CURVATURES AND SIMILARITY OF OPERATORS WITH HOLOMORPHIC EIGENVECTORS

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ABSTRACT. The curvature of the holomorphic vector bundle generated by eigenvectors of operators is estimated, and the necessary and sufficient conditions for contractions to be similar or quasi-similar with unilateral shifts are given.

1. INTRODUCTION

Let $H$ be a separable complex Hilbert space, $g_r(n, H)$ the set of all $n$-dimensional subspaces of $H$, and $\gamma$ a mapping from an open connected set $\Omega$ in the complex plane $\mathbb{C}$ to $g_r(n, H)$. Then $\gamma$ is called a holomorphic curve over $\Omega$, if for each $w_0$ in $\Omega$, there is a nbhd $\Delta$ of $w_0$ and vector valued holomorphic functions $\gamma_{iw}$ on $\Delta$ ($i = 1, \ldots, n$) satisfying $\gamma_w = \sqrt{\{\gamma_{iw} : i = 1, \ldots, n\}}$ for $w$ in $\Delta$. In this case, the Hermitian holomorphic vector bundle $(E_\gamma, \Omega, \pi)$ is defined as

$$E_\gamma = \{(x, w) \in H \times \Omega : x \text{ in } \gamma_w\}, \quad \pi(x, w) = w,$$

and hence for this bundle, the canonical connection and curvature $\mathcal{H}_\gamma$ are well defined [19]. We call $\gamma_{1w}, \ldots, \gamma_{nw}$ a frame for $E_\gamma$ on $\Delta$. The matrix form of $\mathcal{H}_\gamma(w)$ with respect to the above frame is

$$(1.1) \quad -\frac{\partial}{\partial w} \left( G^{-1} \frac{\partial G}{\partial w} \right),$$

where $G_j(w)$ is the Gram matrix whose $(i, j)$ component is $(\gamma_j(w), \gamma_i(w))$ (cf. [4]).

In case of $n = 1$, we have especially

$$\mathcal{H}_\gamma(w) = -\frac{\partial^2}{\partial w \partial w} \log \|\gamma_{1w}\|^2.$$

We explain some notations about relations between given bounded operators $T_1, T_2$. Suppose there is an intertwining bounded operator $X$ such that $XT_1 = T_2X$, then we denote by $T_1 \preceq T_2$, $T_1 \lessdot T_2$, $T_1 \lessdot T_2$, $T_1 \approx T_2$, and $T_1 \cong T_2$,
with dense range, \( X \) injective, quasi-affinity (that is, \( X \) is injective and has dense range), invertible, and unitary, respectively. Moreover we write \( T_1 \sim T_2 \) and say that \( T_1 \) and \( T_2 \) are quasi-similar, if \( T_1 < T_2 \) and \( T_2 < T_1 \). In [4], Cowen-Douglas defined the class \( B_n(\Omega) \) consisting of bounded operator \( T \) satisfying

(a) \( \Omega \subset \sigma(T) \),
(b) range\( (T - w) = H \) for each \( w \) in \( \Omega \),
(c) \( \bigvee_{w \in \Omega} \ker(T - w) = H \),
(d) \( \dim \ker(T - w) = n \) for \( w \) in \( \Omega \).

Now we introduce the class \( B_n^h(\Omega) \) as

\textbf{Definition.} \( T \) belongs to \( B_n^h(\Omega) \) if there is a holomorphic curve \( \gamma: \Omega \to g(n, H) \) such that \( \gamma(w) \subset \ker(T - w) \), and \( \bigvee_{w \in \Omega} \gamma(w) = H \). It is known that \( B_n(\Omega) \subset B_n^h(\Omega) \). If \( T \) in \( B_n^h(\Omega) \), then the bundle is well defined by the curve \( \gamma(w) \). We denote it and its curvature by \( E_T \) and \( \mathcal{K}_T \).

The purpose of this paper is to estimate \( \mathcal{K}_T \) of \( T \) in \( B_n^h(\Omega) \) and to research what kind of operator is similar or quasi-similar to the shifts.

Now we show some examples. Let \( \{e_n\}_{n=0}^{\infty} \) be a C.O.N.B. of \( H \) and \( A \) a weighted shift with positive weight \( \{a_n\}_{n=0}^{\infty} \), that is \( Ae_n = a_{n+1}e_{n+1} \). Set \( b_n = a_1 \cdots a_n \) and \( r_1(A) = \lim_{n \to \infty} (\inf_{k} b_{k+n}/b_k)^{1/n} \). Then we have \( A^* \in B_1(\{w: |w| < r_1(A)\}) \), (see [13 or 12]). Especially, the adjoint of unilateral shift \( S \) corresponding to \( a_n = 1 \) for all \( n \) and the adjoint of the Bergman shift \( B \) corresponding to \( a_n = \sqrt{n/(n+1)} \) for all \( n \) are both in \( B_1(D) \), where \( D \) is the open unit disk. And \( \mathcal{K}_S^*(w) = -1/(1 - |w|^2)^2 \) and \( \mathcal{K}_B^*(w) = -2/(1 - |w|^2)^2 \).

In [17, 18] we studied a contraction \( T \) with \( I - T^*T \) in the trace class, and showed that \( S_n^* < T^* \) if and only if \( T \) is in \( C_{10} \) (that is, \( T^n x \to 0 \), \( T^n x \to 0 \) as \( n \to \infty \) for every \( x \neq 0 \)) [17], and that these are equivalent with \( T^* \in B_n^h(D) \) [18]. We should notice that \( B_n^h(\Omega) \subset B_n^h(\Delta) \) for \( \Delta \subset \Omega \) (cf. p. 193 of [4]).

2. CURVATURES

It was shown that the curvature of a vector bundle generated by a holomorphic curve was nonpositive, and if \( T \) is in \( B_1(\Omega) \), then

\begin{equation}
\mathcal{K}_T(w)^{-1} = -\text{trace} N_w^* N_w,
\end{equation}

where \( N_w = (T - w)|_{\ker(T - w)} \) [4]. Let \( \Omega \) be a finitely connected Jordan region and \( \text{cl} \Omega \) (closure of \( \Omega \)) is a spectral set for \( T \), that is \( \sigma(T) \subset \text{cl} \Omega \) and \( \|f(T)\| \leq \|f\|_{\infty} \) for every rational function \( f \) with no poles in \( \text{cl} \Omega \). Then the curvature of \( T \) in \( B_1(\Omega) \) was estimated by Misra [9] as

\begin{equation}
\mathcal{K}_T(w) \leq -\hat{K}_\Omega(w, w)^2,
\end{equation}

where \( \hat{K}_\Omega \) is the Szegö kernel of \( \Omega \). His proof is based on (2.1). In this section we will extend (2.2) to the case of the \( B_n^h(\Omega) \) by virtue of the canonical model.
theory of contraction due to Sz.-Nagy and Foias [14]; let $T$ be a contraction on $H$ in $C_0$, that is $T^* w \to 0$ for $w$ in $H$. Then there is the characteristic function $\theta(z)$, which is a $B(F_1, F_2)$-valued holomorphic contractive function defined on $D$ and $\theta(z)$ is isometric from $F_1$ to $F_2$ a.e. on the unit circle, where $F_1$ and $F_2$ are the subspaces of $H$ called defect spaces of $T$. And then $T$ on $H$ is unitarily equivalent to $S(\theta)$ on $H(\theta)$ given as the following:

$$(2.3)~ H(\theta) = H^2(F_2) \ominus \theta H^2(F_1), \quad S(\theta)^* = M^*_z|_{H(\theta)},$$

where $M_z$ is the multiplication by $z$ on $H^2(F_2)$, which is the Hardy class of $F_2$-valued holomorphic functions on $D$. We remark that $S_n := S \oplus \cdots \oplus S \cong M_z$ on $H^2(C_0)$.

**Theorem 2.1.** Let $\gamma : \Omega \to g_r(n, H)$ be a holomorphic curve such that $\Omega \subset D$, $\Omega$ is open, $\forall w \in \Omega, \gamma(w) = H$. Suppose there is a contraction $T$ such that $\gamma(w) \subset \ker(T^* - w)$ for $w \in \Omega$. Then $H_\gamma(w) = H_T(w)$ $\leq -I_n/(1 - |w|^2)^2$ for $w$ in $\Omega$.

**Proof.** Since $T^k \gamma(w) = w^k \gamma(w) \to 0$ ($k \to \infty$), $\|T^*\| \leq 1$ implies $T \in C_0$. So we may consider $S(\theta)$ of (2.3) instead of $T$. For any $w_0 \in \Omega$, there is a nbhd $\Delta$ of $w_0$ and a frame $\gamma_1, \ldots, \gamma_n$ for $\gamma_w$ on $\Delta$. Then, since $M^*_z \gamma_{iw} = w \gamma_{iw}$, we can represent $\gamma_{iw}$ as the function in $H(\theta)$:

$$(2.4) \quad \gamma_{iw}(z) = \frac{\gamma_{iw}(z)}{1 - wz} \quad \text{for } z \in D.$$ 

Thus we have

$$(2.5) \quad \gamma_{iw}(0) \perp (\theta(w)F_2)$$

and

$$(2.6) \quad \langle \gamma_{jw}, \gamma_{iw} \rangle_{H(\theta)} = \frac{1}{2\pi} \int_{\partial D} \langle \gamma_{jw}(z), \gamma_{iw}(z) \rangle_{F_2} dz$$

which implies $\gamma_{iw}(0), \ldots, \gamma_{nw}(0)$ are linearly independent. Hence, if we set $\gamma^0_{iw} = \sqrt{\langle \gamma_{iw}(0) : i = 1, \ldots, n \rangle}$ for each $w \in \Delta$, then $\gamma^0 : \Delta \to g_r(n, F_2)$ is a holomorphic curve. From (1.1) and (2.6), it follows that

$$(2.7) \quad H_\gamma(w) = -I_n/(1 - |w|^2)^2 + H^0_\gamma(w) \quad \text{for } w \in \Delta.$$ 

Since $H_\gamma(w) \leq 0$, we can conclude the proof.

**Proposition 2.2.** If $T$ is a contraction in $B^h_n(D)$ and $H_T(w) = I_n/(1 - |w|^2)^2$ on an open set $\Delta \subset D$, then $T \cong S^*_n$.

**Proof.** Since $H_T(w) = H_{S^*_n}(w)$ for $w$ in $\Delta$, from Proposition 3.3 of [4], there is a holomorphic isometric bundle map $U(w)$ satisfying $U(w) \ker(T - w) =$
ker(S*\_n - w) for w in \( \Delta \). Since \( T \) is in \( B^h_n(\Delta) \), by the rigidity theorem (cf. p. 202 of [4]), there is a unitary \( U \) on \( H \) such that \( U \ker(T - w) = \ker(S*\_n - w) \) and hence \( UT = S*\_n U \). Thus the proof is complete.

Let \( \Omega_1, \Omega_2 \) be connected open sets, \( \gamma: \Omega_2 \to \mathcal{g}(n, H) \) a holomorphic curve, and \( \phi \) an injective holomorphic mapping from \( \Omega_1 \) to \( \Omega_2 \). Then by the chain rule and (1.1) we have

\[
(2.8) \quad \mathcal{H}_{\gamma \circ \phi}(w) = |\phi'(w)|^2 \mathcal{H}_\gamma(\phi(w)) \quad \text{for } w \in \Omega_1.
\]

**Proposition 2.3.** If \( T \) is a bounded operator in \( B_n(\Omega) \), where \( \Omega \) is an open connected set, then

\[
\mathcal{H}_T(w) \leq -\frac{I_n}{(\|T\|^2 - |w|^2)^2} \quad \text{for } w \in \Omega.
\]

**Proof.** From (2.8) \( \mathcal{H}_{T/\|T\|}(w/\|T\|) = \|T\|^2 \mathcal{H}_T(w) \) follows. Since \( \Omega/\|T\| \subset D \), Theorem 2.1 implies the above inequality.

**Theorem 2.5.** Let \( \Omega \) be a p-ply connected Jordan region, and \( T \in B^h_n(\Delta) \) for some \( \Delta \subset \Omega \). Suppose \( \text{cl}\Omega \) is a spectral set of \( T \). Then we have

\[
\mathcal{H}_r(w) \leq -\bar{K}_\Omega(w, w)^2 I_n \quad \text{for } w \in \Delta.
\]

**Proof.** For each \( w_0 \) in \( \Delta \) there is a holomorphic function \( F \) from \( \Omega \) to a p-sheeted disc such that \( F(w_0) = 0 \), \( F'(w_0) \neq 0 \), and \( F \) is continuous on \( \text{cl}\Omega \) (cf. [7, 2]). From Mergerlyan’s theorem there is a sequence of rational functions with no poles in \( \text{cl}\Omega \) which uniformly converges to \( F \) on \( \text{cl}\Omega \). We denote it by \( \{R_n\} \). Then Riesz functional \( R_n(T) \) is well defined and \( \{R_n(T)\} \) converges uniformly. We represent its limit by \( R(T) \). Then for a holomorphic curve \( \gamma(w) \subset \ker(T - w) \) on \( \Delta, \|F(T)\| \leq \|F\| = 1 \), \( \{F(T) - F(w)\} \gamma(w) = 0 \) follows, because \( \{R_n(T) - R_n(w)\} \gamma(w) = 0 \). From \( F'(w_0) \neq 0 \) we can take neighbourhoods \( \Omega_1 \) of \( w_0 \) and \( \Omega_2 \) of \( 0 \) such that \( F|_{\Omega_1}: \Omega_1 \to \Omega_2 \) is bijective. Let \( \phi \) be the inverse of \( F|_{\Omega_1} \). Then we have \( \{F(T) - Z\} \gamma(\phi(z)) = 0 \) for \( z \) in \( \Omega_2 \). Since

\[
\bigvee \{\gamma(\phi(z)) : z \in \Omega_2 \} = \bigvee \{\gamma(w) : w \in \Omega_1 \} = \bigvee \{\gamma(w) : w \in \Omega \} = H
\]

follows from p. 194 of [4], a contraction \( F(T) \) and curve \( \gamma \circ \phi \) satisfy the conditions of Theorem 2.1. Thus at the origin \( \mathcal{H}_{\gamma \circ \phi}(0) \leq -I_n \), from which, using (2.8), we get

\[
\mathcal{H}_\gamma(w_0) \leq -|F'(w_0)|^2 I_n = -\bar{K}_\Omega(w_0, w_0)^2 I_n,
\]

because the second equality follows from p. 118 of [2]. Consequently we can conclude the proof.

At the end of this section we consider the question proposed on p. 329 of [5], that is, if \( T_1 \) and \( T_2 \) are contractions in \( B_1(D) \) such that \( \mathcal{H}_{T_1} \leq \mathcal{H}_{T_2} \), then does there exist a bounded operator \( X \) such that \( XT_1 = T_2 X \)? Corollary 2.2 shows \( \mathcal{H}_r \leq \mathcal{H}_{S*} \) for any contraction \( T \) in \( B_1(D) \), and the existence of \( X \).
with dense range satisfying \( XT = S^* X \) is well known (cf. [16], or see the proof of Proposition 3.6). Hence the question is true in the case of \( T_2 = S^* \). In [10] Misra showed that a contraction \( T \) in \( B_1(D) \) is unitarily equivalent to \( \phi(T) \) for every Möbius transformation \( \phi \) of \( D \) if and only if \( \mathcal{H}_T(w) = -\alpha/(1 - |w|^2)^2 \), where \( \alpha \) is a constant and \( \alpha \geq 1 \).

**Proposition 2.6.** Let \( T_1, T_2 \) be contractions in \( B_1(D) \) with curvature \( \mathcal{H}_{T_i}(w) = -\alpha_i/(1 - |w|^2)^2 \) (\( \alpha_i \geq 1 \)). Then next conditions are equivalent: (i) \( \mathcal{H}_{T_2} \leq \mathcal{H}_{T_1} \), (ii) there is a bounded operator \( X \) such that \( XT_2 = T_1X \), and (iii) \( T_2 \prec T_1 \).

**Proof.** Let \( A_i \) be the weighted shift with weight \( a_{ni} = \sqrt{n/(\alpha_i + n - 1)} \) for \( i = 1, 2 \). Then we have \( r_1(A_i) = 1 \) and hence \( A_i^* \in B_1(D) \). Since the square of the norm of a holomorphic eigenvector of \( A_i^* - w \) is \( (1 - |w|^2)^{\alpha_i} \), \( \mathcal{H}_{A_i^*}(w) = \mathcal{H}_{A_i}(w) \), and hence \( A_i^* \equiv T_i \) (see [5]). Thus we may identify \( A_i^* \) with \( T_i \). Assume (i). Then diagonal quasi-affinity \( Y \) defined by \( Ye_n = \{(a_{12}, \ldots, a_{n2})/(a_{11}, \ldots, a_{n1})\}e_n \) satisfies \( YA_i = A_i Y \) and hence \( Y^*T_2 = T_1Y^* \), which implies (iii). Assume (ii). Since \( X^*A_1 = A_2X^* \), setting \( b_{mn} = (X^*e_n, e_m) \), we obtain

\[
 b_{mn+1}a_{n+11} = \begin{cases} 
 0 & (m = 0), \\
 b_{m-1n}a_{mn} & (m \geq 1).
\end{cases}
\]

Since there is a nonvanishing \( b_{ij} \) \( (i \geq j) \), boundedness of \( X \) implies that \( \prod_{k=1}^\infty a_{i+k+2}/a_{i+k+1} \) is bounded. To show (i), suppose \( \alpha_1 > \alpha_2 \), then each term of the infinite product is larger than 1. Hence

\[
\sum_{k=1}^\infty \left( \frac{(\alpha_1 + j + k - 1)}{j + k} / \frac{\alpha_2 + i + k - 1}{i + k} \right) - 1
\]

must converge, however this is impossible. Consequently (i) follows. (iii) obviously implies (ii), and the proof is complete.

We can apply the previous result to show that \( S \prec B \), where \( B \) is the Bergman shift, but there is not a bounded operator \( X \) such that \( XB = SX \), though it is possible to get them by another simple method.

### 3. Exact sequence and intertwining operators

In this section we give the conditions for a contraction \( T \) to be \( T \prec S_n \) or \( T \equiv S_n \). At the beginning we will refer to a result about exact sequence of Hardy classes and use it to show that if \( T \prec S_n \), then \( T^* \in B_n(D) \). A \( B(F_1, F_2) \)-valued holomorphic function \( \Gamma(z) \) on \( D \) is called bounded if \( \sup_{z \in D} \|\Gamma(z)\| < \infty \). In this case a bounded operator \( \Gamma \) from \( H^2(F_1) \) to \( H^2(F_2) \) is determined by \( (\Gamma f)(z) = \Gamma(z)f(z) \).

**Theorem 3.1.** Let \( \Gamma_1, \Gamma_2 \) be operator-valued bounded holomorphic functions on \( D \), and suppose

\[
 H^2(F_1) \xrightarrow{\Gamma_1} H^2(F_2) \xrightarrow{\Gamma_2} H^2(C_n)
\]
is exact and $\Gamma_2$ has the dense range. Then the next sequence is exact for every $z$ in $D$:

$$F_1 \xrightarrow{\Gamma_1(z)} F_2 \xrightarrow{\Gamma_2(z)} C_n \to 0.$$ 

**Proof.** Since $\Gamma_2(z)\Gamma_1(z) = 0$, we have only to show $\ker \Gamma_2(z) \subset \Gamma_1(z)F$. Since $\Gamma_2$ has the dense range, from the Cauchy integral formula, the range of $\Gamma_2(z)$ is dense and hence coincident with $C_n$. Thus $\Gamma_2^*(z) := \Gamma_2(z)^*$ is injective with closed range. Fix an arbitrary $z_0$ in $D$. There is an isometry $V$ from $C_n$ to $F_2$ such that $\det V^*\Gamma_2^*(z_0) \neq 0$. Then $\Omega := \{z \in D: \det V^*\Gamma_2^*(z) = 0\}$ is a set of isolated points. In the same way as Theorem 1 of [17] or p. 94 of [8] we can obtain a $B(F, F_2)$-valued bounded holomorphic function $\Phi(z)$ defined on $D$ such that $\Gamma_2^*(z)C_n \oplus \Phi(z)F = F_2$ for $z \in D \setminus \Omega$, where $F$ is an auxiliary Hilbert space. This implies $\ker \Gamma_2(z) = \Phi(z)F$ for $z \in D \setminus \Omega$ and hence $\Gamma_2 \Phi = 0$. Thus we have $\Phi H^2(F) \subset \ker \Gamma_2 = \Gamma_1 H^2(F_1)$. Taking $F$-valued constant functions we get $\Phi F \subset \Gamma_1(z)F_1$ for $z \in D$. Thus we have $\ker \Gamma_2(z_0) = \Phi(z_0)F \subset \Gamma_1(z_0)F_1$. The proof is complete.

**Remark.** The converse assertion of the theorem is false. In fact, set

$$\Gamma_1(z) = \left(\begin{array}{c}
\exp \frac{z+1}{z-1} \\
0
\end{array}\right), \quad \Gamma_2(z) = (0, 1),$$

then

$$C_1 \xrightarrow{\Gamma_1(z)} C_2 \xrightarrow{\Gamma_2(z)} C_1 \to 0$$

is exact for each $z$, but

$$\Gamma_1 H^2(C_1) = \exp \frac{z+1}{z-1} H^2(C_1) \oplus 0 \subsetneq H^2(C_1) \oplus 0 = \ker \Gamma_2.$$ 

**Corollary 3.2** (K. Takahashi [16]). Let $T$ be a contraction with $T \prec S_n$, then $T^* \in B_n(D)$.

**Proof.** Since $T$ is in class $C_0$, we may identify $S(\theta)$ given by (2.3) with $T$. Let $X$ be a quasi-affinity such that $XS(\theta) = S_nX$. Then, from the lifting theorem (see [14]) there is a $B(F_2, C_n)$-valued bounded holomorphic function $\Phi(z)$ defined on $D$ such that $\Gamma \theta = 0$ and $Xh = \Gamma h$ for $h$ in $H(\theta)$. That $X$ is a quasi-affinity implies that $H^2(F_1) \theta \to H^2(F_2) \Gamma \mapsto H^2(C_n)$ is exact, and that $\Gamma$ has the dense range. Thus from the theorem we get $\theta(w)F_1$ is closed and $\dim\{F_2 \oplus \theta(w)F_1\} = n$ for $w$ in $D$. The next equivalent conditions:

1. $\theta(w)F_1$ is closed in $F_2$,
2. $\frac{z-w}{1-\bar{w}z} H^2(F_2) \oplus \theta(w)F_1$ is closed in $H^2(F_2)$,
3. $\frac{z-w}{1-\bar{w}z} H^2(F_2) + \theta H^2(F_1)$ is closed in $H^2(F_2)$,
4. $P_{H(\theta)} \frac{z-w}{1-\bar{w}z} H(\theta)$ is closed in $H(\theta)$,
5. $(S(\theta) - w)(I - \bar{w}S(\theta))^{-1} H(\theta)$ is closed in $H(\theta)$,

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show that the range of \((S(\theta) - w)^*\) is closed for \(w\) in \(D\). Similarly we have \(\dim \ker(S(\theta) - w)^* = n\), hence the proof is complete.

**Remark.** The latter half in the above proof is trivial if we notice that \(\theta\) is the characteristic function of \(S(\theta)\) [14]. But we showed it directly.

**Theorem 3.3.** Let \(T\) be a contraction. Then \(T < S_n\) if and only if \(T^* \in B_n^h(D)\) and there is a frame \(\{\gamma_1w, \ldots, \gamma_{nw}\}\) for \(\ker(T^* - w)\) on \(D\) such that

\[
\sup_{w \in D} (1 - |w|^2) \|\gamma_{iw}\|^2 < \infty \quad \text{for each } i.
\]

**Proof.** Let \(\{e_1, \ldots, e_n\}\) be the O.N.B. of \(C_n\). Then eigenvectors of \((S_n^* - w)\) are \(e_i/(1 - wz), \ldots, e_n/(1 - wz)\). If \(X\) is the quasi-affinity such that \(XT = S_nX\), then \(\gamma_{iw} = X^*e_i/(1 - wz)\) satisfies the norm condition. The rest of “only if” part is clear. In order to show “if” part, we consider \(S(\theta)\) instead of \(T\). Then \(\gamma_{iw}\) is given by (2.4). By the norm condition and (2.6), \(\|\gamma_{iw}(0)\|\) is uniformly bounded for \(w\) in \(D\). For each \(z\) in \(D\), we determine the operator \(r(z) : F_2 \to C_n\) by

\[
r(z) = \sum_{i=1}^n (y, \gamma_{iz}(0))e_i.
\]

Then from (2.5) we have \(\Gamma(z)\theta(z) = 0\), and clearly \(\sup_{z \in D} \|\Gamma(z)\| < \infty\). Let us determine the bounded operator \(\Gamma(z) : H(\theta) \to H^2(C_n)\) by \(Xh = \Gamma h\) for \(h\) in \(H(\theta)\). Then it clearly follows that \(XS(\theta) = S_nX\). For any \(i, k\), and any \(\zeta\), \(w\) in \(D\), since \(z\) is the variable of a function, we have

\[
\begin{align*}
\left( \frac{X^* e_i}{1 - wz}, \frac{\gamma_k\zeta(0)}{1 - \zeta z} \right)_{H^2(C_n)} &= \left( \frac{e_i}{1 - wz}, \sum_j \frac{(\gamma_j\zeta(0), \gamma_jz(0))e_j}{1 - \zeta z} \right)_{H^2(C_n)} \\
&= \left( \frac{\gamma_{iz}(0)}{1 - wz}, \frac{\gamma_{k}\zeta(0)}{1 - \zeta z} \right)_{L^2(F_2)} = \left( P_{H^2(F_2)} \frac{\gamma_{iz}(0)}{1 - wz}, \frac{\gamma_{k}\zeta(0)}{1 - \zeta z} \right)_{H^2(F_2)} \\
&= \left( \frac{\gamma_{iw}(0)}{1 - wz}, \frac{\gamma_{k}\zeta(0)}{1 - \zeta z} \right)_{H(\theta)} = (\gamma_{iw}, \gamma_{k}\zeta)_{H(\theta)},
\end{align*}
\]

which shows that \(X^*e_i/(1 - wz) = \gamma_{iw}\), because \(\bigvee_{k\zeta} \gamma_{k}\zeta = H(\theta)\), and hence that \(X^*\) has the dense range. Thus \(X\) is injective. Since the rank of \(\Gamma(z)\) is \(n\), \(S_n|_{cl \Gamma H(\theta)} = S_n|_{cl \Gamma H^2(F_2)}\) is unitarily equivalent to \(S_n\). To accomplish the proof, it suffices to take \(PX\) to be the intertwining quasi-affinity, where \(P\) is the projection from \(H^2(C_n)\) to \(cl \Gamma H(\theta)\). The proof is complete.

Suppose \(T\) be a completely nonunitary (c.n.u.) contraction. In [1], Alexander called vectors \(h_1, \ldots, h_n\) analytically independent under \(T\) if a relation \(\phi_i(T)h_1 + \cdots + \phi_n(T)h_n = 0\) with \(\phi_i \in H^\infty\) implies \(\phi_1 = \cdots = \phi_n = 0\), and showed that \(S_n < T\) if and only if \(T\) has \(n\) cyclic vectors which are analytically independent under \(T\). We remark that a contraction \(T\) with the adjoint in \(B_n^h(D)\) satisfies \(T^{*n} \to 0\) so that \(T\) is c.n.u.
Corollary 3.4. Let $T$ be a contraction. Then $T \sim S_n$ if and only if $T$ has $n$-cyclic vectors, $T^* \in B^h_n(D)$ and there is a frame $\{\gamma_{1w}, \ldots, \gamma_{nw}\}$ for $\ker(T^* - w)$ on $D$ such that
\[
\sup_{w \in D} (1 - |w|^2) \|\gamma_{1w}\|^2 < \infty \quad \text{for each } i.
\]

Proof. We have only to show "if" part. From above theorem $T < S_n$ follows. Let $X$ be a quasi-affinity satisfying $XT = S_nX$, and $h_1, \ldots, h_n$ cyclic vectors for $T$. Then $Xh_1, \ldots, Xh_n$ are cyclic vectors for $S_n$. It is trivial to show that for each $z$ in $D$ $(Xh_1)(z), \ldots, (Xh_n)(z)$ span $C_n$ and hence $\det((Xh_1)(z), \ldots, (Xh_n)(z)) \neq 0$. Thus, from [1], $Xh_1, \ldots, Xh_n$ are analytically independent under $S_n$. Since $X\phi_i(T)h_i = \phi_i(S_n(Xh_i)), h_1, \ldots, h_n$ are analytically independent under $T$. Thus we obtain $S_n < T$ and hence $S_n \sim T$.

In [20], P. Y. Wu gave a necessary and sufficient condition for the characteristic function of $T$ to be $T \sim S_n$. That $S_n^*$ has a cyclic vector was shown by D. Sarason. Now we can extend it as follows:

Theorem 3.5. If $\Omega$ is a connected open set and $T^* \in B^h_n(\Omega)$, then $T^*$ has a cyclic vector. Especially if $T$ is a contraction with $T^* \in B^h_n(D)$, then $S < T^*$.

Proof. Fix an arbitrary $w_0$ in $\Omega$, then there is a nbhd $\Delta$ of $w_0$, and a frame $\gamma_{1w}, \ldots, \gamma_{nw}$ for $\ker(T^* - w)$ on $\Delta$. Since $B^h(\Omega) \subset B^h_n(\Delta)$,
\[
\sqrt{\{\gamma_{1w} : 1 \leq i \leq n, w \in \Delta\}} = H
\]
follows. By the Taylor expansion we have $\sqrt{\{\gamma_i^{(k)} : 1 \leq i \leq n, 1 \leq k < \infty\}} = H$, where $\gamma_i^{(k)} = (a^k \gamma_{1w}/dw^k)_{w=w_0} \subset H$. From $(T^* - w)\gamma_{1w} = 0$, it follows that $(T^* - w_0)^{1/2} = k^{1/2}$. Setting $a_k = 1/k!$, clearly $\sum_{k=0}^\infty \gamma_i^{(k)} a_k/k! < \infty$. In case of $n = 1$, $x = \sum_{k=0}^\infty \gamma_1^{(k)} a_k/k!$ is a cyclic vector. In fact, $\gamma_1^{(k)} a_k/k!$ implies that
\[
\frac{\langle (T^* - w_0)^m x - \gamma_1^{(0)} \rangle}{a_m} \leq \frac{a_{m+1}}{a_m} \sum_{k=1}^{\infty} \frac{\|\gamma_i^{(k)}\| a_{m+k}}{k!} \leq \frac{a_{m+1}}{a_m} \left( \sum_{k=1}^{\infty} \frac{\|\gamma^{(k)}\| a_k}{k!} \right) \rightarrow 0
\]
as $m \rightarrow \infty$. Thus $\gamma_1^{(0)} \in \sqrt{\infty}_{m=0} (T^* - w_0)^m x$. From
\[
\left\| \frac{1}{a_m} ((T^* - w_0)^{m-1} x - a_{m-1} \gamma_1^{(0)}) - \gamma_1^{(1)} \right\|
\]
\[
\leq \frac{a_{m+1}}{a_m} \sum_{k=2}^{\infty} \frac{\|\gamma_i^{(k)}\| a_k}{k! a_2} \rightarrow 0 \quad (m \rightarrow \infty),
\]
we have $\gamma_1^{(1)} \in \sqrt{\infty}_{m=0} (T^* - w_0)^m x$. Similarly we get $\gamma_1^{(k)} \in \sqrt{\infty}_{m=0} (T^* - w_0)^m x$, consequently $\sqrt{\infty}_{m=0} (T^* - w_0)^m x = H$, and hence $\sqrt{\infty}_{m=0} T^m x = H$. In case of
\[ x = \gamma_1^{(0)} a_0 + \frac{\gamma_2^{(1)}}{1!} a_1 + \frac{\gamma_3^{(2)}}{2!} a_2 + \cdots + \frac{\gamma_n^{(n-1)}}{(n-1)!} a_{n-1} + \frac{\gamma_1^{(n)}}{n!} a_n + \frac{\gamma_2^{(n+1)}}{(n+1)!} a_{n+1} + \cdots \]

is a cyclic vector for \( T^* \). To show the rest, suppose \( \phi(T^*)x = 0 \) for \( \phi \in H^\infty \).

Since \( \phi(T^*)T^m x = T^{*m} \phi(T^*)x = 0 \), we have \( \phi(T^*) = 0 \). From \( T^* \gamma_{iw} = w \gamma_{iw} \), it follows that \( \phi(T^*) \gamma_{iw} = \phi(w) \gamma_{iw} \) for every \( w \) in \( D \) and hence \( \phi(w) = 0 \), which implies that \( x \) is analytically independent under \( T^* \). Consequently we get \( S \prec T^* \).

**Proposition 3.6.** If \( T \) is a contraction and \( T \prec S_n \), then there is an invariant subspace \( L \) for \( T \) such that \( T|_L \sim S_n \).

**Proof.** Let us consider \( S(\theta) \) instead of \( T \). Then the eigenvector \( \gamma_{i0} \) of \( T^* \) is given by (2.4). Since it is constant vector valued, we can determine a bounded operator \( Y \) from \( H^2(C^n) = \bigoplus_i H^2(C_i) \) to \( H^2(C) \) by

\[
Y(h_1 \oplus \cdots \oplus h_n) = P_{H(\theta)}(h_1 \gamma_{i0} + \cdots + h_n \gamma_{n0}).
\]

Suppose \( Y(h_1 \oplus \cdots \oplus h_n) = 0 \). Then \( \sum h_i \gamma_{i0} \in \theta H^2(F_1) \) so that there is \( f \) in \( H^2(F_1) \) such that \( \sum h_i \gamma_{i0} = \theta f \). By (2.5) and linear independence of \( \gamma_{10}(0), \ldots, \gamma_{n0}(0) \), we have \( h_i(0) = 0 \) and \( f(0) = 0 \). Since

\[
\sum h_i'(0) \gamma_{i0}(0) = \theta'(0) f(0) + \theta(0) f'(0) = \theta(0) f'(0),
\]

we have \( h_i'(0) = 0 \) and \( f'(0) = 0 \) too. Thus to show \( h_i = 0 \) it suffices to continue this process. Set \( L = \text{cl} YH^2(C_n) \). Then \( TL \subset L \) and \( S_n \prec T|_L \).

Let \( X \) be a quasi-affinity satisfying \( XT = S_n X \). Then \( XY \) is injective and commutes with \( S_n \). From the characterizations of invariant subspaces for \( S_n \), it follows that \( S_n|_{\text{cl} X L} = S_n|_{\text{cl} X Y H^2(C_n)} \cong S_n \), and hence \( T|_L \prec S_n \). Thus we have \( T|_L \sim S_n \) and the proof is complete.

Next we will give the conditions for contractions to be similar to \( S_n \) by using the Rosenblum’s infinite corona theorem [11]. Suppose

\[
\sup_{z \in D} \sum_{j=1}^{n} \sum_{i=1}^{\infty} |h_{ij}(z)|^2 < \infty, \quad \text{where} \quad h_{ij} \in H^\infty.
\]

Then a \( B(C_n, l^2) \)-valued holomorphic function \( A(z) = (h_{ij}(z)) \) is bounded on \( D \). Under this setting we have

**Proposition 3.7.** There is a \( B(l^2, C_n) \)-valued bounded holomorphic function \( B(z) \) such that \( B(z) A(z) = I \) for \( z \) in \( D \), if and only if there is a positive constant \( \delta \) such that \( \|A(z) x\| \geq \delta \|x\| \) for every \( x \) in \( C_n \) and every \( z \) in \( D \).

**Proof.** Suppose \( \|A(z) x\| \geq \delta \|x\| \). Then \( A(z)^* A(z) \geq \delta^2 \) and hence

\[
\delta^{2n} \leq \det(A(z)^* A(z)) = \sum_{i_1 < \cdots < i_n} |\det A_{i_1 \cdots i_n}(z)|^2,
\]
where $A_{i_1 \cdots i_n}$ is the $n \times n$ submatrix of $A$. Since $\det(A(z)^*A(z))$ is upper bounded, by the infinite corona theorem, there are $b_{i_1 \cdots i_n} \in H^\infty$ such that

$$ \sup_{z \in D} \sum_{i_1 < \cdots < i_n} |b_{i_1 \cdots i_n}(z)|^2 < \infty, \quad \sum_{i_1 < \cdots < i_n} b_{i_1 \cdots i_n} \det A_{i_1 \cdots i_n} = 1 \quad \text{on } D. $$

Thus we can construct a bounded holomorphic function $B(z)$ such that $B(z)A(z) = I$ in the same way as Fuhrmann [6]. The converse is trivial, so we can conclude the proof.

**Theorem 3.8.** Let $T$ be a contraction. Then $T$ is similar to $S_n$ if and only if $T^* \in B^h_n(D)$, and there is a holomorphic frame $\gamma_{1w}, \ldots, \gamma_{nw}$ for $\ker(T^* - w)$ and positive constants $M, \delta$ such that for any $x_i \in \mathbb{C}$ and $w \in D$

$$ M \sum_{i=1}^n |x_i|^2 \geq (1 - |w|^2) \left\| \sum_{i=1}^n x_i \gamma_{iw} \right\|^2 \geq \delta \sum_{i=1}^n |x_i|^2. $$

**Proof.** We use the notations in the proof of Theorem 3.3. Let $Y$ be an invertible operator satisfying $YT = S_n Y$. Then $\gamma_{iw} = Y^* e_i / (1 - wz)$ satisfies (3.1). It is clear that $T^*$ is in $B^h_n(D)$. Thus we must only show “if” part. We represent $\gamma_{iw}$ as (2.4), and determine $\Gamma(z) : F_2 \rightarrow C_n$ by $\Gamma(z) = \sum_{i=1}^n (y, \gamma_{iz}(0)) e_i$. Then we have $\Gamma^*(z)x = \sum_{i=1}^n (x, e_i) \gamma_{iz}(0)$ for $x \in \mathbb{C}^n, z \in D$. Thus, since

$$ \|\Gamma^*(z)x\|^2 = \left\| \sum_i (x, e_i) \gamma_{iz}(0) \right\|^2 = (1 - |z|^2) \left\| \sum_i (x, e_i) \gamma_{iz}(0) \right\|^2 \text{ for every } z \in D, $$

applying Proposition 3.7, $\Gamma(z)$ has the bounded right inverse. Therefore we have $H^2(C_n) = \Gamma H^2(F_2) = \Gamma H(\theta)$, because $\Gamma \theta = 0$. Consequently $X$ given by $Xh = \Gamma h$ is an invertible operator from $H(\theta)$ to $H^2(C_n)$ satisfying $XT = S_n X$ (see the proof of Theorem 3.3). Hence the proof is complete.

We observe that we can substitute $(1 - |w|^2)G(w)$ for the middle term of (3.1), where $G(w)$ is the Gram matrix of $\gamma_{iw}, \ldots, \gamma_{nw}$.

**Proposition 3.9.** The contraction $T$ is similar to the isometry if and only if $T$ satisfies one of the following equivalent conditions:

(a) there is a positive constant $\delta$ such that $\|T^n x\| \geq \delta \|x\|$ for $x$ in $H$.

(b) There is a power-bounded operator $B$ satisfying $BT = I$.

(c) There is a bounded operator $B$ such that $BT = I$ and for any $w$ in $D$ $(I - wB^*)^{-1}$ exists and $\sup_{w \in D} (1 - |w|) \|(I - wB^*)^{-1}\| < \infty$

**Proof.** In [15], Sz.-Nagy and Foias showed that $T$ satisfies (a) if and only if $T$ is similar to isometry. (a) $\iff$ (b) is trivial. Moreover it is clear that (c) follows from similarity of $T$ and isometry, and its converse is able to be shown in the same way as Eastern [3], by considering

$$ \sum_{n=1}^\infty r^n e^{int} B^n + \sum_{n=1}^\infty r^n e^{-int} T^n $$

instead of $\sum_{n=-\infty}^\infty r^n e^{int} S^n$ on p. 191 of [3].
At the end of this section we remark that from the above proposition we can get conditions for $T$ to be similar to $S_n$. For instance it suffices to add $T \in C_0$ and $\dim \ker T^* = n$ to each condition of the above.

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References


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