ENDOMORPHISM RINGS OF FORMAL $A_0$-MODULES

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ABSTRACT. Let $A_0$ be the valuation ring of a finite extension $K_0$ of $Q_p$ and $A \supset A_0$ be a complete discrete valuation ring with the perfect residue field. We consider the endomorphism rings of $n$-dimensional formal $A_0$-modules $\Gamma$ over $A$ of finite $A_0$-height with reduction absolutely simple up to isogeny. Especially we prove commutativity of $End^\Gamma_{A_0}(\Gamma)$. Given an arbitrary finite unramified extension $K_1$ of $K_0$, a variety of examples (different dimensions and different $A_0$-heights) is constructed whose absolute endomorphism rings are isomorphic to the valuation ring of $K_1$.

Let $K_0$ be a finite extension of $Q_p$ and $A_0$ the valuation ring of $K_0$; let $K \supset K_0$ be a complete discrete valuation field with the perfect residue field $k$ of characteristic $p > 0$; let $A$ be the valuation ring of $K$.

It is known that the fraction field of the endomorphism ring of a one-dimensional formal group of height $h$ over $A$ is a finite extension of $Q_p$ of degree dividing $h$ (cf. Lubin [7]).

In Theorem 1 and Proposition 2, we prove a higher-dimensional analogue of the above fact: if an $n$-dimensional formal $A_0$-module $\Gamma$ over $A$ satisfies an assumption that the reduction $\Gamma_k = \Gamma \otimes_A k$ of $\Gamma$ is an absolutely simple formal $A_0$-module up to isogeny and of finite $A_0$-height $h \geq n$ ($h$ is relatively prime to $n$), then we prove that the fraction field $\Lambda$ of the endomorphism ring of $\Gamma$ over $A$ as a formal $A_0$-module is a finite extension field of $K_0$ of degree dividing $h$ such that $e(\Lambda/K_0)$ divides $e(K/K_0)$ and that $f(\Lambda/K_0)$ divides $f(K/K_0)$ if $f(K/K_0)$ is finite.

In the corollary of Theorem 2, we give examples: for any positive integers $h$ and $n$ with $h \geq n + 1$ ($h \geq 1$ if $n = 1$) and for any positive divisor $g$ of $h$, there exists an $n$-dimensional formal $A_0$-module over $A_0$ of $A_0$-height $h$ whose absolute $A_0$-endomorphism ring is the valuation ring of the unramified extension of $K_0$ of degree $g$ (cf. Cox [1] and Yamasaki [11]).

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1. Notations

In this paper, a field means a commutative field and we use the following notations.

- $p$ = a prime number.
- $\mathbb{Z}_p$ = the ring of $p$-adic integers.
- $\mathbb{Q}_p$ = the field of $p$-adic numbers.
- $K_0$ = a finite extension of $\mathbb{Q}_p$ with the residue field $k_0$.
- $A_0$ = the valuation ring of $K_0$.
- $K (\supset K_0)$ = a complete discrete valuation field with the perfect residue field $k$ of characteristic $p > 0$.
- $\pi$ = a prime element of $K$.
- $A$ = the valuation ring of $K$.
- $\overline{k}$ = an algebraic closure of $k$.

For an extension $E/E'$ of complete discrete valuation fields,

- $e(E/E')$ = the ramification index of $E$ over $E'$.
- $f(E/E')$ = the residue degree of $E$ over $E'$.
- $M_f(T)$ = the total matrix ring of degree $t$ over a ring $T$.
- $(a, b)$ = the (positive) greatest common divisor of integers $a$ and $b$.

2. Endomorphism rings

In this paper, a formal group means a formal group law.

An $n$-dimensional formal $A_0$-module over a commutative $A_0$-algebra $S$ is an $n$-dimensional commutative formal group over $S$ such that there is an endomorphism $[a]$ of a formal group over $S$ for each $a \in A_0$ whose Jacobian matrix is $aI_n$ ($I_n$ = the unit matrix of degree $n$) and that $a \rightarrow [a]$ is a ring homomorphism. If a formal $A_0$-module is of height $H$ as a formal group, we define the $A_0$-height of $\Gamma$ as the number $H/[K_0 : \mathbb{Q}_p]$ (cf. [2], [3, III. 4.3, 5.5] and [5, V. 29.7.2]). A formal $A_0$-homomorphism over $S$ between formal $A_0$-modules is a homomorphism over $S$ of the formal groups which commutes with $[a]$ for all $a \in A_0$. We write $\text{End}_{S,A_0}(\Psi)$ the formal $A_0$-endomorphism ring of a formal $A_0$-module $\Psi$ over $S$. An isogeny over $S$ between formal $A_0$-modules is a formal $A_0$-homomorphism that, as a homomorphism of formal groups, is an isogeny.

Let $\Gamma$ be an $n$-dimensional formal $A_0$-module over $A$ of finite $A_0$-height. Let $\Gamma_k = \Gamma \otimes_A k$ be the formal $A_0$-module over $k$ obtained by reducing the coefficients of $\Gamma$ modulo the maximal ideal of $A$.

We put $\Lambda = Q_p \otimes \mathbb{Z}_p \text{End}_{A,A_0}(\Gamma)$. $\Lambda$ is a $K_0$-algebra. As $\Gamma$ is of finite $A_0$-height, we identify $\Lambda$ with its image in $Q_p \otimes \mathbb{Z}_p \text{End}_{k,A_0}(\Gamma_k)$ through reduction (cf. [5, IV.21.8.19]).

Let $\text{END}_{*,\Lambda_0}(\Gamma)$, the absolute $A_0$-endomorphism ring of $\Gamma$, be the union of $\text{End}_{B,A_0}(\Gamma \otimes_A B)$ where $B$ runs over all the valuation rings of finite extensions of $K$. ($\Gamma \otimes_A B$ is the scalar extension of $\Gamma$ to $B$.)
We assume,

(*) \( \Gamma_k \) is an \( n \)-dimensional formal \( A_0 \)-module such that the scalar extension \( \Gamma_{\overline{k}} = \Gamma_k \otimes_k \overline{k} \) of \( \Gamma_k \) to \( \overline{k} \) is simple as a formal \( A_0 \)-module up to isogeny (i.e. \( \Gamma_{\overline{k}} \) is absolutely simple up to isogeny) and of finite \( A_0 \)-height \( h \) (\( \geq n \)).

Remark 1. (i) \( (h, n) = 1 \) by [5, V.29.8.3] (cf. [3, III.4, Corollary 2 of Proposition 8]).

(ii) For examples of \( \Gamma \) and \( \Gamma_k \) satisfying (*), see §4 and [9, Proposition 5].

We put \( D = Q_p \otimes_{\mathbb{Z}_p} \text{End}_{A_0} (\Gamma_{\overline{k}}) \). By assumption (*), \( D \) is a division algebra over \( K_0 \). We put \( \Omega = \Lambda \otimes_{K_0} K \).

The following simple proof of Proposition 1 is due to the referee.

**Proposition 1.** Under our assumption (*), \( D \) is a central division algebra over \( K_0 \) of dimension \( h^2 \).

**Proof.** By [8, II] (or [5, V.28.5.9]), \( \Gamma_{\overline{k}} \) is isogeneous to a product \( \Gamma_1^i \times \Gamma_2^j \times \cdots \times \Gamma_m^m \), where each \( \Gamma_i \) is simple up to isogeny of finite height \( H_i \), \( \Gamma_i \) is not isogeneous to \( \Gamma_j \) for \( i \neq j \), and the decomposition is unique up to isogeny. All this is taking place in the category of formal groups, not of formal \( A_0 \)-modules. \( \Delta \) denotes the scalar extension \( Q_p \otimes_{\mathbb{Z}_p} \text{End}_{\overline{k}, A_0} (\Gamma_{\overline{k}}) \) of the endomorphism ring of \( \Gamma_{\overline{k}} \) as a formal group. It follows that \( \Delta \cong \bigoplus M_{t_i}(\Delta_i) \), where each \( \Delta_i \) is a central division algebra over \( Q_p \) of dimension \( H_i^2 \). From the definition of \( \Gamma \) as a formal \( A_0 \)-module, we see that \( K_0 \) injects into \( \Delta \) and that \( D \) is the commutant of \( K_0 \) in \( \Delta \). Since \( D \) is a division algebra, the center of \( \Delta \) must be a field. Thus \( \Gamma \) is isogeneous to \( \Gamma_1^i \) and \( \Delta \cong M_{t_i}(\Delta_i) \). From standard theorems about central division algebras, we see that \( D \) is a central division algebra over \( K_0 \) (double commutant theorem) and that

\[
\]

Thus we have

\[
[D : K_0] = \left( \frac{t_i H_i}{[K_0 : Q_p]} \right)^2 = h^2,
\]

where the last equality follows from the definition of \( A_0 \)-height.

Let \( L \) be the tangent space (or the Lie algebra) of the scalar extension \( \Gamma \otimes_A K \) of \( \Gamma \) to \( K \). Then \( L \) is an \( n \)-dimensional vector space over \( K \), a faithful \( \Lambda \)-module and a bimodule over \( \Lambda \) and \( K \). \( L \) is thus a nontrivial module over \( \Omega \) (cf. [5, II.14.2]).

Theorem 1 is a higher-dimensional analogue of [7, Theorem 2.3.2] (or [5, IV. 23.2.6]).
Theorem 1. Under our assumption (*), $\Lambda$ is a finite extension field of $K_0$.

Proof. By Proposition 1, the $K_0$-subalgebra $\Lambda$ of $D$ is a division algebra over $K_0$ of dimension dividing $h^2$. Let $Z$ be the center of $\Lambda$. Then we have $[\Lambda : Z] = h^2$ with $h'$ dividing $h$. $Z \otimes_{K_0} K$ is a finite direct sum of finite extensions $K_i$ ($\supset Z$) of $K$. Hence every minimal left ideal of $\Lambda \otimes_Z K_i$ has dimension over $K$ divisible by $h'$, and so does every minimal ideal of $\Omega \cong \Lambda \otimes_Z (Z \otimes_{K_0} K)$. $\Omega$ is a semisimple $K$-algebra. Since $L$ is a nontrivial $\Omega$-module of dimension $n$ over $K$, $h'$ divides $n$.

On the other hand, $(h, n) = 1$ and $h'$ divides $h$. Thus $h' = 1$ and $\Lambda$ is a field.

Remark 2. If $\Gamma_k$ is absolutely simple up to isogeny and of $A_0$-height $\infty$, then we have $\dim \Gamma = \dim \Gamma_k = \dim \Gamma_{\overline{k}} = 1$ (cf. [5, V.29.8.3]) and so $\text{End}_{A, A_0}(\Gamma)$ is commutative.

Proposition 2. Under our assumption (*), $[\Lambda : K_0]$ divides $h$. Furthermore $e(\Lambda/K_0)$ divides $e(K/K_0)$ and $f(\Lambda/K_0)$ divides $f(K/K_0)$ if $f(K/K_0)$ is finite.

Proof. By Theorem 1, $\Lambda$ ($\supset K_0$) is a subfield of $D$. Therefore, by Proposition 1, $[\Lambda : K_0]$ divides $h$.

Let $F$ be a minimal ideal of $\Omega$. By Theorem 1, $F$ is a composite of $K$ and $\Lambda$ over $K_0$. We have

$$e(F/\Lambda)e(\Lambda/K_0) = e(F/K)e(K/K_0).$$

Then $e(\Lambda/K_0)/(e(\Lambda/K_0), e(K/K_0))$ divides $e(F/K)$ and so $[F : K]$. Therefore $e(\Lambda/K_0)/(e(\Lambda/K_0), e(K/K_0))$ divides $n = [L : K]$, since $\Omega$ is semisimple and $L$ is a nontrivial $\Omega$-module.

On the other hand, $e(\Lambda/K_0)$ divides $[\Lambda : K_0]$ and so $h$. Hence, by $(h, n) = 1$, $e(\Lambda/K_0)$ divides $e(K/K_0)$.

For the residue degrees, the same argument holds if $f(K/K_0)$ is finite.

Corollary (of Theorem 1 and Proposition 2). Under our assumption (*), the absolute $A_0$-endomorphism ring of $\Gamma$ is commutative. Its fraction field is an extension of $K_0$ of degree dividing $h$ and has the ramification index dividing $e(K/K_0)$.

Proof. Let $K^*$ be the composite of $K$ and the fraction field of the Witt vector ring over $\overline{k}$. Let $A^*$ be the valuation ring of $K^*$. We remark $e(K^*/K_0) = e(K/K_0)$. By [10, Theorem 3.2] (or [5, IV.23.2.2]) and [5, IV.21.1.4, Remarks (ii)], $\text{END}_{A^*, A_0}(\Gamma)$ is contained in $\text{End}_{A^*, A_0}(\Gamma \otimes_A A^*)$. Hence our result follows from Theorem 1 and Proposition 2.

3. A LEMMA

Let $K'$ ($\subset K$) be the composite of $K_0$ and the fraction field (in $K$) of the Witt vector ring over $k$. Then $K$ is a totally ramified finite extension of $K'$ and
Let $e(K'/K_0) = 1$. Let $A'$ be the valuation ring of $K'$. Let $\tau'$ be the Frobenius of $K'$ over $K_0$ (i.e. the $K_0$-automorphism of $K'$ satisfying $a^{\tau'} \equiv a^{p/(K_0/Q_p)}$ (mod the maximal ideal of $A'$) for all $a \in A'$).

In §§3 and 4, we assume that there exists an extension $\tau$ of $\tau'$ to an automorphism of $K$.

We write $R$ the $A$-module of formal power series
\[ x = a_h t^h + a_{h+1} t^{h+1} + \cdots + a_n t^n + \cdots \]
in an indeterminant $t$, with coefficients $a_n \in A$, where the exponent $h$ is arbitrary. The $A$-module $R$ is made into a ring by the multiplication law
\[ (a^t)(b^t) = (ab^t)^{t+j} \quad \text{for all } a, b \in A. \]

We also write $A_1[[t]]$ the subring of $R$ with only terms of nonnegative exponents (cf. Hilbert-Witt ring and localized Hilbert-Witt ring in [3, III.4.1]).

**Lemma** (a generalization of the claim in [3, III.5, Proof of Theorem 3]). Let $x = \sum a_i t^i \in A_1[[t]]$ with $a_i \in A$ be such that $v(a_i) > 0$ for all $0 \leq i \leq s-1$ and $v(a_s) = 0$, where $v$ is the normalized discrete valuation of $K$ with $v(\pi) = 1$. Suppose that $u = b_0 + b_1 t + b_2 t^2 + \cdots + b_{s-1} t^{s-1}$ with $b_i \in A$ belongs to the left $A_1[[t]]$-ideal $A_1[[t]]x$ generated by $x$. Then we have $u = 0$, i.e. $b_i = 0$ for all $0 \leq i \leq s-1$.

**Proof.** We remark that $v$ is invariant under $\tau$. We take $\sum c_i t^i \in A_1[[t]]$ such that
\[ \sum b_i t^i = \left( \sum c_i t^i \right) \left( \sum a_j t^j \right) = \sum c_i a_j t^{i+j}. \]
Then we have, for all integers $h \geq 0$,
\[ 0 = c_{s+h} a_0^{i+h} + c_{s+h-1} a_1^{i+h-1} + \cdots + c_0 a_{s+h}. \]

Hence we have, for all integers $h \geq 1$,
\[ v(c_h) = v(c_h) + v(a_s^{i+h}) = v(c_h a_s^{i+h}) = v(-c_{s+h} a_0^{i+h} - c_{s+h-1} a_1^{i+h-1} - \cdots - c_{s+1} a_{s+1}^{i+h-1} - c_0 a_{s+h}) \geq \text{Min}\{v(c_{s+h}), v(a_0), v(c_{s+h-1}) + v(a_1), \ldots, v(c_{h+1}) + v(a_{h-1}) + v(c_{h-1}) + v(a_{h-1}), \ldots, v(c_{s+h}) + v(a_{s+h})\} \geq \text{Min}\{v(c_{s+h}), v(c_{s+h-1}), v(c_{s+h-1}) + 1, \ldots, v(c_{s+h}) + 1\} \]
and, for $h = 0$,
\[ v(c_0) \geq \text{Min}\{v(c_1), v(c_2), \ldots, v(c_s)\} + 1. \]

Therefore if $v(c_{h'}) \geq q$ for all $0 \leq h' \leq h$ and $v(c_{h'}) \geq q - 1$ for all $h'' \geq h + 1$, then $v(c_{h+1}) \geq q$. Also if $v(c_h) \geq q$ for all integers $h \geq 0$, then $v(c_0) \geq q + 1$. 

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Using induction on $h$ and $q$, we have $v(c_h) \geq q$ for all integers $h \geq 0$ and $q \geq 1$. Hence we have $c_h = 0$ for all integers $h \geq 0$ and therefore $u = 0$.

**Corollary.** Let $x$ be as in the lemma. Suppose that $a_0 \neq 0$. If $u = b_0 + b_1 t + b_2 t^2 + \ldots + b_{s-1} t^{s-1}$ with $b_i \in A$ belongs to $R_x$, then $u = 0$.

4. Examples

Let $\tau$ be as in §3. $\sigma = \tau$ and $q = p^{f(K_0/Q_p)}$ satisfy the assumption (F) in [6, §2]. Let $n$ and $m$ be positive integers ($m \geq 0$ if $n = 1$) and $d$ an integer with $0 \leq d \leq m+n-1$. Let $\Gamma_{n,m,d}$ be the $n$-dimensional commutative formal group over $A$ obtained by the following special element $u_{n,m,d}$ as was done in [6]. $u_{n,m,d}$ commutes with $\text{diag}(a, a, \ldots, a)$ for all $a \in A_0$. Hence $\Gamma_{n,m,d}$ is an $n$-dimensional formal $A_0$-module over $A$. By [5, IV.21.1.4, Remarks (ii)], $\text{End}_{A_0}(\Gamma_{n,m,d})$ coincides with the endomorphism ring $\text{End}_{A,Z_p}(\Gamma_{n,m,d})$ of $\Gamma_{n,m,d}$ as a formal group over $A$. $u_{1,m,d} = \pi - t^{m+1}(1 + t^d)$, and for $n \geq 2$,

$$u_{n,m,d} = \begin{pmatrix}
\pi & -t & 0 & \ldots & 0 \\
0 & \pi & -t & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & -t \\
-t^{m+1}(1 + t^d) & 0 & 0 & \ldots & \pi
\end{pmatrix}$$

We have the following generalization of [11, Theorem 2] for $K$.

**Theorem 2.** Suppose that there exists an extension $\tau$ of $\tau'$ to an automorphism of $K$. Then we have

$$\text{End}_{A_0}(\Gamma_{n,m,d}) \cong \{ \text{diag}(a^{n-1}, a^{n-2}, \ldots, a) | a^{t^d} = a^{m+n} = a \in A \}.$$ 

Therefore $\text{End}_{A_0}(\Gamma_{n,m,d})$ is isomorphic to the valuation ring of invariants of $\tau^{(m+n,d)}$ in $A$.

**Proof.** For a ring $T$, $T^n$ denotes the left free $T$-module of the $n$-dimensional row vectors over $T$.

Let $\{e_i\} (1 \leq i \leq n)$ be the images of the canonical basis $\{e_i\} (1 \leq i \leq n)$ of $A^n$ under the composition of the inclusion $A^n \rightarrow R^n$ and the canonical surjection $R^n \rightarrow R^n/R^n u_{n,m,d}$.

First we assume $n \geq 2$. The left $R$-module $R^n u_{n,m,d}$ is generated by $t e_2 - \pi e_1, t e_3 - \pi e_2, \ldots, t e_n - \pi e_{n-1}$ and $\pi e_n - t^{m+1}(1 + t^d) e_1$. Then we have the relations $t e_2 = \pi e_1, t^2 e_3 = \pi e_2, \ldots, t^{n-1} e_n = \pi^{n-2} \pi^{n-3} \pi^{n-4} \ldots \pi^{n-1} e_1$ and the annihilator of $e_1$ is the left $R$-ideal

$$R(\pi^{n-1} \pi^{n-2} \ldots \pi^{n-1} - t^{m+n}(1 + t^d))$$
of $R$. Especially

$$R^n/R^n u_{n,m,d} \cong R/R(\pi \pi^{n-1} \cdots \pi \pi - t^{m+n}(1 + t^d))$$

is a monogenic left $R$-module (cf. [3, III.5.5]).

We suppose $C = (c_{ij}) \in M_n(A)$ be such that

$$f_{n,m,d}^{-1}(C f_{n,m,d}) \in \text{End}_{A,A_0}(\Gamma_{n,m,d}),$$

where $f_{n,m,d}$ is the transformer of $\Gamma_{n,m,d}$. The left $A^r[[t]]$-module $(A^r[[t]])^n u_{n,m,d} C$ is contained in $(A^r[[t]])^n u_{n,m,d}$ by [6, Theorem 3]. Then $C$ gives an $R$-endomorphism of $R^n/R^n u_{n,m,d}$ which stabilizes $\sum_{1 \leq i \leq n} A e_i$. $(\sum_{1 \leq i \leq n} A e_i)$ is a bimodule over $\text{End}_A(Y_{n,m,d})$ and $A$.) Therefore, we have

$$t(c_{11} \bar{e}_1 + c_{22} \bar{e}_2 + \cdots + c_{2n} \bar{e}_n) = \pi(c_{11} \bar{e}_1 + \cdots + c_{1n} \bar{e}_n),$$

$$\vdots$$

$$t(c_{n1} \bar{e}_1 + c_{n2} \bar{e}_2 + \cdots + c_{nn} \bar{e}_n) = \pi(c_{n-11} \bar{e}_1 + \cdots + c_{n-1,n} \bar{e}_n),$$

$$\pi(c_{11} \bar{e}_1 + c_{n2} \bar{e}_2 + \cdots + c_{nn} \bar{e}_n) = t^{m+1}(1 + t^d)(c_{11} \bar{e}_1 + \cdots + c_{1n} \bar{e}_n).$$

By representing $e_i$'s with $e_1$, we have

$$\begin{cases}
\{t(c_{21} + c_{22} \pi + \cdots + c_{2n} \pi^{n-2} \cdots \pi \pi) \\
- \pi(c_{11} + \cdots + c_{1n} \pi^{n-2} \cdots \pi \pi)\} \bar{e}_1 = 0,
\end{cases}
$$

(\text{**})

$$\begin{cases}
\{t(c_{n1} + c_{n2} \pi + \cdots + c_{nn} \pi^{n-2} \cdots \pi \pi) \\
- \pi(c_{n-11} + \cdots + c_{n-1,n} \pi^{n-2} \cdots \pi \pi)\} \bar{e}_1 = 0,
\end{cases}
$$

and

$$\begin{cases}
\{\pi(c_{n1} + c_{n2} \pi + \cdots + c_{nn} \pi^{n-2} \cdots \pi \pi) \\
- t^{m+1}(1 + t^d)(c_{11} + \cdots + c_{1n} \pi^{n-2} \cdots \pi \pi)\} \bar{e}_1 = 0.
\end{cases}
$$

(\text{***})

We multiply (\text{**}) by $t^{n-1}$ from the left.

Since $n \leq m + n - 1$, by the corollary of the lemma we have $c_{ii} = 0$ ($2 \leq i \leq n$), $c_{ik} = c_{i+1k+1}$ ($1 \leq i, k \leq n - 1$), and $c_{in} = 0$ ($1 \leq i \leq n - 1$). Hence we have $c_{ij} = 0$ if $i \neq j$ and $c_{ii} = c_{nn}^{n-1}$ for $1 \leq i \leq n - 1$ and so

$$C = \text{diag}(c_{nn}^{n-1}, c_{nn}^{n-2}, \ldots, c_{nn}).$$

Since the annihilator of $e_1$ is $R(\pi \pi^{n-1} \cdots \pi \pi - t^{m+n}(1 + t^d))$, from (\text{***}) we have

$$\{c_{nn}^{-(m+1)}(1 + t^d) - t^{m+1}(1 + t^d)c_{11}\} \bar{e}_1 = 0.$$
Then, by dividing the above equation by \( t^{m+1} \), we have
\[
\left\{ c_{nn}^{\tau^{-(m+l)}} + c_{nn}^{\tau^{-(m+l)}} t^d - c_{nn}^{\tau^{n-1}} - c_{nn}^{\tau^{n+d-1}} t^d \right\} \tau_1 = 0.
\]
From \( 0 \leq d \leq m + n - 1 \), by the corollary of the lemma we have \( c_{nn}^{\tau^{m+n}} = c_{nn}^{\tau^d} = c_{nn} \).

Conversely if \( C = (c_{ij}) \) satisfies the above conditions, then \( u_{n,m,d} = C \) and so \( u_{n,m,d}(Cf_{n,m,d}) \in \text{End}_{A,A_0}(\Gamma_{n,m,d}) \).

Hence \( \text{End}_{A,A_0}(\Gamma_{n,m,d}) \) is isomorphic to the invariants of \( \tau^{(m+n,d)} \) in \( A \).

Finally, for \( n = 1 \), the analogous argument holds since the annihilator of \( \tau_1 \) is \( R(\pi - t^{m+1}(1 + t^d)) \).

**Remark 3.** (i) The field consisting of the invariants of \( \tau^{(m+n,d)} \) in \( K \) has been determined more explicitly in [11, Theorem 3].

(ii) Suppose that \( e(K/K_0) = 1 \), \( (n, m) = 1 \), and \( k \) is algebraically closed for simplicity. By [3, III.5.2, Proof of Theorem 2], we have
\[
R/R(\pi^{n+1} \cdots \pi \pi - t^{m+n}(1 + t^d)) \cong R/R(\pi^m - t^{m+n}),
\]
where \( \pi_0 \) is a prime element of \( K_0 \). Hence \( \Gamma_{n,m,d} \otimes_A k \) is absolutely simple up to isogeny (cf. Proof of the corollary below).

The following corollary is a higher-dimensional analogue of [1, Theorem 5.2.2] (or [5, IV.23.2.16]).

**Corollary.** For any positive integers \( n \) and \( h \) with \( h \geq n + 1 \) (\( h \geq 1 \) if \( n = 1 \)) and for any positive divisor \( g \) of \( h \), there exists an \( n \)-dimensional formal \( A_0 \)-module over \( A_0 \) of \( A_0 \)-height \( h \) whose absolute \( A_0 \)-endomorphism ring is the valuation ring of the unramified extension of \( K_0 \) of degree \( g \).

**Proof.** Let \( K = K_0 \) and \( m = h - n \). Put \( d = g \) if \( g < h \) and \( d = 0 \) if \( g = h \). Let \( K_0^* \) be the completion of the maximal unramified extension of \( K_0 \) and \( A_0^* \) the valuation ring of \( K_0^* \). As in the proof of the corollary in §2, we have
\[
\text{END}^*_{*,A_0}(\Gamma_{n,m,d}) \subset \text{End}_{A_0^*}(\Gamma_{n,m,d} \otimes_{A_0} A_0^*).
\]
We apply Theorem 2 to \( \Gamma_{n,m,d} \otimes_{A_0} A_0^* \). Thus \( \text{END}^*_{*,A_0}(\Gamma_{n,m,d}) \) is contained in
\[
\{ \text{diag}(a^{n-1}, a^{n-2}, \ldots, a) | a^r = a \in A_0^* \}.
\]

The invariants of \( \tau^g \) in \( A_0^* \) coincide with the valuation ring of the unramified extension of degree \( g \) over \( K_0 \). Especially, any \( A_0 \)-endomorphism of \( \Gamma_{n,m,d} \otimes_{A_0} A_0^* \) is defined over the valuation ring of the finite extension of \( K_0 \).

Hence the converse inclusion follows.

For the functor \( M \) in [2], we have
\[
M(\Gamma_{n,m,d} \otimes_{A_0} k_0) \cong A_{0\tau}[[t]]/A_{0\tau}[[t]]u_{n,m,d}.
\]
as in [4, V.2]. Thus we have

\[ [k_0 \otimes_{A_0} M(\Gamma_{n,m,d} \otimes_{A_0} k_0) : k_0] = m + n \]

and therefore \( \Gamma_{n,m,d} \) is of \( A_0 \)-height \( m + n = h \).

**Remark 4.** If \( (n, m) = 1 \), then \( \Gamma_{n,m,d} \otimes_{A_0} k_0 \) is absolutely simple up to isogeny as in Remark 3(ii) (cf. [3, III.5.5]).

**REFERENCES**

11. Y. Yamasaki, *On the endomorphism rings of Honda groups \( H_{n,m} \) over \( p \)-adic integer rings*, Osaka J. Math. 12 (1975), 457–472.

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