zero. Thus, it must be that $\bar{z}$ acts nonnilpotently on the strings in G_{-1} through multiples of δ. Since $\bar{z}$ commutes with every element of C', we can repeat this argument to show that every root vector of C' acts nonnilpotently on the strings. If there were two linearly independent elements in some root space of C', then some nonzero linear combination of them would annihilate a string of G_{-1}, contrary to what we have just shown. It follows that each root space of C' has dimension no more than one, so that $\dim_{\mathbf{F}} C' \leq p - 1$. Since every root vector of C' has a common eigenvector with $\bar{z}$ on each string in G_{-1}, C' has a common eigenvector on each string, and from that it follows that C' is abelian. □

Lemma 3.5. *With the same hypotheses as in Lemma* 3.3, *case* (a) *of part* (iv) *of that lemma cannot hold.*

Proof. Suppose that $K/H + N$ consists of α-strings through γ and $-\gamma$ plus elements in $(K/H + N)_{i\alpha}$ for each $i \neq 0$, the core of $K^{(\alpha)}$ is a Witt algebra with a group basis, and S_0 contains at most one root space of the form $K_{i\alpha}$. By (v) of Lemma 3.3, G_0 is semisimple. Let I be a minimal ideal of G_0 so that $I \subseteq [G_0, G_0]$. Since $(G_0)^{(\alpha)}$ is solvable, and $(S_0)_{i\alpha+j\gamma} = N_{i\alpha+j\gamma}$ for all $j \neq 0$ except possibly $j = -1$, we see that $H + S_1/S_1 \not\subseteq [G_0, G_0]$. Thus, I is a proper ideal of G_0. By Proposition 2.4, I must contain a root vector $\bar{y}$ acting nonnilpotently on G_{-1}, and necessarily $\bar{y}$ must be in $I_{l\alpha}$ for some l. Since by Lemma 3.3(i), $\bar{y}$ acts nonnilpotently on the root spaces of G_0 corresponding to roots of the form $i\alpha + j\gamma$ for $j \neq 0$, $I \supset (G_0)_{i\alpha+j\gamma}$ for all $j \neq 0$ and all i. Such root spaces must exist in G_0 because $G_0^{(\alpha)}$ is solvable, and in fact, it must be true for some j that $(G_0)_{i\alpha\pm j\gamma} \neq (0)$ for all i by the nonnilpotence of $\bar{y}$. The space $I^{(\alpha)}$ is a toral rank one Cartan subalgebra of I, and $\sum_{i\in P}(G_0)_{i\alpha+j\gamma}$ is a root space of $I^{(\alpha)}$ for each $j \neq 0$. Now $\mathfrak{A} = I/\operatorname{Rad} I$ is a simple Lie algebra by Block [Bl2], and $\mathfrak{H} = I^{(\alpha)} + \operatorname{Rad} I/\operatorname{Rad} I$ is a toral rank one Cartan subalgebra of $\mathfrak{A}$. Thus, by [Wil2], $\mathfrak{A}$ is isomorphic to sl(2) or to one of the Cartan type Lie algebras $W(1 : \underline{n})$ or $H(2 : \underline{m} : \Phi)^{(2)}$. Since $\bar{y} \in I - \operatorname{Rad} I$, and I cannot be spanned modulo $\operatorname{Rad} I$ by elements of $I^{(\alpha)}$, $\dim_{\mathbf{F}} \mathfrak{A} \geq 2p + 1$. Thus, $\mathfrak{A}$ must be one of the Cartan type Lie algebras. It follows from [Wil2] that either (i) $\mathfrak{A}$ is a Witt algebra and $\mathfrak{H}$ is a one-dimensional Cartan subalgebra outside the subalgebra of codimension 1 in $\mathfrak{A}$ or (ii) $\mathfrak{H}$ is contained in a maximal subalgebra of codimension 1 or 2, or in a maximal subalgebra $\mathfrak{M}_0$ of codimension $p-1$ such that $\mathfrak{A}/\mathfrak{M}_0$ is irreducible, and $\mathfrak{M}_0/\mathfrak{M}_1 \cong W(1:\underline{1})$.

Each of the listed Cartan type Lie algebras contains a unique maximal subalgebra of codimension 1 or 2 (see [BO3]). A Cartan subalgebra contained in such a maximal subalgebra must have toral rank one and its roots form a vector space over P of [Be]. This is impossible for $\mathfrak{H} = I^{(\alpha)} + \operatorname{Rad} I/\operatorname{Rad} I$ because $(G_0)_{i\alpha+j\gamma} = (0)$ for $j \neq \pm 2, \pm 1$, or 0 since G_0 acts faithfully on G_{-1}. Hence, $\mathfrak{H}$ must lie outside the maximal subalgebra of codimension 1 or 2. If case (i) or (iii) held, then $\mathfrak{H}$ would have $p-1$ different roots, a contradiction. Thus, case (a) of Lemma 3.3(iv) cannot occur. □

Proposition 3.6. *Assume that at least one* 1*-section of* K *has zero core. Then either* $K = H + N$ *or there is an ideal* $I \subseteq S_0$ *such that* K/I *is nonsolvable and has toral rank one relative to* $H + I/I$.

Proof. Let K be an algebra of minimal dimension satisfying the hypotheses of Proposition 3.6 but not the conclusion. Then $K \neq H + N$ and K contains two root vectors whose product is an element of H not in H_0. By the minimality of $\dim_{\mathbf{F}} K$, there is no nonzero ideal in K modulo which K is still of toral rank two. We recall that $H + N$ is a subalgebra by Lemma 2.1, and so it can be embedded in a maximal subalgebra S_0 of K with corresponding filtration $\{S_i\}$. We assume that $K^{(\alpha)}$ has core zero, so that then $K^{(\alpha)} \subseteq S_0$.

Suppose first that no root vector of S_0 acts nonnilpotently on S_{-1}/S_0. Then, by Proposition 2.4 either there is an ideal I of K in S_0 such that K/I has toral rank one relative to $H + I/I$ or S_0/S_1 contains no proper ideals, so is simple. In the first case, I cannot contain either of the root vectors whose product gives an element of $H - H_0$, for if that happened then I would contain K_γ for all roots γ. Since $I \subseteq S_0$, and $H \subseteq S_0$ we would have $K = S_0$, a contradiction. Thus K/I is nonsolvable in this case and K satisfies the conclusion of the proposition, contrary to the way it was chosen. Hence, we may assume that G_0 is simple and so for some root $\beta \notin P\alpha$, there exist $e_1 \in (S_0)_\beta$ and $e_{-1} \in (S_0)_{-\beta}$ such that $[e_{-1}, e_1] = e_0 \notin H_0$. But then S_0 is an algebra of lower dimension satisfying the hypotheses of the proposition. Since S_0 has a section with nonzero core, it must be that there is an ideal J of S_0 such that S_0/J has toral rank one relative to $H + J/J$. Then $J + S_1/S_1$ is an ideal of $S_0/S_1 = G_0$, which is assumed to be simple, and so $J \subseteq S_1$ and $G_0 = \sum_{i \in P} (G_0)_{i\beta} \cong \mathrm{sl}(2)$ or $W(1:\underline{1})$. If the roots of $H + S_1/S_1$ on G_{-1} lie in $P\beta$, then there is an ideal of K, namely $B = \bigcap S_j$, such that K/B has toral rank one. But then K satisfies the conclusion of the proposition, contrary to the way it was chosen. It must be that G_{-1} is a p-dimensional module for G_0 having weights of the form $i\alpha + j\beta$ for some i and for all $j \in P$. But this is impossible since $K^{(\alpha)} \subseteq S_0$. Thus, the case when G_0 is simple cannot occur. It must be then that in our minimal counterexample G_0 contains a proper ideal $\overline{J}$ and a root vector $\overline{x}$ which acts nonnilpotently on G_{-1}. Since $S_0 \supseteq K^{(\alpha)}$, it follows from Lemma 3.1 that $\overline{x} \in \overline{J}_{q\alpha}$ for some nonzero $q \in P$, and hence the hypotheses of Lemma 3.3 are satisfied. But by (iv) of Lemma 3.3, $K_{i\alpha} - (S_0)_{i\alpha} \neq (0)$ for some i. This contradicts the fact that $S_0 \supseteq K^{(\alpha)}$ and completes the proof. □

Lemma 3.7. *Let S_0 be a maximal subalgebra of K containing $H + N$, and suppose that the S_0-module K/S_0 is irreducible. Assume that the associated graded algebra G satisfies*

(i) *$G_0 \subseteq G^{(\alpha)}$ and G_0 is isomorphic to* $\mathrm{sl}(2)$ *or* $W(1:\underline{1})$,

(ii) *G_{-1} is a complete α-string of elements through β where $\beta \notin P\alpha$.*

Then $\dim_{\mathbf{F}} H = 1$.

Proof. If $G_0 \cong W(1:\underline{1})$, then since K has toral rank two relative to H, it follows from Lemma 2.12 that $\dim_{\mathbf{F}} H = 1$. Hence, it suffices to show that $G_0 \cong W(1:\underline{1})$ must hold. To accomplish this, we work with the associated graded algebra G and argue that $(G_0)_{i\alpha} \neq (0)$ for all $i \in P$. Consider $S(\beta, i\alpha) = \sum_{j\in P} G_{j\beta+i\alpha}$ for $i \neq 0$, and let U be the largest $G^{(\beta)}$-submodule of $S(\beta, i\alpha)$ contained in $\sum_{l\geq 0} G_l$. Then $U \neq S(\beta, i\alpha)$ because $(G_{-1})_{\beta+i\alpha} \neq (0)$. Choose m maximal such that there exists a $\overline{v} \in (G_m)_{-m\beta+i\alpha}$ with $\overline{v} \notin U$. Let $\overline{e}_{-1}, \overline{e}_0, \overline{e}_1, \ldots$ be a basis for $G^{(\beta)}$ modulo $\mathrm{Rad}\, G^{(\beta)}$ with $\overline{e}_j \in G_{-j\beta}$. (Either this is a basis for $G^{(\beta)}/\mathrm{Rad}\, G^{(\beta)} \cong \mathrm{sl}(2)$ or it is a proper basis for $W(1:\underline{1}) \cong G^{(\beta)}/\mathrm{Rad}\, G^{(\beta)}$.) Since $\mathrm{Rad}\, G^{(\beta)}$ and $\overline{e}_i$ for $i \geq 1$ belong to $\sum_{j\geq 1} G_j$, they send $\overline{v}$ to U. We may suppose that $\overline{e}_0\overline{v} = \lambda\overline{v}$, where $\lambda = (-m\beta + i\alpha)(\overline{e}_0) \notin P$. Let V be the $G^{(\beta)}$-submodule of $S(\beta, i\alpha)$ generated by U and $\overline{v}$. It is easy to verify that V is spanned modulo U by the elements $(\mathrm{ad}_{\overline{e}_{-1}})^n\overline{v} \in (G_{m-n})_{-(m-n)\beta+i\alpha}$, and that $\dim_{\mathbf{F}} V/U \geq p$. Since $\overline{e}_{-1}$ acts nilpotently on G, there is a least integer $n \geq p$ with $(\mathrm{ad}_{\overline{e}_{-1}})^n\overline{v} \notin \sum_{l\geq 0} G_l$ and $(\mathrm{ad}_{\overline{e}_{-1}})^{n+1}\overline{v} = 0$. But then $0 \neq (\mathrm{ad}_{\overline{e}_{-1}})^{n-1}\overline{v} \in (G_0)_{i\alpha}$. Thus we have $(G_0)_{i\alpha} \neq (0)$ for all i, and $G_0 \cong W(1:\underline{1})$. □

Lemma 3.8. *Let α be a fixed root in K relative to H, and suppose for each $\gamma \notin P\alpha$ and each nonzero $i \in P$ that the core of $K^{(\gamma)}$ is a Witt algebra with a group basis, and that $[\mathrm{Rad}\, K^{(\gamma)}, S(\gamma, i\alpha)] \subseteq N$. Then K has an ideal B such that K/B is simple and $\dim_{\mathbf{F}} H + B/B = 1$.*

Proof. Let $B = \sum_{\zeta\in\Delta} \mathrm{Rad}\, K^{(\zeta)}$. We want to show first that $[\mathrm{Rad}\, K^{(\gamma)}, K_{i\alpha+j\gamma}] \subseteq B$ for all $\gamma \notin P\alpha$ and all roots $i\alpha + j\gamma$. If $i = 0$, the result is clear since $\mathrm{Rad}\, K^{(\gamma)}$ is an ideal of $K^{(\gamma)}$. Hence we assume $i \neq 0$, and consider the product $[(\mathrm{Rad}\, K^{(\gamma)})_{l\gamma}, K_{i\alpha+j\gamma}]$. If $l \neq -j$, then

$$[(\mathrm{Rad}\, K^{(\gamma)})_{l\gamma}, K_{i\alpha+j\gamma}] \subseteq N_{(j+l)\gamma+i\alpha} \subseteq \mathrm{Rad}\, K^{((j+l)\gamma+i\alpha)}$$

since $K^{((j+l)\gamma+i\alpha)}$ has as its core a Witt algebra with a group basis. Suppose now that $l=-j$ so that by our assumptions $[(\operatorname{Rad} K^{(\gamma)})_{-j\gamma}, K_{i\alpha+j\gamma}] \subseteq N_{i\alpha}$. If the core of $K^{(\alpha)}$ is zero, sl(2), or a Witt algebra with a group basis, then $N_{i\alpha} = (\operatorname{Rad} K^{(\alpha)})_{i\alpha}$, so the result is obvious in this case. Otherwise, we may suppose that $e_{-1}, e_0, \dots, e_{p-2}$ is a proper basis of $K^{(\alpha)}$ modulo $\operatorname{Rad} K^{(\alpha)}$. If $i=\pm 1$, then again $N_{i\alpha} = (\operatorname{Rad} K^{(\alpha)})_{i\alpha}$, so we may assume that $i \geq 2$. We proceed by induction on i simultaneously for all $\gamma \notin P\alpha$. Let $y \in (\operatorname{Rad} K^{(\gamma)})_{-j\gamma}$ and $z \in K_{i\alpha+j\gamma}$. If $[y, z] \notin \operatorname{Rad} K^{(\alpha)}$, then applying $E_{-1} = \operatorname{ad}_{e_{-1}}$ gives $[E_{-1}y, z] + [y, E_{-1}z] \notin \operatorname{Rad} K^{(\alpha)}$. Thus, either $[E_{-1}y, z] \notin \operatorname{Rad} K^{(\alpha)}$ or $[y, E_{-1}z] \notin \operatorname{Rad} K^{(\alpha)}$. Since $E_{-1}y \in \operatorname{Rad} K^{(-\alpha-j\gamma)}$ by the hypotheses and $z \in K_{(\alpha+j\gamma)+(i-1)\alpha}$, either of these would contradict the statement for i replaced by $i-1$. Therefore, it follows that $[\operatorname{Rad} K^{(\gamma)}, K] \subseteq B$ for all $\gamma \notin P\alpha$.

It remains to be shown that $[\operatorname{Rad} K^{(\alpha)}, K_\beta] \subseteq B$ for all roots β. Clearly, we may assume that $\beta \notin P\alpha$. Consider the $K^{(\alpha)}$-module $S(\alpha, \beta)$, and let T be the submodule generated by $\sum_{i\in P} N_{i\alpha+\beta}$. By the hypotheses, T is contained in N, and $U = S(\alpha, \beta)/T$ has dimension one in each root space. Let $R = \operatorname{Rad} K^{(\alpha)}$, and let $R_{(\beta)}$ be the intersection of R with the kernel of the representation of $K^{(\alpha)}$ on U. If for every $\beta \notin P\alpha$, $R_{(\beta)} = R$, then $[\operatorname{Rad} K^{(\alpha)}, K_\beta] \subseteq B$ and we are done. Thus, we may assume that $R_{(\beta)} \neq R$, and since $R_{(\beta)}$ is an ideal of $K^{(\alpha)}$, we suppose further that $R_{(\beta)}$ has been factored out of $K^{(\alpha)}$. Since $H_0 \subseteq R_{(\beta)}$, we see now that $\dim_{\mathbf{F}} H + R_{(\beta)}/R_{(\beta)} = 1$. Observe that the core of $K^{(\alpha)}$ cannot be zero by Proposition 3.6 since the core of each 1-section $K^{(\gamma)}$ for $\gamma \notin P\alpha$ is a Witt algebra. Hence there exist elements e_{-1}, e_0, e_1, where $e_0 \in H - H_0$, which span a copy of sl(2) modulo R. The algebra A generated by e_{-1}, e_0, e_1 and R is a Yermolaev algebra. Let V be an irreducible A-submodule of U. If R acts nonnilpotently on V, then by [BO1, Theorems 2.12 and 2.16] $\dim_{\mathbf{F}} V \geq p^2$. Thus R must act nilpotently, hence trivially on V. Then V is an irreducible sl(2)-module. Since $\beta \notin P\alpha$, e_0 has nonintegral eigenvalues on V. Therefore, V has dimension p, and $V = U$. But then R annihilates U, contrary to assumption. This contradiction shows that it must be the case that $R_{(\beta)} = R$ for all $\beta \notin P\alpha$. Thus $[\operatorname{Rad} K^{(\alpha)}, K] \subseteq B$, and B is an ideal of K. It is easy to see that K/B is simple and $\dim_{\mathbf{F}} H + B/B = 1$. □

Lemma 3.9. *Assume that the core of each* 1*-section of* K *relative to* H *is either* sl(2) *or a Witt algebra. Then* K *has a maximal ideal* M *such that* $M \cap K^{(\gamma)} = \operatorname{Rad} K^{(\gamma)}$ *for all* $\gamma \in \Delta$, *and* K/M *is a simple Lie algebra,* $\dim_{\mathbf{F}} H + M/M = 1$ *and* K/M *has toral rank two relative to* $H + M/M$.

Proof. Let K be an algebra of minimal dimension satisfying the hypotheses of Lemma 3.9 but not containing an ideal B with the property that $\dim_{\mathbf{F}} H + B/B = 1$. For convenience of notation in the remainder of the proof let $R^{(\gamma)} = \operatorname{Rad} K^{(\gamma)}$. Suppose first that K contains a proper ideal Q not contained in $\sum_{\gamma\in\Delta} R^{(\gamma)}$. Then Q contains an element of $K_\zeta - (R^{(\zeta)})_\zeta$ for some $\zeta \in \Delta$, and hence an element of $H - H_0$. This implies that $Q \supset \sum_{\gamma\in\Delta} K_\gamma$. Let Q^∞ be the intersection of the terms in the lower central series of Q. Then $Q^\infty \supset \sum_{\gamma\in\Delta} K_\gamma$ and Q is spanned modulo Q^∞ by elements in H_0. Since $[Q^\infty, Q^\infty] \supset \sum_{\gamma\in\Delta} K_\gamma$, $Q/[Q^\infty, Q^\infty]$ is nilpotent so that $Q^\infty = [Q^\infty, Q^\infty]$. Now Q^∞ satisfies all the hypotheses of the lemma. Since $\dim_{\mathbf{F}} Q^\infty < \dim_{\mathbf{F}} K$, Q^∞ must contain an ideal B' such that Q^∞/B' has rank one. If B' is a maximal ideal with this property, then Q^∞/B' is either simple or one-dimensional. In the latter case, $Q^\infty = [Q^\infty, Q^\infty] \subseteq B'$, a contradiction. Thus, we may suppose that Q^∞/B' is simple and $\dim_{\mathbf{F}}(H \cap Q^\infty + B'/B') = 1$. By [BO2, Corollary 3.8] $(B')_\gamma = (R^{(\gamma)})_\gamma$ for all $\gamma \in \Delta$. Therefore, $[(R_\gamma^{(\gamma)}), K_\beta] = [(B')_\gamma, (Q^\infty)_\beta] \subseteq (B')_{\gamma+\beta} = (R^{(\gamma+\beta)})_{\gamma+\beta}$ for all $\gamma, \beta \in \Delta$ with $\beta \neq -\gamma$. It follows that $B = H_0 + B' = \sum_{\gamma\in\Delta} R^{(\gamma)}$ is an ideal of K such that $H + B/B$ is one-dimensional. This contradiction shows that any proper ideal Q of K must be contained in $\sum_{\gamma\in\Delta} R^{(\gamma)}$. But then K/Q satisfies the hypotheses of Lemma 3.9 and does not contain an ideal B modulo which $\dim_{\mathbf{F}} H + B/B = 1$. By the minimality of K, we conclude that K is simple.

Since $H + N \neq K$, there exists a maximal subalgebra S_0 of K containing $H + N$. Since the core of each 1-section is sl(2) or $W(1 : \underline{1})$, case (iii) of Proposition 2.4 cannot hold. Assume then case (i) holds: G_0 is a simple algebra. Since $H_0 \subseteq S_1$, we see that $\dim_{\mathbf{F}} H + S_1/S_1 = 1$. If G_0 has toral rank two, then from [BO2, Lemma 2.5 and §4, paragraph 1] we know that each nonzero element of $P\gamma + P\alpha$ is a root of G_0, and each 1-section of G_0 is a Witt algebra. Hence, S_0 contains in each root space K_γ an element not in $R^{(\gamma)}$. Since $R^{(\gamma)} \subseteq N$ for each root γ, we see that each root space is wholly contained in S_0, implying that $S_0 = K$. This contradiction shows that G_0 cannot have toral rank two.

Thus, G_0 has toral rank one, so that $G_0 \subseteq G^{(\alpha)}$ for some root α. Then G_{-1} must consist either of the rest of the elements of $K^{(\alpha)}$ which are not in S_0, or of a single complete α-string through a root, say β. In the first case K would have toral rank one, contrary to assumption. So it must be that G_{-1} is a complete α-string through β. If $S_{-1} = K$, then by Lemma 3.7 $\dim_{\mathbf{F}} H = 1$, contrary to our choice of K. Thus we may suppose that $S_{-1} \neq K$. Then since $G_{-2} = [G_{-1}, G_{-1}]$, there are elements in at least one of the root spaces $K_{i\alpha+2\beta}$ which are not in S_0. Now K/S_0 must also consist of complete α-strings, so that K/S_0 must contain a complete α-string through 2β. It follows that the core of every section of K except possibly $K^{(\alpha)}$ is a Witt algebra with group basis. We may apply Lemma 3.2 to conclude that $[R^{(\gamma)}, S(\gamma, i\alpha)] \subseteq N$ for each $\gamma \notin P\alpha$. But then by Lemma 3.8, K would have an ideal B with $\dim_{\mathbf{F}} H + B/B = 1$.

This rules out the case when G_0 is simple.

We may suppose then that case (ii) of Proposition 2.4 holds so that there exists a proper ideal $\overline{J} \subseteq G_0$. By Lemma 3.1 we may assume that there is a root vector $\overline{x} \in \overline{J}_\alpha$ for some root α which acts nonnilpotently on G_{-1}. Then by Lemma 3.5 and 3.3(iv), $S_0 = H+N$, G_{-1} has dimension p^2-1, for some $\beta \notin P\alpha$ there is a $\overline{y} \in (G_0)_\beta$ which acts nonnilpotently on G_{-1}, and every 1-section has as its core a Witt algebra with a group basis. Since N_γ has codimension at most 1 in K_γ for each root γ, the fact that $\dim_{\mathbf{F}} G_{-1} = p^2 - 1$ implies $G_{-2} = (0)$. We want to show next that $G_0 = S_0/S_1$ has one-dimensional root spaces relative to $H + S_1/S_1$. By Lemma 3.5, case (b) of Lemma 3.3(iv) must hold. Therefore, by Lemma 3.3(i), the elements of $[G_0, G_0] = \sum_{\zeta\in\Delta}(G_0)_\zeta$ are arranged in α-strings, and so there exists a nonzero $j \in P$ with $(G_0)_{i\alpha+j\beta} \neq (0)$ for all $i \in P$. But then, using Lemma 3.3(i) with α and β interchanged, we see that $(G_0)_{i\alpha+j\beta} \neq (0)$ for all $i, j \in P$ with $j \neq 0$. Switching back to the point of view of α, Lemma 3.3(i) says that $(G_0)_{j\beta} \neq (0)$ for each nonzero j, and since each of these spaces can have at most dimension 1 by Lemma 3.4, we also see that $\dim_{\mathbf{F}}(G_0)_{i\alpha+j\beta} = 1$ for all $i \neq 0$. Since each of the spaces $(G_0)_{i\alpha}$ for $i \neq 0$ has at most dimension 1, it also will have dimension exactly one. Finally, $(G_0)_0$ has dimension 1.

For each root ζ, we let $\overline{u}_\zeta$ span $(G_0)_\zeta$ and $\overline{v}_\zeta$ span $(G_{-1})_\zeta$. If $[\operatorname{Rad} K^{(\zeta)}, S(\zeta, i\alpha)] \subseteq N \subseteq S_0$, then $\operatorname{Rad} K^{(\zeta)} \subseteq S_1$ and $(G_0)_{l\zeta} = (0)$ for all $l \neq 0$ to give a contradiction. Thus, by Lemma 3.2(iii) $\operatorname{Rad} K^{(\zeta)}$ acts nonnilpotently on K/N for each root ζ. It then follows from Lemma 3.4 that each root space of G_0 acts nonnilpotently on G_{-1}.

We have proved that both G_0 and G_{-1} have each of their weight spaces relative to $H + S_1/S_1$ one-dimensional. We now consider what happens for G_1. Let $\overline{w}_\alpha$ be a nonzero element of $(G_1)_\alpha$. By the definition of G_1, for some root β there exists $\overline{v}_\beta$ with $[\overline{w}_\alpha, \overline{v}_\beta] \in G_0$ and $[\overline{w}_\alpha, \overline{v}_\beta] \neq 0$. If $\beta \notin P\alpha$, then $[[\overline{w}_\alpha, \overline{v}_\beta], \overline{v}_{-\alpha}] \neq 0$. However,

$$[[\overline{w}_\alpha, \overline{v}_\beta], \overline{v}_{-\alpha}] = [[\overline{w}_\alpha, \overline{v}_{-\alpha}], \overline{v}_\beta] + [\overline{w}_\alpha, [\overline{v}_\beta, \overline{v}_{-\alpha}]] = 0,$$

to give a contradiction. Suppose then that there exists $i \in P$ with $[\overline{w}_\alpha, \overline{v}_{i\alpha}] \neq 0$. For any root $\beta \notin P\alpha$ and any $\overline{v}_\beta$ we have $[[\overline{w}_\alpha, \overline{v}_{i\alpha}], \overline{v}_\beta] \neq 0$. However,

$$[[\overline{w}_\alpha, \overline{v}_{i\alpha}], \overline{v}_\beta] = [[\overline{w}_\alpha, \overline{v}_\beta], \overline{v}_{i\alpha}] + [\overline{w}_\alpha, [\overline{v}_{i\alpha}, \overline{v}_\beta]] = 0,$$

again giving a contradiction. This shows that $(G_1)_\alpha = (0)$ for all roots α. If $G_1 = (0)$, then S_1 is an ideal of K, which must then be (0). But then $\dim_{\mathbf{F}} H = 1$. Hence we may assume that $G_1 = (G_1)_0 \neq (0)$. We want to prove next that $\dim_{\mathbf{F}} G_1 = 1$. Suppose that $\overline{h} \in G_1$ satisfies $[\overline{h}, \overline{v}_\alpha] = 0$ for some root α, and let β be a root with $\beta \notin P\alpha$. Setting $\gamma = \beta - \alpha$, we note that $[\overline{u}_\gamma, \overline{v}_\alpha]$ is a nonzero multiple of $\overline{v}_\beta$. But

$$[\overline{h}, [\overline{u}_\gamma, \overline{v}_\alpha]] = [[\overline{h}, \overline{u}_\gamma], \overline{v}_\alpha] + [\overline{u}_\gamma, [\overline{h}, \overline{v}_\alpha]] = 0,$$

so that $[\bar{h}, \bar{v}_\beta] = 0$ for any $\beta \notin P\alpha$. Switching the roles of α and β, we see that $[\bar{h}, \bar{v}_{i\alpha}]$ for all $i \neq 0$ showing that $\mathrm{ad}_{\bar{h}}$ sends G_{-1} into G_1. But then $\bar{h} = 0$. Now if $\bar{h}_1, \bar{h}_2 \in G_1$, then some nonzero linear combination of them will send $\bar{v}_\alpha$ to 0 and by what we have just proved, that linear combination will be zero. Thus, $\dim_{\mathbf{F}} G_1 = 1$.

If $G_2 \neq (0)$, then since $[G_{-1}, G_2] = G_1$, there must exist a nonzero root vector $\bar{x}_\alpha \in (G_2)_\alpha$ for some root α. Then $[\bar{v}_{-\alpha}, \bar{x}_\alpha]$ is a nonzero multiple of $\bar{h}$, and for $\beta \notin P\alpha$, $[\bar{v}_\beta, [\bar{v}_{-\alpha}, \bar{x}_\alpha]]$ is a nonzero multiple of $\bar{u}_\beta$. But, on the other hand,

$$[\bar{v}_\beta, [\bar{v}_{-\alpha}, \bar{x}_\alpha]] = [[\bar{v}_\beta, \bar{v}_{-\alpha}], \bar{x}_\alpha] + [\bar{v}_{-\alpha}, [\bar{v}_\beta, \bar{x}_\alpha]] = 0.$$

This contradiction shows that $G_2 = (0)$. In this case, let $D \neq (0)$ be an ideal of G. Then D must contain some nonzero $\bar{d} \in G_\alpha$ for some root α. Write $\bar{d} = \bar{d}_{-1} + \bar{d}_0$ where $\bar{d}_i \in (G_i)_\alpha$. Suppose $\beta \notin P\alpha$. If $\bar{d}_0 \neq 0$, then $[\bar{v}_\beta, \bar{d}] = [\bar{v}_\beta, \bar{d}_0]$ is a nonzero multiple of $\bar{v}_{\beta-\alpha}$ and $[\bar{v}_\beta, \bar{d}] \in D$. Since G_{-1} is an irreducible G_0-module, we see that $D \supseteq G_{-1}$. If $\bar{d}_0 = 0$, then $\bar{d} = \bar{d}_{-1} \in D \cap G_{-1}$, and again $D \supseteq G_{-1}$. But then $[G_1, G_{-1}] = [G_0, G_0] \subseteq D$. Since $[[G_0, G_0], [G_0, G_0]] = [G_0, G_0]$, D is nonsolvable. Thus, G is semisimple, and $D' = G_{-1} \oplus [G_0, G_0]$ is the unique minimal ideal of G. However, Block [Bl] has shown that each minimal ideal in a semisimple algebra has the form $A \otimes \mathbf{O}_n$, where A is a simple algebra and $\mathbf{O}_n$ is a truncated polynomial algebra in n indeterminates. Since $\dim_{\mathbf{F}} D' = 2p^2 - 2$, it must be $n = 0$ and D' is simple. But G_{-1} is an ideal of D', a contradiction.

We conclude that no such minimal counterexample K exists. Hence each algebra K satisfying the hypotheses must have an ideal B such that $\dim_{\mathbf{F}} H + B/B = 1$. Let K be such an algebra. Then K^∞ satisfies the hypotheses and so must have a maximal ideal M' such that K^∞/M' is simple and $(H \cap K^\infty + M')/M'$ is one-dimensional. (Compare the argument in the first paragraph of the proof.) Then $M = H_0 + M'$ is a maximal ideal of K such that K/M is simple, $\dim_{\mathbf{F}} H + M/M = 1$, and K/M is toral rank two. Since relative to a one-dimensional Cartan subalgebra the radical of each 1-section is zero [BO2, Corollary 3.8] in a simple algebra, we see that $M \cap K^{(\gamma)} = \operatorname{Rad} K^{(\gamma)}$ for all $\gamma \in \Delta$. □

Proof of Theorem 1.1′. We assume that each section has as its core (0), sl(2) or $W(1 : \underline{1})$. If L has toral rank one, then $L = L^{(\alpha)}$ for some root α, L is isomorphic to sl(2) or $W(1 : \underline{1})$, and every Cartan subalgebra is one-dimensional, so that the theorem holds in this case. Thus, we may suppose that the toral rank is at least two. For each $\gamma \in \Delta$, if $L^{(\gamma)} \neq \operatorname{Rad} L^{(\gamma)}$, let $R^{(\gamma)} = \operatorname{Rad} L^{(\gamma)}$. Otherwise the core of $L^{(\gamma)}$ is zero, and we set $R^{(\gamma)} = H_0 + \sum_{i \neq 0} L_{i\gamma}$. We argue that $B = \sum_{\gamma \in \Delta} R^{(\gamma)}$ is an ideal of L by showing that $[R^{(\alpha)}, K^{(\beta)}] \subseteq B$ for all $\alpha, \beta \in \Delta$. Since $R^{(\alpha)}$ is an ideal of $L^{(\alpha)}$ for all $\alpha \in \Delta$ we may assume that $\beta \notin P\alpha$. Suppose first that $L^{(\alpha)}$ has core zero, and consider the 2-section $K = L^{(\alpha,\beta)}$ determined by α and β. By Proposition 3.6, either $K = H + N$ or K has an ideal I modulo which it is nonsolvable and has toral rank one. In the first case the core of each 1-section in the 2-section is zero, so that clearly $[R^{(\alpha)}, L^{(\beta)}] \subseteq B$. In the second case, there is an ideal $J \supseteq I$ such that $J/I = \mathrm{Rad}(K/I)$. Then $\sum_{i\in P} J_{i\gamma} = R^{(\gamma)}$ for all $\gamma \in \Delta \cap (P\alpha + P\beta)$, so that the result holds in this case also. Suppose then that the core of $L^{(\alpha)}$ is sl(2) or $W(1:\underline{1})$. If the 2-section $K = L^{(\alpha,\beta)}$ has a 1-section with core zero, then as in the previous case, $[R^{(\alpha)}, L^{(\beta)}] \subseteq B$. We may suppose then that the core of each 1-section of K is either sl(2) or a Witt algebra. Then by Lemma 3.9, there is an ideal M such that

$$M \cap K^{(\gamma)} = \mathrm{Rad}\, K^{(\gamma)} = R^{(\gamma)} \quad \text{for all } \Delta \cap (P\alpha + P\beta).$$

But then $[R^{(\alpha)}, L^{(\beta)}] \subseteq B$ must indeed hold. Hence, B is an ideal of L. Since L is simple, $B = (0)$, and $\dim_{\mathbf{F}} H = 1$, as claimed. □

Bibliography

[Be] G. M. Benkart, *Cartan subalgebras in Lie algebras of Cartan type*, Lie Algebras and Related Topics, Canad. Math. Soc. Conf. Proc. 5 (D. J. Britten, F. W. Lemire, and R. V. Moody, eds.), Amer. Math. Soc., Providence, R.I., 1986, pp. 157–187.

[BO1] G. M. Benkart and J. M. Osborn, *Representations of rank one Lie algebras of characteristic p*, Lie Algebras and Related Topics (D. J. Winter, ed.), Lecture Notes in Math., vol. 933, Springer-Verlag, Berlin, Heidelberg and New York, 1982, pp. 1–37.

[BO2] ——, *Rank one Lie algebras*, Ann. of Math. (2) **119** (1984), 437–463.

[BO3] ——, *Toral rank one Lie algebras*, J. Algebra **115** (1988), 238–250.

[BIO] G. M. Benkart, I. M. Isaacs and J. M. Osborn, *Lie algebras with self-centralizing* ad-*nilpotent elements*, J. Algebra **57** (1979), 279–309.

[Bl1] R. E. Block, *On the Mills-Seligman axioms for Lie algebras of classical type*, Trans. Amer. Math. Soc. **121** (1966), 378–392.

[Bl2] ——, *Determination of the differentiably simple rings with a minimal ideal*, Ann. of Math. (2) **90** (1969), 433–459.

[BW] R. E. Block and R. L. Wilson, *The simple Lie p-algebras of rank two*, Ann. of Math. (2) **115** (1982), 93–168.

[Br] G. Brown, *Cartan subalgebras in Zassenhaus Lie algebras*, Canad. J. Math. **27** (1975) 1011–1021.

[H] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Math., Springer-Verlag, New York, 1972.

[J] N. Jacobson, *Lie algebras*, Wiley, New York, 1962.

[K] A. I. Kostrikin, *Irreducible graded Lie algebras with the component $L_0 = W_1$*, Ural Gos. Univ. Mat. Zap. **7** (1969/70), no. 3, 92–103. (Russian)

[We] B. Ju. Weisfeiler, *On subalgebras of simple Lie algebras of characteristic $p > 0$*, Trans. Amer. Math. Soc. **286** (1984), 471–503.

[Wil1] R. L. Wilson, *A structural characterization of the simple Lie algebras of generalized Cartan type over fields of prime characteristic*, J. Algebra **40** (1976), 418–465.

[Wil2] ——, *Simple Lie algebras of toral rank one*, Trans. Amer. Math. Soc. **236** (1978), 287–295.

[Win] D. J. Winter, *Cartan decompositions and Engel subalgebra triangulability*, J. Algebra **62** (1980), 400–417.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706