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A CONTINUOUS LOCALIZATION AND COMPLETION

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Dedicated to Professor Hirosi Toda on his 60th birthday

ABSTRACT. The main goal of this paper is to construct a localization and completion of Bousfield-Kan type as a continuous functor for a virtually nilpotent CW-complex. Then the localization and completion of an A_n-space is given to be an A_n-homomorphism between A_n-spaces. For any general compact Lie group, this gives a continuous equivariant localization and completion for a virtually nilpotent G-CW-complex. More generally, we have a continuous localization with respect to a system of core rings for a virtually nilpotent **D**-CW-complex for a polyhedral category **D**.

The simplicial construction of Bousfield and Kan [2] automatically generalizes to the homology localization of a **D**-CW-complex [6] for a discrete category **D**, including the equivariant case for finite groups. But, this fails for general compact Lie groups.

A. Elmendorf [8] constructs functorially an equivariant Eilenberg-Mac Lane space by using May's method and the iteration of a bar construction. Using this, May et al. [16, 17] show the existence of an equivariant localization and completion using Theorems 3 and 4 of [8] for nilpotent G-CW-complexes, which is given with the Arithmetic Square Theorem for nilpotent G-spaces. On the other hand, T. Sumi [24] gives an equivariant localization of a 1-connected G-CW-complex with respect to a system of local rings by using the method of [20]. But, the functors are not continuous.

In this paper, we construct a generalized Eilenberg-Mac Lane space $R(X)$ by using the symmetric product [4]. Then by the methods of a triple (an algebra functor [1]) and a cosimplicial space, we construct a nilpotent tower, and a completion and a localization as continuous functors. Using this, we show the Arithmetic Square Theorem for a virtually nilpotent CW-complex (see Dror-Dwyer-Kan [5]).

As the localization is continuous, the localization of an A_n-space (mapping) is an A_n-space (mapping) by the explicit definition of an A_n-space in Stasheff [22] and an A_n-mapping between A_n-spaces in [12, 13].

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Also we have an equivariant Eilenberg-Mac Lane space by using the method of the generalized Bar construction in [8, 18] with respect to a system of core rings. This gives an equivariant localization and completion with the Arithmetic Square Theorem for a virtually nilpotent G-CW-complex in a slightly general situation in §4.

In §1, we study the properties of a homology functor $R(X)$ as a continuous functor. In §2, we prepare the notion of a cosimplicial space, and in §3, we define the continuous localization and completion. In §4, we discuss a **D**-CW-complex version which includes the equivariant case.

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1. Construction of the continuous functor $R(\ )$

Because the category of all topological spaces is a *large* category, we need to treat large categories. So, we adopt weaker definitions for a large topological category and a continuous functor between them. More precisely, we say that a large category is topological if the set of morphisms between any two objects is topological and the composition is continuous. Also we say that a functor between large topological categories is continuous if it induces a continuous mapping between their morphism spaces.

We always assume, in this section, that a space X is a CW-complex with base point $*$ and that a mapping preserves base points. Also we assume that a discrete abelian group R is a space with the neutral element 0 as the base point. We also denote by 0 the base point of $R \wedge X$. We define $R(X)$ to be the inductive limit of an identification space of the symmetric products $\mathrm{SP}^n(R \wedge X)$ in [4].

Definition 1.1. Let $R_m(X)$ be the identification space of $\coprod_{n<m}(R \wedge X)^n$ by the coordinate transformation of symmetric groups Σ_n (of n letters) and the equivalence relations $\sim$:

$$(r_1 \wedge x_1, \dots, r_n \wedge x_n) \sim (r_1 \wedge x_1, \dots, r_n \wedge x_n, 0)$$

and

$$\begin{aligned}&(r_1 \wedge x_1, \dots, r_{j-1} \wedge x_{j-1}, r_j \wedge x, r_{j+1} \wedge x, r_{j+2} \wedge x_{j+2}, \dots, r_n \wedge x_n)\\ &\qquad \sim (r_1 \wedge x_1, \dots, r_{j-1} \wedge x_{j-1}, (r_j + r_{j+1}) \wedge x, r_{j+2} \wedge x_{j+2}, \dots, r_n \wedge x_n).\end{aligned}$$

Let $R(X)$ be the inductive limit of $R_m(X)$.

Remark 1.2. (1) G. Segal has constructed the above space $R(X^+)$ more generally in [21] by what we call the Segal machinery.

(2) $R(X)$ may be regarded as the identification space of $\mathrm{SP}(R \wedge X)$ with the relation $r_1 \wedge x + r_2 \wedge x = (r_1 + r_2) \wedge x$.

To show the detailed properties about the functor $R(\ )$, we need the precise description of $R(X)$. We can say that $R(X)$ is an infinite sum of R's indexed by the set $X - \{*\}$. Actually, an element of $R(X)$ is described as the following form:

$$\sum_{i\,:\,\text{finite}} r_i x_i = \text{ the class of } (r_1 \wedge x_1, \dots, r_n \wedge x_n, \dots). \tag{1.3}$$

Also we define $R(f)$ by

$$R(f)\left(\sum_{i\,:\,\text{finite}} r_i x_i\right) = \sum_{i\,:\,\text{finite}} r_i f(x_i). \tag{1.4}$$

Proposition 1.5. *If (X, A) is a CW-pair, then so is $(R(X), R(A))$, and $R(\ )$ is a continuous functor.*

Proof. Under the assumption that X is a CW-complex, we see that X^n is a Σ_n-equivariant CW-complex. Let $X^{\langle n\rangle}$ be the subspace of all points whose orbit is Σ_n/H for some subgroup $H \neq \{e\}$ of Σ_n. Then we may assume $(X^n, X^{\langle n\rangle})$ is a Σ_n-equivariant CW-pair and also $(A^n, A^{\langle n\rangle})$ is a Σ_n-equivariant sub-CW-pair. We remark that the Σ_n-equivariant CW-pair

$$(R^n \times X^n, R^n \times (X^{\langle n\rangle} \cup X^{[n]}) \cup R^{[n]} \times X^n)$$

is relatively a Σ_n-covering of the pair $(R_n(X), R_{n-1}(X))$, where $Y^{[n]}$ denotes the union of subspaces $Y^i \times \{*\} \times Y^{n-i-1}$. By the induction on n, we have that $(R(X)_n, R(X)_{n-1})$ is a CW-pair and also that $(R(A)_n, R(A)_{n-1})$ is a sub-CW-pair. By Steenrod [23], $R(X)$ is hence a CW-complex with this weak topology. To complete the proof of this proposition, it is sufficient to show the following lemma.

Lemma 1.6. *The functor $R(\ )$ induces a continuous mapping R:* $\mathrm{Map}_*(X, Y) \to \mathrm{Map}_*(R(X), R(Y))$.

Proof. It is sufficient to show that the adjoint mapping

$$\mathrm{ad}(R)\colon R(X) \times \mathrm{Map}_*(X, Y) \to R(Y)$$

is continuous. The mapping is described as $\mathrm{ad}(R)(w, f) = R(f)(w)$. Since the image of $R_n(X) \times \mathrm{Map}_*(X, Y)$ is in $R_n(Y)$, we can consider the following commutative diagram:

$$\begin{array}{ccc}
(R \wedge X)^n \times \mathrm{Map}_*(X, Y) & \longrightarrow & (R \wedge Y)^n \\
\downarrow & & \downarrow \\
R_n(X) \times \mathrm{Map}_*(X, Y) & \longrightarrow & R_n(Y),
\end{array}$$

where the upper mapping of the diagram is induced from the evaluation, the vertical mappings are the identification, and the lower mapping is the restriction of $\mathrm{ad}(R)$. Then all the mappings with the lower one removed are continuous and the left mapping is an identification mapping. Therefore, the bottom mapping must be continuous. On the other hand, the topology of $R(X)$ is the weak topology filtered by the $R_n(X)$'s. Therefore, $\mathrm{ad}(R)$ is continuous. This implies the lemma and completes the proof of Proposition 1.5. □

By the definition of $R(X)$, we may make the following

Remark 1.7. (1) $\pi_0(R_m(X)) = \{\sum_{i=1}^k r_i\alpha_i;\ 0 \neq r_i \in R,\ * \neq \alpha_i \in \pi_0(X),\ k \leq m\}$ and, hence, $\pi_0(R(X)) = \overline{H}_0(X; R)$ and $\mathrm{Hom}_R(R(X), R) = \overline{H}^0(X, R)$.

(2) If $R = \mathbf{Z}/m\mathbf{Z}$ or $\mathbf{Z}$, then $R(X)$ is homeomorphic with $AG(X, 0; m)$ or $AG(X, 0)$ as in Dold-Thom [4], by the definition.

(3) A homomorphism $h: R \to R'$ induces a natural transformation $h(X): R(X) \to R'(X)$.

(4) When R is a unit ring, $R(X)$ is homeomorphic to $\overline{R}(X) = \{w \in R(X^+);\ \varepsilon(w) = 1\}$, where ε is induced from the trivial mapping $X \to \{*\}$ (see [2]). But the latter space fails to have natural group structure without assuming the existence of a base point.

2. Algebra functor

In this section we further assume that the abelian group R is a ring with unit 1.

Definition 2.1. The action $\overline{m}$ of R on $\mathrm{SP}_n(R \wedge X)$ is defined by the formula

$$\overline{m}\left(r, \sum\langle r_i, a_i\rangle\right) = \sum\langle rr_i, a_i\rangle.$$

By the definition, $\overline{m}$ induces a well-defined continuous mapping $m: R \times R(X) \to R(X)$. Then m is an action of R on $R(X)$ and $R(X)$ is a free R-module. Moreover, the action m induces a mapping $m': R \wedge R(X) \to R(X)$ and hence a natural mapping $\mu_X: R(R(X)) \to R(X)$ given by

$$\mu_X\left(\sum_i \left[r_i, \sum_j [s_j, a_{ij}]\right]\right) = \sum_{i,j} [r_i s_j, a_{ij}]. \tag{2.2}$$

On the other hand, the inclusion $X = \{1\} \times X \subseteq R \wedge X \subseteq \mathrm{SP}(R \wedge X)$ induces a natural mapping $\eta_X: X \to R(X)$ defined by

$$\eta_X(a) = [1, a]. \tag{2.3}$$

Hence we obtain the following

Proposition 2.4. *The functor $R(\ )$ together with the above two natural transformations μ and η is a triple or an algebra functor (see Adams* [1]).

Remark. As in Remark 1.7(4), the above algebra functor $R(\ )$ is naturally equivalent to $\overline{R}(\ )$ for a CW-complex with base point.

Let us recall that a cosimplicial resolution is automatically obtained by an algebra functor (see [2]). Actually, $R(\ )$ induces the following cosimplicial space functor $R^{\cdot}(\ )$:

$$(R^{\cdot}(X))^n = R^{n+1}(X), \quad d^i = R^i\eta R^{n-i}(X), \quad s^j = R^j\mu R^{n-j}(X), \tag{2.5}$$

where $R^k(\ )$ is the composition of k copies of $R(\ )$. Then $(R^{\cdot}(X))^n$ is a space and the cosimplicial identities hold.

Therefore, the triple $(\{(R^{\cdot}(X))^n,\ n \geq 0\},\ \{d^i: (R^{\cdot}(X))^{n-1} \to (R^{\cdot}(X))^n,\ 0 \leq i \leq n\},\ \{s^j: (R^{\cdot}(X))^{n+1} \to (R^{\cdot}(X))^n,\ 0 \leq j \leq n\})$ is a cosimplicial space.

We define as in [11] the R-completion $R_\infty(X)$ and the R-nilpotent tower $\{R_s(X)\}$ for X by the total spaces $\mathrm{tot}(R^{\cdot}(X))$ and $\{\mathrm{tot}_s(R^{\cdot}(X))\}$, $s \geq 0$, of $R^{\cdot}(X)$, respectively:

$$\begin{aligned} \mathrm{tot}(R^{\cdot}(X)) &= \{\{f^n: \Delta^n \to (R^{\cdot}(X))^n\} | d^i f^n = f^{n+1} d^i,\ s^j f^n = f^{n-1} s^j\} \\ &\subseteq \prod_{n>-1} \mathrm{Map}(\Delta^n, (R^{\cdot}(X))^n), \end{aligned} \tag{2.6}$$

$$\begin{aligned} \mathrm{tot}_s(R^{\cdot}(X)) &= \{\{f^n: \Delta^{[s]n} \to (R^{\cdot}(X))^n\} | d^i f^n = f^{n+1} d^i,\ s^j f^n = f^{n-1} s^j\} \\ &\subseteq \prod_{n>-1} \mathrm{Map}(\Delta^{[s]n}, (R^{\cdot}(X))^n), \end{aligned} \tag{2.7}$$

with the sequences of the constant mappings as the base points.

If X is a good CW-complex, i.e., $H_*(X, R) \cong H_*(R_\infty(X), R)$, and $\{R_s(X)\}$ is an R-nilpotent tower for X in [2], then $R_\infty(X)$ is an R-completion of X. This will be seen in §3.

Remark. $R_s(X)$ is weakly equivalent to the realization of $\mathrm{tot}_s(R^{\cdot}(\mathrm{Sing}(X)))$, where Sing means the functor taking the singular simplicial set of a space X.

3. Continuous localization and completion

In this section, we study the homotopy properties of $R(\ )$ and define a continuous localization of a CW-complex. First, we show

Proposition 3.1. *If (X, A) is a CW-pair, then $R(q): R(X) \to R(X/A)$ is a fiber bundle with fiber $R(A)$, where $q: X \to X/A$ is the contraction.*

Proof. It is sufficient to show the existence of the local cross-section on the neighborhood of the neutral element 0. By the assumption, there is a deformation of the identity $h: [0, 1] \times X \to X$ such that the restriction of h to $[0, 1] \times A$ is pr_A and $\mathrm{ad}(h)(1)$ sends a neighborhood of A into A. So we get a homotopy $h_1 = \mathrm{ad}((R\wedge\ )\mathrm{ad}(h)): [0, 1] \times R \wedge X \to R \wedge X$. We construct a sequence of mappings $h_n: [0, 1] \times R_n(X) \to R_n(X)$ by induction.

From the observations in Proposition 1.5, it follows that the pair $(R^n \times X^n, R^n \times (X_A^{[n]} \cup X^{\langle n\rangle}) \cup R^{[n]} \times X^n)$ is a Σ_n-equivariant CW-complex, where $X_A^{[n]}$ is the union of subspaces $X^i \times A \times X^{n-i}$, and hence has an equivariant homotopy extension property. Hence we can take a Σ_n-equivariant deformation $\overline{h}_n: [0, 1] \times R^n \times X^n \to R_n(X)$ of the canonical projection whose restrictions to $[0, 1] \times R^n \times X^{\langle n\rangle}$ and to $[0, 1] \times R^n \times X^i \times A \times X^{n-i}$ are the compositions of h_{n-1} and of $(\mathrm{id} \times h_{n-1})$ with the appropriate identifications to $R_{n-1}(X)$ and to $(R \wedge A) \times R_{n-1}(X)$, given in Proposition 1.5, and $\mathrm{ad}(\overline{h}_n)(1)$ sends an equivariant neighborhood of A^n into $R_n(A)$. Thus we obtain a deformation h_n of the identity of $R_n(X)$ whose restrictions to $R_{n-1}(X)$ and to $[0, 1] \times R_n(A)$ are h_{n-1} and the projection to $R_n(A)$ respectively, and $\mathrm{ad}(h_n)(1)$ sends a neighborhood of $R_n(A)$ into $R_n(A)$ where the restriction to $R_{n-1}(X)$ is that of $R_{n-1}(A)$ for h_{n-1}.

Therefore, the sequence $\{h_n\}$ gives a deformation $h' : [0, 1] \times R(X) \to R(X)$ such that $\mathrm{ad}(h')(t)$ is an $R(A)$-module mapping for all $t \in [0, 1]$ where restriction of h' to $[0, 1] \times R(A)$ is $\mathrm{pr}_{R(A)}$, and $\mathrm{ad}(h')(1)$ sends a neighborhood of $R(A)$ into $R(A)$. Hence the mapping $S: R(X/A) \to R(X)$ defined by

$$S([w]) = w - h'(1, w) \quad \text{for } w \in R(X) \tag{3.2}$$

gives a continuous local cross-section, which is easily verified by using the similar procedure given in Lemma 1.6. □

Corollary 3.3. *Let X be a CW-complex with base point and $X^+ = X \amalg \{p\}$. Then $R(X^+) \cong R \times R(X)$. In particular, $\pi_q(R(X^+)) \cong \pi_q(R(X))$ for $q > 0$, and $\pi_0(R(X^+)) \cong R \oplus \pi_0(R(X))$.*

Proof. Take the canonical inclusion $S^0 = \{*\}^+ \hookrightarrow X^+$ and the projection $X^+ \to X$, which induces a fiber bundle $R(X^+) \to R(X)$ with the fiber $R(S^0) \cong R$. On the other hand, there is a splitting $X^+ \to S^0$ of the inclusion. Therefore the bundle has a splitting and there is a homeomorphism $R(X^+) \cong R \times R(X)$. This implies the corollary. □

Hence we obtain the following (see Dold and Thom [4]).

Theorem 3.4. *Let X be a CW-complex. Then*

$$\pi_q R(X^+) \cong H_q(X; R) \quad \text{and} \quad \pi_q R(X) \cong \overline{H}_q(X; R).$$

Proof. Let us define an additive generalized homology theory h_* by the following formulas as a functor of the category of pairs of CW-complexes to the category of graded R-modules:

$$h_q(X) = \pi_q(R(X^+)), \qquad h_q(X, A) = \pi_q(R(X/A)).$$

To prove that h_* is an additive generalized homology theory, we need to show the homotopy axiom, the exact sequence axiom, the excision axiom, and the additivity axiom.

Homotopy axiom. Suppose f_0 and f_1 are homotopic mappings in $\mathrm{Map}_*(X, Y)$. Then there is a homotopy $f: [0, 1] \to \mathrm{Map}_*(X, Y)$ such that $f(0) = f_0$ and $f(1) = f_1$. By Lemma 1.6, $R(\ )$ induces a continuous mapping from $\mathrm{Map}_*(X, Y)$ to $\mathrm{Map}_*(R(X), R(Y))$, and $R \circ f$ gives a homotopy of $R(f_0)$ to $R(f_1)$. This implies that h_* satisfies the homotopy axiom.

Exact sequence axiom. Proposition 3.1 tells us that the functor h_* satisfies the exact sequence axiom.

Excision axiom. Assume that (X, A) is a CW-pair and an open set U satisfies $U \subseteq \mathrm{Interior}(A)$. Then $(X - U)/(A - U)$ is homeomorphic with X/A, and hence h_* satisfies the excision axiom.

Additivity axiom. Suppose that X is a wedge sum of X_a at the base point for all $a \in A$, where A is not necessarily finite. Then, as a topological abelian group, $R(X)$ is the direct sum of $R(X_a)$. Actually, the natural projections $q_a: X \to X_a$ and inclusions $j_a: X_a \to X$ induce the structural homomorphisms $R(q_a)$ and $R(j_a)$ of the direct sum decomposition of $R(X)$. These mappings also induce the direct sum decomposition of $h_*(X, \{*\})$. This implies that h_* satisfies the additivity axiom.

Hence, h_* is an additive generalized homology theory. Moreover, Corollary 3.3 tells us that h_* also satisfies the dimension axiom.

The theorem follows by Eilenberg-Steenrod [7] and Milnor [19]. □

Corollary 3.5 (J. C. Moore). *For a CW-complex X and a discrete abelian group R, $R(X)$ has the homotopy type of the generalized Eilenberg-Mac Lane complex $\prod_{i>0} K(\overline{H}_i(X, R), i)$.*

Together with Proposition 2.4, we have

Corollary 3.6. *If R is a discrete ring with unit, then there is a continuous algebra functor $R(\ )$ such that $R(X)$ is homotopy equivalent to $\prod_{i>0} K(\overline{H}_i(X, R), i)$.*

Let us recall the constructions (2.6) and (2.7) of the total space $R_s(X)$ of the simplicial space $R^{\bullet}(X)$ given in §2. By using a parallel argument to [2], we have the following

Theorem 3.7. *Let R be a core ring such as a subring of the field of rational numbers $\mathbf{Q}$ or $\mathbf{F}_p$, the prime field of characteristic p. If X is a good CW-complex, then $\{R_s(X)\}$ gives a nilpotent tower for X and hence gives a continuous localization (or completion) $\eta_X: X \to R_\infty X$ of Bousfield-Kan type.*

We will show this later in this section.

Remark. If, further, X is a G-space, then so is $R_\infty X$. But we do not know about the fixed points $(R_\infty X)^H$.

Corollary 3.8. *There is an associated unstable Adams spectral sequence of Bousfield-Kan type:*

$$E_2^{s,t}(X, Y) \cong \mathrm{Ext}^s_{\mathbf{CA}}(\overline{H}_*(\Sigma^t X; \mathbf{F}_p), \overline{H}_*(Y; \mathbf{F}_p))$$

and

$$E^{s,t}(X, Y) \cong \pi_{t-s}(\mathrm{Map}_*(X, Y), *).$$

Then from a result due to Dror, Dwyer, and Kan [5], the following follows.

Corollary 3.9 (Arithmetic Square Theorem). *There is the following continuous functor from the category of virtually nilpotent CW-complexes to the category of weak pull-back diagrams called the "arithmetic square," where a virtually nilpotent space means a base space of a finite covering space with a nilpotent total* *space*:

$$\begin{array}{ccc} \mathbf{Z}_\infty X & \longrightarrow & \prod(\mathbf{F}_p)_\infty X \\ \downarrow & & \downarrow \\ \mathbf{Q}_\infty X & \longrightarrow & \mathbf{Q}_\infty(\prod(\mathbf{F}_p)X) \end{array}$$

Let us turn our attention to A_n-spaces and A_n-mappings. By [22, 12, 13] the A_n-form of a space X or a mapping $f: X \to Y$ is given by a series of mappings $K_i \to \mathrm{Map}_*(X^i, X)$ or $J_i \to \mathrm{Map}_*(X^i, Y)$, where K_i is a complex isomorphic with $(i-2)$-disk and J_i is also a complex isomorphic with $(i-1)$-disk. On the other hand, the mapping $R(X) \times R(Y) \to R(X \times Y)$ given by

$$\left(\sum_i r_i x_i, \sum_j s_j y_j\right) \mapsto \sum_{i,j} r_i s_j (x_i, y_j)$$

gives rise to a natural transformation $R^j(X) \times R^j(Y) \to R^j(X \times Y)$ and $(R^j(X))^i \to R^j(X^i)$ by the similar argument given in the proof of Lemma 1.6. This gives a natural transformation $(R_\infty X)^i \to R_\infty(X^i)$.

Corollary 3.10. *Let X and Y be CW-complexes and $f: X \to Y$ a mapping. If X has an A_n-structure, so does $R_\infty X$ and the localization mapping of X to $R_\infty X$ strictly preserves the A_n-forms (an A_n-homomorphism, see* [22]*). If f is an A_n-mapping, so is $R_\infty f$. If f is an A_n-homomorphism, so is $R_\infty f$.*

Proof. R gives a continuous mapping $\mathrm{Map}_*(X^i, Y) \to \mathrm{Map}_*(R_\infty(X^i), R_\infty Y)$. Composing this with the mapping induced by composition with the mapping $(R_\infty X)^i \to R_\infty(X^i)$, we obtain the continuous mapping

$$\mathrm{Map}_*(X^i, Y) \to \mathrm{Map}_*(R_\infty(X^i), R_\infty Y) \to \mathrm{Map}_*((R_\infty X)^i, R_\infty Y).$$

By composing this mapping with the A_n-forms for X or f, we obtain the A_n-forms for $R_\infty X$ or $R_\infty f$. The latter part is a trivial consequence of the construction of A_n-forms. □

So we are left to show the proof of Theorem 3.7. We show that, by the assumption, $\{R_s(X)\}$ is an R-nilpotent tower for a CW-complex X in the sense of Bousfield and Kan [2]: The inclusion $\Delta^{[m-1]n} \to \Delta^{[m]n}$ induces a restriction

$$P_m: R_m(X) \to R_{m-1}(X), \qquad m > 0.$$

The inverse image of the constant mapping 0 in $R_m(X)$ by P_m is

$$\Omega^m N^m R^{\cdot}(X),$$

where $N^m R^{\cdot}(X) = \mathrm{Ker}\, S$ and the continuous mapping

$$S: R^{m+1}(X) \to \prod^m R^m(X)$$

is defined as a homomorphism by the formula $S(a) = (s^0(a), \ldots, s^{m-1}(a))$.

As is seen in May [15], S has a right inverse C which is described by a composition of continuous mappings and hence is continuous. By the fact that S is a continuous homomorphism, it follows that S is a fibration with a cross section C whose fiber is $N^m R^\cdot(X)$. In addition, we obtain

$$\pi_q(N^m R^\cdot(X)) = \pi_q(R^{m+1}(X)) \cap \operatorname{Ker} s^0 \cap \cdots \cap \operatorname{Ker} s^{m-1}$$

and $N^m R^\cdot(X)$ is a generalized Eilenberg-Mac Lane complex. On the other hand, for any space Y the restriction mapping

$$\operatorname{Map}(\Delta^m, Y) \to \operatorname{Map}(\Delta^{[m-1]m}, Y)$$

is a fibration. Hence one sees that the mapping

$$\begin{aligned} P\colon & \operatorname{Map}(\Delta^m, R^{m+1}(X)) \\ & \to \operatorname{Map}\left(\Delta^m, \prod^m R^m(X)\right) \times_B \operatorname{Map}(\Delta^{[m-1]m}, R^{m+1}(X)) \end{aligned}$$

with $B = \operatorname{Map}(\Delta^{[m-1]m}, \prod^m R^m(X))$ given by the formula $P(f) = (Sf, \overline{f})$ is a fibration with fiber $\Omega^m N^m R^\cdot(X)$, where $\overline{f}$ is the restriction of f to $\partial\Delta^m$ and $\times_B$ denotes the fiber product over B. Let us consider the following commutative diagram:

$$\begin{array}{ccc} R_m(X) & \xrightarrow{\quad F \quad} & \operatorname{Map}(\Delta^m, R^{m+1}(X)) \\ \downarrow{\scriptstyle P_m} & & \downarrow{\scriptstyle P} \\ R_{m-1}(X) & \xrightarrow{\quad G \quad} & \operatorname{Map}(\Delta^m, \prod^m R^m(X)) \times_B \operatorname{Map}(\Delta^{[m-1]m}, R^{m+1}(X)) \end{array}$$

where F and G are inclusions given by $F(\{f_j\}) = (f_m)$ and $G(\{g_j\}) = ((g_{m-1} \times \cdots \times g_{m-1})\overline{S}, g_m)$ and $\overline{S}\colon \Delta^m \to \prod^m \Delta^{m-1}$ is given by $\overline{S}(x) = (s^0(x), \ldots, s^{m-1}(x))$.

One can also see the image of F is just the inverse image by P of the image of G. Hence P_m is a fibration whose fiber is $P^{-1}(0, 0) = \Omega^m N^m R^\cdot(X)$. Clearly the fiber of P_m acts on the total space $R_m(X)$ and P_m is principal (see [11] for its classifying mapping). Thus $R_m(X)$ is an R-nilpotent space with the homotopy type of a CW-complex. So the arguments given in [2] show that $\{R_m(X)\}$ is an R-nilpotent tower for X, when R is a core ring, and the inverse limit $R_\infty X$ gives an R-completion (or localization) of X. This implies the theorem.

4. Equivariant homology and localization

From now on, we assume that G is a compact Lie group. Note that G/H and G/K are G-homeomorphic if H and K are conjugate subgroups in G. We fix a representative set F of the set of all G-homeomorphism classes of G-orbits to satisfy $H < K$ when there is a G-mapping from G/H to G/K, i.e., K includes a conjugate of H, while G/H and G/K are in F. We may regard F as a discrete set. A G-connected G-space X is called F-orbital if the G-orbit G/H of any point of X is G-homeomorphic with one of the elements of F. In this section, we also assume that a G-space is an F-orbital G-CW-complex with a base point $*$, and hence G/G is in F.

Let $\mathbf{O}_F$ be the full subcategory with objects in F of the topological category of all G-orbits and G-mappings. A continuous contravariant functor from $\mathbf{O}_F$ to a topological category is called an $\mathbf{O}_F$-object in the category. Note that, for any G-space X, one can take an associated $\mathbf{O}_F$-object $I^G(X)$ by putting $I^G(X)(G/H) = \mathrm{Map}^G(G/H, X) = X^H$ and $I^G(X)(f)(x) = xf$. We remark that, in [10], Illman shows that, for any closed subgroup H, X^H has a homotopy type of a CW-complex.

By using a generalized bar construction due to May [18], Elmendorf [8] shows

Theorem 4.1 (May [18], Elmendorf [8]). *There is a continuous functor C^G from the category of $\mathbf{O}_F$-spaces to the category of F-orbital G-spaces. Moreover, there are natural transformations $\eta\colon C^G I^G \to \mathrm{id}$ and $\varepsilon\colon I^G C^G \to \mathrm{id}$ such that for any F-orbital G-CW-complex X, the natural projection $\eta_X\colon C^G I^G(X) \to X$ is a G-equivalence and for any $\mathbf{O}_F$-space k, $\varepsilon_k\colon (C^G k)^H \to k(G/H)$ is the natural system of homotopy equivalent projections.*

We remark that our continuous functors R and R_∞ automatically give the generalized Eilenberg-Mac Lane complex and the localization (or completion) of an $\mathbf{O}_F$-object in the category of CW-complexes, by taking compositions with it.

Illman's equivariant homology [9] gives a functor to the category of abelian groups. On the other hand, Bredon's homology [3] is a functor from the category of pairs of an $\mathbf{O}_F$-abelian group and an $\mathbf{O}_F$-space to that of $\mathbf{O}_F$-abelian groups. We will construct a localization (completion) with respect to Bredon's homology.

Now let us introduce a slightly general notion, $\mathbf{D}$-space, due to Dror and Zabrodsky [6]. The category $\mathbf{D} = \mathbf{O}_F$ satisfies the following conditions:

(1) $\mathbf{D}$ is a small topological category and the space of all objects is discrete.
(2) The morphism space of any two objects is a finite polyhedron.
(3) $\mathbf{D}$ has the terminal object.

We will call such a category $\mathbf{D}$ a polyhedral category, and a contravariant functor from $\mathbf{D}$ to the category of topological spaces (rings, etc.) will be called a $\mathbf{D}$-space ($\mathbf{D}$-ring, etc.). In the remainder of this section, we work with $\mathbf{D}$-spaces rather than G-spaces.

A $\mathbf{D}$-CW-complex X is defined, in accordance with Dror-Zabrodsky [6], as:

(1) X has a weak topology with respect to its filtration $\{X_n\}$.

(2) X_{n+1} is obtained by attaching $(n+1)$-cells $B_a^{(\,)} \times D^{n+1}$ on X_n through natural transformations

$$h_a^n\colon B_a^{(\,)} \times S^n \to X_n,$$

where we denote by $B_a^{(\ )}$ the contravariant functor taking values as follows

$$B_a^{(A)} = \mathrm{Mor}_{\mathbf{D}}(A, B),$$
$$B_a^{(f)} = f^{\#}\colon \mathrm{Mor}_{\mathbf{D}}(A', B) \to \mathrm{Mor}_{\mathbf{D}}(A, B)$$

for $f\colon A \to A'$.

Remark 4.2. The above definition of a **D**-CW-complex when $\mathbf{D} = \mathbf{O}_F$ coincides with the definition of the G-CW-complex given in Matumoto [14].

Then a **D**-CW-complex satisfies the following condition:

(*) For any object A in $\mathbf{D}$, $X(A)$ has a filtration $\{X_n(A)\}$ and each $X_{n+1}(A)$ is obtained by attaching polyhedra $K_a(A)$ on $X_n(A)$ through mappings $h_a^n\colon L_a(A) \to X_n(A)$, where $L_a(A)$ is a subpolyhedron of $K_a(A)$.

By deforming the attaching mappings, we obtain

Proposition 4.3. *For a given* **D**-*space* X *with property* $(*)$, $X(A)$ *has a homotopy type of a CW-complex.*

Let $R = \{R^A, R^f\}$ be a **D**-abelian group and let X be connected, i.e., $X(A)$ is connected for all $A \in \mathbf{D}$. Also we need a base point in a **D**-space, that is, a natural inclusion in X of the trivial **D**-space $*$. Then the homology of Bredon's type for X can be defined as the following **D**-abelian group:

$$\overline{H}_*(X; R)^A = \overline{H}_*(X(A); R^A).$$

Let X be a connected **D**-CW-complex with base point. We define $R(X)$ as the **D**-space

$$R(X)(A) = R^A(X(A)), \qquad R(X)(f) = R^f(X(A)) \circ R^{A'}(X(f)),$$

where $f\colon A \to A'$ is a morphism in $\mathbf{D}$. Since an n-fold product of a polyhedron has a Σ_n-equivariant triangular decomposition, we can apply the same argument as in the proof of Proposition 1.5. Hence $R(X)$ is a **D**-space with the property $(*)$ above. Then by the proof of Theorem 3.4, we obtain

Proposition 4.4. $R(X)$ *is a* **D**-*space with the property* $(*)$, *and it satisfies*

$$\pi_q(R(X)(A)) = \overline{H}_q(X(A); R^A).$$

Hence, for the functor $R^G = C^G R I^G$, we obtain

Corollary 4.5. *Let* R *be an abelian group. Then* R^G *is a continuous functor from the category of* O_F-G-*CW-complexes to the category of* $\mathbf{O}_F$-*spaces. Moreover,* $\pi_i(R^G(X)^H) = \overline{H}_i(X^H; R)$ *for any* $G/H \in F$.

We introduce some notions for a **D**-space.

Definition 4.6. (1) A **D**-CW-complex X is said to be virtually nilpotent if $X(A)$ is virtually nilpotent for each object A in $\mathbf{D}$.

(2) A **D**-space X is said to be R-local if each $X(A)$ is R-local for each object A in $\mathbf{D}$.

(3) A natural transformation $X \to Y$ of **D**-spaces is said to be an R-localization if it is an R-homology equivalence and Y is R-local.

For an $\mathbf{O}_F$-CW-complex X, X is R-local in our sense if and only if X is equivariantly R-local in the ordinary sense (by Sumi [24, Theorem 3.3]). By Theorem 3.7, we obtain the following

Proposition 4.7. *Let X be a good nilpotent* **D**-*CW-complex, i.e., X^H is good for all A in $\mathbf{D}$. For any R-homology equivalence $f\colon Y \to Z$, the homotopy set of G-mappings from Y to $R_\infty X$ is in one-to-one correspondence with that from Z to $R_\infty X$. Hence $R_\infty X$ is an R-localization of X.*

Proof. It is a direct consequence of the fact that $R_s^A(X(A)) \to R_{s-1}^A(X(A))$ is a principal fibration for any $A \in \mathbf{D}$ whose fiber has the homotopy type of a connected generalized Eilenberg-Mac Lane complex. □

Theorem 4.8. *Let R be a* **D**-*core ring (a system of core rings) and let X be a virtually nilpotent* **D**-*CW-complex. Then there is an R-nilpotent tower $R_s(X)$ for X and hence an R-localization $R_\infty X$.*

Hence, for the functor $R_\infty^G = C^G R_\infty I^G$, we obtain

Corollary 4.9 (G-localization and G-completion). *Let R be an $\mathbf{O}_F$-core ring. Then $R^G(\ )$ is a continuous functor from the category of $\mathbf{O}_F$-CW-complexes to the category of R-local G-spaces. Moreover, $(R_\infty^G X)^H \simeq (R^{(G/H)})_\infty(X^H)$ for $G/H \in F$.*

By Corollary 3.9 we obtain

Theorem 4.10. *There is the following continuous functor from the category of virtually nilpotent* **D**-*CW-complexes to the category of the* **D**-*weak pull-back diagram of* **D**-*spaces: "***D**-*arithmetic square" among the localizations with respect to the constant coefficient rings $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{F}_p$:*

$$\begin{array}{ccc} \mathbf{Z}_\infty X & \longrightarrow & \prod(\mathbf{F}_p)_\infty X \\ \big\downarrow & & \big\downarrow \\ \mathbf{Q}_\infty X & \longrightarrow & \mathbf{Q}_\infty(\prod(\mathbf{F}_p)_\infty X) \end{array}$$

Corollary 4.11 (G-Arithmetic Square Theorem). *There is the following continuous functor from the category of virtually nilpotent $\mathbf{O}_F$-CW-complexes to the* *category of the $\mathbf{O}_F$-weak pull-back diagram of $\mathbf{O}_F$-spaces:*

$$\begin{array}{ccc} \mathbf{Z}^G_\infty X & \longrightarrow & \prod(\mathbf{F}_p)^G_\infty X \\ \downarrow & & \downarrow \\ \mathbf{Q}^G_\infty X & \longrightarrow & \mathbf{Q}^G_\infty(\prod(\mathbf{F}_p)^G_\infty X), \end{array}$$

where $\mathbf{O}_F$-weak pull-back means that the restriction of the diagram to the fixed point set by H is weak pull-back for any $G/H \in F$.

Corollary 4.12. *Let X and Y be $\mathbf{O}_F$-CW-complexes and $f\colon X \to Y$ an equivariant mapping. If X is an equivariant A_n-space, so is $R^G_\infty X$ and the R-localization $X \to R^G_\infty X$ is an equivariant A_n-mapping. If f is an equivariant A_n-mapping, so is $R^G_\infty f$.*

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