WEIGHTED INEQUALITIES FOR ONE-SIDED MAXIMAL FUNCTIONS

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ABSTRACT. Let $M^+_g$ be the maximal operator defined by

$$M^+_g f(x) = \sup_{h>0} \left( \int_x^{x+h} |f(t)| g(t) \, dt \right) \left( \int_x^{x+h} g(t) \, dt \right)^{-1},$$

where $g$ is a positive locally integrable function on $\mathbb{R}$. We characterize the pairs of nonnegative functions $(u, v)$ for which $M^+_g$ applies $L^p(v)$ in $L^p(u)$ or in weak-$L^p(u)$. Our results generalize Sawyer's (case $g = 1$) but our proofs are different and we do not use Hardy's inequalities, which makes the proofs of the inequalities self-contained.

1. INTRODUCTION

In this paper we will study the operator $M^+_g$ acting on measurable real functions on $\mathbb{R}$ defined by

$$M^+_g f(x) = \sup_{h>0} \left( \int_x^{x+h} |f(t)| g(t) \, dt \right) \left( \int_x^{x+h} g(t) \, dt \right)^{-1},$$

where $g$ is a locally integrable and positive function. If $g = 1$ we obtain the one-sided Hardy-Littlewood maximal operator which has been studied by Sawyer [7].

We will characterize the pairs of weights $(u, v)$ such that $M^+_g$ is of weak and strong type $(p, p)$ with respect to the measures $v dx$ and $u dx$. Our results include Sawyer's as particular cases, but with different proofs. The proof of the theorem about the weak type $(p, p)$ ($p > 1$) is adapted from [1]. On the other hand, the proof of the theorem about the strong type $(p, p)$ is simpler than the corresponding one in [7] (our proof follows the pattern of the proof in [6]) and besides we do not use Hardy's inequalities which makes the proofs of the inequalities self-contained. We also include the weak type $(1,1)$ that is not
studied in [7]. Finally we give several results about the good weights for $M^+_g$ such as relations with Muckenhoupt's classes, factorization, and extrapolation.

2. Notation and main results

Throughout this paper, $g$ will be a positive locally integrable function and $C$ a positive constant not necessarily the same at each occurrence. If $p > 1$, then its conjugate exponent will be denoted by $p'$, and for a Lebesgue measurable set $A$, $\chi_A$ will be its characteristic function and $|A|$ its measure.

We will say that a pair of nonnegative functions $(u, v)$ satisfies condition $A^+_p(g)$, $p > 1$, if there exists a constant $C > 0$ such that for every $y, x, b$ with $y \leq x \leq b$,

\[ \int_y^x u \left( \int_x^b g^p \sigma \right)^{p-1} \leq C \left( \int_y^b g \right)^p, \]

where $\sigma = v^{-1/p-1}$ (as usual, we consider $0 \cdot \infty = 0$).

Condition $A^+_1(g)$ is given by

\[ M_v^{-1}(g^{-1} u) \leq C g^{-1} v \quad \text{a.e.}, \]

where $M_v^{-1}$ is the left maximal operator defined in the obvious way.

A pair of nonnegative functions $(u, v)$ satisfies condition $S^+_p(g)$, $p > 1$, if there exists a constant $C > 0$ such that for every interval $I = (a, b)$ with $\int_{(-\infty, a)} u > 0$,

\[ \int_a^b (M_g^+(\chi_I g^{1/p-1} \sigma))^{p} u \leq C \int_a^b g^p \sigma < \infty. \]

Our main results are the following three theorems.

**Theorem 1.** $M^+_g$ is of weak type $(p, p)$, $p > 1$, with respect to the measures $v \, dx$ and $u \, dx$ if and only if $(u, v)$ satisfies $A^+_p(g)$.

**Theorem 2.** $M^+_g$ is of strong type $(p, p)$, $p \geq 1$, from $L^p(v)$ to $L^p(u)$ if and only if $(u, v)$ satisfies $S^+_p(g)$.

**Theorem 3.** If $u = v$ and $p > 1$, $A^+_p(g)$ and $S^+_p(g)$ are equivalent conditions, that is, the weak type $(p, p)$ is equivalent to the strong type $(p, p)$.

3. Proof of Theorem 1 for $p = 1$

We will need two lemmas:

**Lemma 1.** Let $w$ be a positive increasing function defined on $I = [a, b]$ (i.e., $s \leq t$ implies $w(s) \leq w(t)$). Let $f$ be a positive function on $I$. Suppose for some positive number $\lambda$

\[ \int_t^b g f \geq \lambda \int_t^b g \quad \text{for every } t \in I. \]

Then $\lambda \int_a^b g f w \leq \int_a^b g f w$. 

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Proof of Lemma 1. Let $B > 1$. Let
\[ E = \left\{ t \in [a, b] : \lambda \int_a^b g w \leq \lambda \int_a^t g w + B w(t) \left[ \lambda \int_t^b g - \int_t^b f g \right] + B \int_t^b f g w \right\}. \]
Let $\tau = \inf E$ ($E$ is nonempty). We claim that $\tau = a$. If $a < \tau$, let $n \in (a, \tau)$ such that $B w(n) > \text{ess sup}\{w(t) : a < t \leq \tau\}$. We will prove that $n \in E$, which will contradict that $\tau = \inf E$. Since $\tau \in E$ we have
\[ \lambda \int_a^b g w \leq \lambda \int_a^\tau g w + B w(\tau) \left[ \lambda \int_\tau^b g - \int_\tau^b f g \right] + B \int_\tau^b f g w. \]
Now, the fact that $w$ is increasing, the assumptions of the lemma and $n < \tau$ give
\[ \lambda \int_a^b g w \leq \lambda \int_a^\eta g w + \lambda \int_\eta^\tau g w + B w(\eta) \left[ \lambda \int_\tau^b g - \int_\tau^b f g \right] + B \int_\eta^\tau f g w. \]
If we use again that $w$ is increasing and the election of $\eta$, we obtain
\[ \lambda \int_a^b g w \leq \lambda \int_a^\eta g w + B w(\eta) \left[ \lambda \int_\eta^b g - \int_\eta^b f g \right] + B \int_\eta^\tau f g w. \]
This means that $\eta \in E$, a contradiction. Hence, $\tau = a$ and then $a \in E$, that is
\[ \lambda \int_a^b g w \leq B w(a) \left[ \lambda \int_a^b g - \int_a^b f g \right] + B \int_a^b f g w. \]
Since the expression in brackets is nonpositive, we obtain $\lambda \int_a^b g w \leq B \int_a^b f g w$. Letting $B$ tend to 1, we have the result.

Lemma 2. If $(u, v)$ satisfies $A_1^+(g)$ and $[a, b]$ is an interval, then there exists an increasing function $w$ on $[a, b]$ such that
(i) $w(s) \leq C g^{-1}(s)v(s)$ a.e. $s \in [a, b]$.
(ii) $\int_a^b u \leq \int_a^b g w$.

Proof of Lemma 2. Let $G(y) = M_g^-(g^{-1}u\chi_{[a, b]})(y)$. The function $G$ is lower semicontinuous and finite a.e. by $A_1^+(g)$. 

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Let $w(x) = \min_{x \leq y \leq b} G(y)$. It is obvious that $w$ is increasing and verifies (i). To see (ii), let $0 < B < 1$ and

$$A = \left\{ t \in [a, b]: \int_y^b g w \geq B \int_y^b u \text{ for every } y \in [t, b] \right\}.$$ 

It is clear that $A$ is a closed interval $[\tau, b]$. We will prove that $\tau = a$.

Suppose $\tau > a$. Since $G$ is lower semicontinuous, there exists $\delta > 0$ such that $G(x) \geq BG(\tau)$ if $x \in [\tau - \delta, \tau)$. For such an $x$,

$$w(x) = \min_{x \leq y \leq b} G(y) = \min \left\{ \min_{x \leq y \leq \tau} G(y), \ min_{\tau \leq y \leq b} G(y) \right\} \geq \min\{BG(\tau), w(\tau)\} \geq Bw(\tau).$$

By the definition of $w$, there exists $\gamma$ with $\tau \leq \gamma \leq b$ such that $w(\tau) = G(\gamma)$. For every $x \in [\tau, \gamma]$, $w(x) = w(\tau) = G(\gamma)$, and for every $x \in [\tau - \delta, \tau)$, $w(x) \geq Bw(\tau) = BG(\gamma)$. Therefore, if $x \in [\tau - \delta, \gamma]$ then $w(x) \geq BG(\gamma)$. Hence

$$\int_x^\gamma g w \geq BG(\gamma) \int_x^\gamma g \geq B \int_x^\gamma u \text{ for every } x \in [\tau - \delta, \gamma].$$

This means that $\tau - \delta \in A$, which contradicts that $\tau$ is the infimum of $A$. Therefore $\tau = a$ and then $\int_a^b g w \geq B \int_a^b u$. Letting $B$ tend to 1 the proof is finished.

Now, it is easy to prove that $A^+(g)$ is sufficient for the weak $(1,1)$ inequality. Let $f$ be a positive function with support bounded from above, and let $\lambda, N > 0$. Let $O_{\lambda, N} = (-N, \infty) \cap \{x: M^+g(x) > \lambda\}$. $O_{\lambda, N}$ is a bounded open set and therefore there exists a sequence of maximal pairwise disjoint finite intervals $\{(a_j, b_j)\}$ such that $O_{\lambda, N} = \bigcup (a_j, b_j)$ and $\int_{a_j}^{b_j} f g \geq \lambda \int_{a_j}^{b_j} g$ for every $x \in (a_j, b_j)$. For each $j$, by Lemma 2, there exists an increasing function $w_j$ on $[a_j, b_j]$ such that

$$w_j(t) \leq C g^{-1}(t)v(t) \text{ a.e. } t \in [a_j, b_j]$$

and

$$\int_{a_j}^{b_j} u \leq \int_{a_j}^{b_j} g w_j.$$

If we apply Lemma 1 to each $w_j$, we obtain

$$\lambda \int_{a_j}^{b_j} g w_j \leq \int_{a_j}^{b_j} g f w_j.$$

Now (3.2), (3.3), and (3.1) give

$$\int_{O_{\lambda, N}} u = \sum_j \int_{a_j}^{b_j} u \leq \sum_j \int_{a_j}^{b_j} g w_j \leq \lambda^{-1} \sum_j \int_{a_j}^{b_j} g f w_j \leq C \lambda^{-1} \int_{O_{\lambda, N}} f v = C \lambda^{-1} \int_{O_{\lambda, N}} f v.$$
Letting $N$ tend to infinity we obtain $\int_{\{x : M^+_g f(x) > \lambda\}} u \leq C \lambda^{-1} \int_{-\infty}^{+\infty} f v$.

Conversely, let us suppose that $M^+_g$ is of weak type $(1,1)$ with respect to the measures $v dx$ and $u dx$. For every natural number $N$ we consider the set $E_N = \{ x : g^{-1}(x)v(x) \leq N \}$ and the function $v_N = v \chi_{E_N}$. Let $F_N$ and $H_N$ be the Lebesgue sets of $g^{-1}v_N$ and $\chi_{E_N}$ respectively. It is clear that if $F = \bigcap_N F_N \cap H_N$ then $|R - F| = 0$. Let $x$ be in $F$, and let $\delta, \varepsilon > 0$ such that $\int_{x-\delta}^x g \leq 2 \int_{x-\delta}^x g$. Now consider $N$ with $g^{-1}(x)v(x) \leq N$. If $f_N = g^{-1} \chi_{E_N \cap (x, x+\varepsilon)}$ and $y \in (x-\delta, x)$ then

$$M_g f_N(y) \geq \int_{x}^{x+\varepsilon} \chi_{E_N} \left( 2 \int_{x-\delta}^x g \right)^{-1}.$$  

Therefore, by the weak type inequality,

$$\int_{x-\delta}^x u \leq 2C \left( \int_{x-\delta}^x g \right) \left( \int_{x}^{x+\varepsilon} g^{-1}v_N \right) \left( \int_{x}^{x+\varepsilon} \chi_{E_N} \right)^{-1}.$$  

If we let $\varepsilon$ tend to zero and then $N$ to infinity we get

$$\int_{x-\delta}^x u \leq 2C \left( \int_{x-\delta}^x g \right) (g^{-1}v)(x).$$

Since $\delta$ is an arbitrary positive number we obtain $M_g^{-1}u(x) \leq 2C(g^{-1}v)(x)$ for all $x$ in $F$ and thus for almost every $x$ in $R$.

4. Proof of Theorem 1 for $p > 1$

Suppose that $(u, v)$ satisfies $A^+_p(g)$ and $\int_{(-\infty, b)} g = \infty$ for every $b$ in $R$. Then if $y \leq x \leq b$

$$(4.1) \quad \left( \int_y^x u \right) \left( \int_x^b g^{p-1} \sigma \right)^{p-1} \leq C \left( \int_y^b g \right)^p,$$  

with $C$ independent of $x$, $y$, and $b$. Let $\alpha > 0$. Multiplying both sides of (4.1) by $g(y)(\int_y^b g)^{-p-\alpha-1}$ and integrating with respect to $y$ on $(-\infty, x)$ we get

$$(4.2) \quad \left( \int_x^b g^{p-1} \sigma \right)^{p-1} \int_{-\infty}^x g(y) \left( \int_y^x u \right) \left( \int_y^b g \right)^{-p-\alpha-1} dy$$

for every $x$. Computing the right-hand side of (4.2), we obtain

$$(4.3) \quad \left( \int_x^b g^{p-1} \sigma \right)^{p-1} \int_{-\infty}^x g(y) \left( \int_y^x u \right) \left( \int_y^b g \right)^{-p-\alpha-1} dy \leq C \alpha^{-1} \left( \int_x^b g \right)^{-\alpha}.$$
Besides

\[ \int_{-\infty}^{x} g(y) \left( \int_{-\infty}^{y} u(t) \, dt \right) \left( \int_{t}^{b} g \right)^{-p-a-1} \, dy \]

(4.4)

\[ = \int_{-\infty}^{x} u(t) \left( \int_{-\infty}^{t} g(y) \left( \int_{y}^{b} g \right)^{-p-a-1} \, dy \right) \, dt \]

\[ = \int_{-\infty}^{x} u(t)(p+\alpha)^{-1} \left( \int_{t}^{b} g \right)^{-p-\alpha} \, dt. \]

(4.3) and (4.4) give

\[ \left( \int_{x}^{b} g^{p} \sigma \right)^{p-1} \int_{-\infty}^{x} u(t) \left( \int_{t}^{b} g \right)^{-p-\alpha} \, dt \leq C(p+\alpha)\alpha^{-1} \left( \int_{x}^{b} g \right)^{-\alpha}. \]

(4.5)

It is interesting to note that (4.5) holds even if \( g^{p} \sigma \) is not locally integrable, since \( A_{p}^{+}(g) \) implies that if the integral of \( g^{p} \sigma \) on \([x_1, x_2]\) is infinite then \( u(t) = 0 \) for a.e. \( t < x_1 \).

Let \( f \) be a positive function with support bounded from above. For \( \lambda > 0 \) and natural \( N \), let \( O_{\lambda, N} = \{x: M_{f}(x) > \lambda\} \cap (-N, \infty) \). Let \( (a, b) \) be a connected component of \( O_{\lambda} \). We have

\[ \int_{x}^{b} g < \int_{x}^{b} f g \quad \text{for every } x \text{ in } (a, b). \]

(4.6)

Let \( A = \{x \in [a, b]: \int_{x}^{b} g^{p} \sigma = \infty\} \). If \( A \neq \emptyset \), let \( x_0 = \sup A \); if \( A = \emptyset \), let \( x_0 = a \). Then \( \int_{x}^{b} g^{p} \sigma < \infty \) for every \( x > x_0 \) and it follows from \( A_{p}^{+}(g) \) that \( u(x) = 0 \) a.e. in \([a, x_0]\). Thus \( \int_{a}^{b} u = \int_{x_0}^{b} u \).

Let \( H \) and \( h \) be the functions defined on \((a, b)\) by

\[ H(x) = \int_{a}^{x} u(t) \left( \int_{t}^{b} g \right)^{-p-\alpha} \, dt \quad \text{and} \quad h(x) = \left( \int_{x}^{b} g^{p} \sigma \right)^{1/p'}. \]

(4.7)

It is clear that \( H(x) = 0 \) and \( h(x) = \infty \) if \( x < x_0 \) and from (4.5) we get

\[ (h(x))^{p} H(x) \leq C(p+\alpha)\alpha^{-1} \left( \int_{x}^{b} g \right)^{-\alpha} \quad \text{for every } x \in [a, b]. \]

On the other hand we have

\[ \int_{a}^{b} u(x) \, dx = \int_{x_0}^{b} H'(x) \left( \int_{x}^{b} g \right)^{p+\alpha} \, dx. \]

(4.8)
Integration by parts in the right-hand side of (4.8) gives

\[ \int_{a}^{b} u(x) \, dx = (p + \alpha) \int_{x_0}^{b} H(x)g(x) \left( \int_{x}^{b} g \right)^{p + \alpha - 1} \, dx \]

(4.9) \[ \leq C(p + \alpha)^2 \alpha^{-1} \int_{x_0}^{b} h^{-p}(x)g(x) \left( \int_{x}^{b} g \right)^{p - 1} \, dx, \]

by (4.7). Again integration by part gives

\[ \int_{a}^{b} u \leq C(p + \alpha)^2 \alpha^{-1} \left( \int_{x_0}^{b} h^{-p}(x_0) \left( \int_{x_0}^{b} g \right)^{p} \right)^{p - 1} \int_{x_0}^{b} h(x)h^{-p} \left( \int_{x}^{b} g \right)^{p} \, dx. \]

(4.10) On the other hand, if we raise both sides of (4.6) to the \( p \)th power and apply Hölder's inequality with exponents \( p \) and \( p' \), after introducing suitable factors, we obtain

\[ \lambda^{p} \leq \left( \int_{x}^{b} g \right)^{-p} \left( \int_{x}^{b} g f \right)^{p} \]

(4.11) \[ \leq \left( \int_{x}^{b} g \right)^{-p} \left( \int_{x}^{b} f^{p} h v \right) \left( \int_{x}^{b} g^{p'} \sigma h^{-p'/p} \right)^{p/p'}. \]

Computing the last integral of the above inequality gives us

\[ \int_{x}^{b} g^{p'} \sigma h^{-p'/p} = \int_{x}^{b} g^{p'}(t)\sigma(t) \left( \int_{t}^{b} g^{p'} \sigma \right)^{-1/p} \, dt \]

(4.12) \[ = p' \left( \int_{x}^{b} g^{p'} \sigma \right)^{1/p'} = p' h(x). \]

Then (4.11) becomes

\[ \left( \int_{x}^{b} g \right)^{p} \leq \lambda^{-p} p^{p-1} \left( \int_{x}^{b} f^{p} h v \right) h^{-1}(x). \]

(4.13) If we set \( x = x_0 \) in (4.13), we obtain an inequality that allows us to majorize the first addend of the right-hand side in (4.10), i.e.,

\[ h^{-p}(x_0) \left( \int_{x_0}^{b} g \right)^{p} \leq \lambda^{-p} p^{p-1} \left( \int_{x_0}^{b} f^{p} h v \right) h^{-1}(x_0). \]

(4.14) To majorize the second addend we will use (4.13), the positivity of \( -h' \), and
integration by parts:

\[- \int_{x_0}^{b} h^{-p-1}(x) h'(x) \left( \int_{x}^{b} g \right)^{p} \, dx\]

(4.15)

\[\leq -\lambda^{-p} p^{p-1} \int_{x_0}^{b} h^{-2}(x) h'(x) \left( \int_{x}^{b} f^{p} h \right) \, dx\]

\[= \lambda^{-p} p^{p-1} \left( -h^{-1}(x_{0}) \int_{x_{0}}^{b} f^{p} h + \int_{x_0}^{b} f^{p} v \right).\]

Finally, (4.14), (4.15), and (4.10) give

(4.16) \[\int_{a}^{b} u \leq C \lambda^{-p} (p + \alpha)^{2} p^{p-1} \alpha^{-1} \int_{x_0}^{b} f^{p} v \leq C \lambda^{-p} (p + \alpha)^{2} p^{p-1} \alpha^{-1} \int_{a}^{b} f^{p} v.\]

We have proved that

\[\int \int_{O_{x,N}} u \leq C \lambda^{-p} \int \int_{O_{x,N}} f^{p} v.\]

Letting N tend to infinity we obtain the weak inequality.

Everything we have just done is based on the assumption \( \int_{(-\infty, b)} g = \infty \) for every b. If it is not true we define \( g_{n} = g \) if \( x \geq -n \) and \( g_{n} = \max\{g, 1\} \) if \( x < -n \), for every natural n. Every \( g_{n} \) verifies \( \int_{\infty}^{b} g_{n} = \infty \) for every b, and since \( (u, v) \in A_{p}^{+}(g) \) we have that \( (u, g^{-p} g_{n}^{p} v) \) satisfies \( A_{p}^{+}(g_{n}) \). Then, by what we have already shown

(4.17) \[\int_{\{x : M_{f_{n}}^{+} f(x) > \lambda\}} u \leq C \lambda^{-p} \int_{-\infty}^{+\infty} |f|^{p} g_{n}^{p} g^{-p} v\]

for every f, where C depends only on the constant of the \( A_{p}^{+}(g) \) condition. Now if we apply (4.17) to the functions \( f_{x,-\infty} \) we have

\[\int_{\{x \geq -n : M_{f_{n}}^{+} f(x) > \lambda\}} u \leq C \lambda^{-p} \int_{-n}^{+\infty} |f|^{p} v.\]

Letting n tend to infinity we obtain the weak type inequality.

Conversely, suppose that \( M_{g}^{+} \) is of weak type \((p, p)\) with respect to the measures \( vd x \) and \( ud x \). Let \( x, y, b \) be given with \( x \leq y \leq b \). For every natural n, let \( h_{n} = g^{p} \sigma \chi_{\{x : g^{p} \sigma(x) < n\}} \). \( \{h_{n}\} \) is an increasing sequence with limit \( g^{p} \sigma \). Let \( f = \chi_{(y,b)}(g^{-1}v)^{-1/p-1} \chi_{\{x : g^{p} \sigma < n\}} \) and \( B_{n} = \int_{y}^{b} h_{n}(f_{x} g)^{-1} \). If \( z \in [x, y] \) we have \( M_{g}^{+} f(z) \geq B_{n} \). Then, the weak type inequality gives

\[\int_{x}^{y} u \leq C B_{n}^{-p} \int_{y}^{b} h_{n},\]

or equivalently

\[\left( \int_{x}^{y} u \right) \left( \int_{y}^{b} h_{n} \right)^{p-1} \leq C \left( \int_{x}^{b} g \right)^{p}.\]
Now the $A_{p}^{+}(g)$ condition follows from the monotone convergence theorem.

5. PROOF OF THEOREM 2

The necessity of $S_{p}^{+}(g)$ for the two-norm inequality is trivial. For the converse it will suffice to prove the strong type inequality for bounded positive $f$ in $L^{p}(v)$ with support bounded from above.

Let $N$ be a positive integer. For $k > 0$ let

$$O_{k} = \{ x \in \mathbb{R} : M_{g}^{+} f(x) > 2^{k} \} \cap (-N, +\infty).$$

Each $O_{k}$ is an open set and, therefore, there exists a sequence $\{ I_{jk} \}_{j}$ of open pairwise disjoint intervals with $O_{k} = \bigcup_{j} I_{jk}$ and such that

$$\int_{x}^{b_{jk}} g f \geq 2^{k} \int_{x}^{b_{jk}} g \quad \text{for every } x \in I_{jk} = (a_{jk}, b_{jk}).$$

(5.1)

It is clear that $\sup_{j,k} |I_{jk}| < \infty$. For every $j$ and $k$ let $A_{jk} = \{ x \in I_{jk} : \int_{x}^{b_{jk}} g^{p'} \sigma = \infty \}$. If $A_{jk} \neq \emptyset$, let $x_{jk} = \sup A_{jk}$; if $A_{jk} = \emptyset$, let $x_{jk} = a_{jk}$.

It is clear that $\int_{x}^{b_{jk}} g^{p'} \sigma < \infty$ if $x > x_{jk}$ and $u = 0$ a.e. $x$ in $(a_{jk}, x_{jk})$ by $S_{p}^{+}(g)$. For every $j$ and $k$ let

$$E_{jk} = I_{jk} \cap \{ x : M_{g}^{+} f(x) \leq 2^{k+1} \} \quad \text{and} \quad F_{jk} = (x_{jk}, b_{jk}) \cap E_{jk}.$$

The sets $E_{jk}$ are pairwise disjoint and for every $k$

$$\bigcup_{j} E_{jk} = \{ x : 2^{k} < M_{g}^{+} f(x) \leq 2^{k+1} \} \cap (-N, \infty).$$

(5.2)

Then

$$\int_{-N}^{+\infty} (M_{g}^{+} f)^{p} u = \sum_{k,j} \int_{(-N, +\infty) \cap \{ x : 2^{k} < M_{g}^{+} f(x) \leq 2^{k+1} \}} (M_{g}^{+} f)^{p} u$$

$$= \sum_{k,j} \int_{E_{jk}} (M_{g}^{+} f)^{p} u = \sum_{k,j} \int_{F_{jk}} (M_{g}^{+} f)^{p} u.$$

By the definition of $F_{jk}$ and by (5.1) we have that the last term is smaller than or equal to

$$2^{p} \sum_{k,j} \int_{F_{jk}} u(x) \left( \int_{x}^{b_{jk}} f g \right)^{p} \left( \int_{x}^{b_{jk}} g \right)^{-p} dx.$$

Therefore

$$\int_{-N}^{+\infty} (M_{g}^{+} f)^{p} u \leq 2^{p} \sum_{k,j} \int_{F_{jk}} u(x) \left( \int_{x}^{b_{jk}} f g \right)^{p} \left( \int_{x}^{b_{jk}} g^{p'} \sigma \right)^{-p}$$

$$\times \left( \int_{x}^{b_{jk}} g^{p'} \sigma \right)^{p} \left( \int_{x}^{b_{jk}} g \right)^{-p} dx.$$  

(5.3)
Let $X = \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}$ and let $\omega$ be the product measure $\nu \times \nu \times m$ where $\nu$ is the counting measure on $\mathbb{Z}$ and $m$ is the Lebesgue measure on $\mathbb{R}$. Let $\varphi$ be the real function defined on $X$ by

$$
\varphi(j, k, x) = \chi_{F_{jk}}(x)u(x) \left( \int_X g^{p'}\sigma \right)^p \left( \int_X g \right)^{-p}
$$

and let $T$ be the linear operator

$$
T_h(j, k, x) = \int_X h g^{p'}\sigma \left( \int_X g \right)^{-1}.
$$

With these notations inequality (5.3) can be written in the following way:

$$
\int_{-\infty}^{+\infty} (M^+_g f)^p u \leq 2^p \int_X [T(f(g^{-1}v)^{1/p-1})]^p \varphi d\omega.
$$

If we prove that the operator $T$ is bounded from $L^p(g^{p'}\sigma \, dx)$ to $L^p(X, \varphi d\omega)$, we will get

$$
\int_{-\infty}^{+\infty} (M^+_g f)^p u \leq C 2^p \int_{-\infty}^{+\infty} (f(g^{-1}v)^{1/p-1})^p g^{p'} \sigma = C 2^p \int_{-\infty}^{+\infty} f^p v,
$$

and, letting $N$ tend to infinity, the proof will be finished.

To prove the boundedness of $T$ we observe that it is obviously bounded in $L^\infty$ and by Marcinkiewicz's interpolation theorem it will be enough to prove the weak type $(1,1)$, i.e., $\int \{ (j, k, x) \in X : Th(j, k, x) > \lambda \} \varphi d\omega \leq C \lambda^{-1} \int f^p g^{p'} \sigma$ with $C$ the constant of condition (2.3).

Let $A_{jk}(\lambda) = F_{jk} \cap \{ x : Th(j, k, x) > \lambda \}$. The sets $A_{jk}$ are pairwise disjoint. For each pair $j, k$, let $s_{jk}(\lambda) = \inf A_{jk}(\lambda)$ and $J_{jk} = J_{jk}(\lambda) = [s_{jk}(\lambda), b_{jk})$. If we pick up two of these intervals $J_{jk}$ and $J_{lm}$, then they are either disjoint or one of them is contained in the other. Also it is clear that each $J_{jk}$ verifies

$$
\int_{J_{jk}} h g^{p'} \sigma \geq \lambda \int_{J_{jk}} g^{p'} \sigma.
$$

Let $\{J_i\}$ be the maximal elements of the family $\{J_{jk}\}$. These maximal elements exist since the intervals $J_{jk}$ have uniformly bounded lengths. Also the
intervals $J_i$ verify (5.5). Then

$$\int_{\{j,k,x\} : \text{Th}(j,k,x) > \lambda} \varphi(j,k,x) \, d\omega$$

$$= \sum_{k,j} \sum_{A_{jk}(\lambda)} u(x) \left( \int_x^{b_{jk}} g^p \sigma \left( \int_x^{b_{jk}} g \right)^{-p} \, dx \right)$$

$$\leq \sum_{i} \sum_{\{i,j\} : J_i \supset J_{jk}} \int_{A_{jk}(\lambda)} u(x) \left( \int_x^{b_{jk}} g^p \sigma \left( \int_x^{b_{jk}} g \right)^{-p} \, dx \right)$$

$$\leq \sum_{i} \int_{J_i} (M_g^+(\chi_{J_i} g^{1/p-1} \sigma))(x))^p u(x) \, dx$$

$$\leq C \sum_{i} \int_{J_i} g^p(x) \sigma(x) \, dx \quad \text{by (2.3)},$$

and by (5.5) the last term is smaller than or equal to

$$C \lambda^{-1} \sum_{i} \int_{J_i} h(x) g^p(x) \sigma(x) \, dx \leq C \lambda^{-1} \int_{-\infty}^{\infty} h(x) g^p(x) \sigma(x) \, dx.$$

This proves the weak $(1,1)$ inequality for $T$ and hence the proof of Theorem 2 is finished.

6. PROOF OF THEOREM 3

Suppose that $u \in A_p^+(g)$. Let $I = (a, b)$ be an interval such that $\int_a^b u > 0$. This implies that $\int_a^b g^p \sigma < \infty$ where $\sigma = u^{-1/p-1}$. Let $x \in I$ then there exist $h > 0$ with $x + h \in I$ such that

$$(6.1) \quad \frac{3}{4} M_g^+(\chi_I g^{1/p-1} \sigma)(x) \leq \int_x^{x+h} g^p \sigma \left( \int_x^{x+h} g \right)^{-1}.$$

For this $h$ there exists $t$ with $0 < t < h$ such that $\int_x^{x+t} g = \int_x^{x+h} g$. This $t$ verifies

$$(6.2) \quad \int_x^{x+t} g^p \sigma \left( \int_x^{x+t} g \right)^{-1} \leq M_g^+(\chi_I g^{1/p-1} \sigma)(x).$$

(6.1) and (6.2) give

$$(6.3) \quad M_g^+(\chi_I g^{1/p-1} \sigma)(x) \leq 4 \int_{x+t}^{x+h} g^p \sigma \left( \int_x^{x+h} g \right)^{-1}.$$

On the other hand, condition $A_p^+(g)$ for $u$ gives

$$(6.4) \quad \int_{x+t}^{x+h} g^p \sigma \left( \int_x^{x+h} g \right)^{-1} \leq C \left( \int_x^{x+h} g \right)^{p'-1} \left( \int_x^{x+t} u \right)^{1-p'}.$$
Now (6.3) together with (6.4) gives

\[ M^+_g(\chi^+_l g^{1/p-1} \sigma)(x) \leq C \left( \int_x^{x+h} g \right)^{p'-1} \left( \int_x^{x+t} u \right)^{1-p'} \]
\[ \leq C(M^+_u(\chi^+_l g u^{-1})(x))^{p'-1}. \]

Raising to $p$ and multiplying by $u(x)$, we get

\[ (M^+_g(\chi^+_l g^{1/p-1} \sigma)(x))^p u(x) \leq C(M^+_u(\chi^+_l g u^{-1})(x))^{p'} u(x). \]

But $M^+_u$ is bounded in $L^p(u)$, because $u \in A^+_r(u)$ for every $r > 1$. Then

\[ \int_I (M^+_g(\chi^+_l g^{1/p-1} \sigma)(x))^p u(x) \, dx \leq C \int_I g^{p'} \sigma. \]

Therefore $u$ satisfies $S^+_p(g)$.

The fact that $S^+_p(g)$ implies $A^+_p(g)$ is a consequence of Theorems 1 and 2, not only for a weight, but for pairs of weights. However, we are going to give a direct proof. Suppose that the pair $(u, v)$ satisfies $S^+_p(g)$. Let $a$, $b$, and $c$ be real numbers with $a \leq b \leq c$. If the integral of $g^{p'} \sigma$ over $[b, c]$ is equal to infinity, then, by $S^+_p(g)$, $u(x) = 0$ a.e. $x$ in $[a, b]$ and the inequality

\[ \int_a^b u \left( \int_b^c g^{p'} \sigma \right)^{p-1} \leq C \left( \int_a^c g \right)^p \]

is trivially satisfied.

Now suppose that the integral of $g^{p'} \sigma$ over $[b, c]$ is finite. We define a possibly finite decreasing sequence by $x_0 = b$ and $x_{k+1}$ the real number such that

\[ 2^{k+1} \int_b^{x_{k+1}} g^{p'} \sigma = \int_{x_{k+1}}^c g^{p'} \sigma \]

if

\[ \int_{-\infty}^{x_k} g^{p'} \sigma \geq 2^k \int_b^c g^{p'} \sigma \]

otherwise the sequence finishes in $x_k$.

Suppose first that the sequence is finite and $x_r$ is its last term. If $r = 0$, then

\[ \int_a^b u(x) \left( \int_x^c g \right)^{-p} \left( \int_b^c g^{p'} \sigma \right)^p \, dx \leq \int_a^b u(x)(M^+_g(\chi^+_l g^{1/p-1} \sigma)(x))^p \, dx \]
\[ \leq C \int_a^c g^{p'} \sigma \leq 2C \int_b^c g^{p'} \sigma. \]

This implies trivially $A^+_p(g)$.
If $r > 0$, let $a' < x$, and $a' < a$. Then,

$$
\int_a^b u(x) \left( \int_x^c g \right)^{-p} \left( \int_b^c g^{p'} \sigma \right)^p dx 
\leq 2^{-rp} \int_a^{x'} u(x) \left( \int_x^c g \right)^{-p} \left( \int_x^c g^{p'} \sigma \right)^p dx 
+ \sum_{k=0}^{r-1} 2^{-kp} \int_{x_k}^{x_{k+1}} u(x) \left( \int_x^c g \right)^{-p} \left( \int_x^c g^{p'} \sigma \right)^p dx 
\leq 2^{-rp} \int_a^{x'} u(x)(M^+(x',c)g^{1/p-1}\sigma)(x))^p dx 
+ \sum_{k=0}^{r-1} 2^{-kp} \int_{x_k}^{x_{k+1}} (M^+(x_{k+1},c)g^{1/p-1}\sigma)(x))^p u(x) dx 
\leq C \left( \sum_{k=0}^r 2^{-kp+k+1} \right) \int_b^c g^{p'} \sigma \leq C \int_b^c g^{p'} \sigma.
$$

Finally, suppose that the sequence is infinite and let $d = \lim x_k$. If $d$ is finite, then $u = 0$ a.e. in $(-\infty, d)$ by $S_p^+(g)$. So, whether $d$ is finite or not we have

$$
\int_a^b u(x) \left( \int_x^c g \right)^{-p} \left( \int_b^c g^{p'} \sigma \right)^p dx \leq \int_d^b u(x) \left( \int_x^c g \right)^{-p} \left( \int_b^c g^{p'} \sigma \right)^p dx.
$$

Using the reasoning above with sum from 0 to $\infty$ completes the proof.

7. Further results

(A) Relations with Muckenhoupt's $A_p(g)$ classes and Sawyer's $S_p(g)$ classes.

Consider the weighted two-sided Hardy-Littlewood maximal operator defined by

$$
M_g f(x) = \sup_{h, s > 0} \left( \int_{x-s}^{x+h} |f| g \right) \left( \int_{x-s}^{x+h} g \right)^{1-1}.
$$

It is clear that the following relation holds:

$$
\frac{1}{2} (M_g^+ + M_g^-) \leq M_g \leq M_g^+ + M_g^-.
$$

We have the following results for $M_g$ (see e.g. [5, 6]):

(i) Let $1 \leq p < \infty$. $M_g$ is of weak type $(p, p)$ with respect to the measures $vdx$ and $udx$ if and only if the pair $(u, v)$ satisfies $A_p(g)$, i.e. $A_p(g)$: There exists $C > 0$ such that

$$
\left( \int_a^b u \right) \left( \int_a b \right)^{p-1} \leq C \left( \int_a^b g \right)^p
$$

for every interval $(a, b)$ and $p > 1$. 
$A_1(g)$: There exists $C > 0$ such that $M_g(g^{-1}u) \leq Cg^{-1}v$ a.e.

(ii) Let $1 < p < \infty$. $M_g$ is of strong type $(p, p)$ with respect to the measures $vdx$ and $udx$ if and only if the pair $(u, v)$ satisfies $S_p(g)$, i.e.,

$S_p(g)$: There exists $C > 0$ such that for every interval $(a, b)$

$$\int_a^b |M_g(\chi_{(a,b)}g(\cdot)^{1/p-1})|^p u \leq C \int_a^b g^p \sigma < \infty.$$  

Of course, if $u = v$ then $A_p(g)$ and $S_p(g)$ are equivalent conditions.

It follows from these results, our theorems, and (7.1) that $A_p(g) = A_p^+(g) \cap A_p^-(g) (1 \leq p < \infty)$ and $S_p(g) = S_p^+(g) \cap S_p^-(g) (1 < p < \infty)$. We will now give direct proofs of these equalities and so results (i) and (ii) will be consequences of the results in this paper.

**Theorem 4.** (a) $A_p(g) = A_p^+(g) \cap A_p^-(g) (1 \leq p < \infty)$.

(b) $S_p(g) = S_p^+(g) \cap S_p^-(g) (1 < p < \infty)$.

**Proof of Theorem 4.** (a) For $p = 1$ the equality is trivial by (7.1). Let $1 < p$. Since it is clear that $A_p^+(g) \cap A_p^-(g) \supset A_p(g)$ we only have to prove $A_p(g) \supset A_p^+(g) \cap A_p^-(g)$. Let $(u, v)$ be in $A_p(g) \cap A_p^{-}(g)$, let $a$ and $c$ be real numbers with $a \leq c$, let $N$ be a natural number, define $G^N(x) = (g^{-1}a)(x)$ if $g^{-1}a(x) < N$ and $G^N(x) = 0$ otherwise. There exists $h$ such that

$$\int_a^c G_N = 2 \int_a^h G_N = 2 \int_h^c G_N.$$  

Then

$$\int_a^c u \left(\int_a^c G_N \right)^{p-1} = 2^{p-1} \int_a^h u \left(\int_h^c G_N \right)^{p-1} + 2^{p-1} \int_h^c u \left(\int_a^h G_N \right)^{p-1} \leq 2^p C \left(\int_a^c g \right)^p \text{ by } A_p^+(g) \text{ and } A_p^-(g).$$

Letting $N$ tend to infinity we get $A_p(g)$.

Finally, (b) follows clearly from (7.1).

**B) Factorization.** We will give here a result that generalizes the theorem of Coifman, Jones and Rubio de Francia [2] (see also [4]). As consequences, we will obtain the factorization of $A_p^+(g)$ and $A_p^-(g)$ weights.

**Theorem 5.** Let $F$ and $G$ be two sublinear operators acting on measurable functions of a measure space $(X, \mathcal{M}, \mu)$. For $p > 1$ let $W_p = \{w: F \text{ is bounded in } L^p(ud\mu)\}$ and $U_p = \{u: G \text{ is bounded in } L^p(ud\mu)\}$. Let $g$ be a positive function, and let $W_1 = \{w: G(g^{-1}w) \leq Cg^{-1}w \text{ a.e.}\}$ and $U_1 = \{u: F(g^{-1}u) \leq Cg^{-1}u \text{ a.e.}\}$. Then $g^{p-1}W_1U_1^{-1-p} \supset W_p \cap g^p U_p^{-1-p}$, i.e., if $w \in W_p$ and $g^p w^{-1/p-1} \in U_p$ then there exist $w_0 \in W_1$ and $u_0 \in U_1$ such that $w = g^{p-1} w_0 u_0^{1/p-1}$.
If \( F = G \) and \( g = 1 \) we obtain the above-mentioned result of Coifman, Jones, and Rubio de Francia.

**Proof.** This proof follows the proof of Theorem 5.2 in [4], with the obvious changes. Suppose \( 1 < p < 2 \). Let \( w \in W_p \cap g^p U_p^{1-p} \). We have to find \( v \) such that

(i) \( vw \in W_1 \), i.e., \( G(g^{-1} vw) \leq C g^{-1} vw \) a.e.,

(ii) \( gv^{1/p-1} \in U_1 \), i.e., \( F(v^{1/p-1}) \leq C v^{1/p-1} \) a.e.

Let us define an operator \( S \) by \( S(u) = |G(g^{-1} uw)|w^{-1} g + (F(|u|^{1/p-1}))^{p-1} \).

The operator \( S \) is positive, sublinear, and bounded on \( L^p(w) \). So, \( S \) verifies the conditions of Lemma 5.1 in [4], and it ensures the existence of such a \( v \). Then \( w_0 = vw \) and \( u_0 = g^{1/p-1} \).

**Corollary 1.** \( w \in A_+^p(g) \) if and only if \( w = g^{p-1} w_0 w_1^{1-p} \) with \( w_0 \in A_+^p(g) \) and \( w_1 \in A_-^1(g) \).

**Proof.** If in Theorem 5, we take \( F = M_g^+ \) and \( G = M_g^- \), the classes of good weights are, respectively \( W_p = A_+^p(g) \) and \( U_p = A_-^p(g) \). Then Theorem 5 assures

\[
g^{p-1} A_+^p(g)(A_-^1(g))^{1-p} \supset A_+^p(g) \cap g^p(A_-^p(g))^{1-p}.
\]

But \( A_+^p(g) \cap g^p(A_-^p(g))^{1-p} = A_+^p(g) \), and this proves the factorization of a weight in \( A_+^p(g) \).

Conversely, take \( w_0 \in A_+^p(g) \) and \( w_1 \in A_-^1(g) \), and let \( w = g^{p-1} w_0 w_1^{1-p} \). If \( a \leq b \leq c \),

\[
\int_a^b w \left( \int_b^c g^p w \left( w_1^{1-p} \right)^{p-1} \right)^{p-1} = \int_a^b w_0 \left( \int_b^c w_1^{1-p} \right)^{p-1} \leq C \left( \int_a^b w_0(x) \left( \int_x^{x+h} g \right)^{p-1} \left( \int_x^{x+h} w_1 \right)^{1-p} \right) \times \left( \int_b^c w_1(x) \left( \int_x^{x-s} g \right)^{p-1} \left( \int_x^{x-s} w_0 \right)^{1-p} \right)
\]

for every \( h, s > 0 \) by condition \( A_+^p(g) \) for \( w_0 \) and \( A_-^1(g) \) for \( w_1 \). In partic-
If \( h = c - x \) and \( s = x - a \) we obtain
\[
\int_a^b w \left( \int_b^c w_{p-1} \right)^{p-1} dx 
\leq C \left( \int_a^c g \left( \int_a^b w_{0} \left( \int_x^c w_1 \right)^{-p} \right)^{1-p} \right) 
\times \left( \int_b^c w_1(x) \left( \int_a^x w_0 \right)^{-p} \right)^{p-1} 
\leq C \left( \int_a^c g \right)^p \left( \int_a^c w_1 \right)^{1-p} \left( \int_a^b w_0 \right)^{p-1} \left( \int_a^c w_0 \right)^{(1-p)(p-1)} 
= C \left( \int_a^c g \right)^p .
\]

(C).

**Theorem 6.** If \( w \) is in \( A_1^+(g) \) then there exists \( \delta > 0 \) such that
\[
\int_a^b w^{-\delta} \left( \int_a^b g \right)^{-1} \leq C_\delta \int_a^b w \left( \int_a^b g \right)^{-\delta} \left( \int_a^b g^{-\delta} \right)w^\delta(b) 
\]
for every \( a \) and a.e. \( b \). For this \( \delta \), \( g^{-\delta} w^{1+\delta} \) is in \( A_1^+(g) \).

**Proof.** Let \( a \) and \( b \) be real numbers with \( a < b \) and with \( b \) verifying
\[
M_{\overline{g}}(g^{-1}w)(b) \leq C(g^{-1}w)(b).
\]

Let \( O_\lambda = \{ x: M_{\overline{g}}(g^{-1}w)(x) > \lambda \} \) be open. Then there exists a sequence of pairwise disjoint open intervals \( I_j = (a_j, b_j) \) such that \( O_\lambda = \bigcup I_j \) with
\[
\int_a^x w \chi_{(a, b)} \left( \int_a^x g \right)^{-1} > \lambda \quad \text{for every } x \in (a_j, b_j) 
\]

and with
\[
\int_{a_j}^{b_j} w \chi_{(a, b)} \left( \int_{a_j}^{b_j} g \right)^{-1} = \lambda \quad \text{for every } j.
\]

It is clear that each \( a_j \) is bigger than \( a \). Then, if \( \lambda > C(g^{-1}w)(b) \), where \( C \) is the \( A_1^+(g) \) constant of \( w \), each \( I_j \) verifies either \( (a, b) \supset I_j \) or \( I_j \cap (a, b) = \emptyset \), since if \( I_j \) is not contained in \( (a, b) \) and \( I_j \cap (a, b) \neq \emptyset \), then \( b \in I_j \) and therefore
\[
\int_{a_j}^{b_j} w \chi_{(a, b)} \left( \int_{a_j}^{b_j} g \right)^{-1} > \lambda > C(g^{-1}w)(b)
\]
which goes against the election of \( b \).
By Lebesgue's differentiation theorem we have that \( \{x \in (a, b) : (g^{-1}w)(x) > \lambda \} \) is contained in \( O_\lambda \). This relation, (7.3), and condition \( A_1^+(g) \) for \( w \) imply
\[
\int_{\{x \in (a, b) : (g^{-1}w)(x) > \lambda \}} w \leq \lambda \sum_{\{ j : (a, b) \ni I_j \}} \int_{I_j} g \leq \lambda \int_{\{x \in (a, b) : (g^{-1}w)(x) > \lambda \}} g.
\]

Let \( \delta > 0 \). Multiplying the last inequalities by \( \lambda^{\delta-1} \) and then integrating with respect to \( \lambda \) from \( C(g^{-1}w)(b) \) to \( +\infty \) we get
\[
\int_{C(g^{-1}w)(b)}^{+\infty} \lambda^{\delta-1} \left( \int_{\{x \in (a, b) : (g^{-1}w)(x) > \lambda \}} w(x) \, dx \right) \, d\lambda
\leq C^{\delta+1}(1 + \delta)^{-1} \int_a^b (g^{-1}w)^{1+\delta}(x)g(x) \, dx.
\]

On the other hand, the first item of (7.5) is equal to
\[
\int_a^b \left( \int_{C(g^{-1}w)(b)} (g^{-1}w)(x) \lambda^{\delta-1} \, d\lambda \right) w(x) \, dx
= \delta^{-1} \int_a^b g^{-\delta} w^{1+\delta} - C^\delta \delta^{-1} (g^{-1}w)^{\delta}(b) \int_a^b w.
\]
(7.5) together with (7.6) gives
\[
(\delta^{-1} - C^{1+\delta}(1 + \delta)^{-1}) \int_a^b g^{-\delta} w^{1+\delta} \leq C^\delta \delta^{-1} (g^{-1}w)^{\delta}(b) \int_a^b w.
\]
Choosing \( \delta \) such that \( \delta^{-1} - C^{1+\delta}(1 + \delta)^{-1} > 0 \), we obtain the result.

**Corollary 2.** Let \( 1 < p < \infty \). If \( w \) is in \( A_p^+(g) \) then there exists \( \varepsilon > 0 \) such that \( p - \varepsilon > 1 \) and \( w \) is in \( A_{p-\varepsilon}^+(g) \).

**Proof.** Let \( w \in A_p^+(g) \). By factorization, there exist \( w_0 \) in \( A_1^+(g) \) and \( w_1 \) in \( A_1^{-}(g) \) such that \( w = g^{p-1}w_0w_1^{1-p} \). By Theorem 6 there exist \( \delta > 0 \) such that \( g^{-\delta}w_1^{1+\delta} \in A_1^+(g) \). Then
\[ w = g^{p-1}w_0w_1^{1-p} = g^{p-\varepsilon}w_0(g^{-\varepsilon}w_1^{1+\delta})^{1-(p-\varepsilon)} \] with \( \varepsilon = \delta(p-1)(1 + \delta)^{-1} \) and the result follows from Corollary 1.

**Corollary 3.** If \( w \in A_1^+(g) \) then there exists \( \gamma \) with \( 0 < \gamma < 1 \), a function \( k \) with \( k \) and \( k^{-1} \) in \( L^\infty \), and a function \( f \) such that \( w = kg(M_g^{-\gamma}f) \).

**Proof.** By Theorem 6, there exists \( \delta > 0 \) such that \( M_g^{-\delta}(g^{-1}w)^{1+\delta})^{1/(1+\delta)} \leq Cg^{-1}w \) a.e. On the other hand, Lebesgue's differentiation theorem gives \( g^{-1}w \leq (M_g^{-\delta}(g^{-1}w)^{1+\delta})^{1/(1+\delta)} \). Let \( k(x) = g^{-1}(x)w(x)(M_g^{-\delta}(g^{-1}w)^{1+\delta}(x))^{-1/(1+\delta)} \). Then \( C^{-1} \leq k \leq 1 \) and \( w = kg(M_g^{-\gamma}f) \) where \( \gamma = (1 + \delta)^{-1} \) and \( f = (g^{-1}w)^{1+\delta} \).
(D) Extrapolation. We can also state the following theorem.

**Theorem 7.** Let $T$ be a sublinear operator acting on measurable functions on $\mathbb{R}$. Suppose that for a certain $p_0$, $1 \leq p_0 < \infty$, and for every $w$ in $A^+_p(g)$, $T$ is of weak type $(p_0, p_0)$ with respect to the measure $wdx$. Then for every $p$ with $1 < p < \infty$ and every $w$ in $A^+_p(g)$, $T$ is bounded on $L^p(wdx)$.

The proof follows that of [3] with the obvious changes, which are essentially the definition of $G$ in Lemma 1 in [3] (now $G = (gM_g^{-}(g^{-1}h^{1/t}w)w^{-1})$) and the fact that $w$ is in $A^+_p(g)$ if and only if $g^{p'}\sigma$ is in $A^-_p(g)$.

**References**