WEAKLY ALMOST PERIODIC FUNCTIONS
AND THIN SETS IN DISCRETE GROUPS

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ABSTRACT. A subset $E$ of an infinite discrete group $G$ is called (i) an $R_w$-set
if any bounded function on $G$ supported by $E$ is weakly almost periodic, (ii)
a weak $p$-Sidon set ($1 \leq p < 2$) if on $l^1(E)$ the $l^p$-norm is bounded by a
constant times the maximal $C^*$-norm of $l^1(G)$, (iii) a $T$-set if $xE \cap E$ and
$Ex \cap E$ are finite whenever $x \neq e$, the identity of $G$. It was first
proved by W. Rudin [17] that $T$-sets and hence finite unions of $T$-sets are $R_w$-
sets. However, they seem to constitute the only known $R_w$-sets in the literature.
In §3 we show that every infinite group contains an $R_w$-set which is not a finite
union of $T$-sets. $R_w$-sets have already been studied by W. Ruppert [18]. We
need the following characterization of $R_w$-sets which is similar to a result of
his: a subset $E$ of $G$ is an $R_w$-set if and only if it does not contain a set of
the form $\{x_iy_j: i = 1, 2, \ldots, 1 \leq j \leq i\}$ or $\{x_iy_j: j = 1, 2, \ldots, 1 \leq i \leq j\}$
where $\{x_i\}$ and $\{y_j\}$ are two sequences of distinct elements in $G$.

For $1 \leq p < 2$, a subset $E$ of $G$ is called a weak $p$-Sidon set if there is a
finite constant $\tau$ such that $\|f\|_p \leq \tau \|f\|_*$ whenever $f \in l^1(E)$ where
$\|\cdot\|_*$ denotes the maximal $C^*$-algebra norm on $l^1(G)$. Weak 1-Sidon sets are called
weak Sidon sets in Picardello [15] and, for abelian $G$, weak $p$-Sidon sets are just
$p$-Sidon sets as defined in Edwards and Ross [7]. We show that if $1 \leq p < 4/3$
and $E$ is a weak $p$-Sidon set then $E$ contains no large squares. This generalizes
a result in [7] for abelian groups. Déchamps-Gondim [5] proved that countable Sidon sets in abelian groups are finite unions of $T$-sets. We are able to adopt her proof to show in §4 that if $1 ≤ p < 4/3$ then countable weak $p$-Sidon sets are finite unions of $T$-sets. J. Bourgain [2] showed that Sidon sets, countable or not, in abelian groups are always finite unions of $T$-sets. It does not seem to be known whether his result holds for $p$-Sidon sets if $1 < p < 4/3$.

On the other hand, $T$-sets can be quite large. Indeed, §4 also contains the following result which improves a result of ours in [4]: every infinite $G$ contains a $T$-set $E$ such that, for each positive integer $k$, $E$ has a subset $A$ of the form $A = A_1 \cdots A_k = \{x_1 \cdots x_k : x_i ∈ A_i, i = 1, \ldots, k\}$ where $|A_i| = k$ and $|A| = k^k$. By a result of Johnson and Woodward [12], we then conclude that every infinite abelian group contains a $T$-set which is not a $p$-Sidon set for any $1 ≤ p < 2$.

Definitions and general results on $R_w$-sets and weak $p$-Sidon sets are contained in §2.

2. PRELIMINARIES AND GENERAL RESULTS

Throughout this paper, $G$ denotes an infinite discrete group, $N$ the set of positive integers, and for a set $A$, $|A|$ the cardinality of $A$. 

**Definition 2.1.** (a) If $\{x_i : i ∈ N\}$ and $\{y_j : j ∈ N\}$ are two sequences in $G$ such that $(i, j) → x_i y_j$ is a one-one mapping from $N^2$ into $G$ then $S = \{x_i y_j : i, j ∈ N\}$ is called an infinite square in $G$ and the sets $\{x_i y_j : i ∈ N, 1 ≤ j ≤ l\}$ and $\{x_i y_j : j ∈ N, 1 ≤ i ≤ j\}$ are called infinite triangles.

(b) If $A, A_1, i = 1, \ldots, k$, are subsets of $G$, $A = A_1 \cdots A_k$, $|A_i| = n$ and $|A| = n^k$ then $A$ is called a $k$-cube of length $n$.

(c) If $C = AB$ or $BA$ where $A$ is infinite, $|B| = n$, and $(a, b) → ab$ or $ba$ is a one-one mapping from $A × B$ to $AB$ or $BA$, then $C$ is called a strip of width $n$.

A subset $E$ in $G$ is said to contain large $k$-cubes if, for any given $n ∈ N$, $E$ contains a $k$-cube of length $n$. $E$ is said to contain wide strips if, for any given $n ∈ N$, $E$ contains a strip of width $n$. A 2-cube is called a square in [4] and a 1-cube of length $n$ is just a set with $n$ elements. A $k$-cube of length 2 is also called a parallelepiped of dimension $k$; see Hare [11].

**Lemma 2.2.** (a) If $E = AB$ where $A$ and $B$ are infinite subsets of $G$ then $E$ contains an infinite square.

(b) Suppose that $E ⊂ G$. If, for each $n ∈ N$, there exist subsets $A_1, \ldots, A_k$ of $G$ such that $|A_i| = n, i = 1, \ldots, k$, and $A_1 \cdots A_k ⊂ E$ then $E$ contains large $k$-cubes.

(c) If $E = AB$ where $A, B ⊂ G$, $A$ is infinite and $|B| = n$, then there exists an infinite set $A_i ⊂ A$ such that $A_i B$ is a strip (of width $n$).

**Proof.** (a) Suppose that we have chosen $A_n = \{a_1, \ldots, a_n\} ⊂ A$, $B_n = \{b_1, \ldots, b_n\} ⊂ B$ such that $|A_n B_n| = n^2$. Choose $a_{n+1} ∈ A \setminus A_n B_n B_n^{-1}$ and
then choose $b_{n+1} \in B \setminus A_{n+1}^{-1} A_{n+1} B_n$ where $A_{n+1} = \{a_1, \ldots, a_{n+1}\}$. Let $B_{n+1} = \{b_1, \ldots, b_{n+1}\}$. Then $|A_{n+1} B_{n+1}| = (n+1)^2$. Thus, by induction, $E$ contains an infinite square $\{a_i b_j : i, j \in N\}$.

(b) For $k = 2$, this result was proved in [13, p. 8]. In general, using the method of [13], it is not hard to show, by induction on $k$, that if $B_i \subset G$, $|B_i| = n^{2i-1} + 1$, $i = 1, \ldots, k$, then there exist $A_i \subset B_i$ such that $|A_i| = n$ and $|A_1 \cdots A_k| = n^k$.

We omit the simple proof of (c).

Note that the proof of (a) also shows that if $\{a_i\}$ and $\{b_j\}$ are two sequences of distinct elements in $G$ then $E = \{a_i b_j : i \in N, 1 \leq j \leq n\}$ contains an infinite triangle.

As usual, $l^\infty(G)$ denotes the space of bounded complex-valued functions on $G$ with sup norm. For $f \in l^\infty(G)$ and $x \in G$, $xf \in l^\infty(G)$ is defined by $xf(y) = f(xy)$, $y \in G$. $f \in l^\infty(G)$ is said to be weakly almost periodic (w.a.p.) if the left orbit $O_L(f) = \{xf : x \in G\}$ of $f$ is relatively weakly compact in $l^\infty(G)$. WAP$(G)$, the space of w.a.p. functions on $G$, is a translation invariant $C^*$-subalgebra of $l^\infty(G)$ and, by Ryll-Nardzewski's fixed point theorem [19], it has a unique two-sided invariant mean $m_G$. The following result of Grothendieck [10] is the basic tool in our study of w.a.p. functions.

**Lemma 2.3** (Grothendieck's criterion). $f \in l^\infty(G)$ is w.a.p. if and only if whenever $\{x_i\}$ and $\{y_j\}$ are two sequences in $G$ and $\lim_i \lim_j f(x_i y_j)$ and $\lim_j \lim_i f(x_i y_j)$ exist, then they are equal.

If $E$ is a subset of $G$ and $A$ a subalgebra of $l^\infty(G)$ then $l^\infty(E)$ is said to reside in $A$, or, in short, $E$ is an $R_A$-set, if whenever $f \in l^\infty(G)$ and $f$ vanishes off $E$ then $f \in A$. Clearly, the union of two $R_A$-sets is an $R_A$-set. For convenience, $R_{WAP(G)}$-sets will be called $R_W$-sets. Since $T$-sets are $R_W$-sets (see [4, Lemma 3.2]), finite unions of $T$-sets are $R_W$-sets.

**Proposition 2.4.** Let $E$ be a subset of $G$. Then the following two conditions are equivalent:

1. $E$ is an $R_W$-set;
2. if $\{a_i : i \in N\}$ is a sequence of distinct elements in $G$ then both the sets $A = \{x \in G : xa_i$ is eventually in $E\}$, $B = \{x \in G : a_i x$ is eventually in $E\}$ are finite.

**Proof.** (1) $\Rightarrow$ (2). Suppose that $B$ is infinite. Then there exists a sequence of distinct elements $\{b_j : j \in N\}$ in $G$ such that for each $j$, $\{a_i b_j : i \in N\}$ is eventually in $E$. By replacing $\{a_i\}$ and $\{b_j\}$ by subsequences, we may assume that $\{a_i b_j : i, j \in N\}$ is an infinite square; see Lemma 2.2(a). Define $f \in l^\infty(G)$ by setting $f(a_i b_j) = 1$ if $a_i b_j \in E$ and $i \geq j$ and $f(x) = 0$ for all
other \( x \in G \). Then
\[
\lim_{i} \lim_{j} f(a_i b_j) = 0, \quad \lim_{j} \lim_{i} f(a_i b_j) = 1,
\]
and hence, by Lemma 2.3, \( f \notin \text{WAP}(G) \). Since \( f \) vanishes off \( E \), by definition, \( E \) is not an \( R_w \)-set. Similarly, if \( A \) is infinite then \( E \) is not an \( R_w \)-set.

(2) \( \Rightarrow \) (1). Suppose that (2) holds. Since \( \text{WAP}(G) \) is a norm closed linear space, to show that \( E \) is an \( R_w \)-set, it suffices to show that \( \chi_A \in \text{WAP}(G) \) for each \( A \subset E \). By Lemma 2.3, it suffices to show that whenever \( \{a_i\} \) and \( \{b_j\} \) are two sequences in \( G \) such that
\[
L_1 = \lim_{i} \lim_{j} \chi_A(a_i b_j), \quad L_2 = \lim_{j} \lim_{i} \chi_A(a_i b_j)
\]
exist then \( L_1 = L_2 \). It is easy to see that if either \( \{a_i\} \) or \( \{b_j\} \) is eventually a constant then \( L_1 = L_2 \). Therefore, we only have to consider the case that \( \{a_i\} \) and \( \{b_j\} \) are sequences of distinct elements. We claim that in this case \( L_1 = L_2 = 0 \). Indeed, if, say, \( L_1 = 1 \) then there exists \( j_0 \) such that if \( j \geq j_0 \) then \( \lim_i \chi_A(a_i b_j) = 1 \), i.e., for \( j \geq j_0 \), \( \{a_i b_j ; i \in N\} \) is eventually in \( E \). Therefore \( B \) is infinite, a contradiction.

Remarks. (1) In order to show that weak Sidon sets are \( R_w \)-sets, we presented the above proposition at the 1982 Summer Meeting of the American Mathematical Society in Toronto; see Abstracts Amer. Math. Soc. 3 (1982), p. 353. Meanwhile, Ruppert has obtained, independently, several characterizations of \( R_w \)-sets in [18]. His condition (ii) in Theorem 7 of [18] is equivalent to our condition (2) above. For the sake of completeness, we include a proof of our proposition here.

(2) As usual, if \( \{A_i\} \) is a sequence of sets then \( \liminf A_i = \bigcup_{n=1}^{\infty} (\bigcap_{i=n}^{\infty} A_i) \). The above proposition states that \( E \) is an \( R_w \)-set if and only if \( \liminf a_i E \) and \( \liminf E a_i \) are finite for any sequence \( \{a_i\} \) of distinct elements in \( G \).

(3) It is not hard to see that the above proposition can be also stated as follows: a subset \( E \) of \( G \) is an \( R_w \)-set if and only if it does not contain infinite triangles.

Lemma 2.5. If \( E \) is an \( R_w \)-set in an infinite group \( G \), then \( m_G(\chi_E) = 0 \).

Proof. Since \( \chi_E \in \text{WAP}(G) \), \( m_G(\chi_E) \) is well defined. By Ryll-Nardzewski’s fixed point theorem [19], \( m_G(\chi_E) = c \) is the unique constant in the closed convex hull of \( O_L(\chi_E) \). If \( c > 0 \), then there exists \( \sum_{i=1}^{n} \lambda_i \chi_{x_i E} \in \text{co} O_L(\chi_E) \) (the convex hull of \( O_L(\chi_E) \)) such that
\[
\left\| \sum_{i=1}^{n} \lambda_i \chi_{x_i E} - c \right\|_{\infty} < \frac{c}{2}.
\]
This implies that \( \bigcup_{i=1}^{n} x_i E = G \). Since \( E \) is an \( R_w \)-set, so are \( x_i E \), \( i = 1, \ldots, n \). Therefore, \( G = \bigcup_{i=1}^{n} x_i E \) is also an \( R_w \)-set and hence \( \text{WAP}(G) = l^{\infty}(G) \). This contradicts the well-known fact that \( \text{WAP}(G) \not\subseteq l^{\infty}(G) \); see [3, p.
A group \( G \) is said to be amenable if \( l^\infty(G)^* \) has a left invariant mean \( \mu: \mu \in l^\infty(G)^* \), \( \|\mu\| = 1 \), \( \mu \geq 0 \) and \( \mu(l_x f) = \mu(f) \) for all \( f \in l^\infty(G) \) and \( x \in G \). For example, solvable groups are amenable but nonabelian free groups are not amenable; see Pier [14]. If \( G \) is amenable, let \( \text{LIM}(G) \) be the set of all left invariant means on \( G \) and, for \( E \subset G \), let \( \overline{d}_f(E) = \sup\{\mu(\chi_E): \mu \in \text{LIM}(G)\} \), the left upper density of \( E \). For amenable \( G \), if \( \chi_E \in \text{WAP}(G) \) then \( m_G(\chi_E) = \overline{d}_f(E) \). Therefore, by the above lemma, if \( \overline{d}_f(E) > 0 \) then \( E \) is not an \( R_w \)-set. By Proposition 2.4, we obtain the following.

**Corollary 2.6.** If \( G \) is an infinite amenable group, \( E \subset G \) and \( \overline{d}_f(E) > 0 \) then \( E \) contains infinite triangles.

**Remark.** Let \( Z \) be the additive group of integers. It is easy to construct a subset \( E \) of \( Z \) such that (i) \( E \) contains infinite triangles, (ii) \( \overline{d}_f(E) = 0 \) and (iii) \( E \) does not contain arithmetic progressions of length 3. Note that a celebrated result of E. Szemirédi [20] states that if \( E \subset Z \) and \( \overline{d}_f(E) > 0 \) then \( E \) contains arbitrarily long arithmetic progressions; see Furstenberg [9] for an ergodic theoretical proof of this result.

Let \( C^*(G) \) be the completion of \( l^1(G) \) with respect to the maximal \( C^* \)-norm \( \| \cdot \|_*: f \in l^1(G) \),

\[
\|f\|_* = \sup\{\|\pi(f)\|: \pi \text{ a unitary representation of } G\},
\]

where \( \pi(f) = \sum \{f(x)\pi(x): x \in G\} \). Then the dual Banach space of \( C^*(G) \) can be identified with \( B(G) \) (the Fourier-Stieltjes algebra of \( G \)) which consists of coefficient functions of unitary representations of \( G \). Let \( B_\lambda(G) \) be the algebra of coefficient functions of unitary representations of \( G \) which are weakly contained in the left regular representation \( \lambda \). Then \( B_\lambda(G) \) can be identified with the dual Banach space of \( C_\lambda^*(G) \), the \( C^* \)-algebra generated by \( \{\lambda(f): f \in l^1(G)\} \). See Eymard [8], for definitions and results mentioned in this paragraph.

If \( E \) is a subset of \( G \) and \( f \in l^1(E) \) then \( f \) will be identified with the function on \( G \) which equals \( f \) on \( E \) and is identically zero off \( E \). For \( 1 \leq p < 2 \), a subset \( E \) of \( G \) is called a weak \( p \)-Sidon (\( p \)-Sidon) set if there is a finite constant \( \tau \) such that \( \|f\|_p \leq \tau\|f\|_* \) \( (\|f\|_p \leq \tau\|\lambda(f)\|) \) for each \( f \in l^1(E) \). Note that \( p \)-Sidon sets are always weak \( p \)-Sidon and if \( G \) is amenable then weak \( p \)-Sidon sets are \( p \)-Sidon, since, in this case \( \|f\|_* = \|\lambda(f)\|, f \in l^1(G) \); see [8]. For abelian \( G \), a weak \( p \)-Sidon set is just a \( p \)-Sidon set as defined by Edwards and Ross [7]. Note also that (weak) \( 1 \)-Sidon sets are just (weak) Sidon sets as defined by Picardello [15]. Furthermore, a subset \( E \) of \( G \) is a weak Sidon set (Sidon set) if and only if \( B(G)|E = l^\infty(E) \) \( (B_\lambda(G)|E = l^\infty(E)) \). We showed in [4] that weak Sidon sets do not contain large squares. This result can be strengthened somewhat with a minor change of the proof.
**Proposition 2.7.** If $1 \leq p < 4/3$ and $E$ is a weak $p$-Sidon set in $G$ then $E$ does not contain large squares.

**Proof.** The proof is similar to that of Proposition 3.4 of [4]. We will give only an outline here. Suppose that $E$ contains large squares. Then for each $n$, choose a square $S = AB$ in $E$ of length $n$, where $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. Let $(u_{ij})$ be an $n \times n$ unitary matrix with complex entries and with $|u_{ij}| = 1/\sqrt{n}$. Let $g = \sum_{i,j=1}^{n} u_{ij} \delta_{a_i,b_j}$ where for $t \in G$, $\delta_t$ denotes the function on $G$ which equals 1 at $t$ and zero elsewhere. Then

$$\|g\|_p = \left(\sum_{i,j=1}^{n} |u_{ij}|^p\right)^{1/p} = n^{2/p-1/2}.$$

On the other hand, as proved in [4], $\|g\|_\infty \leq n$. Therefore, if $E$ is a weak $p$-Sidon set, then $2/p - 1/2 \leq 1$ or $p \geq 4/3$, a contradiction.

When $G$ is abelian, the above result is due to Edwards and Ross [7, Corollary 2.7].

**Corollary 2.8.** For $1 \leq p < 4/3$, if $E$ is a weak $p$-Sidon set in $G$ then $E$ is an $R_w$-set; in particular, $\chi_E \in WAP(G)$.

**Proof.** By Proposition 2.7, $E$ does not contain large squares and hence it does not contain infinite triangles. By Proposition 2.4, $E$ is an $R_w$-set.

**Remarks.** (1) If $G$ is an infinite abelian group, then it contains an infinite square $S$ such that $S$ is a $4/3$-Sidon set; see [7, Corollary 5.5]. By Proposition 2.4, $S$ is not an $R_w$-set. Therefore the above result does not hold if $p \geq 4/3$.

(2) If the set $E$ in the above corollary is countable, one can actually conclude that $E$ is a finite union of $T$-sets; see §4.

(3) If $G$ is abelian, a well-known result of Drury [6] states that if $E$ is a Sidon set then $\chi_E \in B(G)^-$, the uniform closure of $B(G)$. Note that, for every infinite group $G$, $B(G)^-$ is properly contained in $WAP(G)$; see [4].

**Lemma 2.9.** For $E \subset G$ and $n \in \mathbb{N}$, consider the following conditions:

(i) $E$ is a union of $n$ $T$-sets;

(ii) $E$ contains no strips of width $n+1$;

(iii) for any finite set $F$ in $G$, $\{x \in G : |xF \cap E| > n\}$ and $\{x \in G : |F \cap E| > n\}$ are finite.

Then (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii).

**Proof.** (i) $\Rightarrow$ (ii). Let $E = E_1 \cup \cdots \cup E_n$ where $E_1, \ldots, E_n$ are $T$-sets. Suppose that $E$ contains a strip $C$ of width $n+1$, say $C = BA$ where $B = \{b_1, \ldots, b_{n+1}\}$ and $A = \{a_1, a_2, \ldots\}$. By replacing $A = \{a_j\}$ by a subsequence, if necessary, we may assume that, for each $1 \leq i \leq n+1$, $b_iA$ is contained in some $E_k$, $1 \leq k \leq n$. So there exist $i_1, i_2 \in \{1, 2, \ldots, n+1\}$, $i_1 \neq i_2$, such that $b_{i_1}A \cup b_{i_2}A \subset E_{k_0}$ for some $1 \leq k_0 \leq n$. This contradicts the fact that $E_{k_0}$ is a $T$-set.
(ii) $\Rightarrow$ (iii). Suppose that there exist a finite set $F$ and a sequence of distinct elements $\{x_k\}$ in $G$ such that $|x_k F \cap E| > n$ for all $k$. Then for each $k$ there is a set $F_k \subset F$ such that $|F_k| = n + 1$ and $x_k F_k \subset E$. Since $F$ contains only finitely many subsets with cardinality $n + 1$, there exists an infinite subset $I$ of $N$ and $F' \subset F$ such that, if $k \in I$, then $F_k = F'$. Then $\{x_k', k \in I\}F'$ is a strip of width $n + 1$ contained in $E$.

For convenience, we call a set $E$ in $G$ an FT-set if it is a finite union of $T$-sets. It is not hard to see that a subset $E$ of $G$ is a $T$-set if and only if given any finite subset $d$ of $G$ there exists a finite subset $F$ of $E$ such that $x, y \in E \setminus F$ and $x \neq y$ imply $x y^{-1}, x^{-1} y \notin \Delta$. Therefore, in the terminology of [13, p. 112], a set $E$ is a $T$-set if and only if it tends to infinity.

3. EXISTENCE OF $R_w$-SETS WHICH ARE NOT FT-SETS

As in [4], a subset $E$ of $G$ is said to be relatively dense if there exist finite sets $X$ and $Y$ such that $G = X Y$. We need the following result of ours in [4].

Lemma 3.1. Let $S$ be a relatively dense subset of $G$ and $F$ a finite subset of $G, e \notin F$. Then there exists a relatively dense subset $E$ of $S$ such that $(x E \cap E) \cup (Ex \cap E) = \emptyset$ for $x \in F$.

Lemma 3.2. Let $P$ be a relatively dense subset of $G$, say $G = X P Y$ where $X$ and $Y$ are finite. Then for each positive integer $n$ there exists a finite set $E$ such that the set $Q = \{x \in G: |x E b^{-1} \cap P| \geq n$ for some $b \in Y\}$ is relatively dense in $G$. In particular, $Q$ is infinite.

Proof. Choose any finite subset $E = \{z_1, \ldots, z_k\}$ of $G$ such that $k = |E| = |X||Y|n$. Fix $x \in G$. Then, for each $1 \leq i \leq k$, $x z_i$ can be written as $x z_i = a_i p_i b_i$ where $a_i \in X, b_i \in Y$, and $p_i \in P$. For $(a, b) \in X \times Y$, let

$I(a, b) = \{i: 1 \leq i \leq k, a_i = a, b_i = b\}.$

Then $\bigcup\{I(a, b): (a, b) \in X \times Y\} = \{1, 2, \ldots, k\}$, and hence

$k = \sum|I(a, b)|: (a, b) \in X \times Y\}.$

Since $k = |X||Y|n$, there exists $(c, d) \in X \times Y$ such that $|I(c, d)| \geq n$. If $i \in I(c, d)$ then

$c^{-1} x z_i d^{-1} = p_i \in (c^{-1} x E d^{-1}) \cap P$

and the $p_i$'s, $i \in I(c, d)$, are distinct. Therefore, $c^{-1} x \in Q$. Thus $G = X Q$ and hence $Q$ is relatively dense.

We are now ready to give the main result of this section.
Theorem 3.3. Let $G$ be an infinite group. Then there exists a subset $D$ of $G$ such that

(a) $D$ is not an FT-set;
(b) $D$ is an $R_{W}$-set.

Proof. Without loss of generality, we may assume that $G$ is countably infinite. Then there exists a sequence of finite symmetric subsets $\{F_{n}\}$ of $G$ such that

$$ e \in F_{1} \subset F_{2} \subset \ldots, \quad \text{and} \quad G = \bigcup F_{n}. $$

(A set $B \subset G$ is symmetric if $B = B^{-1}$.) By Lemma 3.1, we can find a sequence of relatively dense subsets $S_{n}$ of $G$ such that $S_{1} \supset S_{2} \supset \cdots$, and

$$ (xS_{n} \cap S_{n}) \cup (S_{n}x \cap S_{n}) = \emptyset, \quad \text{if} \quad x \in F_{n} \setminus \{e\}. $$

For each $n$, choose finite sets $X_{n}$ and $Y_{n}$ such that $X_{n}S_{n}Y_{n} = G$. By Lemma 3.2, for each $n \in N$, there exists a finite set $E_{n}$ such that

$$ Q_{n} = \{x \in G: |xE_{n}b^{-1} \cap S_{n}| \geq n \text{ for some } b \in Y_{n}\} $$

is infinite.

Fix infinite subsets $N_{1}, N_{2}, \ldots$ of $N$ such that $N_{i} \cap N_{j} = \emptyset$, if $i \neq j$, and $N_{1} \cup N_{2} \cup \cdots = N$. Then for each $n \in N$ there is a unique positive integer $\sigma(n)$ such that $n \in N_{\sigma(n)}$.

Choose $t_{1} \in Q_{\sigma(1)}$, $b_{1} \in Y_{\sigma(1)}$ such that $|t_{1}E_{\sigma(1)}b_{1}^{-1} \cap S_{\sigma(1)}| \geq \sigma(1)$. Suppose that we have chosen $t_{j}, b_{j}$ in $G$, $j = 1, \ldots, n$, such that the $t_{j}$'s are distinct, $b_{j} \in Y_{\sigma(j)}$ and if $D_{j} = t_{j}E_{\sigma(j)}b_{j}^{-1} \cap S_{\sigma(j)}$, then $|D_{j}| \geq \sigma(j)$ and $F_{j}E_{j}D_{j} \cap (D_{1} \cup \cdots \cup D_{j-1}) = \emptyset$, for $2 \leq j \leq n$. Now since $Q_{\sigma(n+1)}$ is infinite there exists $t_{n+1} \in Q_{\sigma(n+1)}$ such that

$$ t_{n+1} \notin F_{n+1}(D_{1} \cup \cdots \cup D_{n})F_{n+1}Y_{\sigma(n+1)}E_{\sigma(n+1)}^{-1} \cup \{t_{1}, \ldots, t_{n}\}. $$

Since $t_{n+1} \in Q_{\sigma(n+1)}$, there exists $b_{n+1} \in Y_{\sigma(n+1)}$ such that if $D_{n+1} = t_{n+1}E_{\sigma(n+1)}b_{n+1}^{-1} \cap S_{\sigma(n+1)}$ then $|D_{n+1}| \geq \sigma(n+1)$. By (3.2),

$$ F_{n+1}D_{n+1}F_{n+1} \cap (D_{1} \cup \cdots \cup D_{n}) = \emptyset. $$

Therefore, by induction, we can construct two sequences $\{t_{n}\}$ and $\{b_{n}\}$ in $G$ such that the $t_{n}$'s are distinct, $b_{n} \in Y_{\sigma(n)}$ and if $D_{n} = t_{n}E_{\sigma(n)}b_{n}^{-1} \cap S_{\sigma(n)}$ then

$$ |D_{n}| = |D_{n} \cap S_{\sigma(n)}| \geq \sigma(n); $$

(3.4)

$$ F_{n}D_{n}F_{n} \cap (D_{1} \cup \cdots \cup D_{n-1}) = \emptyset, \quad n \geq 2. $$

Since $D_{n} \subset S_{\sigma(n)}$, (3.1) implies that

$$ \text{if } x \in F_{\sigma(n)} \setminus \{e\}, \quad \text{then} \quad (xD_{n} \cap D_{n}) \cup (D_{n}x \cap D_{n}) = \emptyset. $$

Also note that, as a consequence of (3.4), we have

$$ \text{if } x \in F_{n}, \quad m \geq n \text{ and } l \neq m, \quad \text{then} \quad (xD_{m} \cap D_{l}) \cup (D_{m}x \cap D_{l}) = \emptyset. $$
We claim that $D = \bigcup_{n=1}^{\infty} D_n$ satisfies conditions (a) and (b) in the statement of the theorem.

We will first prove that $D$ satisfies (a). Fix $k \in \mathbb{N}$. Note that if $n \in N_k$ then $b_n \in Y_k$. Since $N_k$ is infinite and $Y_k$ is finite, there exist an infinite subset $I_k$ of $N_k$ and an element $b \in Y_k$ such that if $n \in I_k$ then $b_n = b$. Let $F_k = E_k b^{-1}$. Then, for $n \in I_k$,

\[ D \cap t_n F_k = D \cap t_n E_k b_n^{-1} \supset D_n, \]

and hence $|D \cap t_n F_k| \geq |D_n| \geq k$. Since $I_k$ is infinite and \{t_n: n \in \mathbb{N}\} is a sequence of distinct elements in $G$, by (i) $\Rightarrow$ (iii) of Lemma 2.9, we conclude that $D$ is not a union of $k-1$ $T$-sets. Since $k \in \mathbb{N}$ is arbitrary, $D$ is not an $FT$-set, as claimed.

It remains to show that $D$ satisfies (b). To this end, first let $x \in F_k \setminus \{e\}$, $k \geq 2$, be fixed. Then, by (3.6),

\begin{equation}
(3.7) \quad xD \cap D \subset \{(xD_1 \cup \cdots \cup xD_{k-1}) \cap D\} \cup \left\{ \bigcup_{m \geq k} (xD_m \cap D_m) \right\}.
\end{equation}

Note that if $m \notin N_1 \cup \cdots \cup N_{k-1}$, i.e., $\sigma(m) \geq k$, then $x \in F_k \subset F_{\sigma(m)}$, and hence, by (3.5), $xD_m \cap D_m = \emptyset$. Therefore, (3.7) implies that

\begin{equation}
(3.8) \quad xD \cap D \subset F_x \cup \left( \bigcup_{m \in N_1 \cup \cdots \cup N_{k-1}} (xD_m \cap D_m) : m \in N_1 \cup \cdots \cup N_{k-1} \right),
\end{equation}

where $F_x = (xD_1 \cup \cdots \cup xD_{k-1}) \cap D$ is a finite set. Now assume that \{a_i, i \in \mathbb{N}\} is a sequence of distinct elements in $D$, $x \in F_k \setminus \{e\}$ and \{xa_1, xa_2, \ldots\} is eventually contained in $D$; in other words \{a_i\} is eventually contained in $x^{-1}D \cap D$. Since $x^{-1} \in F_k \setminus \{e\}$, by (3.8), \{a_i\} is eventually contained in $\bigcup\{D_m: m \in N_1 \cup \cdots \cup N_{k-1}\}$. Similarly, we can prove that if $x \in F_k \setminus \{e\}$ and \{a_i\} and \{a_jx\} are both eventually contained in $D$ then \{a_i\} is eventually contained in $\{D_m: m \in N_1 \cup \cdots \cup N_{k-1}\}$.

Suppose that $D$ is not an $R_w$-set. Then, by Proposition 2.4, there exist two sequences \{a_i\}, \{y_j\} of distinct elements in $G$ such that either

(I) \{ya_i, i \in \mathbb{N}\} is eventually contained in $D$ for each $j$, or

(II) \{a_iy_j, i \in \mathbb{N}\} is eventually contained in $D$ for each $j$.

By symmetry, we only have to consider case (I). By renaming the two given sequences, we may also assume that $y_1 = e$. Then, as demonstrated in the above paragraph, \{a_i\} is eventually contained in $\bigcup\{D_m: m \in N_1 \cup \cdots \cup N_{k-1}\}$ for some fixed $k \geq 2$. By taking a subsequence, if needed, we may assume that \{a_i\} is contained in $\bigcup\{D_m: m \in N_k \}$ for some fixed $k_0$. Assume that $a_i \in D_{m_i}$ where $m_i \in N_k$. We may further assume that the $m_i$'s are distinct.

For a fixed $j$, $j \neq 1$, since \{a_i\} is eventually contained in $D \cap y_j^{-1}D$, by (3.7), $a_i \in y_j^{-1}D_{m_i} \cap D_{m_i}$ when $i$ is sufficiently large. Thus, for each $l \in \mathbb{N}$ there exists an $i$ such that

\[ \{a_i = y_i a_i, y_2 a_i, \ldots, y_l a_i\} \subset D_{m_i}. \]
This is impossible, since \(|D_{n_i}| \leq |E_{k_i}|\) for each \(i\). Therefore, \(D\) is an \(R_w\)-set as claimed.

**Remarks.** (1) If \(G\) is an abelian group then the above proof can be simplified somewhat. Our result seems to be new even for \(G = \mathbb{Z}\), the additive group of integers. However, for \(\mathbb{Z}\) the set \(D\) can be constructed more explicitly as follows.

Write \(N\) as a disjoint union of infinite sets \(N_k\), \(k = 1, 2, \ldots\), and define \(\sigma(n)\) as before. Define blocks of consecutive positive integers \(C_n\), \(n = 1, 2, \ldots\), inductively so that

\[
\min C_{n+1} > \max C_n + n,
\]

(3.9)

\[
|C_n| = (\sigma(n) + 1)^2.
\]

(3.10)

Assume that \(C_n = \{t_n, t_n + 1, \ldots, t_n + (\sigma(n) + 1)^2 - 1\}\). Let

\[J_n = \{t_n, t_n + (\sigma(n) + 1), t_n + 2(\sigma(n) + 1), \ldots, t_n + \sigma(n)(\sigma(n) + 1)\}\]

and \(D = \bigcup_{n=1}^{\infty} J_n\). Then \(D\) is an \(R_w\)-set but is not an \(FT\)-set.

(2) Let \(G^w\) be the weakly almost periodic compactification of the discrete group \(G\). We can consider \(G\) as a subset of \(G^w\). Then the multiplication on \(G\) can be extended to \(G^w\) which makes \(G^w\) a semigroup with separately continuous multiplication; cf. [3]. In particular, \(G\) acts on the compact space \(G^w\) by left multiplication. From the definition of \(R_w\)-sets, it is easy to see that a subset \(E\) of \(G\) is an \(R_w\)-set if and only if \(\chi_E\) is w.a.p. and \(E^-\) (the closure of \(E\) in \(G^w\)) is the Stone-Čech compactification of \(E\); see Ruppert [18]. \(\omega \in G^w \setminus G\) is said to be strongly \(G\)-discrete if there is a neighborhood \(U\) of \(\omega\) in \(G^w \setminus G\) such that \(xU \cap yU = \emptyset\) if \(x, y \in G\), \(x \neq y\). Note that if \(E\) is an \(FT\)-set and \(\omega \in E^- \setminus G\) then \(\omega\) is strongly \(G\)-discrete. On the other hand, if \(D\) is the \(R_w\)-set constructed in Theorem 3.3, then \(D^-\) is homeomorphic to \(\beta D\) and there exists \(\omega \in D^- \setminus G\) such that \(\omega\) is not strongly \(G\)-discrete.

(3) The set \(D\) constructed in Theorem 3.3 contains large squares, since it contains wide strips. Therefore the \(R_w\)-set \(D\) is not a weak Sidon set. Hence it implies the known result that \(B(G)^-\) is properly contained in \(WAP(G)\) for every infinite group \(G\); see [4].

4. FURTHER RESULTS ON \(FT\)-SETS

Déchamps-Gondim proved in [5] that countable Sidon sets in abelian groups are \(FT\)-sets. She has actually obtained the following result in her proof: if \(E\) is a countable subset of an abelian group \(G\) and if \(E\) does not contain wide strips then \(E\) is an \(FT\)-set. Her proof, with some minor modifications, also works for nonabelian groups.

**Theorem 4.1.** A countable subset \(E\) of a group \(G\) is an \(FT\)-set if and only if it does not contain wide strips.
Proof. The "only if" part of the theorem is true no matter whether \( E \) is countable or not; see (i) \( \Rightarrow \) (ii) in Lemma 2.9. We will now outline the proof of the "if" part in four steps. Assume that \( E \) is a countable subset of \( G \) which does not contain strips of width \( n + 1 \).

(I) Given any finite set \( \Delta \) in \( G \), there exists a finite subset \( F \) of \( E \) such that

\[
|\{x, xt, xt_1t_2, \ldots, xt_1 \cdots t_{k-1} = y\}| \leq n,
\]

for all \( x \in G \). A finite set of the form

\[
\{x, xt_1, xt_1t_2, \ldots, xt_1 \cdots t_{k-1} = y\}
\]
is called a \( \Delta \)-chain in \( E \ \setminus F \) if it is a subset of \( E \setminus F \) and \( t_1, \ldots, t_{k-1} \in \Delta \). For \( x, y \in E \ \setminus F \), we define \( x \sim y \) if there is a \( \Delta \)-chain from \( x \) to \( y \). Then \( \sim \) is an equivalence relation. Note that if \( x \sim y \), \( x \neq y \), then \( x \) and \( y \) can be linked by a \( \Delta \)-chain \( x_1 = x, x_2 = xt_1, \ldots, x_k = xt_1 \cdots t_{k-1} = y \) such that \( x_1, \ldots, x_k \) are distinct. By (4.2), \( k \leq n \). Therefore if \( X \) is an equivalence class and \( x_0 \in X \) then any element of \( X \) is of the form \( x_0t \) for some \( t \in \Delta^n \). By (4.2) again, \( |X| \leq n \). Clearly, if \( x \) and \( y \) are in different equivalence classes then \( x^{-1}y \notin \Delta \). Thus the \( F_i \)'s in (II) can be taken to be the equivalence classes of \( \sim \).

(II) There exist \( E_i, \ i = 1, \ldots, n \), such that \( E = E_1 \cup \cdots \cup E_n \) and each \( E_i \) satisfies the condition that \( E_i \cap xE_i \) is finite if \( x \in G \), \( x \neq e \).

Follow the proof of Theorem 9.1 of [13]. But, unlike the proof there, the set \( E \) is not assumed to be symmetric and we apply (II) instead of Lemma 8.9 of [13]. Note that the countability of \( E \) is needed in the proof of (III).

(IV) \( E \) is an \( FT \)-set.

Write \( E = \bigcup_{i=1}^{n} E_i \) as in (III). By symmetry, each \( E_i \) can be written as \( E_i = \bigcup_{j=1}^{n} E_{ij} \), for each \( (i, j) \), \( E_{ij} \cap E_{ij} \) is finite whenever \( x \neq e \). Therefore, \( E = \bigcup_{i, j=1}^{n} E_{ij} \) and each \( E_{ij} \) is a \( T \)-set.

Corollary 4.2. Assume that \( E \) is a countable weak \( p \)-Sidon set in a group \( G \) where \( 1 \leq p < 4/3 \). Then \( E \) is an \( FT \)-set.

Proof. By Proposition 2.7, \( E \) does not contain large squares and hence does not contain wide strips. By the above theorem, \( E \) is an \( FT \)-set.
Remarks. (1) As mentioned in §2, if $p \geq 4/3$ then the above corollary is not true.

(2) We do not know whether Theorem 4.1 or Corollary 4.2 holds for uncountable sets. Using completely different arguments, Bourgain [2, Corollaire 3.5] proved that Sidon sets in abelian groups are always FT-sets. However, his proof does not carry over to the case of $p$-Sidon sets if $p > 1$. A subset $E$ of an abelian group $G$ is said to be exactly $p$-Sidon if $E$ is $p$-Sidon but is not $q$-Sidon for any $q < p$. Blei [1] proved that for any $p$, $1 < p < 2$, and for any infinite abelian group $G$, there exists a countable subset $E$ of $G$ such that $E$ is exactly $p$-Sidon. If $p < 4/3$, by Corollary 4.2, his countable $p$-Sidon set is an FT-set.

We prove in [4] that, given any infinite group $G$, there exists a $T$-set $E$ in $G$ which contains large squares. This is the key step to show that $B(G) \not\subset WAP(G)$ for any infinite group $G$; see [4]. (For abelian $G$ this result is due to Rudin [17] and Ramirez [16].) It turns out that a $T$-set can even contain large $k$-cubes for any given $k$. To prove this we need the following refinement of Lemma 3.6 of [4].

Lemma 4.3. There is a function $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that whenever $A = A_1 \cdots A_k$ is a $k$-cube of length $\alpha(k, n)$ and is contained in the union of two subsets $E_1$ and $E_2$ of a group $G$ then there exist subsets $B_i$ of $A_i$, such that $|B_i| = n$, $i = 1, \ldots, k$, and $B = B_1 \cdots B_k$ is contained in either $E_1$ or $E_2$.

Proof. Let $\alpha(1, n) = 2n$. For $k \geq 2$, define $\alpha(k, n)$ inductively, by setting

$$\alpha(k, n) = 2n \left( \frac{\alpha(k - 1, n)}{n} \right)^{k-1}$$

(For integers $m, n$, $0 \leq m \leq n$, $\binom{n}{m} = \frac{n!}{m!(n-m)!}$.) Then $\alpha$ is the function we want. We will prove this by induction on $k$.

A 1-cube of length $n$ is just a finite set with $n$ elements. If $A \subseteq E_1 \cup E_2$, $|A| = 2n = \alpha(1, n)$ then clearly there is a subset $B$ of $A$ such that $|B| = n$ and $B$ is contained in either $E_1$ or $E_2$. Suppose that our result holds for $k - 1$. Let $A = A_1 \cdots A_k$ be a $k$-cube of length $\alpha(k, n)$ and $A \subseteq E_1 \cup E_2$. Choose subsets $A'_i$ of $A_i$, $i = 1, \ldots, k - 1$, such that $|A'_i| = \alpha(k - 1, n)$. For each $y \in A_k$,

$$A' = A'_1 \cdots A'_{k-1} \subseteq E_1y^{-1} \cup E_2y^{-1}.$$ 

Therefore, by inductive assumption, for each $y \in A_k$, there exists a $(k-1)$-cube $K(y) = A'_1(y) \cdots A'_{k-1}(y)$ of length $n$ where $A'_i(y) \subseteq A'_i$ and $K(y)$ is contained in either $E_1y^{-1}$ or $E_2y^{-1}$, or equivalently, $K(y)y$ is contained in either $E_1$ or $E_2$. Clearly, there exists a set $A'_k \subseteq A_k$ such that $|A'_k| = (1/2)\alpha(k, n)$ and either (i) $K(y)y \subseteq E_1$ for all $y \in A'_k$ or (ii) $K(y)y \subseteq E_2$ for all $y \in A'_k$. Suppose that (i) holds. Let $\{C_1, \ldots, C_l\}$ be the collection of subsets of $A'$ of the form $A'_{i_1} \cdots A'_{i_l}$ where $A'_{i_i} \subseteq A'_i$ and $|A'_{i_i}| = n$, $i = 1, \ldots, k - 1$. Note
that \( l = \left( \frac{\alpha(k-1,n)}{n} \right)^{k-1} \). Let

\[ D_i = \{ y \in A'_k : K(y) = C_i \}, \quad i = 1, \ldots, l. \]

Then \( \bigcup_{i=1}^l D_i = A'_k \). Therefore, there exists some \( i_0 \) such that \( |D_{i_0}| \geq n \); otherwise,

\[ nl = \frac{1}{2}\alpha(k,n) = |A'_k| < nl, \quad \text{a contradiction.} \]

Choose \( A''_k = \{ y_1, \ldots, y_n \} \subset D_{i_0} \) and write \( C_{i_0} = A''_1 \cdots A''_{k-1} \). Then \( A''_1 \cdots A''_k \subset E_1 \). This completes the proof of the lemma.

The above lemma implies that every relatively dense subset of an infinite group contains large \( k \)-cubes for each \( k \in \mathbb{N} \); see [4, p. 146]. As a consequence, we can follow the proof of Proposition 3.10 of [4] to obtain the following.

**Theorem 4.4.** Let \( G \) be an infinite group. Then for each \( n \in \mathbb{N} \) there exists an \( n \)-cube \( K_n \) of length \( n \) in \( G \) such that \( E = \bigcup_{n=1}^\infty K_n \) is a \( T \)-set.

Johnson and Woodward [12] proved that if a subset \( E \) of an abelian group contains large \( k \)-cubes then it is not a \( p \)-Sidon set for any \( p < 2k/(k + 1) \). Therefore Theorem 4.4 has the following consequence.

**Corollary 4.5.** Let \( G \) be an infinite abelian group. Then there exists a \( T \)-set \( E \) in \( G \) such that \( E \) is not a \( p \)-Sidon set for any \( 1 \leq p < 2 \).

We do not know whether the above corollary holds for general infinite groups.

**References**


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