CURVATURES AND SIMILARITY OF OPERATORS WITH
HOLOMORPHIC EIGENVECTORS

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ABSTRACT. The curvature of the holomorphic vector bundle generated by eigen-
vectors of operators is estimated, and the necessary and sufficient conditions for
contractions to be similar or quasi-similar with unilateral shifts are given.

1. INTRODUCTION

Let $H$ be a separable complex Hilbert space, $g_r(n, H)$ the set of all $n$-
dimensional subspaces of $H$, and $\gamma$ a mapping from an open connected set $\Omega$
in the complex plane $\mathbb{C}$ to $g_r(n, H)$. Then $\gamma$ is called a holomorphic curve
over $\Omega$, if for each $w_0$ in $\Omega$, there is a nbhd $\Delta$ of $w_0$ and vector valued
holomorphic functions $\gamma_{iW}$ on $\Delta$ ($i = 1, \ldots, n$) satisfying $\gamma_w = \sqrt[\gamma]{\gamma_{iw}}$ ($i = 1, \ldots, n$) for $w$ in $\Delta$. In this case, the Hermitian holomorphic vector bundle
$(E_\gamma, \Omega, \pi)$ is defined as

$$E_\gamma = \{(x, w) \in H \times \Omega : x \in \gamma_w\}, \quad \pi(x, w) = w,$$

and hence for this bundle, the canonical connection and curvature $\mathcal{H}_\gamma$ are well
defined [19]. We call $\gamma_{1W}, \ldots, \gamma_{nW}$ a frame for $E_\gamma$ on $\Delta$. The matrix form of
$\mathcal{H}_\gamma(w)$ with respect to the above frame is

\begin{equation}
-\frac{\partial}{\partial w} \left( G^{-1} \frac{\partial G}{\partial w} \right),
\end{equation}

where $G(w)$ is the Gram matrix whose $(i, j)$ component is $(\gamma_j(w), \gamma_i(w))$
(cf. [4]).

In case of $n = 1$, we have especially

$$\mathcal{H}_\gamma(w) = -\frac{\partial^2}{\partial w \partial \bar{w}} \log ||\gamma_{1W}||^2.$$

We explain some notations about relations between given bounded operators
$T_1, T_2$. Suppose there is an intertwining bounded operator $X$ such that $XT_1 =
T_2X$, then we denote by $T_1 \prec T_2$, $T_1 \prec T_2$, $T_1 \prec T_2$, $T_1 \approx T_2$, and $T_1 \equiv T_2$,
with dense range, $X$ injective, quasi-affinity (that is, $X$ is injective and has dense range), invertible, and unitary, respectively. Moreover we write $T_1 \sim T_2$ and say that $T_1$ and $T_2$ are quasi-similar, if $T_1 \prec T_2$ and $T_2 \prec T_1$. In [4], Cowen-Douglas defined the class $B_n(\Omega)$ consisting of bounded operator $T$ satisfying

(a) $\Omega \subset \sigma(T)$,
(b) $\text{range}(T - w) = H$ for each $w$ in $\Omega$,
(c) $\bigwedge_{w \in \Omega} \ker(T - w) = H$,
(d) $\dim \ker(T - w) = n$ for $w$ in $\Omega$.

Now we introduce the class $B^h_n(\Omega)$ as

**Definition.** $T$ belongs to $B^h_n(\Omega)$ if there is a holomorphic curve $\gamma : \Omega \rightarrow \mathbb{C}$ such that $\gamma(w) \subset \ker(T - w)$, and $\bigwedge_{w \in \Omega} \gamma(w) = H$. It is known that $B_n(\Omega) \subset B^h_n(\Omega)$. If $T$ is in $B^h_n(\Omega)$, then the bundle is well defined by the curve $\gamma(w)$. We denote it and its curvature by $E_T$ and $\mathcal{A}_T$.

The purpose of this paper is to estimate $\mathcal{A}_T$ of $T$ in $B^h_n(\Omega)$ and to research what kind of operator is similar or quasi-similar to the shifts.

Now we show some examples. Let $\{e_n\}_{n=0}^{\infty}$ be a C.O.N.B. of $H$ and $A$ a weighted shift with positive weight $\{a_n\}_{n=1}^{\infty}$, that is $Ae_n = a_{n+1}e_{n+1}$. Set $b_n = a_1 \cdots a_n$ and $r_1(A) = \lim_{n \rightarrow \infty} (\inf_k b_{k+n}/b_k)^{1/n}$. Then we have $A^* \in B_1(\{w : |w| < r_1(A)\})$, (see [13 or 12]). Especially, the adjoint of unilateral shift $S$ corresponding to $a_n = 1$ for all $n$ and the adjoint of the Bergman shift $B$ corresponding to $a_n = \sqrt{n}/(n+1)$ for all $n$ are both in $B_1(D)$, where $D$ is the open unit disk. And $\mathcal{A}_S^2(w) = -1/(1 - |w|^2)^2$ and $\mathcal{A}_B^*(w) = -2/(1 - |w|^2)^2$.

In [17, 18] we studied a contraction $T$ with $I - T^*T$ in the trace class, and showed that $S^*_n \prec T^*$ if and only if $T$ is in $C_{10}$ (that is, $T^n x \rightarrow 0$, $T^nx \rightarrow 0$ as $n \rightarrow \infty$ for every $x \neq 0$) [17], and that these are equivalent with $T^* \in B^h_n(D)$ [18]. We should notice that $B^h_n(\Omega) \subset B^h_n(\Delta)$ for $\Delta \subset \Omega$ (cf. p. 193 of [4]).

2. Curvatures

It was shown that the curvature of a vector bundle generated by a holomorphic curve was nonpositive, and if $T$ is in $B_1(\Omega)$, then

$$\mathcal{A}_T(w)^{-1} = -\text{trace} N^*_w N_w,$$

where $N_w = (T - w)_{\ker(T - w)}^2$ [4]. Let $\Omega$ be a finitely connected Jordan region and $\overline{\Omega}$ (closure of $\Omega$) is a spectral set for $T$, that is $\sigma(T) \subset \overline{\Omega}$ and $\|f(T)\| \leq \|f\|_{\infty}$ for every rational function $f$ with no poles in $\overline{\Omega}$. Then the curvature of $T$ in $B_1(\Omega)$ was estimated by Misra [9] as

$$\mathcal{A}_T(w) \leq -\mathcal{K}_\Omega^*(w, \overline{w})^2,$$

where $\mathcal{K}_\Omega$ is the Szegö kernel of $\Omega$. His proof is based on (2.1). In this section we will extend (2.2) to the case of the $B^h_n(\Omega)$ by virtue of the canonical model.
theory of contraction due to Sz.-Nagy and Foias [14]; let \( T \) be a contraction on \( H \) in \( C_0 \), that is \( T^*x \to 0 \) for \( x \) in \( H \). Then there is the characteristic function \( \theta(z) \), which is a \( B(F_1, F_2) \)-valued holomorphic contractive function defined on \( D \) and \( \theta(z) \) is isometric from \( F_1 \) to \( F_2 \) a.e. on the unit circle, where \( F_1 \) and \( F_2 \) are the subspaces of \( H \) called defect spaces of \( T \). And then \( T \) on \( H \) is unitarily equivalent to \( S(\theta) \) on \( H(\theta) \) given as the following:

\[
H(\theta) = F_2^2(\theta) \ominus \theta F_1^2(\theta), \quad S(\theta)^* = M_z^*|_{H(\theta)},
\]

where \( M_z \) is the multiplication by \( z \) on \( H^2(F_2) \), which is the Hardy class of \( F_2 \)-valued holomorphic functions on \( D \). We remark that \( S_n := S \oplus \cdots \oplus S \cong M_z \) on \( H(C_n) \).

**Theorem 2.1.** Let \( \gamma : \Omega \to g_r(n, H) \) be a holomorphic curve such that \( \Omega \subset D \), \( \Omega \) is open, \( \{w \in \Omega : \gamma(w) = H\} \) is open. Suppose there is a contraction \( T \) such that \( \gamma(w) \subset \ker(T^* - w) \) for \( w \in \Omega \). Then \( \mathcal{H}_\gamma(w) := \mathcal{H}_{T^*}(w) \leq -I_n/(1 - |w|^2)^2 \) for \( w \) in \( \Omega \).

**Proof.** Since \( T^k\gamma(w) = w^k \gamma(w) \to 0 \) \( (k \to \infty) \), \( \|T^*\| \leq 1 \) implies \( T \in C_0 \). So we may consider \( S(\theta) \) of (2.3) instead of \( T \). For any \( w_0 \in \Omega \), there is a nbhd \( \Delta \) of \( w_0 \) and a frame \( \gamma_{1w}, \ldots, \gamma_{nw} \) for \( \gamma_w \) on \( \Delta \). Then, since \( M_z^*\gamma_{iw} = w\gamma_{iw} \), we can represent \( \gamma_{iw} \) as the function in \( H(\theta) \):

\[
\gamma_{iw}(z) = \frac{\gamma_{iw}(0)}{1 - wz} \quad \text{for} \quad z \in D.
\]

Thus we have

\[
\gamma_{iw}(0) \perp \theta(w)F_2
\]

and

\[
(\gamma_{jw}, \gamma_{iw})_{H(\theta)} = \frac{1}{2\pi} \int_{\partial D} (\gamma_{jw}(z), \gamma_{iw}(z))_{F_2} |dz|
\]

\[
= \frac{1}{1 - |w|^2} (\gamma_{jw}(0), \gamma_{iw}(0))_{F_2},
\]

which implies \( \gamma_{iw}(0), \ldots, \gamma_{nw}(0) \) are linearly independent. Hence, if we set \( \gamma_{w}^0 = \sqrt{\{\gamma_{iw}(0) : i = 1, \ldots, n\}} \) for each \( w \in \Delta \), then \( \gamma^0 : \Delta \to g_r(n, F_2) \) is a holomorphic curve. From (1.1) and (2.6), it follows that

\[
\mathcal{H}_\gamma(w) = -\frac{I_n}{(1 - |w|^2)^2} + \mathcal{H}_{\gamma^0}(w) \quad \text{for} \quad w \in \Delta.
\]

Since \( \mathcal{H}_{\gamma^0}(w) \leq 0 \), we can conclude the proof.

**Proposition 2.2.** If \( T \) is a contraction in \( B_n^h(D) \) and \( \mathcal{H}_T(w) = I_n/(1 - |w|^2)^2 \) on an open set \( \Delta \subset D \), then \( T \cong S_n^* \).

**Proof.** Since \( \mathcal{H}_T(w) = \mathcal{H}_{S_n^*}(w) \) for \( w \) in \( \Delta \), from Proposition 3.3 of [4], there is a holomorphic isometric bundle map \( U(w) \) satisfying \( U(w)\ker(T - w) = \)
\( \ker(S^*_n - w) \) for \( w \) in \( \Delta \). Since \( T \) is in \( B^h_n(\Delta) \), by the rigidity theorem (cf. p. 202 of [4]), there is a unitary \( U \) on \( H \) such that \( U \ker(T - w) = \ker(S^*_n - w) \) and hence \( UT = S^*_n U \). Thus the proof is complete.

Let \( \Omega_1, \Omega_2 \) be connected open sets, \( \gamma : \Omega_2 \rightarrow g_r(n, H) \) a holomorphic curve, and \( \phi \) an injective holomorphic mapping from \( \Omega_1 \) to \( \Omega_2 \). Then by the chain rule and (1.1) we have

\[
(2.8) \quad \mathcal{H}_{\gamma \circ \phi}(w) = |\phi'(w)|^2 \mathcal{H}_{\gamma}(\phi(w)) \quad \text{for } w \text{ in } \Omega_1.
\]

**Proposition 2.3.** If \( T \) is a bounded operator in \( B_n(\Omega) \), where \( \Omega \) is an open connected set, then

\[
\mathcal{H}_{T}(w) \leq -\frac{I_n}{(\|T\|^2 - |w|^2)^2} \quad \text{for } w \in \Omega.
\]

**Proof.** From (2.8) \( \mathcal{H}_{T/(\|T\|)||T||}(w) = \|T\|^2 \mathcal{H}_{T}(w) \) follows. Since \( \Omega/\|T\| \subset D \), Theorem 2.1 implies the above inequality.

**Theorem 2.5.** Let \( \Omega \) be a \( p \)-ply connected Jordan region, and \( T \in B^h_n(\Delta) \) for some \( \Delta \subset \Omega \). Suppose \( \text{cl} \Omega \) is a spectral set of \( T \). Then we have

\[
\mathcal{H}'_{T}(w) \leq -\mathcal{K}_{\Omega}(w, w)^2 I_n \quad \text{for } w \in \Delta.
\]

**Proof.** For each \( w_0 \) in \( \Delta \) there is a holomorphic function \( F \) from \( \Omega \) to a \( p \)-sheeted disc such that \( F(w_0) = 0 \), \( F'(w_0) \neq 0 \), and \( F \) is continuous on \( \text{cl} \Omega \) (cf. [7, 2]). From Mergerlyan’s theorem there is a sequence of rational functions with no poles in \( \text{cl} \Omega \) which uniformly converges to \( F \) on \( \text{cl} \Omega \). We denote it by \( \{R_n\} \). Then Riesz functional \( R_n(T) \) is well defined and \( \{R_n(T)\} \) converges uniformly. We represent its limit by \( F(T) \). Then for a holomorphic curve \( \gamma(w) \subset \ker(T - w) \) on \( \Delta \), \( \|F(T)\| \leq F \| = 1 \), and \( \{F(T) - F(w)\} \gamma(w) = 0 \) follows, because \( \{R_n(T) - R_n(w)\} \gamma(w) = 0 \). From \( F'(w_0) \neq 0 \) we can take neighbourhoods \( \Omega_1 \) of \( w_0 \) and \( \Omega_2 \) of \( 0 \) such that \( F|\Omega_1 : \Omega_1 \rightarrow \Omega_2 \) is bijective. Let \( \phi \) be the inverse of \( F|\Omega_1 \). Then we have \( \{F(T) - z\} \gamma(\phi(z)) = 0 \) for \( z \) in \( \Omega_2 \). Since

\[
\bigvee \{\gamma(\phi(z)) : z \in \Omega_2\} = \bigvee \{\gamma(w) : w \in \Omega_1\} = \bigvee \{\gamma(w) : w \in \Omega\} = H
\]

follows from p. 194 of [4], a contraction \( F(T) \) and curve \( \gamma \circ \phi \) satisfy the conditions of Theorem 2.1. Thus at the origin \( \mathcal{H}_{\gamma \circ \phi}(0) \leq -I_n \), from which, using (2.8), we get

\[
\mathcal{H}_{\gamma}(w_0) \leq -|F'(w_0)|^2 I_n = -\mathcal{K}_{\Omega}(w_0, w_0)^2 I_n,
\]

because the second equality follows from p. 118 of [2]. Consequently we can conclude the proof.

At the end of this section we consider the question proposed on p. 329 of [5], that is, if \( T_1 \) and \( T_2 \) are contractions in \( B_1(D) \) such that \( \mathcal{H}_{T_1} \leq \mathcal{H}_{T_2} \), then does there exist a bounded operator \( X \) such that \( X T_1 = T_2 X \)? Corollary 2.2 shows \( \mathcal{H}_{T} \leq \mathcal{H}_{S^*}_T \) for any contraction \( T \) in \( B_1(D) \), and the existence of \( X \)
with dense range satisfying $XT = S^*X$ is well known (cf. [16], or see the proof of Proposition 3.6). Hence the question is true in the case of $T_2 = S^*$. In [10] Misra showed that a contraction $T$ in $B_1(D)$ is unitarily equivalent to $\phi(T)$ for every Möbius transformation $\phi$ of $D$ if and only if $\mathcal{H}_T(w) = -\alpha/(1 - |w|^2)^2$, where $\alpha$ is a constant and $\alpha \geq 1$.

**Proposition 2.6.** Let $T_1$, $T_2$ be contractions in $B_1(D)$ with curvature $\mathcal{H}_i(w) = -\alpha_i/(1 - |w|^2)^2$ ($\alpha_i \geq 1$). Then next conditions are equivalent: (i) $\mathcal{H}_2 \leq \mathcal{H}_1$, (ii) there is a bounded operator $X$ such that $XT_2 = T_1X$, and (iii) $T_2 < T_1$.

**Proof.** Let $A_i$ be the weighted shift with weight $a_{ni} = \sqrt{n/(\alpha_i + n - 1)}$ for $i = 1, 2$. Then we have $r_i(A_i) = 1$ and hence $A_i^* \in B_1(D)$. Since the square of the norm of a holomorphic eigenvector of $A_i^* - w$ is $(1 - |w|^2)^{\alpha_i}$, $\mathcal{H}_A(w) = \mathcal{H}_T(w)$, and hence $A_i^* \cong T_i$ (see [5]). Thus we may identify $A_i^*$ with $T_i$. Assume (i). Then diagonal quasi-affinity $Y$ defined by $Ye_n = \{(a_{12}a_{n2})/(a_{11}a_{n1})\}e_n$ satisfies $YA_1 = A_2Y$ and hence $Y^*T_2 = T_1Y^*$, which implies (iii). Assume (ii). Since $X^*A_1 = A_2X^*$, setting $b_{m,n} = (X^*e_n, e_m)$, we obtain

$$b_{m,n+1}a_{n+1,1} = \begin{cases} 0 & (m = 0), \\ b_{m-1,n}a_{m,2} & (m \geq 1). \end{cases}$$

Since there is a nonvanishing $b_{ij}$ ($i \geq j$), boundedness of $X$ implies that $\prod_{k=1}^{\infty}a_{i+k,2}/a_{i+k,1}$ is bounded. To show (i), suppose $\alpha_1 > \alpha_2$, then each term of the infinite product is larger than 1. Hence

$$\sum_{k=1}^{\infty} \left( \frac{(\alpha_1 + j + k - 1)/j + k}{(\alpha_2 + i + k - 1)/i + k} - 1 \right)$$

must converge, however this is impossible. Consequently (i) follows. (iii) obviously implies (ii), and the proof is complete.

We can apply the previous result to show that $S < B$, where $B$ is the Bergman shift, but there is not a bounded operator $X$ such that $XB = SX$, though it is possible to get them by another simple method.

### 3. Exact sequence and intertwining operators

In this section we give the conditions for a contraction $T$ to be $T < S_n$ or $T \approx S_n$. At the beginning we will refer to a result about exact sequence of Hardy classes and use it to show that if $T < S_n$, then $T^* \in B_n(D)$. A $B(F_1, F_2)$-valued holomorphic function $\Gamma(z)$ on $D$ is called bounded if $\sup_{z \in D} ||\Gamma(z)|| < \infty$. In this case a bounded operator $\Gamma$ from $H^2(F_1)$ to $H^2(F_2)$ is determined by $(\Gamma f)(z) = \Gamma(z)f(z)$.

**Theorem 3.1.** Let $\Gamma_1$, $\Gamma_2$ be operator-valued bounded holomorphic functions on $D$, and suppose

$$H^2(F_1) \xrightarrow{\Gamma_1} H^2(F_2) \xrightarrow{\Gamma_2} H^2(C_n)$$
is exact and $\Gamma_2$ has the dense range. Then the next sequence is exact for every $z$ in $D$:

$$F_1 \xrightarrow{\Gamma_1(z)} F_2 \xrightarrow{\Gamma_2(z)} C_n \rightarrow 0.$$  

Proof. Since $\Gamma_2(z)\Gamma_1(z) = 0$, we have only to show $\ker \Gamma_2(z) \subset \Gamma_1(z)F$. Since $\Gamma_2$ has the dense range, from the Cauchy integral formula, the range of $\Gamma_2(z)$ is dense and hence coincident with $C_n$. Thus $\Gamma_2^*(z) = \Gamma_2(z)^*$ is injective with closed range. Fix an arbitrary $z_0$ in $D$. There is an isometry $V$ from $C_n$ to $F_2$ such that $\det V^*\Gamma_2^*(z_0) \neq 0$. Then $\Omega := \{z \in D : \det V^*\Gamma_2^*(z) = 0\}$ is a set of isolated points. In the same way as Theorem 1 of [17] or p. 94 of [8] we can obtain a $B(F, F_2)$-valued bounded holomorphic function $\Phi(z)$ defined on $D$ such that $\Gamma_2^*(z)C_n \oplus \Phi(z)F = F_2$ for $z \in D \setminus \Omega$, where $F$ is an auxiliary Hilbert space. This implies $\ker \Gamma_2(z) = \Phi(z)F$ for $z \in D \setminus \Omega$ and hence $\Gamma_2 \Phi = 0$. Thus we have $\Phi H^2(F) \subset \ker \Gamma_2 = \Gamma_1 H^2(F_1)$. Taking $F$-valued constant functions we get $\Phi(z)F \subset \Gamma_1(z)F_1$ for $z \in D$. Thus we have $\ker \Gamma_2(z_0) = \Phi(z_0)F \subset \Gamma_1(z_0)F_1$. The proof is complete.

Remark. The converse assertion of the theorem is false. In fact, set

$$\Gamma_1(z) = \begin{pmatrix} \exp \frac{z+1}{z-1} \\ 0 \end{pmatrix}, \quad \Gamma_2(z) = (0, 1),$$

then

$$C_1 \xrightarrow{\Gamma_1(z)} C_2 \xrightarrow{\Gamma_2(z)} C_1 \rightarrow 0$$

is exact for each $z$, but

$$\Gamma_1 H^2(C_1) = \exp \frac{z+1}{z-1} H^2(C_1) \oplus 0 \subsetneq H^2(C_1) \oplus 0 = \ker \Gamma_2.$$  

Corollary 3.2 (K. Takahashi [16]). Let $T$ be a contraction with $T < S_n$, then $T^* \in B_n(D)$.

Proof. Since $T$ is in class $C_0$, we may identify $S(\theta)$ given by (2.3) with $T$. Let $X$ be a quasi-affinity such that $XS(\theta) = S_nX$. Then, from the lifting theorem (see [14]) there is a $B(F_2, C_n)$-valued bounded holomorphic function $\Gamma(z)$ defined on $D$ such that $\Gamma \theta = 0$ and $Xh = \Gamma h$ for $h$ in $H(\theta)$. That $X$ is a quasi-affinity implies that

$$H^2(F_1) \theta \rightarrow H^2(F_2) \xrightarrow{\Gamma} H^2(C_n)$$

is exact, and that $\Gamma$ has the dense range. Thus from the theorem we get $\theta(w)F_1$ is closed and $\dim \{F_2 \ominus \theta(w)F_1\} = n$ for $w$ in $D$. The next equivalent conditions:

1. $\theta(w)F_1$ is closed in $F_2$,
2. $\frac{z-w}{1-\bar{w}z} H^2(F_2) \oplus \frac{\theta(w)F_1}{1-\bar{w}z}$ is closed in $H^2(F_2)$,
3. $\frac{z-w}{1-\bar{w}z} H^2(F_2) \oplus \theta H^2(F_1)$ is closed in $H^2(F_2)$,
4. $P_{H(\theta)} \frac{z-w}{1-\bar{w}z} H(\theta)$ is closed in $H(\theta)$,
5. $(S(\theta) - w)(I - \bar{w}S(\theta))^{-1} H(\theta)$ is closed in $H(\theta)$,
show that the range of \((S(\theta) - w)^*\) is closed for \(w\) in \(D\). Similarly we have \(\dim \ker(S(\theta) - w)^* = n\), hence the proof is complete.

**Remark.** The latter half in the above proof is trivial if we notice that \(\theta\) is the characteristic function of \(S(\theta)\) [14]. But we showed it directly.

**Theorem 3.3.** Let \(T\) be a contraction. Then \(T < S_n\) if and only if \(T^* \in B_n^h(D)\) and there is a frame \(\{\gamma_{iw}, \ldots, \gamma_{nw}\}\) for \(\ker(T^* - w)\) on \(D\) such that

\[
\sup_{w \in D} (1 - |w|^2) ||\gamma_{iw}||^2 < \infty \quad \text{for each } i.
\]

**Proof.** Let \(\{e_1, \ldots, e_n\}\) be the O.N.B. of \(C_n\). Then eigenvectors of \((S_n^* - w)\) are \(e_i/(1 - wz), \ldots, e_n/(1 - wz)\). If \(X\) is the quasi-affinity such that \(XT = S_n X\), then \(\gamma_{iw} = X^* e_i/(1 - wz)\) satisfies the norm condition. The rest of “only if” part is clear. In order to show “if” part, we consider \(S(\theta)\) instead of \(T\). Then \(\gamma_{iw}\) is given by (2.4). By the norm condition and (2.6), \(||\gamma_{iw}(0)||\) is uniformly bounded for \(w\) in \(D\). For each \(z\) in \(D\), we determine the operator \(r(z): F^2 \to C_n\) by

\[
\Gamma(z)y = \sum_{i=1}^{n} (y_i, \gamma_{iz}(0)) e_i.
\]

Then from (2.5) we have \(\Gamma(z)\theta(z) = 0\), and clearly \(\sup_{z \in D} ||\Gamma(z)|| < \infty\). Let us determine the bounded operator \(X: H(\theta) \to H^2(C_n)\) by \(Xh = \Gamma h\) for \(h\) in \(H(\theta)\). Then it clearly follows that \(XS(\theta) = S_n X\). For any \(i, k\), and any \(\zeta\), \(w\) in \(D\), since \(z\) is the variable of a function, we have

\[
\left( X^* e_i/(1 - wz), \frac{\gamma_k(\zeta)}{1 - \zeta z} \right)_{H(\theta)} = \left( e_i/(1 - wz), \sum_j \left( \frac{\gamma_k(\zeta)}{1 - \zeta z}, \frac{\gamma_j(\zeta)}{1 - \zeta z} \right) \right)_{H^2(C_n)}
\]

\[
= \left( \frac{\gamma_{iz}(0)}{1 - wz}, \frac{\gamma_k(\zeta)}{1 - \zeta z} \right)_{L^2(F)} = \left( \frac{\gamma_{iz}(0)}{1 - wz}, \frac{\gamma_k(\zeta)}{1 - \zeta z} \right)_{H^2(F)}
\]

\[
= \left( \frac{\gamma_{iw}(0)}{1 - wz}, \frac{\gamma_k(\zeta)}{1 - \zeta z} \right)_{H(\theta)} = \left( \gamma_{iw}, \gamma_{k\zeta} \right)_{H(\theta)},
\]

which shows that \(X^* e_i/(1 - wz) = \gamma_{iw}\), because \(\bigvee_{k<\zeta} \gamma_{k\zeta} = H(\theta)\), and hence that \(X^*\) has the dense range. Thus \(X\) is injective. Since the rank of \(\Gamma(z)\) is \(n\), \(S_n \mid_{\text{cl}XH(\theta)} = S_n \mid_{\text{cl}XH^2(F)}\) is unitarily equivalent to \(S_n\). To accomplish the proof, it suffices to take \(PX\) to be the intertwining quasi-affinity, where \(P\) is the projection from \(H^2(C_n)\) to \(\text{cl}XH(\theta)\). The proof is complete.

Suppose \(T\) be a completely nonunitary (c.n.u.) contraction. In [1], Alexander called vectors \(h_1, \ldots, h_n\) analytically independent under \(T\) if a relation \(\phi_1(T)h_1 + \cdots + \phi_n(T)h_n = 0\) with \(\phi_i \in H^\infty\) implies \(\phi_1 = \cdots = \phi_n = 0\), and showed that \(S_n < T\) if and only if \(T\) has \(n\) cyclic vectors which are analytically independent under \(T\). We remark that a contraction \(T\) with the adjoint in \(B_n^h(D)\) satisfies \(T^*n \to 0\) so that \(T\) is c.n.u.
Corollary 3.4. Let $T$ be a contraction. Then $T \sim S_n$ if and only if $T$ has $n$-cyclic vectors, $T^* \in B_n^h(D)$ and there is a frame $\{\gamma_{1w}, \ldots, \gamma_{nw}\}$ for $\ker(T^* - w)$ on $D$ such that
\[
\sup_{w \in D} (1 - |w|^2) \|\gamma_{iw}\|^2 < \infty \quad \text{for each } i.
\]

Proof. We have only to show "if" part. From above theorem $T \prec S_n$ follows. Let $X$ be a quasi-affinity satisfying $XT = S_n X$, and $h_1, \ldots, h_n$ cyclic vectors for $T$. Then $Xh_1, \ldots, Xh_n$ are cyclic vectors for $S_n$. It is trivial to show that for each $z$ in $D$ $(Xh_1)(z), \ldots, (Xh_n)(z)$ span $C_n$ and hence $\det((Xh_1)(z), \ldots, (Xh_n)(z)) \neq 0$. Thus, from [1], $Xh_1, \ldots, Xh_n$ are analytically independent under $S_n$. Since $X\phi_i(T)h_i = \phi_i(S_n)(Xh_i)$, $h_1, \ldots, h_n$ are analytically independent under $T$. Thus we obtain $S_n \prec T$ and hence $S_n \sim T$.

In [20], P. Y. Wu gave a necessary and sufficient condition for the characteristic function of $T$ to be $T \sim S_n$. That $S_n^*$ has a cyclic vector was shown by D. Sarason. Now we can extend it as follows:

Theorem 3.5. If $\Omega$ is a connected open set and $T^* \in B_n^h(\Omega)$, then $T^*$ has a cyclic vector. Especially if $T$ is a contraction with $T^* \in B_n^h(D)$, then $S_n \sim T^*.$

Proof. Fix an arbitrary $w_0$ in $\Omega$, then there is a nbhd $\Delta$ of $w_0$, and a frame $\gamma_{1w}, \ldots, \gamma_{nw}$ for $\ker(T^* - w)$ on $\Delta$. Since $B_n^h(\Omega) \subset B_n^h(\Delta)$,
\[
\sqrt{\{\gamma_{iw}^2 : 1 \leq i \leq n, w \in \Delta\}} = H
\]
follows. By the Taylor expansion we have $\sqrt{\{\gamma_{iw}^{(k)} : 1 \leq i \leq n, 1 \leq k < \infty\}} = H$, where $\gamma_i^{(k)} = (d^k \gamma_{iw}/dw^k)_{w=w_0} \in H$. From $(T^* - w)\gamma_{iw} = 0$, it follows that $(T^* - w_0)\gamma_{iw}^{(k)} = k\gamma_i^{(k-1)}$. Setting $a_k = 1/k!$, clearly $\sum_{k=0}^{\infty}\|\gamma_i^{(k)}\|a_k/k! < \infty$. In case of $n = 1$, $x = \sum_{k=0}^{\infty}\gamma_1^{(k)}a_k/k!$ is a cyclic vector. In fact, $$(T^* - w_0)^m x = \sum_{k=0}^{\infty} \gamma_1^{(k)}a_ka_{m+k}$$ implies that
\[
\left\|\frac{(T^* - w_0)^m}{a_m}x - \gamma_1^{(0)}\right\| \leq \frac{a_{m+1}}{a_m} \sum_{k=1}^{\infty} \frac{\|\gamma_1^{(k)}\|a_{m+k}}{k!} \frac{a_{m+k}}{a_{m+1}} \leq \frac{a_{m+1}}{a_m} \left( \sum_{k=1}^{\infty} \frac{\|\gamma_1^{(k)}\|a_k}{k!} \frac{a_k}{a_1} \right) \to 0
\]
as $m \to \infty$. Thus $\gamma_1^{(0)} \in \sqrt{\{m=0\}(T^* - w_0)^m} x$. From
\[
\left\|\frac{1}{a_m}((T^* - w_0)^{m-1}x - a_{m-1}\gamma_1^{(0)}) - \gamma_1^{(1)}\right\|
\]
\[
\leq \frac{a_{m+1}}{a_m} \sum_{k=2}^{\infty} \frac{\|\gamma_1^{(k)}\|a_k}{k!} \frac{a_k}{a_2} \to 0 \quad (m \to \infty),
\]
we have $\gamma_1^{(1)} \in \sqrt{\{m=0\}(T^* - w_0)^m} x$. Similarly we get $\gamma_1^{(k)} \in \sqrt{\{m=0\}(T^* - w_0)^m} x$, consequently $\sqrt{\{m=0\}(T^* - w_0)^m} x = H$, and hence $\sqrt{\{m=0\}T^m} x = H$. In case of
The natural text is:

\[ x = y_1(0) a_0 + \frac{y_2(1)}{1!} a_1 + \frac{y_3(2)}{2!} a_2 + \cdots + \frac{y_n(n-1)}{(n-1)!} a_{n-1} + \frac{y_1(n)}{n!} a_n + \frac{y_2(n+1)}{(n+1)!} a_{n+1} + \cdots \]

is a cyclic vector for \( T^* \). To show the rest, suppose \( \phi(T^*)x = 0 \) for \( \phi \in H^\infty \).

Since \( \phi(T^*)T^m x = T^m \phi(T^*)x = 0 \), we have \( \phi(T^*) = 0 \). From \( T^* \gamma_{iw} = w \gamma_{iw} \), it follows that \( \phi(T^*) \gamma_{iw} = \phi(w) \gamma_{iw} \) for every \( w \) in \( D \) and hence \( \phi(w) = 0 \), which implies that \( x \) is analytically independent under \( T^* \). Consequently we get \( S \prec T^* \).

**Proposition 3.6.** If \( T \) is a contraction and \( T \prec S_n \), then there is an invariant subspace \( L \) for \( T \) such that \( T|_L \sim S_n \).

**Proof.** Let us consider \( S(\theta) \) instead of \( T \). Then the eigenvector \( \gamma_{i0} \) of \( T^* \) is given by (2.4). Since it is constant vector valued, we can determine a bounded operator \( Y \) from \( H^2(C_n) = \bigoplus H^2(C_1) \) to \( H(\theta) \) by

\[ Y(h_1 \oplus \cdots \oplus h_n) = P_H(\theta)(h_{i1} \gamma_{i0} + \cdots + h_{i0} \gamma_{i0}) \]

Suppose \( Y(h_1 \oplus \cdots \oplus h_n) = 0 \). Then \( \sum h_i \gamma_{i0} \in \theta H^2(F_1) \) so that there is \( f \) in \( H^2(F_1) \) such that \( \sum h_i \gamma_{i0} = \theta f \). By (2.5) and linear independence of \( \gamma_{10}(0), \ldots, \gamma_{n0}(0) \), we have \( h_{i0} = 0 \) and \( f(0) = 0 \). Since

\[ \sum h_i(0) \gamma_{i0}(0) = \theta f(0) + \theta(0) f'(0) = \theta(0) f'(0) \]

we have \( h_i(0) = 0 \) and \( f'(0) = 0 \) too. Thus to show \( h_i = 0 \) it suffices to continue this process. Set \( L = \text{cl } YH^2(C_n) \). Then \( TL \subset L \) and \( S_n \prec T|_L \).

Let \( X \) be a quasi-affinity satisfying \( XT = S_n X \). Then \( XY \) is injective and commutes with \( S_n \). From the characterizations of invariant subspaces for \( S_n \), it follows that \( S_n|_{\text{cl } XL} = S_n|_{\text{cl } XYH^2(C_n)} \approx S_n \), and hence \( T|_L \prec S_n \). Thus we have \( T|_L \sim S_n \) and the proof is complete.

Next we will give the conditions for contractions to be similar to \( S_n \) by using the Rosenblum's infinite corona theorem [11]. Suppose

\[ \sup_{z \in D} \sum_{i=1}^{\infty} \sum_{j=1}^{n} |h_{ij}(z)|^2 < \infty, \quad \text{where } h_{ij} \in H^\infty. \]

Then a \( B(C_n, l^2) \)-valued holomorphic function \( A(z) = (h_{ij}(z)) \) is bounded on \( D \). Under this setting we have

**Proposition 3.7.** There is a \( B(l^2, C_n) \)-valued bounded holomorphic function \( B(z) \) such that \( B(z)A(z) = I \) for \( z \) in \( D \), if and only if there is a positive constant \( \delta \) such that \( \|A(z)x\| \geq \delta \|x\| \) for every \( x \) in \( C_n \) and every \( z \) in \( D \).

**Proof.** Suppose \( \|A(z)x\| \geq \delta \|x\| \). Then \( A(z)^* A(z) \geq \delta^2 \) and hence

\[ \delta^{2n} \leq \det(A(z)^* A(z)) = \sum_{i_1 < \cdots < i_n} |\det A_{i_1 \cdots i_n}(z)|^2, \]

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where $A_{i_1 \cdots i_n}$ is the $n \times n$ submatrix of $A$. Since $\det(A(z)^*A(z))$ is upper bounded, by the infinite corona theorem, there are $b_{i_1 \cdots i_n} \in H^\infty$ such that

$$\sup_{z \in D} \sum_{i_1 < \cdots < i_n} |b_{i_1 \cdots i_n}(z)|^2 < \infty, \quad \sum_{i_1 < \cdots < i_n} b_{i_1 \cdots i_n} \det A_{i_1 \cdots i_n} = 1 \quad \text{on } D.$$ 

Thus we can construct a bounded holomorphic function $B(z)$ such that $B(z)A(z) = I$ in the same way as Fuhrmann [6]. The converse is trivial, so we can conclude the proof.

**Theorem 3.8.** Let $T$ be a contraction. Then $T$ is similar to $S_n$ if and only if $T^* \in B_n^h(D)$, and there is a holomorphic frame $\gamma_{iw}, \ldots, \gamma_{nw}$ for $\ker(T^* - w)$ and positive constants $M, \delta$ such that for any $x_i \in \mathbb{C}$ and $w \in D$,

$$\sum_{i=1}^n |x_i|^2 \geq (1 - |w|^2) \left\| \sum_{i=1}^n x_i \gamma_{iw} \right\|^2 \geq \delta \sum_{i=1}^n |x_i|^2. \quad (3.1)$$

**Proof.** We use the notations in the proof of Theorem 3.3. Let $Y$ be an invertible operator satisfying $YT = S_n Y$. Then $\gamma_{iw} = Y^*e_i/(1 - wz)$ satisfies (3.1). It is clear that $T^*$ is in $B_n^h(D)$. Thus we must only show “if” part. We represent $\gamma_{iw}$ as (2.4), and determine $\Gamma(z) = F_2 \to C_n$ by $\Gamma(z)v = \sum_{i=1}^n (y, \gamma_{iz}(0))e_i$. Then we have $\Gamma^{-1}(z)x = \sum_{i=1}^n (x, e_i)\gamma_{iz}(0)$ for $x \in \mathbb{C}$, $z \in D$. Thus, since

$$\|\Gamma^{-1}(z)x\|^2 = \left\| \sum_{i=1}^n (x, e_i)\gamma_{iz}(0) \right\|^2 = (1 - |z|^2) \left\| \sum_{i=1}^n (x, e_i)\gamma_{iz}(0) \right\|^2 \quad \text{for every } z \in D,$$

applying Proposition 3.7, $\Gamma(z)$ has the bounded right inverse. Therefore we have $H^2(C_n) = \Gamma H^2(F_2) = \Gamma H(\theta)$, because $\Gamma \theta = 0$. Consequently $X$ given by $Xh = \Gamma h$ is an invertible operator from $H(\theta)$ to $H^2(C_n)$ satisfying $XT = S_n X$ (see the proof of Theorem 3.3). Hence the proof is complete.

We observe that we can substitute $(1 - |w|^2)G(w)$ for the middle term of (3.1), where $G(w)$ is the Gram matrix of $\gamma_{iw}, \ldots, \gamma_{nw}$.

**Proposition 3.9.** The contraction $T$ is similar to the isometry if and only if $T$ satisfies one of the following equivalent conditions:

(a) there is a positive constant $\delta$ such that $\|T^n x\| \geq \delta \|x\|$ for $x$ in $H$.

(b) There is a power-bounded operator $B$ satisfying $BT = I$.

(c) There is a bounded operator $B$ such that $BT = I$ and for any $w$ in $D$ $(I - wB^*)^{-1}$ exists and $\sup_{w \in D} (1 - |w|)(I - wB^*)^{-1} \| < \infty$

**Proof.** In [15], Sz.-Nagy and Foias showed that $T$ satisfies (a) if and only if $T$ is similar to isometry. (a) $\iff$ (b) is trivial. Moreover it is clear that (c) follows from similarity of $T$ and isometry, and its converse is able to be shown in the same way as Caster [3], by considering

$$\sum_{n=1}^\infty r^n e^{int} B^n + \sum_{n=1}^\infty r^n e^{-int} T^n$$

instead of $\sum_{n=-\infty}^\infty r^n e^{int} S^n$ on p. 191 of [3].
At the end of this section we remark that from the above proposition we can get conditions for $T$ to be similar to $S_n$. For instance it suffices to add $T \in C_0$ and $\dim \ker T^* = n$ to each condition of the above.

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