TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 320, Number 2, August 1990

POLYNOMIAL FLOWS ON $\mathbb{C}^n$

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ABSTRACT. We show that polynomial flows on $\mathbb{R}^n$ extend to functions holomorphic on $\mathbb{C}^{n+1}$ and that the group property holds after this extension. Then we give some methods, based on power series, for determining when a vector field has a polynomial flow.

0. INTRODUCTION

Consider the initial value problem

$$\dot{y} \quad (\equiv dy/dt) = \mathbf{V}(y), \qquad y(0) = x \in \mathbb{R}^n, \tag{0.1}$$

where $\mathbf{V}$ is a C^1 vector field on $\mathbb{R}^n$. Let $\phi\colon \Omega \to \mathbb{R}^n$ be the (local) flow associated with (0.1) where Ω, an open subset of $\mathbb{R}\times\mathbb{R}^n$, is the natural domain of ϕ. For each t in $\mathbb{R}$ let U^t be the set of all x in $\mathbb{R}^n$ such that (t, x) is in Ω.

The flow ϕ is said to be a *polynomial flow* and $\mathbf{V}$ is said to be a *p-f vector field* if for each t in $\mathbb{R}$ the t-advance map $\phi^t\colon U^t \to \mathbb{R}^n$ is polynomial. That is, if $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ is the projection map onto the ith coordinate, $\pi_i \circ \phi^t$ is polynomial for $i = 1, \dots, n$.

Meisters [M1] first asked the question: Which vector fields have polynomial flows? Results on polynomial flows can be found in [BM, M2, MO, C1, and C2]. However, Meisters original question remains unanswered. In fact, Meisters and Olech [MO] show that a completely effective method for determining which vector fields have polynomial flows may either confirm the Jacobian conjecture of Algebraic Geometry or lead to a counterexample. See Bass, Connell, and Wright [BCW] for a thorough discussion of the Jacobian conjecture.

We wish to develop methods for recognizing p-f vector fields. That is, we wish to find some of their distinguishing properties and to find ways of checking whether a given (not necessarily p-f) vector field has these properties. In this paper we investigate the properties of polynomial flows when we allow time, t, and the initial condition, x, to be complex. In §1 we show that a polynomial

Received by the editors May 10, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34A20; Secondary 32A05, 32A15.

This paper is derived from the author's Ph.D. thesis at the University of Nebraska-Lincoln, 1988. The author wishes to express his gratitude to his thesis advisor, Professor Gary H. Meisters, for his support and aid.

flow extends to a function holomorphic on $\mathbb{C}^{n+1}$. In §2 we derive a power series form of the flow associated with an analytic vector field and show how this series can be used to calculate the solution of some familiar differential equations. Then in §3 we show how to use the power series derived in §3 to determine whether a vector field has a polynomial flow.

1. Polynomial flows are entire

Consider the linear differential equation

$$\dot{y} = Ay \tag{1.1}$$

where A is an $n \times n$ matrix of real numbers. Notice that the flow of (1.1), $\phi(t, x) = e^{tA}x$, is linear in x and that it extends to a function holomorphic on $\mathbb{C} \times \mathbb{C}^n$. In this section we show that, more generally, a polynomial flow ϕ extends to a holomorphic function $\Phi: \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$. Furthermore, we show that this extension satisfies the group property. That is,

$$\Phi(t+s, x) = \Phi\big(t, \Phi(s, x)\big)$$

for all t and s in $\mathbb{C}$ and for all x in $\mathbb{C}^n$.

Let Ω be an open neighborhood of $\{0\} \times \mathbb{R}^n$ in $\mathbb{R} \times \mathbb{R}^n$ and let

$$B(x; \varepsilon) = \{ z \in \mathbb{C} : |z - x| < \varepsilon \}.$$

Definition 1.1. A flow $\phi: \Omega \to \mathbb{R}^n$ is said to be *entire* or to be an *entire flow* if there exists a holomorphic function $\Phi: \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ such that

(1) $\Phi|_\Omega = \phi$;
(2) $\Phi(t+s, x) = \Phi(t, \Phi(s, x))$ for all t and s in $\mathbb{C}$ and all x in $\mathbb{C}^n$.

Theorem 1.1. *Polynomial flows are entire.*

Proof. If $r = (r_1, \dots, r_n)$ is an n-tuple of nonnegative integers and $x = (x_1, \dots, x_n)$ is a vector in either $\mathbb{R}^n$ or $\mathbb{C}^n$, let $|r| = r_1 + \cdots + r_n$ and $x^r = x_1^{r_1} \cdots x_n^{r_n}$. Let ϕ be a polynomial flow. Then there exist real analytic functions $a_r: \mathbb{R} \to \mathbb{R}^n$ such that for some fixed positive integer d

$$\phi(t, x) = \sum_{|r| \le d} a_r(t) x^r, \qquad t \in \mathbb{R}, \ x \in \mathbb{R}^n. \tag{1.2}$$

(See [BM, Theorem 6.1].) Denote the right-hand side of (1.2) by $\bar{\phi}(t, x)$. We can extend each a_r to a function analytic in some open neighborhood W_r of the real axis (W_r is an open subset of $\mathbb{C}$). Let $W = \bigcap_{|r| \le d} W_r$. Since this is a finite intersection, W is also an open neighborhood of the real axis.

Since $\bar{\phi}$ is polynomial in x for each t, it extends uniquely to a function analytic on $W \times \mathbb{C}^n$. Pick $\delta > 0$ so that $B = B(0; \delta) \subset W$. We claim

(1) $\bar{\phi}$ satisfies the group property on $B \times \mathbb{C}^n$; and,
(2) we can take B to be $\mathbb{C}$.

To prove our first claim, we first show $\bar{\phi}$ satisfies the group property on $\mathbb{R} \times \mathbb{C}^n$, then we show $\bar{\phi}$ satisfies the group property on $B \times \mathbb{C}^n$.

To see that $\bar{\phi}$ satisfies the group property on $\mathbb{R} \times \mathbb{C}^n$, fix t and s in $\mathbb{R}$. Then both $\bar{\phi}(t, \bar{\phi}(s, x))$ and $\bar{\phi}(t+s, x)$ are polynomial. Since they are equal as polynomials on $\mathbb{R}^n$, they must be equal as polynomials on $\mathbb{C}^n$. Since t and s were arbitrary, the group property holds on $\mathbb{R} \times \mathbb{C}^n$.

To see that $\bar{\phi}$ satisfies the group property on $B \times \mathbb{C}^n$, fix $x \in \mathbb{C}^n$. First we note that both $\bar{\phi}(t, \bar{\phi}(s, x))$ and $\bar{\phi}(t+s, x)$ are defined when t, s, and $t+s$ all lie in B. Fix $s \in B \cap \mathbb{R}$. Let $f(t) = \bar{\phi}(t+s, x)$ and $g(t) = \bar{\phi}(t, \bar{\phi}(s, x))$. Then $f(t)$ and $g(t)$ are defined and analytic when $t \in B$ and $t+s \in B$. That is, $f(t)$ and $g(t)$ are defined and analytic when $t \in B \cap (B-s)$. But $B \cap (B-s)$ contains an open interval about zero on the real axis and by the group property $f(t) = g(t)$ on this interval. Therefore $f(t) = g(t)$ on $B \cap (B-s)$ since it is a connected open set and both $f(t)$ and $g(t)$ are analytic there. Hence for $s \in B \cap \mathbb{R}$, $t \in B$, and $t+s \in B$ we have

$$\bar{\phi}(t, \bar{\phi}(s, x)) = \bar{\phi}(t+s, x). \tag{1.3}$$

That is, for all for $s \in B \cap \mathbb{R}$, $t \in B$, and $t+s \in B$ the group property holds.

Next fix $t \in B$. Let $h(s) = \bar{\phi}(t+s, x)$ and $k(s) = \bar{\phi}(t, \bar{\phi}(s, x))$. Then $h(s)$ and $k(s)$ are defined and analytic when $s \in B$ and $t+s \in B$. That is, $h(s)$ and $k(s)$ are defined and analytic when $s \in B \cap (B-t)$. But $B \cap (B-t)$ contains an open interval about zero on the real axis and by the argument in the preceding paragraph $h(s) = k(s)$ on this interval. Therefore $h(s) = k(s)$ on $B \cap (B-t)$ since it is a connected open set and both $h(s)$ and $k(s)$ are analytic there. Hence for $s \in B$, $t \in B$, and $t+s \in B$ we have (1.3). That is, for $s \in B$, $t \in B$, and $t+s \in B$ the group property holds. Since $x \in \mathbb{C}^n$ was arbitrary, our first claim is proved.

To prove our second claim, it suffices to show that we may choose $\delta = \infty$ (that is, $B = B(0; \delta) = \mathbb{C}$). By way of contradiction, suppose that for some finite $\delta > 0$, $\bar{\phi}$ has an analytic extension to $B(0; \delta) \times \mathbb{C}^n$ satisfying (1.3) but $\bar{\phi}$ has no analytic extension to $B(0; \delta_1) \times \mathbb{C}^n$ satisfying (1.3) for any $\delta_1 > \delta$. Define $\tilde{\phi}: B(0; 2\delta) \times \mathbb{C}^n \to \mathbb{C}^n$ via $\tilde{\phi}(t, x) = \bar{\phi}(t/2, \bar{\phi}(t/2, x))$. We will show that $\tilde{\phi}$ is analytic on $B(0; 2\delta) \times \mathbb{C}$ and satisfies the group property there and thus arrive at a contradiction.

To see that $\tilde{\phi}$ is analytic on $B(0; 2\delta) \times \mathbb{C}^n$ note that it is the composition of analytic functions. To see that $\tilde{\phi}$ satisfies the group property on $B(0; 2\delta) \times \mathbb{C}^n$ note that if $x \in \mathbb{C}^n$ and t, s, and $t+s$ all lie in $B(0; 2\delta)$ then $t/2$, $s/2$ and

$(t+s)/2$ all lie in $B(0;\delta)$. Thus

$$\begin{aligned}\tilde{\phi}(t+s,x) &= \bar{\phi}((t+s)/2,\bar{\phi}((t+s)/2,x))\\ &= \bar{\phi}(t/2,\bar{\phi}(s/2,\bar{\phi}((t+s)/2,x)))\\ &= \bar{\phi}(t/2,\bar{\phi}(s/2,\bar{\phi}(t/2,\bar{\phi}(s/2,x))))\\ &= \bar{\phi}(t/2,\bar{\phi}((s+t)/2,\bar{\phi}(s/2,x)))\\ &= \bar{\phi}(t/2,\bar{\phi}(t/2,\bar{\phi}(s/2,\bar{\phi}(s/2,x))))\\ &= \bar{\phi}(t/2,\bar{\phi}(t/2,\tilde{\phi}(s,x)))\\ &= \tilde{\phi}(t,\tilde{\phi}(s,x)).\end{aligned}$$

That is, $\tilde{\phi}(t,x)$ satisfies the group property which is a contradiction. Therefore we may take $B = W = \mathbb{C}$ and ϕ extends to a holomorphic function $\Phi\colon \mathbb{C}\times\mathbb{C}^n \to \mathbb{C}^n$ such that for all $x\in\mathbb{C}^n$, and $t,s\in\mathbb{C}$, $\Phi(t+s,x) = \Phi(t,\Phi(s,x))$. Hence ϕ is an entire flow. □

Bass and Meisters [BM] show that p-f vector fields are polynomial, have constant divergence, and are complete—all solutions are defined for all real t. Notice that entire flows are complete flows. However, as the following example demonstrates, the flow for a polynomial vector field with constant divergence can be entire without being polynomial.

Example 1.1 (G. H. Meisters). *Consider the initial value problem*

$$\begin{aligned}\dot{x} &= ax - xp'(xy) \quad (= ax - H_y), \qquad x(0) = x_0,\\ \dot{y} &= by + yp'(xy) \quad (= by + H_x), \qquad y(0) = y_0,\end{aligned}$$

where p is a polynomial in one variable and $H(x,y) = p(xy)$ serves as a polynomial Hamiltonian if a and b are both zero. The vector field here is polynomial, has constant divergence, and the solution is

$$\begin{aligned}x(t) &= x_0\exp\left(at - \int_0^t p'(x_0y_0e^{(a+b)s})ds\right),\\ y(t) &= y_0\exp\left(bt + \int_0^t p'(x_0y_0e^{(a+b)s})ds\right)\end{aligned}$$

which is entire yet not, in general, polynomial in (x_0,y_0).

If $\mathbf{V}$ is a p-f vector field, we can think of Φ as being the flow for the initial value problem

$$\dot{y} = \mathbf{V}(y), \qquad y(0) = x\in\mathbb{C}^n,$$

where we allow both t and x to take on complex values. That is, since $\mathbf{V}$ is polynomial, it has a natural extension to $\mathbf{V}\colon \mathbb{C}^n\to\mathbb{C}^n$ and with the convention $\Phi_x(t) = \Phi(t,x)$, we have $\frac{d}{dt}\Phi_x(t) = \mathbf{V}(\Phi_x(t))$, $(t,x)\in\mathbb{C}\times\mathbb{C}^n$, and $\Phi_x(0) = x$.

2. A power series form of the flow

Our methods for determining whether a vector field has a polynomial flow (see §3) use a power series representation of the flow. In this section we derive

this representation. We also show how it can be used to calculate the flow of some elementary systems.

Let $\mathbf{X}$ be an analytic vector field on some domain G in $\mathbb{C}^n$. Recall that the solution of initial value problem

$$\dot{y} = \mathbf{X}(y), \qquad y(0) = x \in G, \tag{2.1}$$

has a unique analytic solution $\phi_x(t)$ defined on some disk in $\mathbb{C}$ centered at zero. As always, let $\phi^t(x) = \phi_x(t) = \phi(t, x)$. Define $c_k(x)$ by

$$c_0(x) = x; \quad c_{k+1}(x) = Dc_k(x)\mathbf{X}(x), \qquad k \geq 0, \tag{2.2}$$

where D is the derivative operator. We call the c_k's the *Maclaurin coefficients of* $\mathbf{X}$ because of the following.

Theorem 2.1. *Let ϕ and $\mathbf{X}$ be as above. Then there exists a neighborhood U of $\{0\} \times G$ in $\mathbb{C} \times \mathbb{C}^n$ where*

$$\phi(t, x) = \sum_{k=0}^{\infty} \frac{c_k(x)}{k!} t^k. \tag{2.3}$$

We will give a short formal argument to motivate the proof of this theorem, then we will prove it rigorously.

Our formal argument is as follows: Let

$$\phi(t, x) = \sum_{k=0}^{\infty} \frac{f_k(x)}{k!} t^k$$

be a power series representation of the flow. Set $t = 0$ to see $f_0(x) = x$. By the group property we have

$$\phi(t + s, x) = \phi(t, \phi(s, x))$$

or

$$\sum_{k=0}^{\infty} \frac{f_k(x)}{k!} (t + s)^k = \sum_{k=0}^{\infty} \frac{f_k(\phi(s, x))}{k!} t^k.$$

First, differentiate with respect to s

$$\sum_{k=1}^{\infty} \frac{f_k(x)}{(k-1)!} (t + s)^{k-1} = \sum_{k=0}^{\infty} \frac{Df_k(\phi(s, x))\dot{\phi}(s, x)}{k!} t^k,$$

then set $s = 0$

$$\sum_{k=1}^{\infty} \frac{f_k(x)}{(k-1)!} t^{k-1} = \sum_{k=0}^{\infty} \frac{Df_k(x)\mathbf{V}(x)}{k!} t^k,$$

adjust the summation index on the left-hand side

$$\sum_{k=0}^{\infty} \frac{f_{k+1}(x)}{k!} t^k = \sum_{k=0}^{\infty} \frac{Df_k(x)\mathbf{V}(x)}{k!} t^k,$$

and by equating coefficients of corresponding powers of t, we get the recursion relation for the Maclaurin coefficients.

It seems unlikely that this formula is unknown. Although, since it is quite easy to derive (at least formally), it is surprising that it is not better known.

The rest of this section is devoted to rigorously proving Theorem 2.1. First we introduce some notation

(1) t and s represent complex time;
(2) x is in $\mathbb{C}^n$;
(3) $R = \{ w \in \mathbb{C}^n : |w - x| < b \}$ where b is a positive real number;
(4) $B(t; \varepsilon) = \{ z \in \mathbb{C} : |z - t| < \varepsilon \}$.

We stated above that the solution of (2.1) is analytic in t. More precisely, we have the following.

Lemma 2.1. *Suppose* $\mathbf{X}$ *is analytic and bounded on the open rectangle* R *and let* $M = \sup\{ |\mathbf{X}(w)| : w \in R \}$. *Then there exists a unique function* $\phi_x(t) = \phi(t, x)$, *analytic in* t *on* $B(0; b/M)$, *which is a solution of the initial value problem* (2.1).

This lemma follows immediately from Theorem 8.1 in Chapter 1 of Coddington and Levinson [CL].

Notice that for t and s sufficiently small (but complex) both $\phi_{\phi_x(s)}(t)$ and $\phi_x(t+s)$ are solutions of the initial value problem

$$\dot{y} = \mathbf{X}(y), \qquad y(0) = \phi_x(s).$$

By uniqueness of solutions guaranteed by Lemma 2.1, we have that ϕ satisfies the (local) group property in complex time.

Lemma 2.2. *Suppose* $\sum_{k=0}^{\infty} a_k x^k$ *has radius of convergence* $\rho > 0$ *and suppose* $t, s \in B(0; \rho/2)$. *Then*

$$\sum_{k=0}^{\infty} a_k (t+s)^k = \sum_{k=0}^{\infty} \sum_{l=0}^{k} a_k \binom{k}{l} t^l s^{k-l} = \sum_{l=0}^{\infty} \sum_{k=l}^{\infty} a_k \binom{k}{l} t^l s^{k-l}.$$

Proof. From the binomial theorem it follows that

$$\sum_{k=0}^{\infty} a_k (t+s)^k = \sum_{k=0}^{\infty} \sum_{l=0}^{k} a_k \binom{k}{l} t^l s^{k-l}.$$

To see that we may then switch the order of summation, it suffices to show

$$\sum_{k=0}^{\infty} \sum_{l=0}^{k} \left| a_k \binom{k}{l} t^l s^{k-l} \right| < \infty.$$

Let $M = \max\{|t|, |s|\}$ and note that $M < \rho/2$. Then

$$\begin{aligned} \sum_{l=0}^{k} \left| a_k \binom{k}{l} t^l s^{k-l} \right| &= |a_k| \sum_{l=0}^{k} \binom{k}{l} |t^l| |s^{k-l}| \\ &\le |a_k| \sum_{l=0}^{k} \binom{k}{l} M^l M^{k-l} \\ &= |a_k| (M+M)^k = |a_k| (2M)^k. \end{aligned}$$

Thus

$$\sum_{k=0}^{\infty}\sum_{l=0}^{k}|a_k\binom{k}{l}t^l s^{k-l}| \le \sum_{k=0}^{\infty}|a_k|(2M)^k.$$

Since $2M < \rho$, $\sum_{k=0}^{\infty}|a_k|(2M)^k$ converges. □

Proof of Theorem 2.1. For each z in G we will show that ϕ is of the form (2.3) on a neighborhood of $(0, z)$ in $\mathbb{C}\times\mathbb{C}^n$. The union as z runs through G of such neighborhoods will give us U.

By Lemma 2.1, for each z in G there exists $\varepsilon_z, b_z > 0$ with $R_z = \{x \in \mathbb{C}^n : |x - z| < b_z\}$ such that

(1) ϕ is defined on $B(0; \varepsilon_z)\times R_z$;
(2) for each $x \in R_z$, ϕ_x is analytic in t on $B(0; \varepsilon_z)$;
(3) given $x \in R_z$ there exists ${}_x\delta_z > 0$ such that for $s, t \in B(0; {}_x\delta_z)$, we have $\phi(s, x) \in R_z$ and $\phi(t+s, x) = \phi(t, \phi(s, x))$.

Thus for $(t, x) \in B(0; \varepsilon_z)\times R_z$ we have

$$\phi(t, x) = \sum_{k=0}^{\infty}\frac{f_k(x)}{k!}t^k.$$

Recall that power series are uniquely represented by their coefficients. Hence, since z is arbitrary, we can take $f_k : G \to \mathbb{C}^n$, $k = 0, 1, 2, \dots$, and these functions are well defined.

We notice that

$$\phi(0, x) = f_0(x) = x = c_0(x), \qquad x \in G,$$

and then proceed by induction: Suppose f_m is analytic and equal to c_m on G for some fixed $m \ge 0$. Fix z in G and then fix x in R_z. If $\gamma = \min(\varepsilon_z/2, {}_x\delta_z)$, for t, s in $B(0; \gamma)$ we have

$$\phi(t+s, x) = \phi(t, \phi(s, x)). \tag{2.4}$$

We also have

$$\phi(t+s, x) = \sum_{k=0}^{\infty}\frac{f_k(x)}{k!}(t+s)^k.$$

By Lemma 2.2, since $t, s \in B(0; \varepsilon_z/2)$,

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{f_k(x)}{k!}(t+s)^k &= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\binom{k}{l}\frac{f_k(x)}{k!}t^l s^{k-l}\\ &= \sum_{l=0}^{\infty}\sum_{k=l}^{\infty}\binom{k}{l}\frac{f_k(x)}{k!}t^l s^{k-l}\\ &= \sum_{l=0}^{\infty}t^l\sum_{k=l}^{\infty}\binom{k}{l}\frac{f_k(x)}{k!}s^{k-l}.\end{aligned} \tag{2.5}$$

We also have

$$\phi(t, \phi(s, x)) = \sum_{l=0}^{\infty} \frac{f_l(\phi(s, x))}{l!} t^l. \tag{2.6}$$

Now fix s. From (2.4) we see that the coefficient of t^m in (2.5) must be equal to the coefficient of t^m in (2.6) (we are again using the fact that if two power series are equal, their coefficients must be equal). Hence for all s in $B(0; \gamma)$ we have

$$\frac{f_m(\phi(s, x))}{m!} = \sum_{k=m}^{\infty} \frac{f_k(x)}{k!} \binom{k}{m} s^{k-m}. \tag{2.7}$$

We now differentiate both sides of (2.7) with respect to s and set $s = 0$. Since f_m is analytic on G, from the left-hand side of (2.7) we have

$$\begin{aligned} \frac{d}{ds} f_m(\phi(s, x))/m!\Big|_{s=0} &= \frac{1}{m!} Df_m(\phi(s, x))\dot{\phi}(s, x)\Big|_{s=0} \\ &= \frac{1}{m!} Df_m(\phi(s, x))\mathbf{X}(\phi(s, x))\Big|_{s=0} \\ &= \frac{1}{m!} Df_m(x)\mathbf{X}(x). \end{aligned} \tag{2.8}$$

From the right-hand side of (2.7) we have

$$\begin{aligned} \frac{d}{ds} \sum_{k=m}^{\infty} \binom{k}{m} \frac{f_k(x)}{k!} s^{k-m}\Big|_{s=0} &= \sum_{k=m+1}^{\infty} \binom{k}{m} \frac{f_k(x)}{k!}(k-m)s^{k-m-1}\Big|_{s=0} \\ &= \binom{m+1}{m} \frac{f_{m+1}(x)}{(m+1)!} \\ &= \frac{m+1}{(m+1)!} f_{m+1}(x) \\ &= \frac{1}{m!} f_{m+1}(x). \end{aligned} \tag{2.9}$$

Comparing (2.8) and (2.9) we see that

$$Df_m(x)\mathbf{X}(x) = f_{m+1}(x).$$

Hence f_{m+1} is analytic on R_z. Furthermore, $f_{m+1} = c_{m+1}$ on R_z (because $f_m = c_m$ on G). Since z was arbitrary, f_{m+1} is analytic and equal to c_{m+1} on G. By induction, f_k is analytic and equal to c_k on G for all k.

Take

$$U = \bigcup_{z \in G} (B(0; \varepsilon_z) \times R_z).$$

Notice that

(1) U is a neighborhood of $\{0\} \times G$ in $\mathbb{C} \times \mathbb{C}^n$;
(2) ϕ has the form (2.3) on U. □

Notice that Theorem 2.1 could also be proved using Taylor's theorem. To see this, notice that if ϕ is the flow associated with the vector field $\mathbf{X}$, we can use (2.1) to show that

$$\frac{d^k \phi_x}{dt^k}(0) = c_k(x).$$

for every x in $\mathbb{R}^n$.

The form of the solution given in Theorem 2.1 can be used in solving some familiar differential equations

Example 2.1. *Consider the one-dimensional initial value problem*

$$\dot{y} = y^2, \qquad y(0) = x.$$

In this case $c_k(x) = k!x^{k+1}$ *for* $k \geq 0$. *Therefore, if* ϕ *is the associated flow,*

$$\phi(t, x) = \sum_{k=0}^{\infty} x^{k+1} t^k = \frac{x}{1 - xt}$$

on some open neighborhood of $\{0\} \times \mathbb{C}^n$.

Example 2.2. *Consider the* n*-dimensional system*

$$\dot{y} = Ay, \qquad y(0) = x$$

where A *is an* $n \times n$ *matrix of real numbers. In this case* $c_k(x) = A^k x$ *for* $k \geq 0$. *Therefore, if* ϕ *is the associated flow,*

$$\phi(t, x) = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} x = e^{At} x$$

on some open neighborhood of $\{0\} \times \mathbb{C}^n$ *(namely* $\mathbb{C} \times \mathbb{C}^n$*).*

Example 2.3. *Consider the* 2*-dimensional system*

$$\dot{x} = x, \quad x(0) = x_0, \quad \dot{y} = 3y + x^2, \quad y(0) = y_0. \tag{2.10}$$

If ϕ *is the associated flow,*

$$\phi(t, (x_0, y_0)) = \begin{pmatrix} x_0 e^t \\ (y_0 + x_0^2)e^{3t} - x_0^2 e^{2t} \end{pmatrix}$$

on some open neighborhood of $\{0\} \times \mathbb{C}^n$ *(namely* $\mathbb{C} \times \mathbb{C}^n$*).*

Proof. We will assume a form for the c_k's and then show by induction that the form holds for all k. Let

$$\bar{c}_k(x_0, y_0) = \begin{pmatrix} x_0 \\ 3^k y_0 + R_k x_0^2 \end{pmatrix}, \qquad k \geq 0,$$

where the R_k's are constants which satisfy the difference equation

$$R_{k+1} = 2R_k + 3^k, \qquad k \geq 0, \qquad R_0 = 0. \tag{2.11}$$

Notice that $\bar{c}_0 = c_0$. Assume that $\bar{c}_k = c_k$ for some $k \geq 0$. Then

$$\begin{aligned} c_{k+1}(x_0, y_0) &= Dc_k(x_0, y_0)\mathbf{V}(x_0, y_0) \\ &= \begin{pmatrix} 1 & 0 \\ 2R_k x_0 & 3^k \end{pmatrix} \begin{pmatrix} x_0 \\ 3y_0 + x_0^2 \end{pmatrix} \\ &= \begin{pmatrix} x_0 \\ 3^{k+1} y_0 + (2R_k + 3^k)x_0^2 \end{pmatrix} \\ &= \begin{pmatrix} x_0 \\ 3^{k+1} y_0 + R_{k+1} x_0^2 \end{pmatrix}. \end{aligned}$$

Thus $c_k = \bar{c}_k$ for all $k \geq 0$. We solve (2.11) to get $R_k = 3^k - 2^k$. Thus

$$c_k(x_0, y_0) = \begin{pmatrix} x_0 \\ 3^k(y_0 + x_0^2) - 2^k x_0^2 \end{pmatrix}.$$

By Theorem 2.1, if ϕ is the flow associated with (2.10)

$$\begin{aligned} \phi(t, (x_0, y_0)) &= \sum_{k=0}^{\infty} \begin{pmatrix} x_0 t^k / k! \\ (3^k(y_0 + x_0^2) - 2^k x_0^2) t^k / k! \end{pmatrix} \\ &= \begin{pmatrix} x_0 e^t \\ (y_0 + x_0^2)e^{3t} - x_0^2 e^{2t} \end{pmatrix}. \quad \square \end{aligned}$$

3. Applications to polynomial flows

Which vector fields have polynomial flows? Using the results in §1, we can now narrow down the field of candidates to those vector fields which

(1) are polynomial;
(2) have entire flows;
(3) have constant divergence.

However, as Example 1.1 shows, a vector field may have all three of these properties and still not be a p-f vector field.

In this section we give a condition, based on the Maclaurin coefficients, necessary and sufficient for a vector field to have a polynomial flow. We caution the reader that verification of this condition is tantamount to finding a closed form expression for the flow. Hence, while it is a strong condition, applying it is not practical.

In this section we also give a condition, also based on the Maclaurin coefficients, sufficient for a vector field not to have a polynomial flow. This condition, in contrast, is frequently easy to apply. We will use this condition to show that the Lorenz system does not have a polynomial flow. See Coomes [C2] for another proof of this.

Define the degree of of a polynomial map $P: \mathbb{R}^n \to \mathbb{R}^n$ to be the maximum of the degrees of each of its components. Also define the degree of zero to be $-\infty$. Notice that the system in Example 2.3 has a polynomial flow. Also notice that the c_k's are of bounded degree. This is a property which characterizes p-f vector fields as we show in the following.

Theorem 3.1. *Let* $\mathbf{V}$ *be a polynomial vector field on* $\mathbb{R}^n$ *and let* c_k, $k = 0, 1, 2, \dots$, *be the Maclaurin coefficients of* $\mathbf{V}$. *Then* $\mathbf{V}$ *is a p-f vector field if and only if the sequence* $\{\deg c_k(x)\}_{k=0}^{\infty}$ *is bounded above.*

Proof. ($\Leftarrow$) If the sequence is bounded above, say by d, then we have vectors $a_{kr} \in \mathbb{R}^n$ such that $c_k(x) = \sum_{|r|\le d} a_{kr}x^r$ for all $k \ge 0$. If ϕ is the flow associated with $\mathbf{V}$, by Theorem 2.1 we have

$$\phi(t, x) = \sum_{k=0}^{\infty} \frac{c_k(x)}{k!} t^k = \sum_{k=0}^{\infty} \sum_{|r|\le d} a_{kr} x^r t^k / k!. \tag{3.1}$$

Notice that $\mathbf{V}(x) = c_1(x) = \sum_{|r|\le d} a_{1r}x^r$. Let M be the number of n-tuples of nonnegative integers r with $|r| \le d$. If

$$a_{kr} = \begin{pmatrix} a_{kr}(1) \\ \vdots \\ a_{kr}(n) \end{pmatrix}$$

then let $N_k = \max\{\, |a_{kr}(i)| : 1 \le i \le n,\ |r| \le d \,\}$. Since $c_{k+1}(x) = Dc_k(x)\mathbf{V}(x)$, the ith component of c_{k+1} is given by

$$\begin{aligned} \sum_{|r|\le d} a_{k+1,r}(i)x^r &= \sum_{j=1}^{n} \sum_{|r|\le d} a_{kr}(i) r_j \frac{x^r}{x_j} \sum_{|s|\le d} a_{1s}(j) x^s \\ &= \sum_{j=1}^{n} \sum_{|r|\le d} \sum_{|s|\le d} a_{kr}(i) a_{1s}(j) r_j \frac{x^r x^s}{x_j} \end{aligned}$$

which implies that $|a_{k+1,r}(i)| \le nM^2 d N_1 N_k$, $i = 1, \dots, n$. Hence $N_{k+1} \le nM^2 d N_1 N_k$. Notice that $N_0 = 1$. Thus if we let $L = nM^2 d N_1$, we have $N_k \le L^k$ for all $k \ge 0$.

Pick $x \in \mathbb{C}^n$. Let $K = \max\{1, |x_1|, \dots, |x_n|\}$. If $\|\cdot\|$ is the sup norm,

$$\begin{aligned} \sum_{k=0}^{\infty} \sum_{|r|\le d} \|a_{kr} x^r t^k / k!\| &\le \sum_{k=0}^{\infty} \sum_{|r|\le d} \frac{L^k K^d |t|^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{M L^k K^d |t|^k}{k!} \\ &= M K^d e^{L|t|}. \end{aligned}$$

Thus we may switch the order of summation in (3.1) and hence

$$\begin{aligned} \phi(t, x) &= \sum_{|r|\le d} \sum_{k=0}^{\infty} a_{kr} x^r t^k / k! \\ &= \sum_{|r|\le d} x^r \sum_{k=0}^{\infty} a_{kr} t^k / k! \\ &= \sum_{|r|\le d} a_r(t) x^r. \end{aligned}$$

Therefore $\mathbf{V}$ has a polynomial flow.

($\Rightarrow$) If $\mathbf{V}$ is a p-f vector field with flow ϕ, by Theorem 2.1 we have

$$c_k(x) = \frac{d^k \phi_x}{dt^k}(0).$$

But we also have

$$\frac{d^k \phi_x}{dt^k}(0) = \sum_{|r| \le d} \frac{d^k a_r}{dt^k}(0) x^r.$$

Hence all the c_k's are of degree less than or equal to d. □

Notice that contained in the proof of Theorem 3.1 is another proof that solutions associated with p-f vector fields are entire (see Theorem 1.1).

Without explicitly calculating the Maclaurin coefficients of a given polynomial vector field, it is difficult to determine whether they are of bounded degree. But, from Theorem 2.1, a formula for the Maclaurin coefficients gives a formula for the flow. That is, explicitly calculating the Maclaurin coefficients is equivalent to explicitly calculating the flow. This points out the impracticality of applying Theorem 3.1. However, the following theorem does lead to a practical method for determining whether a vector field does not have a polynomial flow.

Theorem 3.2. *Let* $\mathbf{V}$ *be a p-f vector field and let* c_k, $k = 0, 1, 2, \ldots$, *be the Maclaurin coefficients of* $\mathbf{V}$. *Then* $\max_{k \ge 0}\{\deg c_k\} = \max_{t \in \mathbb{R}}\{\deg \phi^t\}$.

Proof. Since $\mathbf{V}$ is a p-f vector field, both $\{\deg c_k : k = 1, 2, 3, \ldots\}$ and $\{\deg \phi^t : t \in \mathbb{R}\}$ are finite sets containing 1 ($\deg c_0 = \deg \phi^0 = 1$). Thus the maxima of these sets are defined. Let $d_1 = \max_{k \ge 0}\{\deg c_k\}$ and $d_2 = \max_{t \in \mathbb{R}}\{\deg \phi^t\}$. It follows that we may write

$$\phi(t, x) = \sum_{|r| \le d_2} a_r(t) x^r.$$

Since

$$c_k(x) = \frac{d^k \phi_x}{dt^k}(0) = \sum_{|r| \le d_2} \frac{d^k a_r}{dt^k}(0) x^r$$

we have $d_1 \le d_2$. It also follows from Theorem 1.1 that we may write

$$\phi(t, x) = \sum_{k=0}^{\infty} \frac{c_k(x)}{k!} t^k.$$

Using the same notation as in the proof of Theorem 3.1, if $c_k(x) = \sum_{|r| \le d_1} a_{rk} x^r$, we have

$$\phi(t, x) = \sum_{k=0}^{\infty} \sum_{|r| \le d_1} a_{rk} x^r t^k / k!. \tag{3.2}$$

In the proof of Theorem 3.1 we showed that we can switch the order of summation in (3.2) and factor out x^r to get

$$\phi(t, x) = \sum_{|r| \le d_1} x^r \sum_{k=0}^{\infty} a_{rk} t^k / k!.$$

Thus the degree of each t-advance map is less than or equal to d_1. Therefore $d_2 \le d_1$. Hence $d_2 = d_1$. □

Call a polynomial map $P: \mathbb{R}^n \to \mathbb{R}^n$ a *p-symmetry* of the vector field $\mathbf{V}$ if P has a polynomial inverse and maps solutions of (0.1) to solutions of (0.1). If $\mathbf{V}$ has a polynomial flow, (see Coomes [C1 or C2]) each t-advance map ϕ^t is a p-symmetry of $\mathbf{V}$.

We now give our method, based on Theorem 3.2 and p-symmetries, for determining when a polynomial vector field does not have a polynomial flow: Given a polynomial vector field we can frequently get a bound on the degree of its p-symmetries. The degree of some c_k exceeds this bound if and only if $\mathbf{V}$ does not have a polynomial flow.

Example 3.1. *The Lorenz system*

$$\left.\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= -\beta z + xy \end{aligned}\right\} \quad \begin{aligned} &(x, y, z) \in \mathbb{R}^3, \\ &\sigma, \rho, \beta > 0 \end{aligned} \tag{3.3}$$

(*see, for example, Lorenz* [L], *Sparrow* [S], *and Guckenheimer and Holmes* [GH]) *does not have a polynomial flow.*

Proof. By way of contradiction, suppose that the Lorenz system does have a polynomial flow. It can be shown (see Coomes [C1 or C2]) that every p-symmetry of (3.3) has degree at most two. By the remarks made above, it follows that every t-advance map associated with (3.3) has degree less than or equal to two. That is, $\max_{t \in \mathbb{R}}\{\deg \phi^t\} \le 2$.

Let $\mathbf{V}$ be the vector field of (3.3) and let c_k, $k = 0, 1, 2, \ldots$, be the Maclaurin coefficients of $\mathbf{V}$

Notice that $c_1 = \mathbf{V}$. Thus

$$\begin{aligned} c_2(x) &= Dc_1(x)\mathbf{V}(x) = D\mathbf{V}(x)\mathbf{V}(x) \\ &= \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{pmatrix} \begin{pmatrix} \sigma(y - x) \\ \rho x - y - xz \\ xy - \beta z \end{pmatrix} \\ &= \begin{pmatrix} (\sigma^2 + \sigma\rho)x - (\sigma^2 + \sigma)y - \sigma xz \\ (\sigma\rho + 1)y - (\sigma\rho + \rho)x + (\beta + \sigma + 1)xz - \sigma yz - x^2 z \\ \rho x^2 - (\beta + \sigma + 1)xy + \sigma y^2 + \beta^2 z - x^2 z \end{pmatrix}. \end{aligned}$$

Thus $\deg c_2 = 3$. Hence $\max_{k \ge 0}\{\deg c_k\} \ge 3$. We have

$$\max_{t \in \mathbb{R}}\{\deg \phi^t\} \le 2 < 3 \le \max_{k \ge 0}\{\deg c_k\}$$

which contradicts Theorem 3.2. □

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