

CLASSIFICATION OF CROSSED-PRODUCT C^* -ALGEBRAS ASSOCIATED WITH CHARACTERS ON FREE GROUPS

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ABSTRACT. We study the classification problem of crossed-product C^* -algebras of the form $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$, where G is a discrete group, χ is a one-dimensional character of G , and α_χ is the unique $*$ -automorphism of $C_r^*(G)$ such that if U is the left regular representation of G , then $\alpha_\chi(U_g) = \chi(g)U_g$, $g \in G$. When $G = F_n$, the free group on n generators, we have a complete classification of these crossed products up to $*$ -isomorphism which generalizes the classification of irrational and rational rotation C^* -algebras. We show that these crossed products are determined by two K -theoretic invariants, that these two invariants correspond to two orbit invariants of the characters under the natural $\text{Aut}(F_n)$ -action, and that these two orbit invariants completely classify the characters up to automorphisms of F_n . The classification of crossed products follows from these results.

We also consider the same problem for G some other groups.

0. INTRODUCTION

Let G be a discrete (not necessarily abelian) group. A character, χ , of G is by definition a group homomorphism from G to the unit circle T . There is a unique $*$ -automorphism, α_χ , on the reduced group C^* -algebras of G , $C_r^*(G)$, such that $\alpha_\chi(U_g) = \chi(g)U_g$, $g \in G$, where U is the left regular representation of G on $l^2(G)$ (see Proposition 1.1 below). This gives an action of \mathbf{Z} , the group of integers, on $C_r^*(G)$ by means of powers of α_χ , and one can then form the crossed-product C^* -algebras $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ as defined in [23]. A natural problem is to classify these crossed products up to $*$ -isomorphism in terms of characters. In the simplest case $G = \mathbf{Z}$, a character χ on \mathbf{Z} is determined by a complex number $\rho = e^{2\pi i\theta}$. The corresponding crossed product, denoted by A_θ , can be identified with the transformation group C^* -algebra associated with a rotation by angle $2\pi\theta$ on the circle T . When θ is irrational, this C^* -algebra A_θ is called an irrational rotation C^* -algebra; these are classified using a combination of the work of Rieffel [32] and Pimsner and Voiculescu [27]. When θ is rational, A_θ is called a rational rotation C^* -algebra; classification has been done by Høegh-Krohn and Skjelbred [14]. The work of Riedel [31] classifies

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(in our terminology) in the case that G be an infinite abelian group, under the additional hypothesis that characters χ are one-to-one. In the present paper, we will completely classify the crossed products $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ when $G = F_n$, the free group on n generators. Many other crossed products will also be classified. Our results generalize all previous work in this direction.

The irrational rotation C^* -algebras, as well as Riedel's minimal rotation C^* -algebras, are classified by one K -theoretical invariant, the image of their K_0 -group under the unique normalized trace. To further explore this invariant, we are led to the study of traces on $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$. The problem is, in general, that these C^* -algebras may have more than one trace. Using ideas of Elliott [12], however, we show that a large class of C^* -algebras, including all $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ with $G = F_n$ or \mathbf{Z}^n , have the property that all of their normalized traces induce the same map from their K_0 -groups to the reals \mathbf{R} (see §2). As a consequence, we see that $\tau_*(K_0(C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}))$ is an isomorphism invariant, where $G = F_n$ or \mathbf{Z}^n and τ is the canonical normalized trace on $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$. Computation with the Pimsner-Voiculescu six term exact sequence then gives the following result (Theorem 3.3)

$$(*) \quad \exp \circ \tau_*(K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})) = \chi(F_n),$$

which also holds when F_n is replaced by \mathbf{Z}^n . In general, the equality $(*)$ does not hold for arbitrary discrete groups G . But we can prove that if G is a discrete amenable group then

$$\bigcap_{\varphi} \exp \circ \varphi_*(K_0(C_r^*(G) \times_{\alpha_\chi} \mathbf{Z})) = \chi(G),$$

where φ runs through all normalized traces on $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ (Theorem 3.5).

The classification theorem for rational rotation C^* -algebras shows that the invariant $\chi(G)$ is not complete for rotation C^* -algebras. A somewhat ad hoc invariant is defined in [14], base on the theory of fibre bundles with finite dimensional C^* -algebras as fibres, and is shown to classify rational rotation C^* -algebras. The approach in [33, 10 and 3] employs, more or less, the same tools. But this approach seems unlikely to work when dealing with $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ ($n \geq 2$), because in this case we would have to deal with fibre bundles where the fibres are infinite dimensional simple C^* -algebras, and the computation of the corresponding homotopy groups would be very difficult.

To overcome the difficulty that $\chi(G)$ is not complete (referred to above), we use the fact that all normalized traces on $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ agree at the K_0 -level, to associated a rational number in the interval $[0, \frac{1}{2}]$ to the C^* -algebras $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$. This is called the twist and is denoted by $t(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})$. We show that the twist is an isomorphism invariant (Proposition 3.8). The two invariants $\tau_*(K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}))$ and $t(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})$ will completely classify all the C^* -algebras $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ (Theorem 5.1).

The classification theorem for rotation C^* -algebras A_θ says that A_{θ_1} is isomorphic to A_{θ_2} if and only if $\theta_1 = \theta_2$ or $\theta_1 = 1 - \theta_2$, where $0 \leq \theta_j < 1$. We remark that the conditions on θ_j should be interpreted as the two characters $\chi_j: \mathbf{Z} \rightarrow T$, $\chi_j(1) = e^{2\pi i\theta_j}$, $0 \leq \theta_j < 1$, differ only by an automorphism of \mathbf{Z} . This observation naturally leads to the study of the action of the automorphism group of G , $\text{Aut}(G)$, on the set \widehat{G} of all characters of G . When $G = \mathbf{Z}^n$ or F_n , we find a complete classification for the orbits in \widehat{G} . We show that, for any character χ in \widehat{G} , its orbit is determined by two invariants, one of which is the set $\chi(G)$, the other is a rational number in the interval $[0, \frac{1}{2}]$ (Theorem 4.5). We call the second invariant the twist of the character χ and denote it by $t(\chi)$. We next show that the twist of the C^* -algebras $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ is always equal to the twist of the character χ (Proposition 5.2).

After bringing together all of these results, we can prove the following classification theorem.

Theorem 5.1. *Let χ_1 and χ_2 be any two characters on F_n . Then the following are equivalent:*

- (i) $C_r^*(F_n) \times_{\alpha_{\chi_1}} \mathbf{Z} \simeq C_r^*(F_n) \times_{\alpha_{\chi_2}} \mathbf{Z}$;
- (ii) $\tau_{1*}(K_0(C_r^*(F_n) \times_{\alpha_{\chi_1}} \mathbf{Z})) = \tau_{2*}(K_0(C_r^*(F_n) \times_{\alpha_{\chi_2}} \mathbf{Z}))$ and $t(C_r^*(F_n) \times_{\alpha_{\chi_1}} \mathbf{Z}) = t(C_r^*(F_n) \times_{\alpha_{\chi_2}} \mathbf{Z})$;
- (iii) $\chi_1(F_n) = \chi_2(F_n)$ and $t(\chi_1) = t(\chi_2)$;
- (iv) χ_1 and χ_2 are in the same orbit under the $\text{Aut}(F_n)$ -action, that is, $\chi_1 = \chi_2 \circ \varphi$ for some φ in $\text{Aut}(F_n)$;
- (v) α_{χ_1} and α_{χ_2} are conjugate in $\text{Aut}(C_r^*(F_n))$;
- (vi) α_{χ_1} and α_{χ_2} are outer conjugate in $\text{Aut}(C_r^*(F_n))$.

Similar classification theorems for G some other groups are proved (see §§5 and 6). In particular, we show that (Theorem 6.6)

$$C^*(\mathbf{Z}^2) \times_{\alpha_{\chi_1}} \mathbf{Z} \simeq C^*(\mathbf{Z}^2) \times_{\alpha_{\chi_2}} \mathbf{Z} \quad \text{iff} \quad \chi_1(\mathbf{Z}^2) = \chi_2(\mathbf{Z}^2).$$

The organization of the paper is as follows. In §1, we prove some preliminary results. In §2, we study when $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ has a unique normalized trace, when all of its traces agree at the K_0 -level, and when it is simple. We also determine the fixed point subalgebra of the $*$ -automorphism α_χ and the set of eigenvalues of α_χ . In §3, we first show that $\chi(G)$ is an isomorphism invariant for $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ when $G = F_n$ or G is a discrete amenable group. Then we define the twist of $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ and compute it in a special case. The computation is to be completed in §5. In §4, we define the twist of characters on $G = \mathbf{Z}^n$ and F_n , and classify the orbits in \widehat{G} under the action of $\text{Aut}(G)$. In §5, we prove our main classification theorem for $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$. Analogous results for $C_r^*(F_\infty) \times_{\alpha_\chi} \mathbf{Z}$ and for $C^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ are also proved. In §6, we prove classification theorems for G some other groups.

The notation and terminology used in this paper are standard. In particular, G and K refer to discrete (not necessarily abelian) groups. Characters are indicated by χ , and the set of all characters of G is denoted by \widehat{G} . The reduced group C^* -algebras of G is denoted by $C_r^*(G)$. The $*$ -automorphism on $C_r^*(G)$ associated with a character χ is denoted by α_χ . Then $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ denotes the corresponding C^* -crossed product. By a trace of a C^* -algebras, we mean a tracial state, that is, we always assume it is normalized. We use τ to denote both the canonical trace on $C_r^*(G)$ and its canonical extension to $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$. Finally, \mathbf{Z} , \mathbf{Q} , \mathbf{R} and \mathbf{C} will denote the groups of integers, rationals, reals and complex numbers, respectively.

1. PRELIMINARIES

In this section we set up some notation and terminology and prove some preliminary results.

Let G be a discrete (not necessarily abelian) group, $l^2(G)$ the Hilbert space of square summable functions on G , and $f_g \in l^2(G)$ the function on G which takes the value one at $g \in G$ and zero elsewhere. Then $\{f_g : g \in G\}$ is an orthonormal basis for $l^2(G)$. The left regular representation U of G on $l^2(G)$ is given by $U_g(f_h) = f_{gh}$, $g, h \in G$, and the reduced group C^* -algebras of G , $C_r^*(G)$, is the C^* -subalgebra of $B(l^2(G))$ generated by $\{U_g : g \in G\}$. It is easy to see that the set $\{U_g : g \in G\}$ is linearly independent, and the set of all finite linear combinations of $\{U_g : g \in G\}$ is a dense $*$ -subalgebra of $C_r^*(G)$.

There exists a faithful (normalized) trace, τ , on $C_r^*(G)$, which is defined via

$$\tau(a) = \langle af_e, f_e \rangle, \quad a \in C_r^*(G),$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $l^2(G)$ and e is the identity element of G . Since linear combinations of $\{U_g : g \in G\}$ are dense in $C_r^*(G)$, the trace τ is characterized by

$$\tau(U_g) = \begin{cases} 0, & \text{if } g \neq e, \\ 1, & \text{if } g = e. \end{cases}$$

We call τ the canonical trace on $C_r^*(G)$.

A character, χ , on the discrete group G is defined to be a group homomorphism from G to the unit circle T . Let \widehat{G} be the set of all characters on G . Then \widehat{G} is an abelian group under pointwise multiplication.

Proposition 1.1. *If χ is a character on G , then there exists a unique $*$ -automorphism α_χ on $C_r^*(G)$ such that*

$$\alpha_\chi(U_g) = \chi(g)U_g, \quad g \in G.$$

Proof. Given χ , we define a unitary operator W_χ on $l^2(G)$ by

$$W_\chi(f_g) = \chi(g)f_g, \quad g \in G.$$

In fact, W_χ is a diagonal operator with entries $\{\chi(g): g \in G\}$ with respect to the orthonormal basis $\{f_g: g \in G\}$. Hence $\text{ad } W_\chi$ is a $*$ -automorphism of $B(l^2(G))$. We have

$$\begin{aligned} W_\chi U_g(f_h) &= W_\chi(f_{gh}) = \chi(gh)f_{gh}, & g, h \in G, \\ U_g W_\chi(f_h) &= U_g(\chi(h)f_h) = \chi(h)f_{gh}, & g, h \in G. \end{aligned}$$

Therefore $(\text{ad } W_\chi)(U_g) = \chi(g)U_g$, $g \in G$. Since $C_r^*(G)$ is generated by $\{U_g: g \in G\}$, $\text{ad } W_\chi$ is the required $*$ -automorphism α_χ on $C_r^*(G)$. The uniqueness is obvious. \square

Remark. The commutation relation $W_\chi U_g = \chi(g)U_g W_\chi$, $g \in G$, $\chi \in \widehat{G}$, has been known for many years [39], and it holds in much general situations. In fact, let G be any locally compact group with left Haar measure μ , and let $\chi: G \rightarrow B(H)$ be any unitary representation of G on some Hilbert space H . We define unitary operators U_g , V_g , and W_χ on $L^2(G, H) \simeq L^2(G, \mu) \otimes H$ by

$$(U_g f)(h) = f(g^{-1}h), \quad (V_g f)(h) = \chi(g)f(h), \quad (W_\chi f)(h) = \chi(h)f(h),$$

where $f \in L^2(G, H)$, $g, h \in G$. Then we have the following commutation relations

$$U_g V_h = V_h U_g, \quad W_\chi U_g W_\chi^* = V_g U_g, \quad g, h \in G.$$

In general, however, $\text{ad } W_\chi$ fails to map the C^* -algebras generated by U_g and V_g , $g \in G$, into itself, if $\chi(G)$ is not abelian.

For work involving the $*$ -automorphism $\alpha_\chi = \text{ad } W_\chi$, we mention Akemann and Ostrand [1], Paschke [21] and John Phillips [24, 25], among many others.

From Proposition 1.1, we obtain a C^* -dynamical system $(C_r^*(G), \alpha_\chi, \mathbf{Z})$, and hence a C^* -crossed product $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$.

Recall that if (A, α, K) is a C^* -dynamical system, then the group action is said to be effective if the homomorphism $\alpha: K \rightarrow \text{Aut}(A)$ is one-to-one. For the dynamical system $(C_r^*(G), \alpha_\chi, \mathbf{Z})$, it is easy to see that the action is effective if and only if $\chi(G)$ is an infinite subgroup of T .

In the investigation of traces on $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$, we will have to consider both effective and ineffective group actions. For that reason, let K be a discrete abelian group and $\rho: K \rightarrow \widehat{G}$ a group homomorphism. We get an action of K on $C_r^*(G)$ via the composition

$$K \xrightarrow{\rho} \widehat{G} \xrightarrow{\alpha} \text{Aut}(C_r^*(G)).$$

This gives a C^* -dynamical system $(C_r^*(G), \alpha \circ \rho, K)$ and a C^* -crossed product $C_r^*(G) \times_{\alpha \circ \rho} K$. When $K = \mathbf{Z}$ and $\rho(1) = \chi \in \widehat{G}$, the crossed product $C_r^*(G) \times_{\alpha \circ \rho} K$ is just $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$.

The crossed product $C_r^*(G) \times_{\alpha \circ \rho} K$ is the C^* -algebra generated by unitaries $\{U_g : g \in G\}$ and $\{W_k : k \in K\}$ such that

$$W_k U_g W_k^* = \langle \rho(k), g \rangle U_g, \quad g \in G, k \in K,$$

with the largest possible C^* -norm, where $\langle \cdot, \cdot \rangle$ is the pairing $\widehat{G} \times G \rightarrow T$. Note that the set of linear combinations of $\{U_g W_k : g \in G, k \in K\}$ is a dense $*$ -subalgebra of $C_r^*(G) \times_{\alpha \circ \rho} K$. Most of the arguments in §2 will concentrate on this dense $*$ -subalgebra.

The following lemma is presumably well known. We give a brief proof for the convenience of the reader.

Lemma 1.2. *Let (A, α, K) be a C^* -dynamical system and K a discrete abelian group. Then every α -invariant trace φ on the C^* -algebra A can be extended to a trace $\tilde{\varphi}$ on $A \times_{\alpha} K$. Moreover, if φ is faithful, then $\tilde{\varphi}$ can be chosen faithful.*

Proof. The crossed product $A \times_{\alpha} K$ is generated by elements of the form aW_k , where $a \in A$ and $W_k, k \in K$, are unitaries such that $W_k a W_k^* = \alpha_k(a)$, $a \in A, k \in K$. By Itoh [15], there exists a conditional expectation E from $A \times_{\alpha} K$ onto A such that

$$E(aW_k) = \begin{cases} 0, & \text{if } k \neq e, \\ A, & \text{if } k = e, \end{cases}$$

where e is the identity element in K . The extension $\tilde{\varphi}$ of φ is defined by $\tilde{\varphi}(z) = \varphi(E(z))$, $z \in A \times_{\alpha} K$. The α -invariance of φ ensures that $\tilde{\varphi}$ is a trace on $A \times_{\alpha} K$. Since the conditional expectation E is faithful, the second statement of the lemma is clear. \square

Corollary 1.3. *There exists a faithful trace $\tilde{\tau}$ on $C_r^*(G) \times_{\alpha \circ \rho} K$ such that*

$$\tilde{\tau}(U_g W_k) = \begin{cases} 0, & \text{if } k \neq 0 \text{ or } g \neq e, \\ 1, & \text{if } k = 0 \text{ and } g = e. \end{cases} \quad \square$$

We call the extension $\tilde{\tau}$ of τ the canonical trace on $C_r^*(G) \times_{\alpha \circ \rho} K$, and we denote it also by τ since no confusion will result.

Not let H be any subgroup of G . By Paschke-Salinas [22], the C^* -subalgebra of $C_r^*(G)$ generated by $\{U_g : g \in H\}$ can be identified with the reduced group C^* -algebra $C_r^*(H)$ in an obvious way, and there exists a conditional expectation E from $C_r^*(G)$ onto $C_r^*(H)$ such that

$$E(U_g) = \begin{cases} 0, & \text{if } g \notin H, \\ U_g, & \text{if } g \in H. \end{cases}$$

Futhermore, let χ be any character on G . Since

$$\alpha_{\rho(k)}(U_h) = \langle \rho(k), h \rangle U_h, \quad h \in H, k \in K,$$

$C_r^*(H)$, as a C^* -subalgebra of $C_r^*(G)$, is $\alpha \circ \rho$ -invariant. Therefore, $C_r^*(H) \times_{\alpha \circ \rho} K$ can be identified with the C^* -subalgebra of $C_r^*(G) \times_{\alpha \circ \rho} K$ generated by

$\{U_g W_k : g \in H, k \in K\}$. In the future, we will use this identification without explicit reference.

Finally, it is easy to see that there is a natural $*$ -homomorphism from $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ onto the C^* -subalgebra, $C^*(U(G), W_\chi)$, of $B(l^2(G))$ generated by U_g , $g \in G$, and W_χ .

Proposition 1.4. *If $\chi(G)$ is infinite, then $C^*(U(G), W_\chi) \simeq C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$; if $\chi(G)$ is finite with order q , then $C^*(U(G), W_\chi) \simeq C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}_q$.*

Proof. We use Landstad's characterization of crossed products [23]. We observe that the right regular representation $g \rightarrow R_g$, $R_g(f_h) = f_{hg^{-1}}$, $g, h \in G$, provides an action dual to $\text{ad } W_\chi$. In fact, for any g in G , $\text{ad } R_g$ is a $*$ -automorphism of $C^*(U(G), W_\chi)$ such that

$$\text{ad } R_g(U_h) = U_h, \quad \text{ad } R_g(W_\chi) = \chi(g)W_\chi, \quad g, h \in G.$$

This gives an action of $\chi(G) \simeq G/\ker \chi$ on $C^*(U(G), W_\chi)$. When $\chi(G)$ is infinite, we can extend this action in an obvious way to the closure of $\chi(G)$ which is the whole circle T . It is easy to verify that the extended action of T on $C^*(U(G), W_\chi)$ is dual to the action $\text{ad } W_\chi$ of \mathbf{Z} on $C_r^*(G)$, and that the fixed point algebra of the T -action is just $C_r^*(G)$. Hence $C^*(U(G), W_\chi)$ and $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ are isomorphic by [23]. When $\chi(G)$ is finite, the situation is similar. \square

2. ANALYTIC ASPECTS

In this section, we study the analytical aspects of the algebras $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$. One of our goals is to find conditions under which either $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ has a unique trace, or all of its traces agree at the K_0 -level.

Lemma 2.1. *Let A and B be unital C^* -algebras and $A \otimes B$ their tensor product equipped with an arbitrary C^* -norm. If φ is a trace on $A \otimes B$ such that the restriction of φ to A is an extremal trace, then*

$$(2.1) \quad \varphi(a \otimes b) = \varphi(a \otimes 1)\varphi(1 \otimes b), \quad a \in A, b \in B.$$

Proof. Let $b \in B$ be such that $0 \leq b < 1 - \varepsilon$ for some $\varepsilon > 0$. If $\varphi(1 \otimes b) = 0$, then the Cauchy-Schwarz inequality implies that

$$\begin{aligned} |\varphi(a \otimes b)|^2 &= |\varphi((a \otimes b^{1/2})(1 \otimes b^{1/2}))|^2 \\ &\leq \varphi((a \otimes b^{1/2})(a \otimes b^{1/2})^*)\varphi((1 \otimes b^{1/2})^*(1 \otimes b^{1/2})) \\ &= \varphi(aa^* \otimes b)\varphi(1 \otimes b) = 0. \end{aligned}$$

Hence the equality (2.1) holds. If $\varphi(1 \otimes b) \neq 0$, we have

$$\varphi(a \otimes 1) = \varphi(1 \otimes b) \frac{\varphi(a \otimes b)}{\varphi(1 \otimes b)} + (1 - \varphi(1 \otimes b)) \frac{\varphi(a \otimes (1 - b))}{1 - \varphi(1 \otimes b)}, \quad a \in A.$$

It is easy to see that

$$a \rightarrow \frac{\varphi(a \otimes b)}{\varphi(1 \otimes b)} \quad \text{and} \quad a \rightarrow \frac{\varphi(a \otimes (1 - b))}{1 - \varphi(1 \otimes b)}$$

are traces on A . By hypothesis, $a \rightarrow \varphi(a \otimes 1)$ is an extremal trace on A . Hence $\varphi(a \otimes 1) = \varphi(a \otimes b)/\varphi(1 \otimes b)$, that is,

$$(2.2) \quad \varphi(a \otimes b) = \varphi(a \otimes 1)\varphi(1 \otimes b), \quad a \in A, \quad b \in B, \quad 0 \leq b < 1 - \varepsilon.$$

Since every element $b \in B$ can be expressed as $b = \sum_1^4 \lambda_i b_i$, $\lambda_i \in \mathbf{C}$, $0 \leq b_i < 1 - \varepsilon$ for some $\varepsilon > 0$, it follows that (2.2) holds for any $a \in A$ and $b \in B$. \square

Theorem 2.2. *Given the C^* -dynamical system $(C_r^*(G), \alpha \circ \rho, K)$, let H be the normal subgroup of G defined by $H = \{g \in G : \langle \rho(k), g \rangle = 1, k \in K\}$. If K acts effectively on $C_r^*(G)$ and $C_r^*(H)$ has a unique trace, then $C_r^*(G) \times_{\alpha \circ \rho} K$ has a unique trace.*

Proof. Let φ be any trace on $C_r^*(G) \times_{\alpha \circ \rho} K$. Recall that linear combinations of $\{U_g W_k : g \in G, k \in K\}$ constitute a dense $*$ -subalgebra of $C_r^*(G) \times_{\alpha \circ \rho} K$. It is enough to show that the values $\varphi(U_g W_k)$ are uniquely determined. If $g \notin H$, then we can find some $l \in K$ such that $\langle \rho(l), g \rangle \neq 1$. Since K is abelian, we get

$$\varphi(U_g W_k) = \varphi(W_l U_g W_k W_l^*) = \langle \rho(l), g \rangle \varphi(U_g W_k), \quad k \in K.$$

This implies that

$$\varphi(U_g W_k) = 0, \quad g \notin H, \quad k \in K.$$

The C^* -subalgebra of $C_r^*(G) \times_{\alpha \circ \rho} K$ generated by $\{U_g W_k : g \in H, k \in K\}$ is isomorphic to $C_r^*(H) \times_{\alpha \circ \rho} K$ (see §1), which in turn is isomorphic to $C_r^*(H) \otimes C^*(K)$ since K acts trivially on $C_r^*(H)$. Applying Lemma 2.1 to $C_r^*(H) \otimes C^*(K)$ we see that if τ is the unique canonical trace on $C_r^*(H)$, then

$$\varphi(U_h W_k) = \tau(U_h) \varphi(W_k), \quad h \in H, \quad k \in K.$$

Since $\tau(U_h) = 0$ if $h \in H \setminus \{e\}$, the only possibly nonzero terms among $\{\varphi(U_h W_k)\}$ are those of the form $\varphi(W_k)$, $k \in K$. The action of K is effective, hence for any $k \in K \setminus \{0\}$ we can find some $g \in G$ such that $\langle \rho(k), g \rangle \neq 1$. Then $\varphi(W_k) = \varphi(U_g^* W_k U_g) = \langle \rho(k), g \rangle \varphi(W_k)$, and it follows that $\varphi(W_k) = 0$, $k \in K \setminus \{0\}$. We have shown that

$$\varphi(U_g W_k) = \begin{cases} 0, & \text{if } g \neq e \text{ or } k \neq 0, \\ 1, & \text{if } g = e \text{ and } k = 0. \end{cases}$$

Therefore φ agrees with the canonical trace on $C_r^*(G) \times_{\alpha \circ \rho} K$. \square

Remark. When G is a discrete abelian group, the crossed products $C_r^*(G) \times_{\alpha_x} \mathbf{Z}$ can be identified with the C^* -algebras studied in [4]. The result in [4] about the traces can be expressed, in our terminology, as traces on $C_r^*(G) \times_{\alpha_x} \mathbf{Z}$ coming from the composition

$$C_r^*(G) \times_{\alpha_x} \mathbf{Z} \xrightarrow{E} C_r^*(H) \otimes C^*(L) \xrightarrow{\varphi} C,$$

where $L = \{l \in \mathbf{Z}: \alpha_\chi^l = 1\}$, E is the obvious conditional expectation and φ is a trace. This result can be deduced from our proof of Theorem 2.2. In fact, when G is abelian, we do not need the hypothesis that $C_r^*(H)$ has a unique trace to spilt $\varphi(U_h W_k)$ into $\tau(U_h)\varphi(W_k)$, instead, we can directly show that $\varphi(U_h W_k)$ vanishes if $k \notin L$, and thus show that φ is supported on $C_r^*(H) \otimes C^*(L)$. We point out that our proof is much simpler than that given in [4].

Corollary 2.3. *Let χ be a character on G . If $\chi(g)$ is infinite and $C_r^*(\ker \chi)$ has a unique trace, then $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ has a unique trace.*

Proof. The crossed product $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ is a special case of $C_r^*(G) \times_{\alpha \circ \rho} K$, where $K = \mathbf{Z}$ and $\rho: K \rightarrow \widehat{G}$ is given by $\rho(1) = \chi$. The subgroup H of Theorem 2.2 is, in the present situation, the subgroup $\ker \chi$. Since \mathbf{Z} acts effectively on $C_r^*(G)$ if and only if $\chi(G)$ is infinite, the corollary is a direct consequence of Theorem 2.2. \square

Given the C^* -dynamical system $(C_r^*(G), \alpha_\chi, \mathbf{Z})$, if $\chi(G)$ is a finite subgroup of T with order q , then the \mathbf{Z} -action α_χ is not effective. However the naturally induced \mathbf{Z}_q -action on $C_r^*(G)$ is effective. Denote this action by $\dot{\alpha}_\chi$.

Corollary 2.4. *If $\chi(G)$ is an finite group with order q and $C_r^*(\ker \chi)$ has a unique trace, then $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}_q$ has a unique trace. \square*

Remark. We will show that $\chi(G)$ is the set of eigenvalues of α_χ and $C_r^*(\ker \chi)$ is the fixed point subalgebra of α_χ (see Propositions 2.18 and 2.19 below).

Proposition 2.5. *If $C_r^*(G) \times_{\alpha \circ \rho} K$ has a unique trace, then K must act effectively on $C_r^*(G)$.*

Proof. Assume the K -action is not effective. Then $L = \ker(\alpha \circ \rho)$ is a nontrivial subgroup of K and there exists an induced action $\widetilde{\alpha \circ \rho}$ of the group K/L on $C_r^*(G)$. By the universal property of crossed products, there exists a natural $*$ -homomorphism

$$\pi: C_r^*(G) \times_{\alpha \circ \rho} K \rightarrow C_r^*(G) \times_{\widetilde{\alpha \circ \rho}} K/L$$

with nontrivial kernel. By Corollary 1.3, $C_r^*(G) \times_{\widetilde{\alpha \circ \rho}} K/L$ has a trace, say φ . Then $\varphi \circ \pi$ is a trace on $C_r^*(G) \times_{\alpha \circ \rho} K$, which is not faithful. But the canonical trace τ on $C_r^*(G) \times_{\alpha \circ \rho} K$ is faithful. Hence $\varphi \circ \pi \neq \tau$. \square

Proposition 2.6. *Let G be a discrete amenable group. The $C_r^*(G) \times_{\alpha \circ \rho} K$ has a unique trace if and only if G is abelian, the subgroup H of Theorem 2.2 is trivial, and K acts effectively on $C_r^*(G)$.*

Proof. First assume that $C_r^*(G) \times_{\alpha \circ \rho} K$ has a unique trace. Since the normal subgroup H is amenable, there exists a trace φ on $C_r^*(G)$ such that $\varphi(U_g) = 0$ if $g \in H$, and $\varphi(U_g) = 1$ if $g \in H$ (see [22]). If H is not trivial,

then φ is different from the canonical trace τ on $C_r^*(G)$. But both φ and τ are α_χ -invariant, and hence they all extend to traces on $C_r^*(G) \times_{\alpha \circ \rho} K$, which contradicts the assumption that $C_r^*(G) \times_{\alpha \circ \rho} K$ has a unique trace. Hence H must be trivial. Since H contains the commutator subgroup of G , the triviality of H implies that G is abelian. The effectiveness of the K -action is shown in Proposition 2.5. The converse is a direct consequence of Theorem 2.2. \square

Corollary 2.7. *Let G be a discrete amenable group. Then $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ has a unique trace if and only if G is an infinite abelian group and the character χ is one-to-one. \square*

We now study the case that there are more than one trace on $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$. The proof of the following theorem is inspired by Elliott [12].

Theorem 2.8. *Let A be a separable unital C^* -algebra, α a $*$ -automorphism of A satisfying $\alpha^q = 1$, and $A \times_{\dot{\alpha}} \mathbf{Z}_q$ be the C^* -crossed product associated with the quotient action $\dot{\alpha}$. If $A \times_{\dot{\alpha}} \mathbf{Z}_q$ has a unique trace, then all traces on $A \times_{\alpha} \mathbf{Z}$ agree on projections in $A \times_{\alpha} \mathbf{Z}$, and moreover, they induce the same map from $K_0(A \times_{\alpha} \mathbf{Z})$ to \mathbf{R} .*

Proof. Recall that the crossed product $A \times_{\alpha} \mathbf{Z}$ is the universal C^* -algebra generated by $a \in A$ and a unitary W with respect to the condition $waw^* = \alpha(a)$. Since $\alpha^q = 1$, the C^* -subalgebra of $A \times_{\alpha} \mathbf{Z}$ generated by $w^q, C^*(w^q)$, is contained in $\mathbf{z}(A \times_{\alpha} \mathbf{Z})$, the centre of $A \times_{\alpha} \mathbf{Z}$. We first want to show that the space of extremal traces on $A \times_{\alpha} \mathbf{Z}$, X , is homeomorphic to the maximal ideal space of $C^*(W^q)$, which is the unit circle T . Given any maximal ideal t in $C^*(W^q)$, let I_t be the closed two-sided ideal of $A \times_{\alpha} \mathbf{Z}$ generated by t , and π_t the quotient map $A \times_{\alpha} \mathbf{Z} \rightarrow A \times_{\alpha} \mathbf{Z}/I_t$. Since t is maximal in $C^*(W^q)$, $\pi_t(W^q)$ is a scalar in T . It is convenient to denote $\pi_t(W^q)$ by t . By the universal property of crossed products, there exists a $*$ -isomorphism $\sigma_t: A \times_{\alpha} \mathbf{Z}/I_t \rightarrow A \times_{\dot{\alpha}} \mathbf{Z}_q$ such that

$$\sigma_t(\pi_t(a)) = a, \quad a \in A, \quad \sigma_t(\pi_t(W)) = t^{1/q} V,$$

where V is the canonical unitary generator of $A \times_{\dot{\alpha}} \mathbf{Z}_q$ such that $V^q = 1$ and $VaV^* = \alpha(a)$, $a \in A$. Let ν be the unique trace on $A \times_{\dot{\alpha}} \mathbf{Z}_q$. Then $\nu \circ \sigma_t \circ \pi_t$ is a trace on $A \times_{\alpha} \mathbf{Z}$, which is denoted by φ_t .

Claim. φ_t is an extremal trace on $A \times_{\alpha} \mathbf{Z}$.

Assume φ_1 and φ_2 are two traces on $A \times_{\alpha} \mathbf{Z}$ such that $\varphi_t = \lambda_1 \varphi_1 + \lambda_2 \varphi_2$, $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$. For any positive element $y \in I_t$, we have $0 = \varphi_t(y) = \lambda_1 \varphi_1(y) + \lambda_2 \varphi_2(y)$. Hence $\varphi_1(y) = \varphi_2(y) = 0$. since I_t is the linear span of its positive elements, we obtain $\varphi_1(I_t) \equiv 0 \equiv \varphi_2(I_t)$. Therefore both φ_1 and φ_2 factor through $A \times_{\alpha} \mathbf{Z}/I_t = A \times_{\dot{\alpha}} \mathbf{Z}_q$. The uniqueness of traces on $A \times_{\dot{\alpha}} \mathbf{Z}_q$ then implies $\varphi_1 = \varphi_2$, and it follows that φ_t is extremal.

Each maximal ideal t in $C^*(W^q)$ thereby yields an extremal trace φ_t . Conversely, let φ be an extremal trace on $A \times_{\alpha} \mathbf{Z}$. Since the restriction of an

extremal trace to the centre is multiplicative, $\varphi|_{C^*(W^q)}$ is multiplicative, and hence the kernel of $\varphi|_{C^*(W^q)}$ is a maximal ideal in $C^*(W^q)$. It is easy to verify that $t \rightarrow \varphi_t$ is a one-to-one correspondence between the maximal ideal space T of $C^*(W^q)$ and the space of extremal traces X . To show this correspondence is a homeomorphism, we prove the following.

Claim. The space X is compact in W^* -topology.

Since the space of traces on a unital C^* -algebra is always compact in the W^* -topology, it is enough to show that the space X is a closed subspace. Let $\{\varphi_\gamma\}$ be a net of extremal traces such that $\varphi_\gamma \rightarrow \varphi$, where φ is a trace on $A \times_\alpha \mathbf{Z}$. Since $\varphi_\gamma|_{C^*(W^q)}$ are each multiplicative, $\varphi|_{C^*(W^q)}$ must be multiplicative, and $\ker(\varphi|_{C^*(W^q)})$ is thus a maximal ideal t in $C^*(W^q)$. As we have shown, from this ideal t we can construct an extremal trace φ_t on $A \times_\alpha \mathbf{Z}$. Since $\varphi(I_t) \equiv 0$ by the Cauchy-Schwarz inequality, φ factors through $A \times_\alpha \mathbf{Z}/I_t = A \times_\alpha \mathbf{Z}_q$. The uniqueness of traces on $A \times_\alpha \mathbf{Z}_q$ implies $\varphi = \varphi_t$. This proves that φ is extremal, and hence X is closed.

It is then easy to check that the space X is homeomorphic to T . As a consequence, we see that X is a connected space.

Now we can prove that all traces on $A \times_\alpha \mathbf{Z}$ agree on projections. The proof mimics Elliott's argument [12]. Let p be a projection in $A \times_\alpha \mathbf{Z}$. Then p can be thought of as a continuous function on the space X via $p(\varphi) \equiv \varphi(p)$, $\varphi \in X$. Since all extremal traces on $A \times_\alpha \mathbf{Z}$ factor through $A \times_\alpha \mathbf{Z}_q$, the range $p(X)$ of the function p is contained in the range of the unique trace ν of $A \times_\alpha \mathbf{Z}_q$ evaluating on the projections in $A \times_\alpha \mathbf{Z}_q$. By hypothesis, A is separable, hence $A \times_\alpha \mathbf{Z}_q$ is separable and it has only countably many unitary equivalence classes of projections. Therefore $\nu(\text{proj}(A \times_\alpha \mathbf{Z}_q))$ is a countable subset of the real line \mathbf{R} . Since X is a connected space and p is continuous on X , $p(X)$ is connected in \mathbf{R} . This forces $p(X)$ to be a single point. In other words, all extremal traces take the same value on p , and hence so do all traces.

To show that all traces on $A \times_\alpha \mathbf{Z}$ induce the same map from $K_0(A \times_\alpha \mathbf{Z})$ to \mathbf{R} , we have to show that all traces on $M_n \otimes (A \times_\alpha \mathbf{Z})$ agree on its projections. Since

$$\begin{aligned} M_n \otimes (A \times_\alpha \mathbf{Z}) &\simeq (M_n \otimes A) \times_{1 \otimes \alpha} \mathbf{Z}, \\ M_n \otimes (A \times_\alpha \mathbf{Z}_q) &\simeq (M_n \otimes A) \times_{1 \otimes \alpha} \mathbf{Z}_q, \end{aligned}$$

and $M_n \otimes (A \times_\alpha \mathbf{Z}_q)$ has a unique trace, we can apply what we have proved to the crossed product $(M_n \otimes A) \times_{1 \otimes \alpha} \mathbf{Z}$ and the desired result follows. \square

Remark 1. If $A \times_\alpha \mathbf{Z}_q$ is simple, then the centre of $A \times_\alpha \mathbf{Z}$ is $C^*(W^q) = C(T)$.

In fact, given a maximal ideal t of the centre $\mathbf{z}(A \times_\alpha \mathbf{Z})$, let $t' = t \cap C^*(W^q)$. Denote by I_t and $I_{t'}$ the closed two-sided ideals of $A \times_\alpha \mathbf{Z}$ generated by t and t' respectively. Since $I_t \supseteq I_{t'}$, the quotient map $\pi_t: A \times_\alpha \mathbf{Z} \rightarrow A \times_\alpha \mathbf{Z}/I_t$ factors

through $A \times_{\alpha} \mathbf{Z}/I_{t'}$:

$$\begin{array}{ccc} A \times_{\alpha} \mathbf{Z} & \xrightarrow{\pi_t} & A \times_{\alpha} \mathbf{Z}/I_t \\ & \searrow & \nearrow \tilde{\pi}_t \\ & A \times_{\alpha} \mathbf{Z}/I_{t'} & \end{array}$$

Since t' is a maximal ideal of $C^*(W^q)$, $A \times_{\alpha} \mathbf{Z}/I_{t'} \simeq A \times_{\alpha} \mathbf{Z}_q$ as indicated in the proof of the theorem. By hypotheses, $A \times_{\alpha} \mathbf{Z}_q$ is simple. Hence $\tilde{\pi}_t$ is an isomorphism since it is also surjective. It follows that $I_t = I_{t'}$. Since t is maximal in $\mathbf{z}(A \times_{\alpha} \mathbf{Z})$,

$$t = I_t \cap \mathbf{z}(A \times_{\alpha} \mathbf{Z}) = I_{t'} \cap \mathbf{z}(A \times_{\alpha} \mathbf{Z});$$

thus t' determines t , that is, the map $t \rightarrow t'$ is one-to-one. Hence $\mathbf{z}(A \times_{\alpha} \mathbf{Z}) = C^*(W_q)$.

Remark 2. If $A \times_{\alpha} \mathbf{Z}_q$ is simple, then the primitive ideal space of $A \times_{\alpha} \mathbf{Z}$ is T . The proof is similar to that for Remark 1.

Remark 3. We have an extension of C^* -algebras:

$$0 \rightarrow I_t \rightarrow A \times_{\alpha} \mathbf{Z} \rightarrow A \times_{\alpha} \mathbf{Z}_q \rightarrow 0,$$

where I_t is determined by a maximal ideal t of $C^*(W^q)$. If φ_t is the extremal trace associated to t , then, by construction,

$$I_t \subseteq \{y \in A \times_{\alpha} \mathbf{Z} : \varphi_t(y^*y) = 0\}.$$

If $A \times_{\alpha} \mathbf{Z}_q$ is simple, then I_t is maximal, and hence

$$I_t = \{y \in A \times_{\alpha} \mathbf{Z} : \varphi_t(y^*y) = 0\}.$$

Now we apply Theorem 2.8 to $C_r^*(G) \times_{\alpha_x} \mathbf{Z}$, where G is a Powers group. This definition is due to de la Harpe [13]. He showed that many groups, including the free product $G_1 * G_2$ of any two groups G_1 and G_2 with $|G_1| \geq 2$ and $|G_2| \geq 3$, are Powers groups. He also showed that Powers groups have many interesting properties, among them: (i) every subgroup of finite index in a Powers group is a Powers group; (ii) the reduced group C^* -algebra of any Powers group is simple and has a unique trace.

Corollary 2.9. *If G is a countable discrete Powers group and $\chi(G)$ is finite, then all traces on $C_r^*(G) \times_{\alpha_x} \mathbf{Z}$ agree at the K_0 -level.*

Proof. Assume $\chi(G)$ is finite with order q . Then $\alpha_x^q = 1$ and the subgroup $\ker \chi$ of G is of finite index q . Therefore $\ker \chi$ is a Powers group and $C_r^*(\ker \chi)$ has a unique trace. By Corollary 2.4 the crossed product $C_r^*(G) \times_{\alpha_x} \mathbf{Z}_q$ has a unique trace. Thus we can apply Theorem 2.8 to $C_r^*(G) \times_{\alpha_x} \mathbf{Z}$. \square

Corollary 2.10. *If G is the free product of two finite groups G_1 and G_2 with $|G_1| \geq 2$ and $|G_2| \geq 3$, then all traces on $C_r^*(G) \times_{\alpha_x} \mathbf{Z}$ agree at the K_0 -level.*

Proof. Since G is a Powers group and $\chi(G)$ is always finite, the conclusion follows from Corollary 2.9. \square

Conjecture 1. If G is a Powers group and $\chi(G)$ is infinite, then $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ has a unique trace. (See Added in proof.)

Conjecture 2. If G is a Powers group and H is a normal subgroup of G containing the commutator subgroup $[G, G]$, then H is a Powers group.

Remark 1. Conjecture 2 implies Conjecture 1. This can be seen as follows: $H = \ker \chi$ is always a normal subgroup of G containing $[G, G]$. If Conjecture 2 is true, then $\ker \chi$ is a Powers group and hence $C_r^*(\ker \chi)$ has a unique trace. It follows from Corollary 2.3 that $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ has a unique trace.

Remark 2. Conjecture 2 is true for many Powers groups. For example, it is true for $G = F_n$, $n \geq 2$.

For abelian groups G , we have necessary and sufficient conditions for all traces on $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ agreeing at the K_0 -level.

Theorem 2.11. *Let G be a countable discrete abelian group. Then the following are equivalent:*

- (i) *all traces on $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ agree at the K_0 -level;*
- (ii) *the restriction of χ to the torsion subgroup of G is one-to-one;*
- (iii) *the maximal ideal space of the centre of $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ is connected.*

Proof. (ii) \Rightarrow (i). The proof is similar to that of Theorem 2.8, so we only indicate some modifications needed in the present situation. Notation, if not explicitly given, is that used in the proof of Theorem 2.8.

Let $H = \ker \chi$ and $K = \{k \in \mathbf{Z} : \alpha_\chi^k = 1\} \simeq q\mathbf{Z}$. We have a naturally induced crossed product $C_r^*(G/H) \times_{\alpha_\chi} \mathbf{Z}_q$, which has a unique trace by Theorem 2.2. This crossed product plays the role $A \times_{\alpha} \mathbf{Z}_q$. The C^* -subalgebra of $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ generated by $\{U_h : h \in H\}$ and $\{W^k : k \in K\}$, $C^*(U_h, W^k)$, is contained in the centre of $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$, which plays the role $C^*(W^q)$ of Theorem 2.8. Since

$$C^*(U_H, W^K) \simeq C_r^*(H) \otimes C^*(K) \simeq C(\widehat{H} \times \widehat{K}),$$

the maximal ideal space of $C^*(U_H, W^K)$ is $\widehat{H} \times \widehat{K}$. Now we use the hypothesis that the restriction of χ to the torsion subgroup of G is one-to-one. This is equivalent to saying that the group $H = \ker \chi$ is torsion-free. Hence the space $\widehat{H} \times \widehat{K}$ is connected. We need this crucial property to prove that all traces agree on projections. Finally, given any point $t = (t_0, t_1) \in \widehat{H} \times \widehat{K}$, we need a $*$ -isomorphism

$$\sigma_t : C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}/I_t \rightarrow C_r^*(G/H) \times_{\alpha_\chi} \mathbf{Z}_q$$

so that we can get an extremal trace on $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$. Note that the quotient map

$$\pi_t : C_r^*(G) \times_{\alpha_\chi} \mathbf{Z} \rightarrow C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}/I_t$$

induces a character r_t on the group H by $r_t(h) = \pi_t(U_h)$, $h \in H$, and this character r_t extends to a character \tilde{r}_t on G . The $*$ -isomorphism σ_t is then defined by

$$\sigma_t(\pi_t(U_g)) = \tilde{r}_t(g)U_{[g]}, \quad g \in G, \quad \sigma_t(\pi_t(W)) = t_1^{1/q}V.$$

The rest of the proof is the same as that of Theorem 2.8.

(i) \Rightarrow (ii). Assume the restriction of χ to the torsion subgroup of G is not one-to-one. Then $H = \ker \chi$ contains a nontrivial cyclic group \mathbf{Z}_q . Note that $C_r^*(H)$, and hence $C_r^*(\mathbf{Z}_q)$, is contained in the centre of $C_r^*(G_n) \times_{\alpha_\chi} \mathbf{Z}$. Since any extremal trace φ restricts to a multiplicative linear functional on the centre, φ kills a maximal ideal in $C_r^*(\mathbf{Z}_q) \simeq \mathbf{C}^q$. But any maximal ideal in \mathbf{C}^q contains nontrivial projections, hence φ kills some nontrivial projections. Since the canonical trace τ on $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ is faithful, we see that τ and φ do not agree on $K_0(C_r^*(G) \times_{\alpha_\chi} \mathbf{Z})$.

(ii) \Leftrightarrow (iii). As seen in Remark 1 after Theorem 2.8, we can prove that the centre of $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ is $C^*(U_H, W^K) \simeq C_r^*(H) \otimes C^*(K) \simeq C(\widehat{H} \times \widehat{K})$, where $\widehat{K} = T$. Since the restriction of χ to the torsion subgroup of G is one-to-one iff $H = \ker \chi$ is torsion-free iff \widehat{H} is connected, the equivalence of (ii) and (iii) is clear. \square

Remark. The centre of $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ (G abelian) was previously determined by M. D. Brabanter and H. Zettl [4].

Now we study the problem when $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ is simple. We need the following well-known result.

Proposition 2.12. *The bounded linear map $\eta: C_r^*(G) \rightarrow l^2(G)$, $\eta(T) = Tf_e$, $T \in C_r^*(G)$, is one-to-one. \square*

Remark. The $*$ -automorphism α_χ on $C_r^*(G)$ and the unitary operator W_χ on $l^2(G)$ (see the proof of Propositions 1.1) are related via η by the following commutative diagram:

$$\begin{array}{ccc} C_r^*(G) & \xrightarrow{\alpha_\chi} & C_r^*(G) \\ \eta \downarrow & & \downarrow \eta \\ l^2(G) & \xrightarrow{W_\chi} & l^2(G) \end{array}$$

Proposition 2.13. *Let χ be a character on a discrete group G . If the number of elements of every finite conjugacy class in G is less than the order of $\chi(G)$ (in particular, if $\chi(G)$ is infinite), then α_χ is an outer $*$ -automorphism on $C_r^*(G)$.*

Proof. Assume $\alpha_\chi = \text{ad } U$ for some unitary $U \in C_r^*(G)$. Let

$$\eta(U) = \sum_{h \in G} \lambda_h f_h, \quad \lambda_h \in \mathbf{C}, \quad \sum_{h \in G} |\lambda_h|^2 < \infty.$$

Since $UU_gU^* = \alpha_\chi(U_g) = \chi(g)U_g$ for any $g \in G$, we have $\chi(g)U_gU = UU_g$. Applying both sides to vector f_e , we get

$$\chi(g)U_gUf_e = \chi(g)U_g \left(\sum_{h \in G} \lambda_h f_h \right) = \sum_{h \in G} \chi(g)\lambda_h f_{gh} = \sum_{h \in G} \chi(g)\lambda_{g^{-1}hg} f_{hg}$$

and

$$UU_gf_e = UR_{g^{-1}}f_e = R_{g^{-1}}Uf_e = R_{g^{-1}} \left(\sum_{h \in G} \lambda_h f_h \right) = \sum_{h \in G} \lambda_h f_{hg},$$

where $R: G \rightarrow B(l^2(G))$ is the right regular representation of G defined by $R_g(f_h) = f_{hg^{-1}}$, $g, h \in G$. Comparing coefficients in the above two equations, we see that

$$\lambda_h = \chi(g)\lambda_{g^{-1}hg}, \quad g, h \in G.$$

Define a unitary operator V_g on $l^2(G)$ by $V_g(f_h) = f_{g^{-1}hg}$. We have

$$V_g(\eta(U)) = V_g \left(\sum_{h \in G} \lambda_h f_h \right) = \sum_{h \in G} \lambda_h f_{g^{-1}hg} = \sum_{h \in G} \chi(g)\lambda_{g^{-1}hg} f_{g^{-1}hg} = \chi(g)\eta(U).$$

Since the map η is one-to-one, $\eta(U) \neq 0$ and hence $\chi(g)$ is an eigenvalue of V_g . Now decompose G into a disjoint union of conjugacy classes: $G = G_1 \cup G_2 \cup \dots$. We have a corresponding decomposition of $l^2(G)$ into a direct sum: $l^2(G) = l^2(G_1) \oplus l^2(G_2) \oplus \dots$. Since conjugation by g is an automorphism of the group G which leaves every conjugacy class G_i invariant, the operator V_g leaves every subspace $l^2(G_i)$ invariant. Let $\eta(U) = y_1 + y_2 + \dots$ be the direct sum decomposition with $y_i \in l^2(G_i)$. Since $\eta(U) \neq 0$, there is some $y_{i_0} \neq 0$. The condition $\lambda_h = \chi(g)\lambda_{g^{-1}hg}$, $g, h \in G$, implies that all the coefficients λ_h with h in the same conjugacy class have the same absolute value. Since $\eta(U)$ is square summable, it follows that if $\lambda_h \neq 0$ then h is of finite conjugacy class in G . Therefore $l^2(G_i)$ is a finite dimensional space. The restriction of V_g to $l^2(G_{i_0})$ is given by a permutation of the basis $\{f_h: h \in G_{i_0}\}$ of $l^2(G_{i_0})$. Since every permutation is a product of disjoint cyclic permutations, V_g is of the form

$$\begin{bmatrix} T_1 & & \\ & \ddots & \\ & & T_l \end{bmatrix}$$

on $l^2(G_{i_0})$, where each T_k is of the form

$$\begin{bmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}.$$

Since $\chi(g)$ is an eigenvalue of V_g , it is easy to see that $\chi(g)$ is also an eigenvalue of each T_k . If T_k is an $N_k \times N_k$ matrix, then its eigenvalues are precisely the N_k th roots of unity. By hypothesis, we can choose g such that the order of $\chi(G)$ is greater than the number of elements in G_{i_0} , the latter equal to the dimension of $l^2(G_{i_0})$. Therefore $\chi(g)$ cannot be an N_k th root of unity, a contradiction. This proves that α_χ is not inner. \square

Remark. The above proposition generalizes an earlier result of Paschke [21], which needs the hypothesis that G is an infinite conjugacy class group.

Proposition 2.14. *If $C_r^*(G)$ is simple and $\chi(G)$ is infinite, then $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ is simple.*

Proof. By Pederson [23], it is enough to show that the Connes spectrum of α_χ is the whole unit circle T . By Olesen [19] this reduces to show that α_χ^n (which is equal to α_{χ^n}) is not inner for each $n \neq 0$. Since $\chi(G)$ is infinite, $\chi^n(G)$ must be infinite if $n \neq 0$. It follows from Proposition 2.13 that α_{χ^n} is not inner for each $n \neq 0$. \square

Proposition 2.15. *If $C_r^*(G)$ is simple and $\chi(G)$ is finite with order q , then $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}_q$ is simple.*

Proof. Along the same lines as in the proof of Proposition 2.14, we can show that each $(\alpha_\chi)^{[n]}$ is not inner if $[n] \neq 0$ in \mathbf{Z}_q . In fact, since $C_r^*(G)$ is simple, its centre must be trivial. This implies that G is an infinite-conjugacy-classes group. Then we apply Proposition 2.13 to α_χ^n . \square

Applying results in this section to the free groups F_n yields the following corollary.

Corollary 2.16. *If $\chi(F_n)$ is infinite, then $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ is a simple C^* -algebra with a unique trace; if $\chi(F_n)$ is finite, then $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ is not simple and has more than one trace, but all of its traces agree at the K_0 -level.*

Proof. First we assume $n > 1$. Then F_n and its subgroup $\ker \chi$ are both free nonabelian. Hence $C_r^*(F_n)$ is simple and $C_r^*(\ker \chi)$ has a unique trace. If $\chi(F_n)$ is infinite, then $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ is a simple C^* -algebras with a unique trace by Proposition 2.14 and Corollary 2.3. If $\chi(F_n)$ is finite, then $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ is not simple since its centre is nontrivial, and it has more than one trace by Proposition 2.5. However, Corollary 2.9 implies that all traces on $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ agree at the K_0 -level.

When $n = 1$, $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ is just the rotation C^* -algebras. The result is well known, and can also be deduced from our Proposition 2.14, Corollary 2.3, and Theorem 2.11. \square

Finally, we want to determine the fixed point subalgebra and the set of eigenvalues of the dynamical system $(C_r^*(G), \alpha_\chi, \mathbf{Z})$.

Lemma 2.17. *Let H be a subgroup of G and let $l^2(H)$ be identified with the closed linear subspace of $l^2(G)$ generated by $\{f_h; h \in H\}$. If $T \in C_r^*(G)$ and $\eta(T) \in l^2(H)$, where η is defined in Proposition 2.12, then $T \in C_r^*(H)$.*

Proof. Let P_H be the projection from $l^2(G)$ onto $l^2(H)$, and E_H the conditional expectation from $C_r^*(G)$ onto $C_r^*(H)$ (see [22]). We have the following commutative diagram:

$$\begin{array}{ccc} C_r^*(G) & \xrightarrow{E_H} & C_r^*(H) \\ \eta \downarrow & & \downarrow \eta \\ l^2(G) & \xrightarrow{P_H} & l^2(H) \end{array}$$

Hence $\eta(E_H(T)) = P_H(\eta(T)) = \eta(T)$ since $\eta(T) \in l^2(H)$. But η is one-to-one by Proposition 2.12; it follows that $T = E_H(T) \in C_r^*(H)$. \square

The idea in the proof of the above lemma was suggested by Professor John Phillips.

Propositions 2.18. *Given the C^* -dynamical system $(C_r^*(G), \alpha_\chi, \mathbf{Z})$, the fixed point subalgebra of α_χ is $C_r^*(\ker \chi)$.*

Proof. It is obvious that $C_r^*(\ker \chi)$ is contained in the fixed point subalgebra. For the converse, assume $T \in C_r^*(G)$ and $\alpha_\chi(T) = T$. We further assume that $\eta(T) = \sum_{g \in G} C_g f_g \in l^2(G)$, $C_g \in \mathbf{C}$. Since $\eta \circ \alpha_\chi = W_\chi \circ \eta$, we have

$$\sum_{g \in G} C_g f_g = \eta(T) = \eta(\alpha_\chi(T)) = W_\chi(\eta(T)) = W_\chi \left(\sum_{g \in G} C_g f_g \right) = \sum_{g \in G} \chi(g) C_g f_g.$$

Hence $C_g = \chi(g) C_g$, $g \in G$. This implies that if $g \in G$ is such that $C_g \neq 0$, then $\chi(g) = 1$ and hence $g \in \ker \chi$. Therefore $\eta(T) \in l^2(\ker \chi)$. It follows from Lemma 2.17 that $T \in C_r^*(\ker \chi)$. \square

Remark. Proposition 2.18 is also true when $C_r^*(G)$ is replaced by twisted group C^* -algebras with the same proof.

Proposition 2.19. *Given the C^* -dynamical system $(C_r^*(G), \alpha_\chi, \mathbf{Z})$, the set of eigenvalues of α_χ is $\chi(G)$.*

Proof. Use the map $\eta: C_r^*(G) \rightarrow l^2(G)$ and argue as above. \square

3. K -THEORETICAL INVARIANTS

This section is devoted to the K -theory of the C^* -algebras $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$.

We shall find some classifying K -theoretic invariants. If G is the free group F_n or any (discrete) amenable group, we will show that the set $\chi(G)$ is an isomorphism invariant for $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$. The second invariant, only for $G = F_n$, is a rational number in the interval $[0, \frac{1}{2}]$. We call it the twist of

the C^* -algebra $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$, and denote it by $t(\cdot)$. This name comes from Rieffel's work on the classification of rational rotation C^* -algebras [33], though our definition is entirely different from his. We will show in §5 that these two invariants completely classify $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$.

For results on the K -theory of operator algebras, the standard references are Atiyah [2], Karoubi [16], and J. L. Taylor [35]. We also recommend Cuntz [7, 8]. An important result we are going to employ is the Pimsner-Voiculescu six term exact sequence [28]. We will call it the P-V sequence for short. We will also use their computations of the K -theory of $C_r^*(F_n)$ [29] (see also [6, 17]).

We first remind the reader of the work of Rieffel and Pimsner-Voiculescu on the irrational rotation C^* -algebra. Recall that a rotation C^* -algebra is the C^* -crossed product $C(T) \times_\theta \mathbf{Z}$, where the action θ is given by a rotation of angle $2\pi\theta$ on the unit circle T . This C^* -algebra is denoted A_θ . The normalized Lebesgue measure on T induces a trace τ on A_θ . Rieffel showed in [32] that if $0 < \theta < 1$, then there is a projection p in A_θ such that $\tau(p) = \theta$. This projection p is then called the Rieffel projection. When $\theta = 0$ or 1 , A_θ is isomorphic to the C^* -algebra $C(T^2)$, and there are no nontrivial projections in $C(T^2)$. However we can find a special nontrivial projection p in $M_2 \otimes C(T^2)$ such that $\tau(p) = 1$ and $\{[1], [p]\}$ generate $K_0(C(T^2)) \simeq \mathbf{Z}^2$. This projection is called the Bott projection. Pimsner-Voiculescu showed in [28], using their six term exact sequence, that $K_0(A_\theta) \simeq \mathbf{Z}^2$ with generators $[1]$ and $[p]$, where p is a Rieffel projection or a Bott projection, and that $\partial([p]) = -[u]$, where $\partial: K_0(A_\theta) \rightarrow K_1(C(T))$ is the boundary map in the P-V sequence and u is the function on T such that $u(z) = z$ for all z in T . Although the proofs of these results in [32 and 28] are for irrational θ 's, they remain valid for rational θ 's.

The rotation C^* -algebras are relevant to C^* -crossed products $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$, because

$$C(T) \times_\alpha \mathbf{Z} \simeq C_r^*(\mathbf{Z}) \times_{\alpha_\chi} \mathbf{Z}$$

when $C_r^*(\mathbf{Z})$ is identified with $C(T)$ by Gelfand transform and α_χ is then identified with the rotation θ on T , where χ and θ are related by $\chi(1) = e^{2\pi i\theta}$. Furthermore, if g is any element of infinite order in a discrete group G , then $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ contains the C^* -subalgebra $C_r^*(U_g) \times_{\alpha_\chi} \mathbf{Z}$, where $C_r^*(U_g)$ is the C^* -subalgebra of $C_r^*(G)$ generated by U_g . The later is isomorphic to the rotation C^* -algebras A_θ with $e^{2\pi i\theta} = \chi(g)$.

Lemma 3.1. *Suppose $G = \mathbf{Z}_q$ and $\chi \in \widehat{\mathbf{Z}}_q$. Then*

$$\chi(\mathbf{Z}_q) \subseteq \exp \circ \varphi_*(K_0(C_r^*(\mathbf{Z}_q))),$$

where φ is any α_χ -invariant trace on $C_r^*(\mathbf{Z}_q)$.

Proof. Assume the order of $\chi(\mathbf{Z}_q)$ is k . Hence

$$\chi(\mathbf{Z}_q) = \{e^{2\pi im/k} : m \in \mathbf{Z}\}.$$

By duality theory, $C_r^*(\mathbf{Z}_q) \simeq C(\widehat{\mathbf{Z}}_q)$, α_χ corresponds to the translation M_χ on the dual group $\widehat{\mathbf{Z}}_q$: $M_\chi(\rho) = \chi \cdot \rho$, $\rho \in \widehat{\mathbf{Z}}_q$, and α_χ -invariant traces φ are in one-to-one correspondence to M_χ -invariant probability measures μ on $\widehat{\mathbf{Z}}_q$. Since the order of χ in $\widehat{\mathbf{Z}}_q$ is k , every M_χ -orbit in $\widehat{\mathbf{Z}}_q$ has k points, and there are q/k orbits. Since every M_χ -invariant measure μ takes the same values at points in the same orbit, we may assume that μ assigns weights C_i , $i = 1, 2, \dots, q/k$, to each point in the i th orbit. The normalization condition for μ is now

$$kC_1 + kC_2 + \dots + kC_{q/k} = 1.$$

Since $K_0(C_r^*(\mathbf{Z}_q)) \simeq K_0(C(\widehat{\mathbf{Z}}_q)) \simeq K_0(\mathbf{C}^q) \simeq \mathbf{Z}^q$, and since each generator of \mathbf{Z}^q comes from a minimal projection in \mathbf{C}^q , we see that $\varphi_*(K_0(C_r^*(\mathbf{Z}_q)))$ is the subgroup of \mathbf{R} generated by $C_1, C_2, \dots, C_{q/k}$, which contains $C_1 + \dots + C_{q/k} = \frac{1}{k}$. Hence $\exp \circ \varphi_*(K_0(C_r^*(\mathbf{Z}_q)))$ contains $e^{2\pi i/k}$ which generates $\chi(\mathbf{Z}_q)$. \square

Lemma 3.2. *For any discrete group G , any character χ on G , and any trace φ on $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$, we have*

$$\chi(G) \subseteq \exp \circ \varphi_*(K_0(C_r^*(G) \times_{\alpha_\chi} \mathbf{Z})).$$

Proof. Assume $\lambda = \chi(g) \neq 1$ for some $g \in G$. Write $\lambda = e^{2\pi i\theta}$ with $0 < \theta < 1$. If g is of infinite order in G , then the C^* -subalgebra $C^*(U_g) \times_{\alpha_\chi} \mathbf{Z}$ of $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ is the rotation C^* -algebra A_θ . Hence we can find a Rieffel projection p in $C^*(U_g) \times_{\alpha_\chi} \mathbf{Z}$ such that $\tau(p) = \theta$, where τ is the canonical trace on A_θ . By Theorem 2.11, all traces on A_θ agree on projections in A_θ . This implies that the restriction of φ to $C^*(U_g) \times_{\alpha_\chi} \mathbf{Z} \simeq A_\theta$ agrees with τ on the projection p : $\varphi(p) = \tau(p) = \theta$. Therefore

$$\lambda = e^{2\pi i\theta} \in \exp \circ \varphi_*(K_0(C_r^*(G) \times_{\alpha_\chi} \mathbf{Z})).$$

If g is of finite order q in G , then the subgroup generated by g , $\langle g \rangle$, is isomorphic to \mathbf{Z}_q , and we have the inclusions

$$C_r^*(\langle g \rangle) \hookrightarrow C_r^*(\langle g \rangle) \times_{\alpha_\chi} \mathbf{Z} \hookrightarrow C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}.$$

Since the restriction of φ to $C_r^*(\langle g \rangle)$ is an α_χ -invariant trace, we obtain from Lemma 3.1 that

$$\begin{aligned} \lambda &= \chi(g) \in \chi(\langle g \rangle) \subseteq \exp \circ \varphi_*(K_0(C_r^*(\langle g \rangle))) \\ &\subseteq \exp \circ \varphi_*(K_0(C_r^*(G) \times_{\alpha_\chi} \mathbf{Z})). \quad \square \end{aligned}$$

Theorem 3.3. *For any character χ on F_n , we have*

$$\exp \circ \tau_*(K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})) = \chi(F_n).$$

Proof. By Lemma 3.2, it is enough to show that

$$\exp \circ \tau_*(K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})) \subseteq \chi(F_n).$$

From the P-V sequence, we have the following exact sequence:

$$\begin{array}{ccccc} K_0(C_r^*(F_n)) & \xrightarrow{1-\alpha_\chi^*} & K_0(C_r^*(F_n)) & \longrightarrow & K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) & \longleftarrow & K_1(C_r^*(F_n)) & \xleftarrow{1-\alpha_\chi^*} & K_1(C_r^*(F_n)). \end{array}$$

Assume the free generators of F_n are g_1, g_2, \dots, g_n , and assume $\chi(g_k) = e^{2\pi i \theta_k}$, $k = 1, 2, \dots, n$. For any t in $[0, 1]$, we define a character χ_t of F_n by

$$\chi_t(g_k) = e^{2\pi i t \theta_k}, \quad k = 1, 2, \dots, n.$$

Then α_{χ_t} is a continuous path of *-automorphisms of $C_r^*(F_n)$ connecting α_χ to the identity map. The homotopy invariance of K -theory yields $\alpha_{\chi^*} = \text{id}$. Thus the P-V sequence breaks up into short exact sequences:

$$0 \rightarrow K_i(C_r^*(F_n)) \rightarrow K_i(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) \rightarrow K_{i+1}(C_r^*(F_n)) \rightarrow 0.$$

Since $K_0(C_r^*(F_n)) \simeq \mathbf{Z}$ and $K_1(C_r^*(F_n)) \simeq \mathbf{Z}^n$, according to [29], it follows that $K_1(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) \simeq \mathbf{Z}^{n+1}$, $i = 1, 2$.

Now let g_k be a free generator of F_n . Since $C^*(U_{g_k})$ is invariant under the action α_χ , naturality of the P-V sequence yields the following commutative diagram, where the rows are exact:

$$\begin{array}{ccccccc} 0 \rightarrow K_0(C_r^*(F_n)) & \longrightarrow & K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) & \xrightarrow{\partial} & K_1(C_r^*(F_n)) & \rightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ 0 \rightarrow K_0(C^*(U_{g_k})) & \longrightarrow & K_0(C^*(U_{g_k}) \times_{\alpha_\chi} \mathbf{Z}) & \xrightarrow{\partial} & K_1(C^*(U_{g_k})) & \rightarrow & 0 \end{array}$$

Since $C^*(U_{g_k}) \times_{\alpha_\chi} \mathbf{Z}$ is a rotation C^* -algebras, we can find a Rieffel projection p_k in $C^*(U_{g_k}) \times_{\alpha_\chi} \mathbf{Z}$ (or a Bott projection p_k in $M_2 \otimes (C^*(U_{g_k}) \times_{\alpha_\chi} \mathbf{Z})$) such that $\tau(p_k) = \theta_k$ and $\partial([p_k]) = -[U_{g_k}]$ in the second row of the above diagram. The commutativity of this diagram implies that $\partial([p_k]) = -[U_{g_k}]$ also holds in the first row. This means that for the generators $[U_{g_1}], \dots, [U_{g_n}]$ of $K_1(C_r^*(F_n))$, we can find projections p_1, p_2, \dots, p_n in $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ or $M_2 \otimes (C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})$ such that

$$\exp \circ \tau(p_k) = \chi(g_k) \quad \text{and} \quad \partial([p_k]) = -[U_{g_k}], \quad k = 1, \dots, n.$$

Since $\{[U_{g_k}]: 1 \leq k \leq n\}$ is a basis for $K_1(C_r^*(F_n))$ and $[1]$ is a basis for $K_0(C_r^*(F_n))$ by [29], $\{[p_k]: 1 \leq k \leq n\}$ together with $[1]$ must be a basis for $K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})$. Then it is clear that

$$\exp \circ \tau_*(K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})) \subseteq \chi(F_n). \quad \square$$

Remark 1. If we only want to prove that all traces on $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ induce the same map on $K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})$, then the following observation provides an easy proof.

Let φ_1 and φ_2 be two traces on $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$. To show that they induce the same map on $K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})$, it is enough to show that they agree on the generators of $K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})$. By the proof of Theorem 3.3, a set of generators for $K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})$ is $\{[1], [p_1], \dots, [p_n]\}$, where each p_i is a Rieffel projection or a Bott projection sitting in a rotation C^* -subalgebra of $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$, hence it is enough to show that when restricted to a rotation C^* -algebra, φ_1 and φ_2 agree on the K_0 -group. But this has already been shown in Elliott [12] (cf. our Theorem 2.11).

This observation also works in some other circumstances, e.g. for the twisted group C^* -algebras associated with the 3-dimensional discrete Heisenberg group.

Remark 2. It would be interesting to decide to which groups Theorem 3.3 can be generalized. It fails, for example, if $G = \mathbf{Z}_q$, as follows. A simple computation shows that

$$\exp \circ \tau_*(K_0(C_r^*(\mathbf{Z}_q) \times_{\alpha_\chi} \mathbf{Z})) = e^{\{e^{2\pi ik/q} : k \in \mathbf{Z}\}}.$$

But $\chi(\mathbf{Z}_q)$ need not be the same set, e.g. if χ is trivial. When χ is trivial, the C^* -algebra $C_r^*(\mathbf{Z}_q) \times_{\alpha_\chi} \mathbf{Z} \simeq \mathbf{C}^q \otimes C(T)$ has more than one trace, and these traces do not agree on the K_0 -group.

In the following, we show that the conclusion of Theorem 3.3 does not hold for $G = \mathbf{Z}_n * \mathbf{Z}_m$ with $n > 1$, $m > 2$, although all traces on $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m) \times_{\alpha_\chi} \mathbf{Z}$ agree on the K_0 -group by Corollary 2.10.

Applying the P-V sequence to $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m) \times_{\alpha_\chi} \mathbf{Z}$, yields an exact sequence:

$$\begin{array}{ccccc} K_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)) & \longrightarrow & K_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)) & \xrightarrow{j_*} & K_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m) \times_{\alpha_\chi} \mathbf{Z}) \\ & & \uparrow & & \downarrow \\ K_1(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m) \times_{\alpha_\chi} \mathbf{Z}) & \longleftarrow & K_1(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)) & \longleftarrow & K_1(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)). \end{array}$$

Since $K_1(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)) = 0$ by [6], this breaks into an exact sequence

$$\rightarrow K_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)) \xrightarrow{j_*} K_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m) \times_{\alpha_\chi} \mathbf{Z}) \rightarrow 0$$

Let φ be any trace on $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m) \times_{\alpha_\chi} \mathbf{Z}$, and denote its restriction to

$C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$ by $\tilde{\varphi}$. Then we have the commutative diagram:

$$\begin{array}{ccccc}
 K_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)) & \xrightarrow{j_*} & K_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m) \times_{\alpha_\chi} \mathbf{Z}) & \longrightarrow & 0 \\
 & \searrow \tilde{\varphi}_* & \swarrow \varphi_* & & \\
 & & \mathbf{R} & &
 \end{array}$$

Since j_* is surjective, any element $\chi \in K_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m) \times_{\alpha_\chi} \mathbf{Z})$ lifts to an element $\tilde{\chi} \in K_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m))$, and $\varphi_*(\chi) = \tilde{\varphi}_*(\tilde{\chi})$. However $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$ has unique trace since $\mathbf{Z}_n * \mathbf{Z}_m$ is a Powers group. This implies that $\varphi_*(\chi)$ does not depend on the choice of the traces on $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m) \times_{\alpha_\chi} \mathbf{Z}$. So we have proved once again that all traces on $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m) \times_{\alpha_\chi} \mathbf{Z}$ agree on K_0 (cf. Corollary 2.10). From the equality $\varphi_*(\chi) = \tilde{\varphi}_*(\tilde{\chi})$ for any $\chi \in K_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m) \times_{\alpha_\chi} \mathbf{Z})$, we get

$$\varphi_*(K_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m) \times_{\alpha_\chi} \mathbf{Z})) = \tilde{\varphi}_*(K_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m))) = \frac{1}{[m, n]} \mathbf{Z},$$

where $[m, n]$ is the least common multiple of m and n , and the last equality is shown in [6]. Since $\frac{1}{[m, n]} \mathbf{Z}$ does not depend on χ , it is clear that the conclusion of Theorem 3.3 does not hold for $G = \mathbf{Z}_n * \mathbf{Z}_m$.

Remark 3. The groups G given in Remark 2 above all contain torsion elements. What happens if G is torsion-free? We have the following observation. If for a discrete torsion-free group G ,

$$(3.1) \quad \exp \circ \tau_*(K_0(C_r^*(G) \times_{\alpha_\chi} \mathbf{Z})) = \chi(G)$$

holds for all $\chi \in \hat{G}$, where τ is the canonical trace, then the generalized Kadison conjecture holds for G , that is, $C_r^*(G)$ has no nontrivial projections. This is because when we take χ to be the trivial character, then (3.1) implies

$$\tau_*(K_0(C_r^*(G) \times_{\alpha_\chi} \mathbf{Z})) \subseteq \mathbf{Z}.$$

Since $C_r^*(G)$ is contained in $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ and τ is faithful, the generalized Kadison conjecture follows.

Although (3.1) does not hold in general even for abelian groups, we can show that $\chi(G)$ is an isomorphism invariant for any amenable group G . This motivates the following.

Definition 3.4. For any unital C^* -algebra A , define

$$T(A) = \bigcap_{\varphi} \exp \circ \varphi_*(K_0(A)),$$

where φ runs through all (normalized) traces on A . If A has no traces at all, define $T(A) = T$, the unit circle.

Remark. $T(\cdot)$ is a covariant functor from the category of unital C^* -algebras with unital $*$ -homomorphisms to the category of subgroups of T with inclusions

as morphisms. In fact, assume $\pi: A \rightarrow B$ is a unital $*$ -homomorphism. Let φ be any trace on B . Then $\varphi \circ \pi$ is a trace on A and we have

$$T(A) \subseteq \exp \circ (\varphi \circ \pi)_*(K_0(A)) = \exp \circ \varphi_*(\pi_*(K_0(A))) \subseteq \exp \circ \varphi_*(K_0(B)).$$

Since φ is arbitrary, we get $T(A) \subseteq T(B)$.

Theorem 3.5. *For any discrete amenable group G and any character χ on G , we have*

$$T(C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}) = \chi(G).$$

Proof. By Lemma 3.2, it is enough to show that $T(C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}) \subseteq \chi(G)$. Let $H = \ker \chi$. The character χ on G induces a quotient character $\dot{\chi}: G/H \rightarrow T$. Since G is amenable, the reduced group C^* -algebra and the full group C^* -algebra coincide. Hence there exists a unital $*$ -homomorphism $\pi: C_r^*(G) \rightarrow C_r^*(G/H)$ such that $\pi(U_g) = U_{[g]}$, $g \in G$. Since π is compatible with the α_χ -action on $C_r^*(G)$ and the $\alpha_{\dot{\chi}}$ -action on $C_r^*(G/H)$, it extends to a unital $*$ -homomorphism, still denoted by π , from $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ to $C_r^*(G/H) \times_{\alpha_{\dot{\chi}}} \mathbf{Z}$. Let τ be the canonical trace on $C_r^*(G/H) \times_{\alpha_{\dot{\chi}}} \mathbf{Z}$. Then $\tau \circ \pi$ is a trace on $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$. Hence

$$\begin{aligned} T(C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}) &\subseteq \exp \circ (\tau \circ \pi)_*(K_0(C_r^*(G) \times_{\alpha_\chi} \mathbf{Z})) \\ &\subseteq \exp \circ \tau_*(K_0(C_r^*(G/H) \times_{\alpha_{\dot{\chi}}} \mathbf{Z})). \end{aligned}$$

Now G/H is an abelian group, $\dot{\chi}$ is one-to-one, and τ is the canonical trace on $C_r^*(G/H) \times_{\alpha_{\dot{\chi}}} \mathbf{Z}$. In this case, Pimsner's computation [26, Theorem 5] shows that

$$\exp \circ \tau_*(K_0(C_r^*(G/H) \times_{\alpha_{\dot{\chi}}} \mathbf{Z})) = \dot{\chi}(G/H) = \chi(G).$$

Therefore $T(C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}) \subseteq \chi(G)$ and we are done. \square

Now we return to the C^* -crossed products $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$. From Theorem 3.3 we see that $\chi(F_n)$ is an isomorphism invariant for $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$. But even when $n = 1$, this invariant does not determine the crossed products. To completely classify all these $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$, we need a second invariant. Our definition is based on the following observation.

Consider the category \mathcal{C} of those unital C^* -algebras which have traces and all of whose traces agree at the K_0 -level. the morphisms in this category are unital $*$ -homomorphisms.

Examples of C^* -algebras in this category are ubiquitous. In addition to unital C^* -algebras with unique traces, and those crossed products of Corollary 2.9 and Theorem 2.11, examples include noncommutative tori [12]. We also have the following result.

Proposition 3.6. *Let X be a compact Hausdorff space. Then $C(X)$ is in the category C if and only if X is connected. \square*

Proposition 3.6 suggests that category C singles out those C^* -algebras which are noncommutatively connected. We will not however pursue this here.

A feature of the category C is that whenever $\varphi: A \rightarrow B$ is a morphism in this category, there exists a commutative diagram

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\varphi_*} & K_0(B) \\ & \searrow \tau_{A^*} & \swarrow \tau_{B^*} \\ & \mathbf{R} & \end{array}$$

where τ_A and τ_B are any two traces on A and B respectively. Furthermore, if H is any subgroup of \mathbf{R} , let

$$H(A) = \{z \in K_0(A) : \tau_{A^*}(z) \in H\}, \quad A \in C.$$

Then $H(\cdot)$ is a covariant functor from the category C to the category of abelian groups, and any morphism $\varphi: A \rightarrow B$ in C induces a commutative diagram:

$$\begin{array}{ccc} H(A) & \xrightarrow{\varphi_*} & H(B) \\ & \searrow \tau_{A^*} & \swarrow \tau_{B^*} \\ & H & \end{array}$$

Using this notion, we can give a very simple proof of the classification of rational rotation C^* -algebras (cf. [37]).

To classify $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$, we take $H = \mathbf{Q}$, the rational numbers, and write $\mathbf{Q}(A)$ for $H(A)$.

Definition 3.7. If A is a C^* -algebra in the category C and $[1]$ generates a free direct summand of $K_0(A)$, we define the twist of A , $t(A)$, by

$$t(A) = \begin{cases} 0, & \text{if } \mathbf{Q}(A) \not\simeq \mathbf{Z}^2, \\ \text{dist}(\tau_{A^*}(e), \mathbf{Z}), & \text{if } \mathbf{Q}(A) \simeq \mathbf{Z}^2, \end{cases}$$

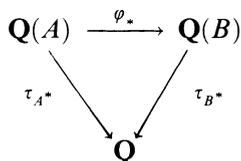
where e is the other generator of $\mathbf{Q}(A) \simeq \mathbf{Z}^2$ and dist denotes the usual distance on the real line \mathbf{R} .

Proposition 3.8. *The definition of the twist does not depend on the choice of the generator e , and the twist is an isomorphism invariant for those C^* -algebras A whose twists are defined.*

Proof. Assume e' and $[1]$ also generate the group $\mathbf{Q}(A) \simeq \mathbf{Z}^2$. Then $e = m[1] + ne'$ for some $m, n \in \mathbf{Z}$. Since e and $[1]$ generate \mathbf{Z}^2 , n must be $+1$ or -1 . Therefore $\tau_{A^*}(e) = \tau_{A^*}(m[1] \pm e') = m \pm \tau_{A^*}(e')$. This shows that $\text{dist}(\tau_{A^*}(e), \mathbf{Z}) = \text{dist}(\tau_{A^*}(e'), \mathbf{Z})$.

To prove the isomorphism invariance of $t(A)$, assume $\varphi: A \rightarrow B$ is an isomorphism. Then B is in the category C , and we have a commutative

diagram:



If $\mathbf{Q}(A) \not\cong \mathbf{Z}^2$, then $\mathbf{Q}(B) \not\cong \mathbf{Z}^2$ and so their twists are both zero. If $\mathbf{Q}(A) \cong \mathbf{Z}^2$ with generators $[1]$ and e , then $\mathbf{Q}(B) \cong \mathbf{Z}^2$ is generated by $[1] = \varphi_*([1])$ and $\varphi_*(e)$. Hence

$$t(B) = \text{dist}(\tau_{B^*}(\varphi_*(e)), \mathbf{Z}) = \text{dist}(\tau_{A^*}(e), \mathbf{Z}) = t(A). \quad \square$$

By Corollary 2.16, $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ is in the category \mathcal{C} , and by the proof of Theorem 3.3, $[1]$ generates a direct summand of $K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})$. Hence $t(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})$ is defined. Computation of the twist for arbitrary characters χ on F_n will have to be deferred to the next section. In the following, we only compute the twist for a special class of characters. However, this computation covers essentially all possibilities. This will become clear in the next section.

We say that the set of real numbers $\{\theta_0, \theta_1, \dots, \theta_k\}$ is \mathbf{Z} -linearly independent, if $M_0\theta_0 + \dots + M_k\theta_k = 0$, $M_i \in \mathbf{Z}$, $0 \leq i \leq k$, implies $M_0 = M_1 = \dots = M_k = 0$.

Proposition 3.9. *Let $\{g_1, \dots, g_n\}$ be free generators of F_n , $\chi \in \widehat{F}_n$, and $\chi(g_j) = e^{2\pi i\theta_j}$, $0 < \theta_j \leq 1$, $j = 1, \dots, n$. Suppose the set $\{1, \theta_1, \dots, \theta_k\}$ is \mathbf{Z} -linearly independent, $\theta_{k+1} = p/q$ with $(p, q) = 1$, $0 \leq p \leq [q/2]$, and $\theta_{k+2} = \dots = \theta_n = 0$. Then*

$$t(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) = \begin{cases} 0, & \text{if } k \neq n - 1, \\ p/q, & \text{if } k = n - 1. \end{cases}$$

Proof. From the proof of Theorem 3.3, we see that

$$K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) = [1]\mathbf{Z} \oplus [p_1]\mathbf{Z} \oplus \dots \oplus [p_n]\mathbf{Z}$$

with $\tau_*([p_j]) = \theta_j$, $1 \leq j \leq n$. Since $\{1, \theta_1, \dots, \theta_k\}$ is \mathbf{Z} -linearly independent,

$$\mathbf{Q}(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) = [1]\mathbf{Z} \oplus [p_{k+1}]\mathbf{Z} \oplus \dots \oplus [p_n]\mathbf{Z}.$$

If $k \neq n - 1$, then $\mathbf{Q}(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) \not\cong \mathbf{Z}^2$, and hence $t(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) = 0$. If $k = n - 1$, then

$$\mathbf{Q}(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) = [1]\mathbf{Z} \oplus [p_n]\mathbf{Z},$$

$$t(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) = \text{dist}(\tau_*([p_n]), \mathbf{Z}) = \text{dist}(p/q, \mathbf{Z}) = p/q. \quad \square$$

Corollary 3.10. *If A_θ is a rotation C^* -algebra with $0 \leq \theta \leq 1$, then*

$$t(A_\theta) = \begin{cases} 0, & \text{if } \theta \text{ is irrational;} \\ \min\{\theta, 1 - \theta\}, & \text{if } \theta \text{ is rational.} \end{cases} \quad \square$$

Remark. The twist defined in this section also works for some other C^* -algebras. For example, let H be the 3-dimensional discrete Heisenberg group, and let $C^*(H, \sigma)$ be the twisted group C^* -algebras associated with a 2-cocycle σ on H . Since generators of $K_0(C^*(H, \sigma))$ are in some rotation C^* -subalgebras of $C^*(H, \sigma)$, Remark 1 of Theorem 3.3 implies that all traces on $C^*(H, \sigma)$ agree on $K_0(C^*(H, \sigma))$. Hence we can define the twist for $C^*(H, \sigma)$ as in Definition 3.7. Then the two isomorphism invariants $t(\cdot)$ and $\exp \circ \tau_*(K_0(C^*(H, \sigma)))$ will completely classify $C^*(H, \sigma)$ for all σ up to $*$ -isomorphism. A classification theorem analogous to Theorem 5.1 can be proved. In her recent work, J. A. Packer [20] has completed the classification of these $C^*(H, \sigma)$, among many other results. Her methods are close to, but different from, ours.

4. ALGEBRAIC ASPECTS

We study the algebraic aspects of the C^* -algebras $C_r^*(G) \times_{\alpha_\chi} \mathbf{Z}$ in this section. Given a discrete group G , the automorphism group of G , $\text{Aut}(G)$, acts naturally on \widehat{G} , the set of all characters on G , via:

$$\varphi(\chi) = \chi \circ \varphi^{-1}, \quad \varphi \in \text{Aut}(G), \chi \in \widehat{G}.$$

We observe that if two characters χ_1 and χ_2 differ only by an automorphism, then the crossed products $C_r^*(G) \times_{\alpha_{\chi_1}} \mathbf{Z}$ and $C_r^*(G) \times_{\alpha_{\chi_2}} \mathbf{Z}$ are $*$ -isomorphic.

This naturally leads to the study of the orbits in \widehat{G} . For the purpose of classifying $C_r^*(\mathbf{Z}^n) \times_{\alpha_\chi} \mathbf{Z}$ and $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$, we restrict ourselves to the cases $G = \mathbf{Z}^n$ and $G = F_n$. We will show that, in these two cases, the orbits in \widehat{G} are completely classified by two invariants. One is the image $\chi(G)$, the other is a rational number in the interval $[0, \frac{1}{2}]$. To be consistent with §3, we call this rational number the twist of the character, and denote it $t(\cdot)$.

We start with the case $G = \mathbf{Z}^n$. Let χ be a character on \mathbf{Z}^n . First, we show that χ can be expressed as a matrix with integer entries, and that this matrix can be reduced to a standard form. We will then define the twist $t(\chi)$ of χ by its standard form, and show that $t(\chi)$ is invariant under the $\text{Aut}(G)$ -action. The classification of orbits in \widehat{G} then follows readily.

For $\chi \in \widehat{\mathbf{Z}^n}$, the image $\chi(\mathbf{Z}^n)$ is a finitely generated subgroup of T . Hence $\chi(\mathbf{Z}^n)$ is isomorphic to some \mathbf{Z}^k , $0 \leq k \leq n$, if it is torsion-free, or isomorphic to some $\mathbf{Z}^k \oplus \mathbf{Z}_q$, $0 \leq k \leq n - 1$, if it has torsion. Fix a basis $\{\lambda_1, \dots, \lambda_k\}$ for the subgroup of $\chi(\mathbf{Z}^n)$ which is isomorphic to \mathbf{Z}^k . The (finite) torsion subgroup \mathbf{Z}_q of $\chi(\mathbf{Z}^n)$ (if it exists) is

$$\{e^{2\pi ip/q} : 0 \leq p \leq q - 1\}.$$

Let $\lambda_{k+1} = e^{2\pi i/q}$ be the unique minimal generator of this cyclic group. Then the character χ , considered as a map from \mathbf{Z}^n to $\chi(\mathbf{Z}^n) = \mathbf{Z}^k$, or $\mathbf{Z}^k \oplus \mathbf{Z}_q$,

can be expressed in the form of a matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,n} \end{bmatrix},$$

where $a_{ij} \in \mathbf{Z}$ and the last row is determined only $\pmod q$. If $\chi(\mathbf{Z}^n)$ is torsion-free, the last row does not appear, but usually we will add this row to make the arguments complete.

Given $\varphi \in \text{Aut}(\mathbf{Z}^n) = \text{GL}(n, \mathbf{Z})$, φ acts on $\widehat{\mathbf{Z}^n}$ via $\varphi(\chi) = \chi \circ \varphi^{-1}$. This new character $\chi \circ \varphi^{-1}$ can be expressed by the matrix $(a_{ij})(t_{ij})$, where $(t_{ij}) \in \text{GL}(n, \mathbf{Z})$ and the multiplication is the usual matrix multiplication.

If two characters χ_1 and χ_2 are in the same orbit, then $\chi_1(\mathbf{Z}^n) = \chi_2(\mathbf{Z}^n)$. So we can use the same basis $\{\lambda_2, \dots, \lambda_k\}$ plus $\{\lambda_{k+1}\}$ for all characters in one orbit. The corresponding matrix expressions for all characters in the orbit then differ by an element in $\text{GL}(n, \mathbf{Z})$.

We now proceed to use multiplication by elements of $\text{GL}(n, \mathbf{Z})$, to reduce the matrix (a_{ij}) into a standard form.

Elementary matrix theory shows that multiplication by elements in $\text{GL}(n, \mathbf{Z})$ on the right permits us to perform the following three types of operations on (a_{ij}) :

- (i) interchange two columns;
- (ii) add an integral multiple of one column to another column;
- (iii) change the sign of every entry in a column.

Since \mathbf{Z} is a Euclidean domain, standard argument with the above three types of operations enables us to reduce (a_{ij}) to the form

$$\begin{bmatrix} r_1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & r_k & 0 & 0 & \cdots & 0 \\ * & \cdots & * & r_{k+1} & 0 & \cdots & 0 \end{bmatrix},$$

where $r_i \in \mathbf{Z}$, $r_i \geq 0$, $i = 1, 2, \dots, k + 1$.

A special condition which the matrices above must satisfy is that these matrices represent surjective homomorphisms from \mathbf{Z}^n to $\chi(\mathbf{Z}^n) = \mathbf{Z}^k$ or $\mathbf{Z}^k \oplus \mathbf{Z}_q$. therefore we get $r_1 = r_2 = \cdots = r_k = 1$ and r_{k+1} a generator of \mathbf{Z}_q .

Further application of matrix operations of type (ii) enable us to reduce (a_{ij}) into the form

$$\begin{bmatrix} 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ d_1 & \cdots & d_k & r_{k+1} & 0 & \cdots & 0 \end{bmatrix},$$

where $d_i \equiv 0 \pmod q$, $1 \leq i \leq k$.

Now consider the following three cases.

(i) $\chi(\mathbf{Z}^n)$ is torsion-free. In this case the last row of the above matrix does not appear. So (a_{ij}) has the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}$$

(ii) $\chi(\mathbf{Z}^n)$ has torsion, but $k \leq n-2$. In this case, since r_{k+1} is a generator of \mathbf{Z}_q , we can find $a, b \in \mathbf{Z}$ such that $ar_{k+1} + bq = 1$. Then

$$\begin{bmatrix} I_k & a & -q & & & & \\ & b & r_{k+1} & & & & \\ & & & I_{n-k-2} & & & \end{bmatrix} \in \text{GL}(n, \mathbf{Z}),$$

and

$$\begin{aligned} & \begin{bmatrix} & & & 0 & 0 & 0 & \cdots & 0 \\ & I_k & & \vdots & \vdots & \vdots & & \vdots \\ & & & 0 & 0 & 0 & & 0 \\ d_1 & \cdots & d_k & r_{k+1} & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} I_k & & & & & & & \\ & a & -q & & & & & \\ & b & r_{k+1} & & & & & \\ & & & I_{n-k-2} & & & & \end{bmatrix} \\ & = \begin{bmatrix} & & & 0 & 0 & 0 & \cdots & 0 \\ & I_k & & \vdots & \vdots & \vdots & & \vdots \\ & & & 0 & 0 & 0 & \cdots & 0 \\ d_1 & \cdots & d_k & ar_{k+1} & -qr_{k+1} & 0 & \cdots & 0 \end{bmatrix}. \end{aligned}$$

Since the last row of the above matrix is only determined mod q , we see that (a_{ij}) has been reduced to the form

$$\begin{bmatrix} I_k & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

(iii) $\chi(\mathbf{Z}^n)$ has torsion and $k = n-1$. In this case, by changing the sign of r_{k+1} if necessary, we can make $r_{k+1} \equiv p \pmod{q}$, with $(p, q) = 1$ and $1 \leq p \leq [q/2]$. After mod q in the last row, (a_{ij}) has the form

$$\begin{bmatrix} I_{n-1} & 0 \\ 0 & p \end{bmatrix}.$$

The above argument almost contains a proof of the following lemma.

Lemma 4.1. *If $(a_{ij})_{(k+1) \times n}$ represents a character χ from \mathbf{Z}^n onto $\chi(\mathbf{Z}^n) = \mathbf{Z}^k$ or $\mathbf{Z}^k \oplus \mathbf{Z}_q$ in the manner described above, then there exists a matrix $(t_{ij}) \in \text{GL}(n, \mathbf{Z})$ such that*

$$(a_{ij})(t_{ij}) = \begin{pmatrix} I_k & 0 & 0 \\ 0 & d_{k+1} & 0 \end{pmatrix},$$

where the last row is given up to mod q , and where

(i) the last row does not appear if $\chi(\mathbf{Z}^n)$ is torsion-free;

- (ii) $d_{k+1} \equiv 1 \pmod{q}$, if $\chi(\mathbf{Z}^n)$ has torsion and $k \leq n - 2$;
- (iii) $d_{k+1} \equiv p \pmod{q}$ with $(p, q) = 1$ and $1 \leq p \leq [q/2]$, if $\chi(\mathbf{Z}^n)$ has torsion and $k = n - 1$.

Moreover, the number p in case (iii) is independent of the choice of the matrix (t_{ij}) and of the choice of the basis $\{\lambda_1, \dots, \lambda_k\}$ in $\chi(\mathbf{Z}^n)$.

We call the matrix resulting from this lemma the *standard form* of the character χ .

Proof. The existence of the matrix (t_{ij}) has been justified.

Now assume that there are (t_{ij}) and (s_{ij}) in $\text{GL}(n, \mathbf{Z})$ such that

$$(a_{ij})(t_{ij}) = \begin{pmatrix} I_{n-1} & 0 \\ 0 & p \end{pmatrix}$$

and

$$(a_{ij})(s_{ij}) = \begin{pmatrix} I_{n-1} & 0 \\ 0 & p' \end{pmatrix},$$

with $(q, p) = 1 = (p', q)$, $1 \leq p \leq [q/2]$, $1 \leq p' \leq [q/2]$. Then we can find $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GL}(n, \mathbf{Z})$ such that

$$\begin{pmatrix} I_{n-1} & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & p' \end{pmatrix}.$$

Taking determinants of both sides of the above equality, we get $p' = \pm p$. Since $0 < p, p'$, it follows that $p = p'$. Hence p does not depend on the choice of the matrix (t_{ij}) .

If we change the basis $\{\lambda_1, \dots, \lambda_k\}$ to a new basis $\{\lambda'_1, \dots, \lambda'_k\}$ by a $k \times k$ matrix (d_{ij}) , then it is straightforward to check that χ is now expressed by the matrix

$$\begin{pmatrix} D^T & 0 \\ 0 & 1 \end{pmatrix} \cdot (a_{ij}),$$

where D^T stands for the transpose of (d_{ij}) . Assume this change of basis takes the standard form of χ from

$$\begin{pmatrix} I_{n-1} & 0 \\ 0 & p \end{pmatrix} \text{ into } \begin{pmatrix} I_{n-1} & 0 \\ 0 & p' \end{pmatrix}.$$

Then we get

$$\begin{pmatrix} D^T & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & p' \end{pmatrix}.$$

So we still have $p = p'$. \square

As a consequence of Lemma 4.1, we have the following corollary. Here $\{e_1, \dots, e_n\}$ is the canonical basis of \mathbf{Z}^n .

Corollary 4.2. *Let χ be a character on \mathbf{Z}^n such that $\chi(\mathbf{Z}^n)$ has free rank k . Then given any k real numbers $\theta_1, \dots, \theta_k$ such that $e^{2\pi i\theta_j} \in \chi(\mathbf{Z}^n)$, $1 \leq j \leq k$, and $\{1, \theta_1, \dots, \theta_k\}$ are \mathbf{Z} -linearly independent, we can find $\varphi \in \text{GL}(n, \mathbf{Z})$ such that $\chi(\varphi(e_j)) = e^{2\pi i\theta_j}$, $1 \leq j \leq n$, $\theta_{k+1} = p/q$, $(p, q) = 1$, $0 \leq p \leq [q/2]$, and $\theta_{k+2} = \dots = \theta_n = 0$. Moreover, the number p/q is independent of the choice of φ and of the choice of $\{\theta_1, \dots, \theta_k\}$. If $\chi(\mathbf{Z}^n)$ is torsion-free, then $p/q = 0$; if $\chi(\mathbf{Z}^n)$ has torsion, then q is the order of the torsion subgroup of $\chi(\mathbf{Z}^n)$, and $p/q = 1/q$ when $k \leq n - 2$. \square*

Now we introduce the following definitions.

Definition 4.3. Let χ be a character on \mathbf{Z}^n . Define its twist by

$$t(\chi) = \begin{cases} 0, & \text{if the free rank of } \chi(\mathbf{Z}^n) \text{ is not } n - 1; \\ p/q, & \text{if the free rank of } \chi(\mathbf{Z}^n) \text{ is } n - 1; \end{cases}$$

where p/q is the number derived in Corollary 4.2.

Definition 4.4. Let χ be a character on F_n . Define its twist by means of $t(\chi) = t(\dot{\chi})$, where $\dot{\chi}$ is the quotient character on \mathbf{Z}^n induced from χ .

Theorem 4.5. *Let $G = \mathbf{Z}^n$ or F_n . Then two characters χ_1 and χ_2 on G are in the same orbit under the $\text{Aut}(G)$ -action if and only if $\chi_1(G) = \chi_2(G)$ and $t(\chi_1) = t(\chi_2)$.*

Proof. (Necessity) Assume $\chi_1 = \chi_2 \circ \varphi$ for some $\varphi \in \text{Aut}(G)$. It is obvious that $\chi_1(G) = \chi_2(G)$. When $G = \mathbf{Z}^n$, $t(\chi_1) = t(\chi_2)$ follows from Corollary 4.2. When $G = F_n$, $\chi_1 = \chi_2 \circ \varphi$ implies $\dot{\chi}_1 = \dot{\chi}_2 \circ \dot{\varphi}$, where $\dot{\varphi} \in \text{GL}(n, \mathbf{Z})$ is induced from φ . Hence $t(\chi_1) = t(\dot{\chi}_1) = t(\dot{\chi}_2) = t(\chi_2)$.

(Sufficiency) We first consider the case $G = \mathbf{Z}^n$. After choosing a basis for $\chi_1(\mathbf{Z}^n) = \chi_2(\mathbf{Z}^n)$, we can reduce χ_1 and χ_2 into their standard forms by Corollary 4.2. Since χ_1 and χ_2 have the same twists, their standard forms are the same. Hence $\chi_2 = \chi_1 \circ \varphi$ for some $\varphi \in \text{GL}(n, \mathbf{Z})$. For the case $G = F_n$, since $\chi_i(F_n) = \dot{\chi}_i(\mathbf{Z}^n)$ and $t(\chi_i) = t(\dot{\chi}_i)$, $i = 1, 2$, we can find some $\varphi \in \text{GL}(n, \mathbf{Z})$ such that $\dot{\chi}_1 = \dot{\chi}_2 \circ \varphi$. It is well known that every automorphism on \mathbf{Z}^n lifts to an automorphism on F_n (cf.[5]). Hence the lifting of φ to $\text{Aut}(F_n)$ takes χ_1 to χ_2 . \square

5. THE CLASSIFICATION OF $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$

In this section we prove our main classification theorem as stated in the introduction. We express the isomorphism conditions for $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ in terms of its K theoretic invariants, of the character χ , and of the *-automorphism α_χ . The theorem says that the isomorphism classes of $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$, the orbits of χ and the conjugacy classes of α_χ are in one-to-one correspondence. It also says that outer conjugacy, conjugacy and algebraic conjugacy of α_χ are identical

relations on the $*$ -automorphisms α_χ . We also prove analogous theorems for the group F_∞ and for the full group C^* -algebra $C^*(F_n)$.

For the convenience of the reader, we restate the classification theorem below, where τ is the canonical trace and $t(\cdot)$ is the twist defined for $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ in 3.7 and for χ in 4.3.

Theorem 5.1. *Let χ_1 and χ_2 be two characters on F_n . Then the following are equivalent:*

- (i) $C_r^*(F_n) \times_{\alpha_{\chi_1}} \mathbf{Z} \simeq C_r^*(F_n) \times_{\alpha_{\chi_2}} \mathbf{Z}$;
- (ii) $\tau_{1*}(K_0(C_r^*(F_n) \times_{\alpha_{\chi_1}} \mathbf{Z})) = \tau_{2*}(K_0(C_r^*(F_n) \times_{\alpha_{\chi_2}} \mathbf{Z}))$ and $t(C_r^*(F_n) \times_{\alpha_{\chi_1}} \mathbf{Z}) = t(C_r^*(F_n) \times_{\alpha_{\chi_2}} \mathbf{Z})$;
- (iii) $\chi_1(F_n) = \chi_2(F_n)$ and $t(\chi_1) = t(\chi_2)$;
- (iv) $\chi_1 = \chi_2 \circ \varphi$ for some $\varphi \in \text{Aut}(F_n)$;
- (v) α_{χ_1} and α_{χ_2} are conjugate in $\text{Aut}(C_r^*(F_n))$;
- (vi) α_{χ_1} and α_{χ_2} are outer conjugate in $\text{Aut}(C_r^*(F_n))$.

Proof. (i) \Rightarrow (ii). By Corollary 2.16, all traces of $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ agree at the K_0 -level. Hence $\tau_*(K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}))$ is an isomorphism invariant. By Proposition 3.8, $t(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})$ is also an isomorphism invariant.

(ii) \Rightarrow (iii). By Theorem 3.3, $\exp \circ \tau_*(K_0(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})) = \chi(F_n)$. In Proposition 5.2 below, we will prove $t(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) = t(\chi)$. These two results show that (ii) \Rightarrow (iii).

(iii) \Leftrightarrow (iv). This is just Theorem 4.5.

Since the proof of (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i) works for any discrete group G , we use G , instead of F_n , in the following proof.

(iv) \Rightarrow (v). Assume $\chi_1 = \chi_2 \circ \varphi$ for some $\varphi \in \text{Aut}(G)$. We can define a unitary operator V_φ on the Hilbert space $l^2(G)$ by $V_\varphi(f_g) = f_{\varphi(g)}$, $g \in G$. Then $\text{ad } V_\varphi$ is the unique $*$ -automorphism α_φ on $C_r^*(G)$ such that $\alpha_\varphi(U_g) = U_{\varphi(g)}$, $g \in G$. It is easy to verify that $\alpha_{\chi_1} = \alpha_{\varphi^{-1}} \circ \alpha_{\chi_2} \circ \alpha_\varphi$.

(v) \Rightarrow (vi). Obvious.

(vi) \Rightarrow (i). Assume $\alpha_{\chi_1} = \text{ad } U \circ (\beta^{-1} \circ \alpha_{\chi_2} \circ \beta)$, where U is unitary in $C_r^*(G)$ and $\beta \in \text{Aut}(C_r^*(G))$. Then α_{χ_1} and $\beta^{-1} \circ \alpha_{\chi_2} \circ \beta$ are exterior equivalent, and $\beta^{-1} \circ \alpha_{\chi_2} \circ \beta$ and α_{χ_2} are conjugate. Hence crossed products of \mathbf{Z} by α_{χ_1} and by α_{χ_2} are isomorphic.

To complete the proof, it remains to finish the proof that (ii) implies (iii). This is shown in the following proposition.

Proposition 5.2. $t(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) = t(\chi)$.

Proof. Let $\dot{\chi}$ denote the quotient character on \mathbf{Z}^n . By Corollary 4.2 we can find some $\dot{\varphi} \in \text{GL}(n, \mathbf{Z})$ such that $\dot{\chi} \circ \dot{\varphi}$ is in its standard form. Lift $\dot{\varphi}$ to some $\varphi \in \text{Aut}(F_n)$ and let $\chi' = \chi \circ \varphi$. Since the twist of character is invariant

under the $\text{Aut}(F_n)$ -action, we have $t(\chi') = t(\chi)$. By the proof of (iv) \Rightarrow (i) in Theorem 5.1, we have $C_r^*(F_n) \times_{\alpha_{\chi'}} \mathbf{Z} = C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$. Hence

$$t(C_r^*(F_n) \times_{\alpha_{\chi'}} \mathbf{Z}) = t(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z})$$

by Proposition 3.8. Since χ' is in its standard form, $t(C_r^*(F_n) \times_{\alpha_{\chi'}} \mathbf{Z})$ is computed in Proposition 3.9. Comparing that computation with the definition of $t(\chi')$ in 4.3, we see that $t(C_r^*(F_n) \times_{\alpha_{\chi'}} \mathbf{Z}) = t(\chi')$. It follows that $t(C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) = t(\chi)$. \square

Now we prove a similar result for F_∞ , the free group with countably many generators.

Theorem 5.3. $C_r^*(F_\infty) \times_{\alpha_{\chi_1}} \mathbf{Z} \simeq C_r^*(F_\infty) \times_{\alpha_{\chi_2}} \mathbf{Z}$ if and only if $\chi_1 = \chi_2 \circ \varphi$ for some $\varphi \in \text{Aut}(F_\infty)$.

Proof. Assume $\varphi: C_r^*(F_\infty) \times_{\alpha_{\chi_1}} \mathbf{Z} \rightarrow C_r^*(F_\infty) \times_{\alpha_{\chi_2}} \mathbf{Z}$ is an isomorphism. Since in Corollary 2.16 we do not require F_∞ be finitely generated, that corollary applies to F_∞ , that is, all traces on $C_r^*(F_\infty) \times_{\alpha_\chi} \mathbf{Z}$ agree at the K_0 -level. Hence the isomorphism φ induces a commutative diagram:

$$\begin{array}{ccc} K_0(C_r^*(F_\infty) \times_{\alpha_{\chi_1}} \mathbf{Z}) & \xrightarrow{\varphi_*} & K_0(C_r^*(F_\infty) \times_{\alpha_{\chi_2}} \mathbf{Z}) \\ & \searrow \tau_{1*} & \swarrow \tau_{2*} \\ & \mathbf{R} & \end{array}$$

By [29], $K_0(C_r^*(F_\infty)) \simeq \mathbf{Z}$ is generated by [1], and $K_1(C_r^*(F_\infty))$ is free abelian with generators $[U_g]$, where g runs through a basis X of F_∞ . In other words, $F_\infty/[F_\infty, F_\infty] \rightarrow K_1(C_r^*(F_\infty))$, $[g] \mapsto [U_g]$, $g \in F_\infty$, is an isomorphism. From computations as in the proof of Theorem 3.3, it follows that

$$K_0(C_r^*(F_\infty) \times_{\alpha_\chi} \mathbf{Z}) = [1]\mathbf{Z} \oplus \left(\bigoplus_{g \in X} [P_g]\mathbf{Z} \right),$$

where the P_g 's, $g \in X$, are Rieffel projections or Bott projections, satisfying $\exp \circ \tau(P_g) = \chi(g)$, $g \in X$. Since $\varphi_*([1]) = [1]$ and $\exp \circ \tau_*([1]) = 1$, we obtain the following commutative diagram:

$$\begin{array}{ccc} K_0(C_r^*(F_\infty) \times_{\alpha_{\chi_1}} \mathbf{Z})/[1]\mathbf{Z} & \xrightarrow{\varphi_*} & K_0(C_r^*(F_\infty) \times_{\alpha_{\chi_2}} \mathbf{Z})/[1]\mathbf{Z} \\ & \searrow \exp \circ \tau_{1*} & \swarrow \exp \circ \tau_{2*} \\ & T & \end{array}$$

However, $K_0(C_r^*(F_\infty) \times_{\alpha_\chi} \mathbf{Z})/[1]\mathbf{Z}$ can be identified with $F_\infty/[F_\infty, F_\infty]$ via

$$K_0(C_r^*(F_\infty) \times_{\alpha_\chi} \mathbf{Z})/[1]\mathbf{Z} \simeq K_1(C_r^*(F_\infty)) \simeq F_\infty/[F_\infty, F_\infty],$$

and under this identification the map $\exp \circ \tau_*$ is just the quotient character $\dot{\chi}$. Therefore the following diagram is commutative:

$$\begin{array}{ccc} F_\infty/[F_\infty, F_\infty] & \xrightarrow{\varphi_*} & F_\infty/[F_\infty, F_\infty] \\ & \searrow \dot{\chi}_1 & \swarrow \dot{\chi}_2 \\ & T & \end{array}$$

Since automorphisms on $F_\infty/[F_\infty, F_\infty]$ lift to automorphisms on F_∞ (see [18]), we get a commutative diagram

$$\begin{array}{ccc} F_\infty & \xrightarrow{\hat{\varphi}_*} & F_\infty \\ & \searrow \chi_1 & \swarrow \chi_2 \\ & T & \end{array}$$

where $\hat{\varphi}_* \in \text{Aut}(F_\infty)$ is the lifting of φ_* .

The proof of the converse is the same as that in Theorem 5.1. \square

In the following we briefly discuss the situation that the $*$ -automorphism α_χ acts on the full group C^* -algebra, $C^*(G)$, instead of on the reduced group C^* -algebra $C_r^*(G)$. By the universal property of the full group C^* -algebra $C^*(G)$, it is easy to see that there exists a unique $*$ -automorphism α_χ on $C^*(G)$ such that $\alpha_\chi(U_g) = \chi(g)U_g$, where U is the universal representation of G . Then we form the C^* -crossed product $C^*(G) \times_{\alpha_\chi} \mathbf{Z}$. The proof of Theorem 3.5 actually proves the following result.

Theorem 5.4. *For any discrete group G and any character χ on G , we have $T(C^*(G) \times_{\alpha_\chi} \mathbf{Z}) = \chi(G)$.*

Proof. If we replace the reduced group C^* -algebras by the full group C^* -algebras (and hence drop the assumption that G be amenable) in the proofs of Lemma 3.1, Lemma 3.2 and Theorem 3.5, all the arguments still hold. \square

Now consider $G = F_n$. Since $C_r^*(F_n)$ is K -amenable [6] and the canonical map $\pi: C^*(F_n) \rightarrow C_r^*(F_n)$ is compatible with the action α_χ , $C^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ and $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ behave the same from the K -theoretic point of view. The classification theorem for $C_r^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ is proved essentially by K -theoretic arguments, hence we can also prove the following theorem.

Theorem 5.5. *The statement of Theorem 5.1 still holds if $C_r^*(F_n)$ is replaced by $C^*(F_n)$.*

Proof. Since $K_0(C^*(F_n)) \simeq \mathbf{Z}$ with generator [1] and since $K_1(C^*(F_n)) \simeq \mathbf{Z}^n$ with generators $[U_{g_1}], \dots, [U_{g_n}]$, where $\{g_1, \dots, g_n\}$ are free generators of F_n , we prove, as in the proof of Theorem 3.3, that $K_0(C^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) \simeq \mathbf{Z}^{n+1}$ with generators $[1], [p_1], \dots, [p_n]$, where each projection p_j is in some

rotation C^* -subalgebra. By Remark 1 after Theorem 3.3, we see that all traces on $C^*(F_n) \times_{\alpha_\chi} \mathbf{Z}$ agree at the K_0 -level. Hence we can define the twist $t(C^*(F_n) \times_{\alpha_\chi} \mathbf{Z})$, and do the same computation as in Proposition 3.9. Thus $t(C^*(F_n) \times_{\alpha_\chi} \mathbf{Z}) = t(\chi)$ still results as in Proposition 5.2. These arguments together with Theorem 5.4 prove the implications (i) \Rightarrow (ii) \Rightarrow (iii) of Theorem 5.1. Other implications are obvious. \square

6. CLASSIFICATION OF OTHER C^* -CROSSED PRODUCTS

In this section various classification theorems, including a complete classification of $C^*(\mathbf{Z}^2) \times_{\alpha_\chi} \mathbf{Z}$, are proved.

Let G be a discrete group. We say a subset E of \widehat{G} has the property $(*)$ if $\chi_1 \in E$, $\chi_2 \in E$ and $\chi_1(G) = \chi_2(G)$ together imply the existence of an automorphism φ of G such that $\chi_1 = \chi_2 \circ \varphi$.

Proposition 6.1. *Let G be a discrete amenable group. If χ_1, χ_2 are in a subset of E of \widehat{G} with the property $(*)$, then $C_r^*(G) \times_{\alpha_{\chi_1}} \mathbf{Z} \simeq C_r^*(G) \times_{\alpha_{\chi_2}} \mathbf{Z}$ if and only if $\chi_1(G) = \chi_2(G)$.*

Proof. This follows from Theorem 3.5. \square

The following result is due to Riedel [31] when G is infinite, and due to Ghatage and Phillips [40] when G is a torsion group.

Corollary 6.2. *Let G be a discrete abelian group. If $\chi_1, \chi_2 \in \widehat{G}$ are one-to-one, then*

$$C^*(G) \times_{\alpha_{\chi_1}} \mathbf{Z} \simeq C^*(G) \times_{\alpha_{\chi_2}} \mathbf{Z} \text{ iff } \chi_1(G) = \chi_2(G).$$

Proof. The characters χ_1 and χ_2 are in a subset E of \widehat{G} with the property $(*)$. \square

Corollary 6.3. *If χ_1 and χ_2 are characters on \mathbf{Z}^n satisfying the conditions that $\chi_i(G)$, $i = 1, 2$, is torsion-free or its free rank is not $n - 1$, then*

$$C^*(\mathbf{Z}^n) \times_{\alpha_{\chi_1}} \mathbf{Z} \simeq C^*(\mathbf{Z}^n) \times_{\alpha_{\chi_2}} \mathbf{Z} \text{ iff } \chi_1(\mathbf{Z}^n) = \chi_2(\mathbf{Z}^n).$$

Proof. The conditions imposed to χ_i are just that the twist $t(\chi_1) = 0 = t(\chi_2)$. It follows from Theorem 4.5 that χ_1 and χ_2 are in a subset E of $\widehat{\mathbf{Z}^n}$ with property $(*)$. \square

Corollary 6.4. *Let χ_1 and χ_2 be any two characters on \mathbf{Z}^n . Then*

$$(C^*(\mathbf{Z}^n) \times_{\alpha_{\chi_1}} \mathbf{Z}) \otimes C(T) \simeq (C^*(\mathbf{Z}^n) \times_{\alpha_{\chi_2}} \mathbf{Z}) \otimes C(T) \text{ iff } \chi_1(\mathbf{Z}^n) = \chi_2(\mathbf{Z}^n).$$

Proof. Since $C(T) \simeq C^*(\mathbf{Z})$, we have

$$(C^*(\mathbf{Z}^n) \times_{\alpha_\chi} \mathbf{Z}) \otimes C(T) \simeq C^*(\mathbf{Z}^{n+1}) \times_{\alpha_{\tilde{\chi}}} \mathbf{Z},$$

where $\tilde{\chi}$ is the trivial extension of χ to \mathbf{Z}^{n+1} : $\tilde{\chi}(\sum_1^{n+1} a_i e_i) = \chi(\sum_1^n a_i e_i)$. Then $\tilde{\chi}$ satisfies the conditions of Corollary 6.3. \square

Propositions 6.5. *Let χ_1 and χ_2 be any two characters on a finite abelian group G . Then*

$$C^*(G) \times_{\alpha_{\chi_1}} \mathbf{Z} \simeq C^*(G) \times_{\alpha_{\chi_2}} \mathbf{Z} \text{ iff } \chi_1(G) = \chi_2(G).$$

Proof. First assume $\chi_1(G) = \chi_2(G)$. We want to show that the two $*$ -automorphisms α_{χ_1} and α_{χ_2} are conjugate in $\text{Aut}(C^*(G))$. This will imply $C^*(G) \times_{\alpha_{\chi_1}} \mathbf{Z} \simeq C^*(G) \times_{\alpha_{\chi_2}} \mathbf{Z}$. By duality theory $C^*(G) \simeq C(\widehat{G})$, the C^* -algebra of continuous functions on G . Since G is a finite abelian group, \widehat{G} , as a topological space, is merely a finite, discrete set of points. Hence a $*$ -automorphism Φ on $C^*(G)$ comes from a bijection φ on \widehat{G} . Since α_{χ_1} and α_{χ_2} correspond to translations by χ_1 and χ_2 on the group \widehat{G} , we only need to define a bijection φ on \widehat{G} such that $\varphi \circ \chi_1 = \chi_2 \circ \varphi$, where $\chi_i: \widehat{G} \rightarrow \widehat{G}$ is the translation defined by $\chi_i(\rho) = \rho\chi_i$, $\rho \in \widehat{G}$. By assumption, $\chi_1(G) = \chi_2(G)$. Hence χ_1 and χ_2 have the same order k in the group \widehat{G} . Now pick up any two points ρ, η in \widehat{G} . The χ_1 -orbit of ρ is $\{\rho, \rho\chi_1, \rho\chi_1^2, \dots, \rho\chi_1^{k-1}\}$, and the χ_2 -orbit of η is $\{\eta, \eta\chi_2, \eta\chi_2^2, \dots, \eta\chi_2^{k-1}\}$. We define $\varphi(\rho\chi_1^i) = \eta\chi_2^i$, $0 \leq i \leq k-1$. Then if ρ' in \widehat{G} is such that $\varphi(\rho')$ has not been defined yet, and η' in \widehat{G} is such that η' is not yet in the range of φ , we define $\varphi(\rho'\chi_1^i) = \eta'\chi_2^i$, $0 \leq i \leq k-1$. Continue this procedure until all the points in \widehat{G} are in the domain of φ . It is obvious that φ is a bijection and $\varphi \circ \chi_1 = \chi_2 \circ \varphi$ holds. Therefore α_{χ_1} and α_{χ_2} are conjugate in $\text{Aut}(C^*(G))$ and we are done. The converse follows from Theorem 3.5. \square

Remark 1. The invariant $\chi(G)$ in this proposition can be recovered by using classical C^* -algebras theory.

Remark 2. If $G = \mathbf{Z}_n$, then $\chi_1(\mathbf{Z}_n) = \chi_2(\mathbf{Z}_n)$ implies $\chi_1 = \chi_2 \circ \varphi$ for some $\varphi \in \text{Aut}(\mathbf{Z}_n)$. This is not true in general, however. For example, let $G = \mathbf{Z}_2 \oplus \mathbf{Z}_4$, and define $\chi_1, \chi_2 \in \widehat{G}$ by $\chi_1((1, 0)) = 1$, $\chi_1((0, 1)) = -1$, and $\chi_2((1, 0)) = -1$, $\chi_2((0, 1)) = 1$. We leave it to the reader to check that $\chi_1 \neq \chi_2 \circ \varphi$ for any $\varphi \in \text{Aut}(G)$.

Theorem 6.6. *Let χ_1 and χ_2 be any two characters on \mathbf{Z}^2 . The following holds: $C^*(\mathbf{Z}^2) \times_{\alpha_{\chi_1}} \mathbf{Z} \simeq C^*(\mathbf{Z}^2) \times_{\alpha_{\chi_2}} \mathbf{Z}$ iff $\chi_1(\mathbf{Z}^2) = \chi_2(\mathbf{Z}^2)$.*

Proof. (Necessity) This is a direct consequence of Theorem 3.5.

(Sufficiency) First we consider χ_1 and χ_2 as characters on \mathbf{Z}^n with $n \geq 2$. If χ_1 and χ_2 have the same twists, then $\chi_1 = \chi_2 \circ \varphi$ for some $\varphi \in \text{GL}(n, \mathbf{Z})$ by Theorem 4.5, and hence α_{χ_1} and α_{χ_2} are conjugate in $\text{Aut}(C^*(\mathbf{Z}^n))$. This implies $C^*(\mathbf{Z}^n) \times_{\alpha_{\chi_1}} \mathbf{Z} \simeq C^*(\mathbf{Z}^n) \times_{\alpha_{\chi_2}} \mathbf{Z}$. So we can assume $t(\chi_1) \neq t(\chi_2)$. Therefore, at least one of $t(\chi_1)$ and $t(\chi_2)$ is not zero. Without loss of generality, we assume $t(\chi_1) = p_1/q$, $(p_1, q) = 1$, $1 \leq p_1 \leq [q/2]$. By the definition of the twist, this implies that $\chi_1(\mathbf{Z}^n)$ has torsion, its torsion subgroup has order q ,

and its free rank is $n - 1$. Since $\chi_2(\mathbf{Z}^n) = \chi_1(\mathbf{Z}^n)$, $\chi_2(\mathbf{Z}^n)$ also has torsion, its torsion subgroup also has order q , and its free rank is also $n - 1$. Therefore $t(\chi_2) = p_2/q$, $(p_2, q) = 1$, $1 \leq p_2 \leq [q/2]$. Now choose a basis $\{\lambda_1, \dots, \lambda_{n-1}\}$ for the free part of $\chi_1(\mathbf{Z}^n) = \chi_2(\mathbf{Z}^n)$. By Corollary 4.2, we can find $\varphi_1, \varphi_2 \in \text{GL}(n, \mathbf{Z})$ such that $\chi_1(\varphi_1(e_j)) = \lambda_j = \chi_2(\varphi_2(e_j))$, $1 \leq j \leq n - 1$, $\chi_1(\varphi_1(e_n)) = e^{2\pi i p_1/q}$ and $\chi_2(\varphi_2(e_n)) = e^{2\pi i p_2/q}$, where $\{e_1, \dots, e_n\}$ is the canonical basis of \mathbf{Z}^n . Since crossed products by α_{χ_i} and by $\alpha_{\chi_i \circ \varphi_i}$ are isomorphic, without loss of generality we can assume that $\chi_1(e_j) = \lambda_j = \chi_2(e_j)$, $1 \leq j \leq n - 1$, $\chi_1(e_n) = e^{2\pi i p_1/q}$, and $\chi_2(e_n) = e^{2\pi i p_2/q}$. The crossed product $C^*(\mathbf{Z}^n) \times_{\alpha_\chi} \mathbf{Z}$ is the universal C^* -algebra generated by unitaries U_1, \dots, U_n and W such that $U_i U_j = U_j U_i$, $W U_j = \lambda_j U_j W$, $1 \leq j \leq n - 1$, and $W U_n = e^{2\pi i p_1/q} U_n W$. The C^* -subalgebra generated by U_1, \dots, U_{n-1} and W can be identified with $C^*(\mathbf{Z}^{n-1}) \times_{\alpha_\chi} \mathbf{Z}$, and $\text{ad } U_n$ acts on this C^* -subalgebra as $*$ -automorphism β :

$$\beta(U_j) = U_j, \quad 1 \leq j \leq n - 1, \quad \beta(W) = e^{2\pi i p_1/q} W.$$

It is easy to see that $C^*(\mathbf{Z}^n) \times_{\alpha_\chi} \mathbf{Z} \simeq (C^*(\mathbf{Z}^{n-1}) \times_{\alpha_\chi} \mathbf{Z}) \times_\beta \mathbf{Z}$. To show $C^*(\mathbf{Z}^n) \times_{\alpha_{\chi_1}} \mathbf{Z} \simeq C^*(\mathbf{Z}^n) \times_{\alpha_{\chi_2}} \mathbf{Z}$, it is then enough to show that $(C^*(\mathbf{Z}^{n-1}) \times_{\alpha_{\chi_1}} \mathbf{Z}) \times_{\beta_1} \mathbf{Z} \simeq (C^*(\mathbf{Z}^{n-1}) \times_{\alpha_{\chi_2}} \mathbf{Z}) \times_{\beta_2} \mathbf{Z}$. Since χ_1 and χ_2 agree on \mathbf{Z}^{n-1} (in the decomposition $\mathbf{Z}^n = \mathbf{Z}^{n-1} \oplus \mathbf{Z}$), it is enough to show that β_1 and β_2 are conjugate in $\text{Aut}(C^*(\mathbf{Z}^{n-1}) \times_{\alpha_\chi} \mathbf{Z})$, where χ is the common restriction of χ_1 and χ_2 to \mathbf{Z}^{n-1} .

Now we assume $n = 2$. In this case $C^*(\mathbf{Z}^{n-1}) \times_{\alpha_\chi} \mathbf{Z}$ is just an irrational rotation C^* -algebra. It is well known [36, 41] that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbf{Z})$, then $U \rightarrow U^a W^c$, $W \rightarrow U^b W^d$ gives arise to a $*$ -automorphism of the irrational rotation C^* -algebras. Since $(p_1, q) = 1 = (p_2, q)$, We can find x, y, z, w in \mathbf{Z} such that $p_1 x + q y = 1 = p_2 z + q w$. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p_2 x & -y - p_1 x w \\ q & p_1 z \end{pmatrix}.$$

Its determinant is $p_1 z p_2 x + q(y + p_1 x w) = p_1 x(1 - q w) + q y + q p_1 x w = p_1 x - q p_1 x w + q y + q p_1 x w = p_1 x + q y = 1$. Hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbf{Z})$. It is easy to verify that the $*$ -automorphism φ , arising from $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, satisfies the condition $\varphi \circ \beta_1 = \beta_2 \circ \varphi$. In fact,

$$\begin{aligned} \varphi \circ \beta_1(U) &= \varphi(U) = U^a W^c, \\ \beta_2 \circ \varphi(U) &= \beta_2(U^a W^c) = U^a \cdot e^{-2\pi i p_2 c/q} W^c = U^a W^c \quad \text{since } c = q, \\ \varphi \circ \beta_1(W) &= \varphi(e^{-2\pi i p_1/q} W) = e^{-2\pi i p_1/q} U^b W^d, \\ \beta_2 \circ \varphi(W) &= \beta_2(U^b W^d) = U^b \cdot e^{-2\pi i p_2 d/q} W^d. \end{aligned}$$

Since $d = p_1 z$, we have

$$\frac{p_2 d}{q} = \frac{p_2 z p_1}{q} = \frac{(1 - qw)p_1}{q} = \frac{p_1}{q} - p_1 w.$$

Hence $\varphi \circ \beta_1 = \beta_2 \circ \varphi$ holds. This completes the proof. \square

Remark. It is shown in [38], by using determinate theory of C^* -algebras applied to nonstable K -theory, that if G is a torsion-free discrete abelian group and if χ_1 and χ_2 are two characters on G , then α_{χ_1} and α_{χ_2} are conjugate in $\text{Aut}(C^*(G))$ iff $\chi_1 = \chi_2 \circ \varphi$ for some $\varphi \in \text{Aut}(G)$. This is why we show in the proof of the above theorem that β_1 and β_2 are conjugate instead of working on α_{χ_i} , $i = 1, 2$.

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