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MAXIMAL FUNCTIONS ON CLASSICAL LORENTZ SPACES AND HARDY'S INEQUALITY WITH WEIGHTS FOR NONINCREASING FUNCTIONS

MIGUEL A. ARIÑO AND BENJAMIN MUCKENHOUPT

ABSTRACT. A characterization is given of a class of classical Lorentz spaces on which the Hardy Littlewood maximal operator is bounded. This is done by determining the weights for which Hardy's inequality holds for nonincreasing functions. An alternate characterization, valid for nondecreasing weights, is also derived.

1. INTRODUCTION

The classical Lorentz spaces $\Lambda_q(W)$ considered here are defined as the set of functions g on R^n such that

$$\|g\|_{\Lambda_q(W)} = \left[\int_0^\infty [g^*(x)]^q W(x)\,dx\right]^{1/q} < \infty,$$

where

$$g^*(y) = \inf\{s\colon \mu(\{t\colon |g(t)| > s\}) \le y\}$$

is the nonincreasing rearrangement of g on $[0, \infty)$, μ is Lebesgue measure, $W(x)$ is nonnegative and $1 \le q < \infty$. For $W(x) = (q/p)x^{(q/p)-1}$, $\Lambda_q(W)$ is the space $L(p, q)$ studied in [3 and 10]. We characterize here the functions W for which a constant D exists such that

$$\|Mg\|_{\Lambda_q(W)} \le D\|g\|_{\Lambda_q(W)}, \tag{1.1}$$

where M is the Hardy Littlewood maximal operator defined as

$$Mg(x) = \sup \frac{1}{|Q|}\int_Q |g(y)|\,dy,$$

and the sup is taken over all cubes Q containing x. As shown in §4, this problem is equivalent to determining the nonnegative functions W for which the Hardy inequality

$$\int_0^\infty \left|\frac{1}{x}\int_0^x f(t)\,dt\right|^q W(x)\,dx \le C\int_0^\infty |f(x)|^q W(x)\,dx \tag{1.2}$$

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holds for all nonnegative nonincreasing f on $[0, \infty)$.

By Theorem 1 of [6], inequality (1.2) holds for all functions f on $[0, \infty)$ if and only if

$$\sup_{r>0} \left[\int_r^\infty \frac{W(x)}{x^q}\,dx\right] \left[\int_0^r W(x)^{-q'/q}\,dx\right]^{q/q'} < \infty, \tag{1.3}$$

where $q' = q/(q-1)$ and the second factor is taken to be $\operatorname{ess\,sup}_{[0,r]} \frac{1}{W(x)}$ if $q = 1$. With the restraint that f is nonnegative and nonincreasing, however, other weights satisfy (1.2). For example, if $1 \le q < \infty$, the function

$$W(x) = \begin{cases} 0, & 1 < x < 2, \\ x^{-1/2}, & 0 \le x \le 1 \text{ or } 2 \le x, \end{cases} \tag{1.4}$$

clearly does not satisfy (1.3). However, for this $W(x)$ the left side of (1.2) is bounded by

$$\int_0^\infty x^{-q-1/2} \left[\int_0^x f(t)\,dt\right]^q dx. \tag{1.5}$$

By the classical Hardy inequality, [10, p. 196], there is a constant B such that (1.5) is bounded by

$$B \int_0^\infty f(x)^q x^{-1/2}\,dx. \tag{1.6}$$

Since f is nonincreasing and nonnegative,

$$\int_1^2 f(x)^q x^{-1/2}\,dx \le \int_0^1 f(x)^q x^{-1/2}\,dx,$$

and, therefore, (1.6) is bounded by the right side of (1.2) with $C = 2B$.

The main results of this paper are as follows.

Theorem (1.7). *If $1 \le q < \infty$ and $W(x) \ge 0$, then (1.2) holds for all nonnegative, nonincreasing f on $[0, \infty)$ if and only if there is a constant B such that for every $r > 0$,*

$$\int_r^\infty \frac{W(x)}{x^q}\,dx \le \frac{B}{r^q} \int_0^r W(x)\,dx. \tag{1.8}$$

Corollary (1.9). *If $1 \le q < \infty$ and $W(x) \ge 0$, then (1.1) holds for all g on R^n if and only if there is a constant B such that (1.8) holds for $r > 0$.*

Theorem (1.10). *If $W(x) \ge 0$, $1 \le q < \infty$ and*

$$\sup_{r>0} \frac{1}{r} \left[\int_0^r W(x)\,dx\right]^{1/q} \left[\int_0^r W(x)^{-q'/q}\,dx\right]^{1/q'} < \infty, \tag{1.11}$$

then (1.2) holds for all nonnegative, nonincreasing functions f, or equivalently, (1.1) holds. The converse is also true for nondecreasing W.

Corollary (1.12). *Condition (1.11) is stronger than (1.8) and for W nondecreasing and nonnegative they are equivalent.*

In §2 a basic lemma is proved that is needed for the proof of Theorem (1.7) in §3. Corollary (1.9) is proved in §4 by showing the equivalence of the two problems. Theorem (1.10) and Corollary (1.12) are proved in §5.

The convention $0 \cdot \infty = 0$ is assumed throughout this paper.

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2. A basic lemma

The proof of Theorem (1.7) will use the following lemma.

Lemma (2.1). *If* $w(x) \geq 0$, $1 \leq q < \infty$ *and* (1.8) *holds, then there is a* $\delta > 0$ *and a constant* D *such that for* $r > 0$,

$$\int_r^\infty \frac{W(x)}{x^{q-\delta}}\,dx \leq \frac{D}{r^{q-\delta}} \int_0^r W(x)\,dx.$$

This follows from the proof of Lemma 21 on page 12 of [11] that B_p implies $B_{p-\varepsilon}$. We give here a simpler proof that can also be used to prove that B_p implies $B_{p-\varepsilon}$. For this proof fix an $r > 0$ and let $A_k = \int_0^{2^k r} W(x)\,dx$. Then for a nonnegative integer n,

$$\sum_{k=n}^\infty \frac{A_k}{2^{kq}} = \sum_{k=n}^\infty \int_0^{2^k r} \frac{W(x)}{2^{kq}}\,dx$$

which equals

$$\int_0^{2^n r} W(x) \left[\sum_{k=n}^\infty \frac{1}{2^{kq}}\right] dx + \int_{2^n r}^\infty W(x) \left[\sum_{2^k r \geq x}^\infty \frac{1}{2^{kq}}\right] dx.$$

Performing the sums gives a bound of

$$\frac{1}{1-2^{-q}} \left[\int_0^{2^n r} \frac{W(x)}{2^{nq}}\,dx + \int_{2^n r}^\infty W(x) \left(\frac{r}{x}\right)^q dx\right].$$

Using (1.8) then shows that

$$\sum_{k=n}^\infty \frac{A_k}{2^{kq}} \leq \frac{B+1}{1-2^{-q}} 2^{-nq} \int_0^{2^n r} W(x)\,dx. \tag{2.2}$$

This is equivalent to

$$\sum_{k=n}^\infty \frac{A_k}{2^{kq}} \leq \frac{B+1}{1-2^{-q}} \left[\sum_{k=n}^\infty \frac{A_k}{2^{kq}} - \sum_{k=n+1}^\infty \frac{A_k}{2^{kq}}\right]$$

from which

$$\sum_{k=n+1}^\infty \frac{A_k}{2^{kq}} \leq S \sum_{k=n}^\infty \frac{A_k}{2^{kq}}$$

with

$$S = \left[\frac{B+1}{1-2^{-q}} - 1\right] \Big/ \left[\frac{B+1}{1-2^{-q}}\right] < 1.$$

By induction for $j \geq 0$,

$$\sum_{k=j}^{\infty} \frac{A_k}{2^{kq}} \leq S^j \sum_{k=0}^{\infty} \frac{A_k}{2^{kq}},$$

and, therefore,

$$A_j \leq (2^q S)^j \sum_{k=0}^{\infty} \frac{A_k}{2^{kq}}. \tag{2.3}$$

Now choose $\delta > 0$ such that $2^\delta S < 1$. Then

$$\int_r^\infty \frac{W(x)}{x^{q-\delta}}\, dx \leq \sum_{j=1}^{\infty} \frac{\int_{2^{j-1}r}^{2^j r} W(x)\, dx}{(2^{j-1}r)^{q-\delta}}.$$

By the definition of A_j, the right side is bounded by

$$\left(\frac{2}{r}\right)^{q-\delta} \sum_{j=1}^{\infty} \frac{A_j}{2^{j(q-\delta)}}.$$

By (2.3) this has the bound

$$\left(\frac{2}{r}\right)^{q-\delta} \sum_{j=1}^{\infty} (2^\delta S)^j \sum_{k=0}^{\infty} \frac{A_k}{2^{kq}}.$$

Performing the first sum and using (2.2) then gives the bound

$$\left(\frac{2}{r}\right)^{q-\delta} \frac{1}{1-2^\delta S} \frac{B+1}{1-2^{-q}} \int_0^r W(x)\, dx.$$

This completes the proof of Lemma (2.1).

3. Proof of Theorem (1.7)

The fact that (1.2) for nonincreasing, nonnegative f implies (1.8) follows immediately by taking $f(x) = \chi_{[0,r]}(x)$ in (1.2). To prove the converse we start by applying Lemma (2.1) to obtain constants D and ε such that $0 < \varepsilon < 1$ and for $r > 0$,

$$\int_r^\infty \left(\frac{x}{r}\right)^{q\varepsilon} \frac{W(x)}{x^q}\, dx \leq \frac{D}{r^q} \int_0^r W(x)\, dx. \tag{3.1}$$

We will assume that f is continuous, has compact support, and is constant on $[0, d]$ with $d > 0$. We can add these restrictions on f without loss of generality by use of the monotone convergence theorem.

Having fixed such an f, we define sequences $\{a_n\}$ and $\{b_n\}$ inductively as follows. Let $b_0 = 0$. Given b_{n-1}, we take a_n to be the infimum of all $x > b_{n-1}$ such that

$$\frac{x f(x)}{\int_0^x f(t)\, dt} \tag{3.2}$$

is less than or equal to $\varepsilon/10$. By this definition we have

$$\int_0^x f(t)\,dt \le \frac{10}{\varepsilon} x f(x), \qquad b_{n-1} < x \le a_n, \tag{3.3}$$

and, since f is continuous,

$$\int_0^{a_n} f(t)\,dt = \frac{10}{\varepsilon} a_n f(a_n). \tag{3.4}$$

Furthermore, since (3.2) equals 1 for $0 < x \le d$, we have $a_1 > d$.

Given a_n, define b_n to be the infimum of all $x > a_n$ such that (3.2) is greater than ε. Then

$$\int_0^x f(t)\,dt \ge \frac{1}{\varepsilon} x f(x), \qquad a_n \le x \le b_n, \tag{3.5}$$

and

$$\int_0^{b_n} f(t)\,dt = \frac{1}{\varepsilon} b_n f(b_n). \tag{3.6}$$

Since f is nonincreasing and $b_n \le a_{n+1}$,

$$\frac{\varepsilon}{a_{n+1}} \int_0^{a_{n+1}} f(t)\,dt \le \frac{\varepsilon}{b_n} \int_0^{b_n} f(t)\,dt;$$

from (3.4) and (3.6) we see that $10 f(a_{n+1}) \le f(b_n)$. It follows that

$$10 f(a_{n+1}) \le f(a_n). \tag{3.7}$$

Similarly, since $a_n \le b_n$ and f is nonnegative,

$$\varepsilon \int_0^{a_n} f(t)\,dt \le \varepsilon \int_0^{b_n} f(t)\,dt,$$

and combining this with (3.4) and (3.6) shows that $10 a_n f(a_n) \le b_n f(b_n)$. Since $b_n f(b_n) \le a_{n+1} f(a_n)$ we have

$$10 a_n \le a_{n+1}. \tag{3.8}$$

Now (3.8), the fact that $a_1 \ge d$ and the fact that all a_n's must lie in the support of f show that there are only a finite number of a_n's; call the last one a_N. If b_N, as defined above, existed, then a_{N+1} would also exist since (3.2) is 0 off the support of f. This contradiction shows that (3.2) is less than or equal to ε for $x \ge a_N$. Therefore, (3.5) remains valid for $n = N$ if we define $b_N = \infty$.

If $a_n \le x \le b_n$ we have by (3.5) that

$$\int_{a_n}^x \left[\frac{f(t)}{\int_0^t f(u)\,du} \right] dt \le \int_{a_n}^x \frac{\varepsilon}{t}\,dt$$

or

$$\log \left[\frac{\int_0^x f(u)\,du}{\int_0^{a_n} f(u)\,du} \right] \le \varepsilon \log \frac{x}{a_n}.$$

From this

$$\int_0^x f(u)\,du \le \left(\frac{x}{a_n}\right)^{\varepsilon} \int_0^{a_n} f(u)\,du, \qquad a_n \le x \le b_n. \tag{3.9}$$

Now to prove (1.2) write the left side as the sum of

$$\sum_{n=1}^{N} \int_{b_{n-1}}^{a_n} \left[\frac{1}{x}\int_0^x f(t)\,dt\right]^q W(x)\,dx \tag{3.10}$$

and

$$\sum_{n=1}^{N} \int_{a_n}^{b_n} \left[\frac{1}{x}\int_0^x f(t)\,dt\right]^q W(x)\,dx. \tag{3.11}$$

By (3.3) we see that (3.10) is bounded by the right side of (1.2) with $C = (10/\varepsilon)^q$. For (3.11) use (3.9) to get the bound

$$\sum_{n=1}^{N} \left[\int_{a_n}^{b_n} \frac{W(x)}{x^{q-q\varepsilon}}\,dx\right] \left[\frac{1}{a_n^{\varepsilon}} \int_0^{a_n} f(u)\,du\right]^q.$$

By (3.1) this is bounded by

$$\sum_{n=1}^{N} \frac{D}{a_n^q} \left[\int_0^{a_n} W(x)\,dx\right] \left[\int_0^{a_n} f(u)\,du\right]^q.$$

By (3.4) this equals

$$\sum_{n=1}^{N} D\left[\frac{10}{\varepsilon}\right]^q \int_0^{a_n} f(a_n)^q W(x)\,dx$$

which can be written

$$D\left[\frac{10}{\varepsilon}\right]^q \int_0^{a_N} \sum_{a_n \ge x} f(a_n)^q W(x)\,dx.$$

From (3.7) and the fact that f is nonincreasing we get the bound

$$D\left[\frac{10}{\varepsilon}\right]^q \frac{10}{9} \int_0^{a_N} f(x)^q W(x).$$

This completes the proof of Theorem (1.7)

4. Equivalence of the problems, proof of Corollary (1.9)

Corollary (1.9) follows immediately from theorem (1.7) and the following lemma.

Lemma (4.1). *If* $1 \le q < \infty$, $n \ge 1$ *and* $W(x)$ *is nonnegative, then* (1.1) *holds for all* g *on* R^n *if and only if* (1.2) *holds for all nonnegative, nonincreasing* f *on* $[0, \infty)$.

To prove that (1.1) for g in R^n implies (1.2) for nonnegative, nonincreasing f on $[0, \infty)$, fix such an f and define $g(x) = f(A|x|^n)$, where A is the

volume of the unit sphere in R^n. Taking Q as the cube with center x and side length $4|x|$ in the definition of M we have

$$Mg(x) \geq \frac{1}{(4|x|)^n} \int_{|y| \leq |x|} g(y)\, dy.$$

Using the definition of g and changing to polar coordinates shows that

$$Mg(x) \geq \frac{nA}{(4|x|)^n} \int_0^{|x|} f(At^n) t^{n-1}\, dt,$$

or, with a change of variables

$$Mg(x) \geq 4^{-n} A \frac{\int_0^{A|x|^n} f(s)\, ds}{A|x|^n}.$$

Since the right side is a radial nonincreasing function of x,

$$(Mg)^*(t) \geq \frac{4^{-n} A}{t} \int_0^t f(s)\, ds.$$

Using this and the fact that $g^*(t) = f(t)$ in (1.1) then proves (1.2) with $C = [4^n D/A]^q$.

Conversely, if W satisfies (1.2) for nonincreasing, nonnegative f on $[0, \infty)$ then

$$\int_0^\infty \left[\frac{1}{x} \int_0^x g^*(t)\, dt\right]^q W(x)\, dx \leq C \int_0^\infty g^*(x)^q W(x)\, dx \tag{4.2}$$

for g in $\Lambda_q(W)$. Now by [12, p. 31] on R^1 or [9, p. 306] on R^n, $(Mg)^*(x) \leq (A/x) \int_0^x g^*(t)\, dt$ with A depending only on n. Combining this with (4.2) proves (1.1).

5. Proof of Theorem (1.10) and Corollary (1.12)

We will use the following lemma.

Lemma (5.1). *If $1 \leq q < \infty$ and $V(x)$ and $W(x)$ are nonnegative, then there is a constant B such that for $0 < r < s$,*

$$\left[\frac{1}{s^q} \int_r^s V(x)\, dx\right]^{1/q} \left[\int_0^r W(x)^{-q'/q}\, dx\right]^{1/q'} \leq B \tag{5.2}$$

if and only if for every f and $\lambda > 0$,

$$\int_{E_\lambda} V(x)\, dx \leq \frac{C}{\lambda^q} \int_0^\infty |f(x)|^q W(x)\, dx, \tag{5.3}$$

where $E_\lambda = \{x : \frac{1}{x} \int_0^x |f(t)|\, dt > \lambda\}$ and C is independent of f and λ.

That (5.3) implies (5.2) is simple; its proof is indicated on p. 16 of [1]. It is also shown there that (5.2) implies that (1.6) of [1] is finite, and Theorem 2 of [1] shows that this implies (5.3). The lemma also follows immediately from

Theorem 4 of [8] and the statement on the following two lines by taking η, p, q, r, w, and v of that theorem to be respectively 1, q, ∞, q, V, and W.

To prove the first part of Theorem (1.10), observe that (1.11) implies (5.2) with $V(x) = W(x)$. By Lemma (5.1), we then have (5.3) with $V(x) = W(x)$. Theorem 3 of [1] and Theorem 1 of [6] then prove (1.2).

For the second part of Theorem (1.10) if $q > 1$, let

$$f(x) = W_n(x)^{-q'/q}\chi_{[0,r]}(x),$$

where W_n is defined by

$$W_n(x) = \begin{cases} W(x), & W(x) > 1/n, \\ 1/n, & W(x) \le 1/n. \end{cases}$$

Since W_n is nondecreasing, we have $f(x)$ nonincreasing and

$$\frac{1}{r}\int_0^r f(t)\,dt \le \frac{1}{x}\int_0^x f(t)\,dt, \qquad 0 < x \le r.$$

Therefore,

$$\left[\frac{1}{r}\int_0^r W_n(x)^{-q'/q}\,dx\right]^q \int_0^r W(x)\,dx \le \int_0^r \left[\frac{1}{x}\int_0^x f(t)\,dt\right]^q W(x)\,dx. \tag{5.4}$$

By (1.2) the right side of (5.4) is bounded by

$$C\int_0^r f(x)^q W(x)\,dx \le C\int_0^r W_n(x)^{-q'/q}\,dx.$$

Therefore,

$$\frac{1}{r^q}\left[\int_0^r W_n(x)^{-q'/q}\,dx\right]^{q-1}\int_0^r W(x)\,dx \le C$$

and the monotone convergence theorem completes the proof. If $q = 1$, then (1.8) shows that the only nondecreasing W's that satisfy (1.2) for nonincreasing, nonnegative f are $W(x) \equiv 0$ and $W(x) \equiv \infty$ for which (1.11) is trivial.

To prove the first part of Corollary (1.12) observe first that Theorem (1.10) and Theorem (1.7) show that (1.11) implies (1.8). That (1.11) is stronger then follows by observing that the $W(x)$ defined in (1.4) satisfies (1.8) by Theorem (1.7) but does not satisfy (1.11). The equivalence for W nondecreasing follows from Theorem (1.7) and the second part of Theorem (1.10).

References

1. K. Andersen and B. Muckenhoupt, *Weighted weak type inequalities with applications to Hilbert transforms and maximal functions*, Studia Math. **72** (1982), 9–26.
2. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.
3. R. A. Hunt, *On $L(p, q)$ spaces*, Enseign. Math. (2) **12** (1966), 249–276.
4. G. G. Lorentz, *Some new functional spaces*, Ann of Math. (2) **51** (1950), 37–55.
5. ——, *On the theory of spaces Λ*, Pacific J. Math. **1** (1951), 411–429.
6. B. Muckenhoupt, *Hardy's inequality with weights*, Studia Math. **44** (1972), 31–38.

7. ——, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.

8. E. Sawyer, *Weighted Lebesgue and Lorentz norm inequalities for the Hardy operator*, Trans. Amer. Math. Soc. **281** (1984), 329–337.

9. E. Stein, *Note on the class* $L \log L$, Studia Math. **32** (1969), 305–310.

10. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, N.J., 1971.

11. J. O. Strömberg and A. Torchinsky, *Weighted Hardy spaces*, Lecture Notes in Math., vol. 1381, Springer-Verlag, New York, 1989.

12. A. Zygmund, *Trigonometric series*, vol. I, Cambridge Univ. Press, London, New York, 1959.

SLOAN SCHOOL OF MANAGEMENT, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, E40-262, CAMBRIDGE, MASSACHUSETTS 02139

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903 (Current address of Benjamin Muckenhoupt)

Current address (M. A. Ariño): I.E.S.E. Universidad de Navarra, Avda Pearson 21, 08034 Barcelona, Spain