ON THE BIHOMOGENEITY PROBLEM OF KNASTER

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ABSTRACT. The author constructs a locally connected, homogeneous, finite-dimensional, compact, metric space which is not bihomogeneous, thus providing a compact counterexample to a problem posed by B. Knaster around 1921.

0. INTRODUCTION

A topological space $X$ is said to be homogeneous if for every two points $p$ and $q$ in $X$ there exists a homeomorphism $h: X \to X$ such that $h(p) = q$. $X$ is said to be bihomogeneous if for every two points $p$ and $q$ in $X$ there exists a homeomorphism $h: X \to X$ such that $h(p) = q$ and $h(q) = p$. Around 1921, B. Knaster asked the question of whether every homogeneous space is bihomogeneous, and shortly after that C. Kuratowski (see [6]) described an example of a non-locally-compact, homogeneous subset of the plane, which is not bihomogeneous. In 1930, D. van Danzig asked whether homogeneity implies bihomogeneity for continua; see [10]. A locally compact, homogeneous, nonbihomogeneous, metric space was found by H. Cook in the early 1980s; see [3].

This paper contains an example of a seven-dimensional, homogeneous, nonbihomogeneous, locally connected, compact metric space. G. S. Ungar proved that certain homogeneity type properties imply local connectedness; see [9]. However, [5] and this paper show that a locally connected homogeneous continuum may lack some stronger but still very simple homogeneity properties.

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1. PRELIMINARIES

All spaces considered in this paper are metric and all maps are continuous. By $S^n$, $E^n$, $B^n$, and $\overline{B}^n$ we mean the $n$-dimensional sphere, the Euclidean $n$-space, the $n$-dimensional open ball, and the $n$-dimensional closed ball respectively.
Let $P$ and $Q$ be two disjoint, closed subsets in a compact space $X$, and let $g: P \to Q$ be a homeomorphism. Let $\sim$ be an equivalence relation on $X$ such that $p \sim q$ iff $p = q$, or if $p \in P$ and $q \in Q$ then $g(p) = q$, or if $q \in P$ and $p \in Q$ then $g(q) = p$. The space of equivalence classes with the quotient topology will be denoted by $X/g$.

In our applications of homology theory, we use either the singular or Čech homology groups with integral coefficients. For basic concepts of homotopy theory we refer the reader to [4].

Throughout this paper, $M$ will denote the universal Menger curve as described in R. D. Anderson's paper [1, p. 321]. $M$ is a subset of the cube \[ \{(x, y, z) \in \mathbb{R}^3 : x, y, z \in [0, 1] \} \] such that the intersection of $M$ with each of the faces of the cube is homeomorphic to Sierpinski's plane curve.

In [1], Anderson proved that $M$ is homogeneous, and that every 1-dimensional continuum with no local cut points and no open subsets embeddable in the plane is homeomorphic to $M$. Furthermore, from results in [1 and 2], it follows that if $U$ is an open connected subset of $M$, and $p, q \in U$, then there exists a homeomorphism $h: M \to M$ such that $h(p) = q$, and $h(v) = v$ for $v \in M - U$.

In [5], the authors employ the fact that continua which are Cartesian products with one or more factors homeomorphic to $M$ admit few homeomorphisms. A similar idea is used here in the form of Lemmas 1 and 2 whose proofs are analogous to those of Theorems 2 and 1 in [5].

**Lemma 1.** Let $X = X_1 \times X_2$, where $X_i$ is homeomorphic to $M$ for $i = 1, 2$. Let $U_i \subset X_i$ be a connected open set for $i = 1, 2$. If $\varphi: U_1 \times U_2 \to X$ is an open embedding, then $\varphi = \varphi_1 \times \varphi_2$, where either (1) $\varphi_1: U_1 \to X_1$ and $\varphi_2: U_2 \to X_2$, or (2) $\varphi_1: U_1 \to X_2$ and $\varphi_2: U_2 \to X_1$.

**Proof.** Let $\pi_i: X \to X_i$ be the projection. Suppose that $(u, v_1)$ and $(u, v_2)$ are two distinct points in $U_1 \times U_2$. Let $\varphi((u, v_1)) = (x_1, y_1)$ and $\varphi((u, v_2)) = (x_2, y_2)$. Suppose that $x_1 \neq x_2$ and $y_1 \neq y_2$. Let $V_1 \subset X_1$ and $V_2 \subset X_2$ be such that $V_1 \times V_2$ is a neighborhood of $(x_1, y_1)$ contained in $\varphi(U_1 \times U_2)$.

There exists a nonsingular loop $f: S^1 \to U_1$ such that $u \in f(S^1)$, and if $f_i: S^1 \to X$ is defined by $f_i(s) = (f(s), v_i)$ for $i = 1, 2$, then $\varphi \circ f_1(S^1) \subset V_1 \times V_2$, and for $i = 1, 2$, we have $\pi_i \circ \varphi \circ f_1(S^1) \cap \pi_i \circ \varphi \circ f_2(S^1) = \emptyset$. Since $f_1(S^1)$ is a retract of $U_1 \times U_2$, then $\varphi \circ f_1(S^1)$ is a retract of $\varphi(U_1 \times U_2)$, and hence a retract of $V_1 \times V_2$. Therefore $\varphi \circ f_1$ is essential, which implies that at least one of the maps $\pi_i \circ \varphi \circ f_1$ is essential. Suppose that $\pi_1 \circ \varphi \circ f_1$ is essential. Since $U_2$ is arcwise connected, then $f_1$ and $f_2$ are homotopic, and hence $\pi_1 \circ \varphi \circ f_1$ and $\pi_2 \circ \varphi \circ f_2$ are homotopic. However, no two essential and disjoint loops in the Menger curve are homotopic. Therefore either $x_1 = x_2$ or $y_1 = y_2$.

If $x_1 = x_2$, then for any $v_3 \in U_2$ we have $\varphi((u, v_3)) = (x_1, y_3)$. Since a similar fact can be shown for points in $U_1 \times U_2$ with equal second coordinate,
then \( \varphi = \varphi_1 \times \varphi_2 \), where \( \varphi_1: U_1 \to X_1 \) and \( \varphi_2: U_2 \to X_2 \).

If \( y_1 = y_2 \) then \( \varphi = \varphi_1 \times \varphi_2 \), where \( \varphi_1: U_1 \to X_2 \) and \( \varphi_2: U_2 \to X_1 \).

**Lemma 2.** Let \( X = X_1 \times X_2 \), where \( X_1 \) is homeomorphic to \( M \times M \), and \( X_2 \) is a continuum whose every point has a closed neighborhood which is an absolute retract. For \( i = 1, 2 \), let \( U_i \subset X_i \) be an open set and let \( U_2 \) be connected. If \( \varphi: U_1 \times U_2 \to X \) is an open embedding, then for every \( u \in U_1 \) there exists an \( x \in X_1 \) such that \( \varphi(\{u\} \times U_2) \subset \{x\} \times X_2 \).

**Proof.** Suppose that \( (u, v_1) \) and \( (u, v_2) \) are points in \( U_1 \times U_2 \). Let \( \varphi((u, v_1)) = (x_1, y_1) \) and let \( \varphi((u, v_2)) = (x_2, y_2) \). Suppose that \( x_1 \neq x_2 \). Let \( V_1 \subset X_1 \) and \( V_2 \subset X_2 \) be such that \( V_1 \times V_2 \) is a neighborhood of \( (x_1, y_1) \) contained in \( \varphi(U_1 \times U_2) \), and \( V_2 \) is an absolute retract. Let \( \pi_1: X \to X_1 \) be the projection.

There exists an embedding \( f: S^1 \times S^1 \to U_1 \) such that \( u \in f(S^1 \times S^1) \), \( f(S^1 \times S^1) \) is a retract of \( X_1 \), and if \( f_i: S^1 \times S^1 \to X \) is defined by \( f_i(s) = (f(s), v_i) \) for \( i = 1, 2 \), then \( \pi_1 \circ f_i(S^1 \times S^1) \cap \pi_1 \circ f_2(S^1 \times S^1) = \emptyset \), and \( \varphi \circ f_1(S^1 \times S^1) \subset V_1 \times V_2 \). Note that \( f_1 \) and \( f_2 \) are homotopic. Since \( f_1(S^1 \times S^1) \) is a retract of \( U_1 \times U_2 \), \( \varphi \circ f_1(S^1 \times S^1) \) is a retract of \( \varphi(U_1 \times U_2) \), and hence \( \varphi \circ f_1(S^1 \times S^1) \) is a retract of \( V_1 \times V_2 \). Let \( g: X_1 \to X \) be an embedding defined by \( g(p) = (p, y_1) \). Using the Čech homology and the induced homomorphism, we have \( 0 \neq (\varphi \circ f_1)_*(a) = (g \circ \pi_1 \circ \varphi \circ f_1)_*(a) = (g \circ \pi_1 \circ \varphi \circ f_2)_*(a) \), where \( a \) is a generator of \( H_2(S^1 \times S^1) \).

Therefore, there are two 2-dimensional nontrivial homologous Čech cycles with disjoint carriers in \( X_1 \). By [7, p. 246], the dimension of \( X_1 \) is greater than 2, which is a contradiction. \( \square \)

2. The twisted products

Denote by \((r, \theta, z)\) the cylindrical coordinates of a point in \( E^3 \). Let \( \mu \) be an embedding of the Menger curve \( M \) in \( E^3 \) defined by \( \mu(x, y, z) = (r, \theta, z) \), where \( r = x + 1, \theta = \frac{2\pi}{3} y \), and \( z = z \), for every \((x, y, z) \in M \). Let \( f(r, \theta, z) = (r, \theta + \frac{2\pi}{3}, z) \) be the rotation about the \( z \)-axis through the angle of \( \frac{2\pi}{3} \). Put \( A_0 = \mu(M), A_k = f^{(k)}(A_0) \), where \( f^{(k)} \) is the \( k \)-th iteration of \( f \), and put \( A = \bigcup_{k=0}^{\infty} A_k \).

Clearly, \( A \) is invariant under \( f \), and \( f_i = f|_{A_i} \) is a periodic homeomorphism of \( A \) onto itself. By [1], \( A \) is homeomorphic to \( M \). Cylindrical coordinates \((r, \theta, z)\) will be used to denote a point in \( A \), and Cartesian coordinates \((\overline{x}, \overline{y}, \overline{z})\) will be used to denote a point in \( M \). If \( p = (a, m) \in A \times M \) is a point, where \( a = (r, \theta, z) \) and \( m = (\overline{x}, \overline{y}, \overline{z}) \), then \( p \) may be denoted by \((r, \theta, z, \overline{x}, \overline{y}, \overline{z})\).

For every \( \alpha \in [0, 1] \), put \( M_\alpha = \{(\overline{x}, \overline{y}, \overline{z}) \in M : \overline{z} = \alpha \} \). Let \( g_1: M_1 \to M_0 \) be the homeomorphism taking \((\overline{x}, \overline{y}, 1)\) onto \((\overline{x}, \overline{y}, 0)\). Let \( g_2: A \times M_1 \to A \times M_0 \) be defined by \( g_2(a, m) = (f_i(a), g_1(m)) \). Define \( B \) as the quotient space \((A \times M)/g_2\).
Thus the continuum $B$, a twisted product of $A$ and $M$, is obtained from the Cartesian product $A \times M$ by pasting the "top" $A \times M_1$ to the "bottom" $A \times M_0$. Points in $B$ will be denoted in the same manner as the corresponding points in $A \times M$ for which $\bar{z} \neq 1$.

Define $f_2: B \to B$ by $f_2(a, m) = (f_1(a), m)$ for $a \in A$ and $m \in M - M_1$. Clearly, $f_2$ is a periodic homeomorphism of period 9.

**Lemma 3.** For every two points $p, q \in B$ there exists a homeomorphism $h: B \to B$ such that $h(p) = q$ and $h \circ f_2 = f_2 \circ h$.

**Proof.** First, we shall show that for every two points $p = (r_p, \theta_p, x_p, \bar{x}_p, \bar{y}_p, \bar{z}_p)$ and $q = (r_q, \theta_q, x_q, \bar{x}_q, \bar{y}_q, \bar{z}_q)$ there exists a homeomorphism $h_i: B \to B$ such that $h_i(p) = (r_p, \theta_p, z_p, \bar{x}_q, \bar{y}_q, \bar{z}_q)$ and $h_i \circ f_2 = f_2 \circ h_i$.

Let $M/g_1$ be the space homeomorphic to $M$ obtained from $M$ by identifying, in a similar fashion as above, the point $(\bar{x}, \bar{y}, 1)$ with the point $(\bar{x}, \bar{y}, 0)$ for $\bar{x}, \bar{y} \in [0, 1]$. For $\alpha \in [0, 1]$, denote by $\tilde{M}_\alpha$ the subset of $M/g_1$ corresponding to $M_\alpha \subset M$. The point in $M/g_1$ corresponding to the point $(\bar{x}, \bar{y}, \bar{z}) \in M$, where $\bar{z} \neq 1$, will be denoted by $(\bar{x}, \bar{y}, \bar{z})$.

For every $\alpha \in [0, 1]$, put $B_\alpha = \{(a, m) \in B: m \in M_\alpha \}$. The map $\Psi_\alpha: B - B_\alpha \to A \times (M/g_1 - \tilde{M}_\alpha)$ defined by

$$\Psi_\alpha(r, \theta, z, \bar{x}, \bar{y}, \bar{z}) = \begin{cases} ( (r, \theta, z), (\bar{x}, \bar{y}, \bar{z}) ) & \text{if } 0 \leq z < \alpha, \\ ( (r, \theta + \frac{2\pi}{9}, z), (\bar{x}, \bar{y}, \bar{z}) ) & \text{if } \alpha < z < 1, \end{cases}$$

is a homeomorphism.

There exists a number $\alpha_0$ and there exists a connected open subset $U \subset M/g_1$ containing $(\bar{x}_p, \bar{y}_p, \bar{z}_p)$ and $(\bar{x}_q, \bar{y}_q, \bar{z}_q)$ such that $U \cap \tilde{M}_{\alpha_0} = \emptyset$. There exists a homeomorphism $k_1: M/g_1 \to M/g_1$ taking $(\bar{x}_p, \bar{y}_p, \bar{z}_p)$ onto $(\bar{x}_q, \bar{y}_q, \bar{z}_q)$, and not moving points outside $U$. Let $\tilde{k}_1: A \times (M/g_1 - \tilde{M}_{\alpha_0}) \to A \times (M/g_1 - \tilde{M}_{\alpha_0})$ be such that

$$\tilde{k}_1((r, \theta, z), (\bar{x}, \bar{y}, \bar{z})) = ((r, \theta, z), k_1(\bar{x}, \bar{y}, \bar{z})), $$

and define $h_1: B \to B$ by setting

$$h_1(v) = \begin{cases} \Psi_\alpha^{-1} \circ \tilde{k}_1 \circ \Psi_\alpha(v) & \text{if } v \notin B_\alpha, \\ v & \text{if } v \in B_\alpha. \end{cases}$$

Hence,

$$h_1(p) = \Psi_\alpha^{-1} \circ k_1 \circ \Psi_\alpha(p) = (r_p, \theta_p + \varepsilon \frac{2\pi}{9}, z_p, \bar{x}_q, \bar{y}_q, \bar{z}_q),$$

where $\varepsilon \in \{0, 1, -1\}$. Put

$$h_1(v) = \begin{cases} \tilde{h}_1(v) & \text{if } \varepsilon = 0, \\ f_2^{-1} \circ \tilde{h}_1(v) & \text{if } \varepsilon = 1, \\ f_2 \circ \tilde{h}_1(v) & \text{if } \varepsilon = -1. \end{cases}$$
For any \( v = (r, \theta, z, \bar{x}, \bar{y}, \bar{z}) \in B \), 
\( h_1(v) = ((r, \theta + \delta \frac{2\pi}{p}, z), k_1(\bar{x}, \bar{y}, \bar{z})) \), 
where \( \delta \in \{0, 1, -1\} \). Hence \( h_1 \circ f_2(v) = ((r, \theta + (\delta + 1) \frac{2\pi}{q}, z), k_1(\bar{x}, \bar{y}, \bar{z})) = f_2 \circ h_1(v) \).

Next, we shall show that there exists a homeomorphism \( h_2: B \to B \) such that 
\( h_2(r_p, \theta_p, z_p, \bar{x}_q, \bar{y}_q, \bar{z}_q) = q \), and \( h_2 \circ f_2 = f_2 \circ h_2 \).

If \( f_2^{(i)}(r_p, \theta_p, z_p) = (r_q, \theta_q, x_q) \) for some \( i \), then set \( h_2 = f_2^{(i)} \). Otherwise, there is an open connected set \( U \subset A \) containing \( (r_p, \theta_p, z_p) \) and \( (r_q, \theta_q, z_q) \), and such that the sets \( U, f(U), \ldots, f^{(8)}(U) \) are pairwise disjoint. There is a homeomorphism \( k_2: A \to A \) which is the identity outside \( U \) taking \( (r_p, \theta_p, z_p) \) onto \( (r_q, \theta_q, z_q) \). Define \( h_2 \) by 
\[
   h_2(v) = \begin{cases} 
   (f^{(i)} \circ k_2 \circ (f^{(i)})^{-1}(a), m) & \text{if } a \in f^{(i)}(U), \\
   v & \text{if } a \not\in \bigcup_{j=1}^{9} f^{(j)}(U),
   \end{cases}
\]
where \( v = (a, m) \) and \( i = 1, \ldots, 9 \).

Clearly, \( h_2 \circ f_2 = f_2 \circ h_2 \).

Finally, put \( h = h_2 \circ h_1 \). \( \Box \)

Let \( n \) be a positive integer, and let \( N \) be an \( n \)-manifold with nonempty boundary such that \( \partial N = N_0 \cup N_1 \), where \( N_0 \cap N_1 = \emptyset \), both \( N_0 \) and \( N_1 \) are closed, and there exists a homeomorphism \( g_3: N_1 \to N_0 \). Let \( g_4: B \times N_1 \to B \times N_0 \) be such that \( g_4(b, s) = (f_2^{(3)}(b), g_3(s)) \), where \( f_2^{(3)} \) is the third iteration of \( f_2 \). Define \( Z_N \) as the quotient space \( (B \times N)/g_4 \).

Points in \( Z_N \) will be denoted in the same way as the corresponding points in the Cartesian products \( B \times N \) or \( A \times M \times N \). Specifically, if \( p \in Z_N \), then \( p = (b, s) \), where \( b \in B \) and \( s \in N - N_1 \), or \( p = (a, m, s) \), where \( a \in A \), \( m \in M - M_1 \), and \( s \in N - N_1 \), or \( p = (r, \theta, z) \), \( (\bar{x}, \bar{y}, \bar{z}) \in M - M_1 \), and \( s \in N - N_1 \).

**Lemma 4.** \( Z_N \) is homogeneous.

**Proof.** Let \( p = (b_p, s_p) \) and \( q = (b_q, s_q) \) be two points in \( Z_N \). To show that there exists a homeomorphism \( h: Z_N \to Z_N \) taking \( p \) onto \( q \), it is enough to show that there are homeomorphisms \( h_1, h_2: Z_N \to Z_N \) such that \( h_1(p) = (b_p, s_p) \) and \( h_2(b_p, s_p) = q \).

Let \( U \) be a neighborhood of \( b_p \) in \( B \) such that \( U, f_2^{(3)}(U), \) and \( f_2^{(6)}(U) \) are pairwise disjoint. The set \( W = \{(b, s) \in Z_N \mid b \in \bigcup_{i=3,6,9} f_2^{(i)}(U)\} \) is homeomorphic to \( U \times Q \), where \( Q \) is an \( n \)-manifold; in fact, \( Q \) is a union of three copies of \( N \). For any two points \( d_1 \) and \( d_2 \) in \( Q \), there exists a homeomorphism \( k: Q \to Q \) isotopic to the identity such that \( k(d_1) = d_2 \). Using the isotopy and the Cartesian product structure of \( W \), it is easy to obtain the homeomorphism \( h_1: Z_N \to Z_N \) with \( h_1(b_p, s_p) = (b_p, s_p) \) and \( h_1(v) = v \) for \( v \not\in W \).

By Lemma 3, there exists a homeomorphism \( \overline{h}: B \to B \) such that \( \overline{h}(b_p) = b_q \) and \( \overline{h} \circ f_2 = f_2 \circ \overline{h} \). In particular, \( \overline{h} \circ f_2^{(3)} = f_2^{(3)} \circ \overline{h} \). Hence, \( h_2: Z_N \to Z_N \),
where \( h_2(b, s) = (\hat{h}(b), s) \), is well defined and \( h_2(b_p, s_q) = q \). □

Let \( p = (a_0, m_0) \) be a point in \( B \). The sets \( A_p \) and \( M_p \) are defined by
\[
A_p = \{(a, m) \in B : m = m_0\},
\]
\[
M_p = \{(a, m) \in B : a = f^i(a_0), \text{ where } i = 1, \ldots, 9\}.
\]
Similarly, if \( p = (b_0, s_0) \in \mathbb{Z}_N \), then the sets \( B_p \) and \( N_p \) are defined by
\[
B_p = \{(b, s) \in \mathbb{Z}_N : s = s_0\},
\]
\[
N_p = \{(b, s) \in \mathbb{Z}_N : b = f_2^{(i)}(b_0), \text{ where } i = 3, 6, 9\}.
\]
Each of the sets \( A_p \), \( M_p \), \( B_p \), and \( N_p \) will be called a fiber.

**Lemma 5.** If \( h: B \to B \) is a homeomorphism, then either (1) \( h(A_p) = A_{h(p)} \) and \( h(M_p) = M_{h(p)} \) for all \( p \in B \), or (2) \( h(A_p) = M_{h(p)} \) and \( h(M_p) = A_{h(p)} \) for all \( p \in B \).

**Proof.** Every point in \( B \) has a closed neighborhood in the form of a Cartesian product \( X_1 \times X_2 \), where \( X_1 \) is a subset of \( A \) homeomorphic to \( M \), and by means of a homeomorphism similar to the homeomorphism \( \Psi_{\alpha} \) of Lemma 3, \( X_1 \) is a subset of \( M/\gamma_1 \) homeomorphic to \( M \). Moreover, for every \( x_1 \in X_1 \) and \( x_2 \in X_2 \), each of the sets \( \{x_1\} \times X_2 \) and \( X_1 \times \{x_2\} \) is contained in a fiber \( A_p \) or \( M_p \). Since \( B \) is compact, there exists a finite collection \( \{V_1, \ldots, V_k\} \) of these neighborhoods such that \( B = \bigcup_{j=1}^k \text{Int}(V_j) \). Similarly, every point in \( B \) has arbitrarily small open neighborhoods in the form \( V_1 \times V_2 \), where \( V_1 \) is homeomorphic to a connected subset of \( M \). Let \( \{W_1, \ldots, W_l\} \) be a finite collection of these neighborhoods covering \( B \), and such that for each \( j = 1, \ldots, l \), there is an \( i \) such that \( h(W_j) \subset V_i \). By Lemma 1, if \( p \in B \), then for each \( j = 1, \ldots, l \), there is an \( i \) such that \( h(A_p \cap W_j) \subset A_{h(p)} \cap V_i \subset A_{h(p)} \) or \( h(A_p \cap W_j) \subset M_{h(p)} \cap V_i \subset M_{h(p)} \). Hence \( h(A_p) \subset A_{h(p)} \) or \( h(A_p) \subset M_{h(p)} \). Since \( B \) is connected, then if \( h(A_p) \subset A_{h(p)} \), then \( h(A_p) \subset A_{h(p)} \) for every \( p \in B \). A similar statement holds for \( M_p \). Since \( h \) is one-to-one, we have \( h(A_p) = A_{h(p)} \) and \( h(M_p) = M_{h(p)} \) for \( p \in B \), or we have \( h(A_p) = M_{h(p)} \) and \( h(M_p) = A_{h(p)} \) for \( p \in B \). □

Let \( p \in B \) be a point. Denote by \( O_p \) the orbit of \( p \) under \( f_2 \), i.e., \( O_p = A_p \cap M_p = \{p, f_2(p), \ldots, f_2^{(8)}(p)\} \). The following lemma is an immediate consequence of Lemma 5.

**Lemma 6.** If \( h: B \to B \) is a homeomorphism, then \( h(O_p) = O_{h(p)} \) for every \( p \in B \).

**Lemma 7.** Let \( p_i = (1, \frac{2\pi i}{9}, 0, 0, 0) \in B \) for \( i = 0, \ldots, 8 \). If \( h: B \to B \) is a homeomorphism such that \( h(p_0) = p_i \) and \( h(p_1) = p_{i+1} \), then \( h(p_j) = p_{\lfloor i + j(i-1) \rfloor \mod 9} \).

**Proof.** Let \( L_i \) be the arc \( \{(1, \theta, 0, 0, 0) \in B : \frac{2\pi i}{9} \leq \theta \leq \frac{2\pi (i+1)}{9}\} \), where \( i = 0, \ldots, 8 \). The set \( L = \bigcup_{i=0}^8 L_i \) is a simple closed curve invariant under
By Lemma 6, \( U^0 \) is a simple closed curve. Furthermore, for \( i = 0, \ldots, 8 \), the end points of the arc \( f_2 \circ h(L_0) \) are the points \( p_{(i_0+i) \mod 9} \) and \( p_{(i_1+i) \mod 9} \). Therefore, since \( h(L) \) is a simple closed curve, we have \( \bigcup_{i=1}^9 f_2 \circ h(L_0) = h(L) \). Hence, the ends of the arc \( h(L_j) \) are \( p_{[i_0+j(i_1-i_0)] \mod 9} \) and \( p_{[i_1+j(i_1-i_0)] \mod 9} \). Thus \( h(p_j) = p_{[i_0+j(i_1-i_0)] \mod 9} \).

**Lemma 8.** If \( h: Z_N \to Z_N \) is a homeomorphism, then \( h(N_p) = N_{h(p)} \) for every \( p \in Z_N \).

**Proof.** Every point in \( Z_N \) has a closed neighborhood in the form of a Cartesian product \( X_1 \times X_2 \), where \( X_1 \) is homeomorphic to \( M \times M \) and \( X_2 \) is homeomorphic to a closed ball \( \overline{B}^n \). We may assume that if \( p(x_1, x_2) \in X_1 \times X_2 \), then \( X_1 \times \{x_2\} \subset N_p \) and \( \{x_1\} \times X_2 \subset B_p \). Since \( Z_N \) is compact, there exists a finite collection \( \{V_1, \ldots, V_k\} \) of these neighborhoods such that \( Z_N = \bigcup_{i=1}^k \text{Int}(V_i) \). Similarly, every point in \( Z_N \) has arbitrarily small neighborhoods in the form \( U_1 \times U_2 \), where \( U_1 \) is homeomorphic to an open subset in \( M \times M \), and \( U_2 \) is homeomorphic to an open ball \( B^n \). Let \( \{W_1, \ldots, W_l\} \) be a finite collection of these neighborhoods covering \( Z_N \) such that for every \( j = 1, \ldots, l \), there is an \( i \) such that \( h(W_j) \subset V_i \). By Lemma 2, if \( p \in Z_N \), then for every \( j = 1, \ldots, l \), there is an \( i \) such that \( h(N_p \cap W_j) \subset N_{h(p)} \cap V_i \subset N_{h(p)} \). Since \( N_p \) is connected, then \( h(N_p) \subset N_{h(p)} \).

Now, we will define a continuum \( C \) by putting \( C = Z_N \), where \( N = [0, 1] \), \( N_1 = \{1\} \), \( N_0 = \{0\} \), and \( g_3(1) = 0 \). Notice that \( C = \{(b, s): b \in B \) and \( s \in [0, 1]\} \). Consider \( B \) to be the subset \( \{(b, s) \in C: s = 0\} \) of \( C \).

Let \((\rho, \alpha)\) denote the polar coordinates in the plane. Assume that \( S^1 = \{(\rho, \alpha) \in E^2: \rho = 1 \) and \( \alpha \in [0, 2\pi]\} \). Let \( \Gamma: C \to S^1 \) be defined by

\[
\Gamma(r, \theta, z, \bar{z}, \bar{\theta}, s) = \left(1, \left(\theta + \frac{2\pi}{3}z + \frac{2\pi}{3}s\right) \mod 2\pi\right).
\]

Clearly, \( \Gamma \) is continuous.

**Lemma 9.** For every point \( p \in C \), \( \Gamma|_{N_p}: N_p \to S^1 \) is a homeomorphism.

**Proof.** If \( p = (r_p, \theta_p, z_p, \bar{z}_p, \bar{\theta}_p, s_p) \), then

\[
N_p = \{(r, \theta + \frac{2\pi}{3}e, z, \bar{z}, \bar{\theta}, s): e \in \{0, 1, 2\} \) and \( s \in [0, 1]\}\}.
\]

\[
\Gamma((r, \theta + \frac{2\pi}{3}e, z, \bar{z}, \bar{\theta}, s)) = (1, (\theta + \frac{2\pi}{3}e + \frac{2\pi}{3}z + \frac{2\pi}{3}s) \mod 2\pi).
\]

Clearly, \( \Gamma \) is one-to-one and therefore \( \Gamma \) is a homeomorphism.

Consider \( H_1(S^1) \) to be the additive group of integers. For every \( p \in C \), denote by \( a_p \) the generator of \( H_1(N_p) \) such that \( (\Gamma|_{N_p})_*(a_p) = 1 \). Just the first homology group determines an orientation on \( S^1 \) and on each fiber \( N_p \).
Definition. A homeomorphism $h: C \to C$ is said to be orientation preserving [reversing] if for every $p \in C$, $h|_{N_p}$ is orientation preserving [reversing].

Lemma 10. If $k: C \to C$ is a map and if for every $p \in C$ there exists a $p' \in C$ such that $k(N_p) \subset N_{p'}$, then for any two points $p_1$ and $p_2$ in $C$, we have $(\Gamma \circ k|_{N_{p_1}})_*(a_{p_1}) = (\Gamma \circ k|_{N_{p_2}})_*(a_{p_2})$.

Proof. There exists a finite open cover $\{V_i\}$ of $C$ such that each $V_i$ is homeomorphic to $(V_i \cap B) \times S^1$, and such that if $V_i \cap V_j \neq \emptyset$, then the two Cartesian product structures coming from $V_i$ and $V_j$ are compatible. Hence, any two simple closed curves $N_{p_1}$ and $N_{p_2}$ bound a singular annulus. Therefore $(\Gamma \circ k|_{N_{p_1}})_*(a_{p_1}) = (\Gamma \circ k|_{N_{p_2}})_*(a_{p_2})$.

Lemma 11 follows immediately from Lemma 10.

Lemma 11. If $h: C \to C$ is a homeomorphism, then $h$ is orientation preserving or $h$ is orientation reversing.

Lemma 12. Let $p_i = (1, \frac{2\pi i}{9}, 0, 0, 0, 0, 0) \in C$, where $i = 0, \ldots, 8$. Let $h: C \to C$ be a homeomorphism such that $h(B) = B$. If $h(N_{p_0}) = N_{p_1}$ and $h(N_{p_1}) = N_{p_0}$, then $h$ is orientation reversing.

Proof. Since $h(\{p_0, p_3, p_6\}) = \{p_1, p_4, p_7\}$, and $h(\{p_1, p_4, p_7\}) = \{p_0, p_3, p_6\}$, we have $h(p_0) = p_{i_0}$, where $i_0 \mod 3 = 1$, and $h(p_1) = p_{i_1}$, where $i_1 \mod 3 = 0$. By Lemma 7, $h(p_j) = p_{(i_0+j(i_1-i_0))\mod 9}$ for $j = 0, \ldots, 8$. Therefore $h(p_3) = p_{(i_0+6)\mod 9}$ and $h(p_6) = p_{(i_0+3)\mod 9}$ which implies that $h$ is orientation reversing. 

3. The Example

Assume the following notation:

\[
J^n = \{(x_1, \ldots, x_n) \in E^n : x_1 \in [0, 1]\}, \\
J^{n-1}_0 = \{(x_1, \ldots, x_n) \in E^n : x_1 = 0\}, \\
J^{n-1}_1 = \{(x_1, \ldots, x_n) \in E^n : x_1 = 1\}.
\]

For $i < n$, consider $E^i$ to be the subset of $E^n$ for which $x_{i+1} = \cdots = x_n = 0$.

Let $T$ be the Möbius strip with $\partial T = T_1$, and let $T_0$ be the middle simple closed curve of $T$. Consider $T$ to be the mapping cylinder of $\gamma: T_1 \to T_0$, where $\gamma$ is a map of degree 2. Since there is a piecewise linear embedding of $T$ in $E^3$, we may assume that $T$ is a piecewise linear subset of $J^4$ with $T_i = T \cap J_i$ for $i = 0, 1$. Let $\Sigma T_i$ be the suspension of $T_i$. Denote by $\Sigma T$ the mapping cylinder of the suspension of $\gamma$. Again, assume that $\Sigma T$ is a piecewise linear subset of $J^5$ with $\Sigma T_i = \Sigma T \cap J_i$ for $i = 0, 1$. Let $V$ be a regular neighborhood of $\Sigma T$ in $J^5$ such that for $i = 0, 1$, $V_i = V \cap J_i$ is a regular neighborhood of $\Sigma T_i$. Let $V', V''$, $V_0'$, $V_0''$, and $V_1'$, $V_1''$ be two copies of $V$, $V_0$, $V_1$, $V_0'$, $V_0''$, $V_1'$, $V_1''$.
and $V^1_i$ respectively. Denote by $\sigma: \partial V' - \text{Int}(V'_0 \cup V'_1) \to \partial V'' - \text{Int}(V''_0 \cup V''_1)$, $\sigma_0: \partial V'_0 \to \partial V''_0$, and $\sigma_1: \partial V'_1 \to \partial V''_1$ the homeomorphisms corresponding to the identity homeomorphisms. Assume that $V$ has an orientation compatible with the orientation of $E^5$, and assume that each $V_i$ has an orientation induced by the orientation of $V$. Let $\overline{\gamma}: V'_1 \to V'_0$ be an orientation reversing homeomorphism, and let $\overline{\gamma}': V'_1 \to V'_0$ and $\overline{\gamma}'': V''_1 \to V''_0$ be the homeomorphisms corresponding to $\overline{\gamma}$. Let $G = (V' \cup V'')/\sigma$. Note that $\partial G = G_0 \cup G_1$, where $G_0 = (V'_0 \cup V''_0)/\sigma_0$ and $G_1 = (V'_1 \cup V''_1)/\sigma_1$ are disjoint sets, each homeomorphic to $S^2 \times S^2$. The homeomorphisms $\overline{\gamma}'$ and $\overline{\gamma}''$ yield a homeomorphism $\gamma: G_1 \to G_0$.

Denote by $D$ the continuum obtained by putting $D = Z_N$, with $N = G$ and $g_3 = \gamma$, where $g_4$ is the map appearing in the definition of $Z_N$ given in §2. Each fiber $N_p$ of $D$ is an orientable 5-manifold $F$ which is a union of three copies of $G$ intersecting along the boundary components.

Let $L$ be a properly embedded arc in $G$ with end points $q$ and $g_3(q)$ on $G_1$ and $G_0$, respectively. There exists a retraction $r: G \to L$ such that $r^{-1}(g_3(q)) = G_1$, $r^{-1}(g_3(q)) = G_0$. We can write $F = G^0 \cup G^1 \cup G^2$ with $G^i = G^{(i+1) \mod 3}$ for $i = 0, 1, 2$, where $G^0$, $G^1$, and $G^2$ are copies of $G$, $G_0$, and $G_1$, respectively. Let $L_i \subset G^i$ be an arc corresponding to the arc $L \subset G$. Put $K = L_0 \cup L_1 \cup L_2$. Note that $K$ is a retract of $F$. Let $\overline{r}: F \to K$ be a retraction such that for $i = 0, 1, 2$, $\overline{r}^{-1}|_{G_i}: G^i \to L_i$ is the retraction corresponding to $r$. Notice that $\overline{r}$ induces an isomorphism of the first homology groups $\overline{r}_* : H_1(F) \to H_1(K)$.

Let $\tau: \tilde{F} \to F$ be a covering map such that $\tau^{-1}(K)$ is homeomorphic to $E^1$. Clearly, for $s_0 \in \tilde{F}$, $\pi_1(\tilde{F}, s_0) \approx 0$. For $i = 0, \pm 1, \pm 2, \ldots$, denote by $F_i$ a subset of $\tilde{F}$ homeomorphic to $G$ such that $\tilde{F} = \bigcup_{i=-\infty}^{\infty} F_{i-1} \cap F_i \neq \emptyset$ and for $k = 0, 1, 2$, $\tau^{-1}(G^k) = \bigcup_{j=-\infty}^{\infty} F_{3j+k}$.

**Definition.** Let $m$ be an integer. A homeomorphism $h: \tilde{F} \to \tilde{F}$ is said to be an $m$-shift homeomorphism if $h(F_i) = F_{i+m}$ for $i = 0, \pm 1, \pm 2, \ldots$.

**Definition.** Let $m = 0, 1, 2$. A homeomorphism $h: F \to F$ is said to be an $m$-shift homeomorphism if $h(G_i) = G^{(i+m) \mod 3}$ for $i = 0, 1, 2$.

Observe that for $i = 0, \pm 1, \pm 2, \ldots$, $F_{i-1} \cap F_i$ is homeomorphic to $S^2 \times S^2$.

From the properties of mapping cylinders and regular neighborhoods, it follows that the fourth homology group $H_4(\tilde{F})$ is generated by $\{b_i\}_{i=-\infty}^{\infty}$, with relations $b_i = 2b_{i-1}$, where $b_i$ is obtained from the 4-manifold $F_{i-1} \cap F_i$ for $i = 0, \pm 1, \pm 2, \ldots$. Moreover, by choosing an appropriate orientation of $F_{i-1} \cap F_i$, we may assume that the cycle representing $b_i$ has coefficient 1 on every simplex of $F_{i-1} \cap F_i$. Then, $b_i$ cannot be represented by a cycle with its carrier contained in $\bigcup_{j=i+1}^{\infty} F_j$ for $i = 0, \pm 1, \pm 2, \ldots$. Also, note that if $h_1, h_2: \tilde{F} \to \tilde{F}$ ($h_1, h_2: F \to F$) are two isotopic $m_1$-shift and $m_2$-shift homeomorphisms, respectively, then $m_1 = m_2$.
Let $a$ be a generator of $H_1(K)$.

**Lemma 13.** If $h: F \to F$ is a homeomorphism, then $(\tilde{r} \circ h|_K)_*(a) = a$.

**Proof.** By [4, pp. 90–91] there exists a homeomorphism $\tilde{h}: \tilde{F} \to \tilde{F}$, and there exists a retraction $\tilde{r}: \tilde{F} \to \tau^{-1}(K)$ such that the diagram

$$
\begin{array}{ccc}
\tilde{F} & \xrightarrow{\tilde{h}} & \tilde{F} \\
\tau \downarrow & & \downarrow \tau \\
F & \xrightarrow{h} & F \\
\end{array}
\xrightarrow{\tau^{-1}(K)}
\begin{array}{ccc}
\tau^{-1}(K) & \to & K \\
\downarrow & & \downarrow \\
\tau_1^{-1}(K) & \to & \tau^{-1}(K)
\end{array}
$$

commutes.

Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of points in $\tilde{F}$. We will say that $\lim_{n \to \infty} p_n = \infty$ if for every integer $n_0$ almost all of the points $p_n$ belong to $\bigcup_{i=n_0}^{\infty} F_i \cup \bigcup_{i=-\infty}^{0} F_i$. To prove Lemma 13, it is enough to show that if $\lim_{n \to \infty} p_n = \infty$, then $\lim_{n \to \infty} \tilde{h}(p_n) = \infty$ (hence $\lim_{n \to \infty} \tilde{r} \circ \tilde{h}(p_n) = \infty$) for every sequence $\{p_n\}_{n=1}^{\infty}$ contained in $\tau^{-1}(K)$.

There exist two sequences of positive integers $\{i_n\}_{n=1}^{\infty}$ and $\{j_n\}_{n=1}^{\infty}$ such that for every $n = 1, 2, \ldots$,

$$
\bigcup_{k=-i_n}^{i_n} F_k \subset \bigcup_{k=-j_n}^{j_n} \tilde{h}(F_k) \subset \bigcup_{k=-i_{n+1}}^{i_{n+1}} F_k.
$$

Note that if $n_0 \geq 0$, then $\bigcup_{k=-n_0}^{n_0} F_k$ separates $\tilde{F}$ between $\bigcup_{k=-n_0-2}^{n_0} F_k$ and $\bigcup_{k=n_0+1}^{n_0} F_k$, and $\bigcup_{k=-n_0}^{n_0} \tilde{h}(F_k)$ separates $\tilde{F}$ between $\bigcup_{k=-n_0-2}^{n_0} \tilde{h}(F_k)$ and $\bigcup_{k=n_0+1}^{n_0} \tilde{h}(F_k)$. Hence, there exists a strictly increasing sequence $\{i_m\}_{m=1}^{\infty}$, and a strictly increasing or decreasing sequence $\{j_m\}_{m=1}^{\infty}$ such that $\tilde{h}(F_{j_m}) \subset \bigcup_{k=i_m}^{i_{m+1}} F_k$. If $\{j_m\}_{m=1}^{\infty}$ is strictly decreasing, then $\tilde{h}(b_{j_0})$ can be represented by a cycle with its carrier in $\bigcup_{k=j_0}^{\infty} \tilde{h}(F_k)$ for some $k_0 > j_0$, which is a contradiction. Hence, $\{j_m\}_{m=1}^{\infty}$ is strictly increasing, and if $\{p_n\}_{n=1}^{\infty}$ is a sequence with $\lim_{n \to \infty} p_n = \infty$, then $\lim_{n \to \infty} \tilde{h}(p_n) = \infty$. \[\square\]

**Lemma 14.** Let $X = S^1 \times F$. Let $h: X \to X$ be a homeomorphism such that for every $\alpha \in S^1$ there exists an $\alpha' \in S^1$ with $h(\{\alpha\} \times F) = \{\alpha'\} \times F$. Let $\rho: \tilde{X} \to X$ be a covering map defined by setting $\tilde{X} = S^1 \times \tilde{F}$ and $\rho = \text{id}_{S^1} \times \tau$. Then there exists a map $\tilde{h}: \tilde{X} \to \tilde{X}$ such that the diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{h}} & \tilde{X} \\
\rho \downarrow & & \downarrow \rho \\
X & \xrightarrow{h} & X
\end{array}
$$

commutes.

**Proof.** As defined before, $S^1 = \{\alpha: 0 \leq \alpha < 2\pi\}$. We may assume that both copies of $S^1$ appearing in $h: S^1 \times F \to S^1 \times F$ are parametrized in such a
way that \( h(\{\alpha\} \times F) = \{\alpha\} \times F \). Let \( x_0 = (0, s_0) \in X \), \( y_0 = (0, r_0) \in X \), \( \tilde{x}_0 = (0, \tilde{s}_0) \in \tilde{X} \), and \( \tilde{y}_0 = (0, \tilde{r}_0) \in \tilde{X} \) be points such that \( \rho(\tilde{x}_0) = x_0 \), \( \rho(\tilde{y}_0) = y_0 \), and \( h(x_0) = y_0 \). By [4, p. 90], it is enough to show that \( (h \circ \rho)_\#(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \rho_\#(\pi_1(\tilde{X}, \tilde{y}_0)) \). Let \( f: [0, 1] \to \tilde{X} \) be a loop defined by \( f(\alpha) = (2\pi \alpha \mod 2\pi, \tilde{s}_0) \); the loop \( f \) represents a generator of \( \pi_1(\tilde{X}, \tilde{x}_0) \).

For \( 0 \leq a \leq b \leq 2\pi \), put

\[
X_{[a, b]} = \begin{cases} \{((\alpha, s) \in X : a \leq \alpha \leq b\} & \text{if } b \neq 2\pi, \\ \{((\alpha, s) \in X : a \leq \alpha < b \text{ or } \alpha = 0\} & \text{if } b = 2\pi. 
\end{cases}
\]

Let \( \tilde{X}_{[a, b]} = \rho^{-1}(X_{[a, b]}) \). Let \( \tilde{h}_1: \tilde{X}_{[0, \pi]} \to \tilde{X}_{[0, \pi]} \) be the unique lifting of \((h \circ \rho)|_{\tilde{X}_{[0, \pi]}}\) with \( \tilde{h}_1(\tilde{x}_0) = \tilde{y}_0 \), and let \( \tilde{h}_2: \tilde{X}_{[\pi, 2\pi]} \to \tilde{X}_{[\pi, 2\pi]} \) be the unique lifting of \((h \circ \rho)|_{\tilde{X}_{[\pi, 2\pi]}}\) such that \( \tilde{h}_2|_{\tilde{X}_{[\pi, \pi]}} = \tilde{h}_1|_{\tilde{X}_{[\pi, \pi]}} \). By [4, p. 86], there exists a path \( \tilde{f}: [0, 1] \to \tilde{X} \) with \( \tilde{f}(0) = \tilde{y}_0 \) such that \( \rho \circ \tilde{f} = h \circ \rho \circ f \). Let \( i_0: \tilde{F} \to \{\alpha\} \times \tilde{F} \) be the inclusion defined by \( i_0(s) = (\alpha, s) \), and let \( \tilde{\pi}_F: S^1 \times \tilde{F} \to \tilde{F} \) be the projection. If \( \tilde{f}(1) \neq \tilde{y}_0 \), then \( k = \pi_F \circ \tilde{h}_2^{-1} \circ \tilde{h}_1 \circ i_0: \tilde{F} \to \tilde{F} \) is a 3\( n \)-shift homeomorphism with \( n \neq 0 \). However, \( \tilde{h}_1 \) and \( \tilde{h}_2 \) yield an isotopy \( H_i: \tilde{F} \to \tilde{F} \) defined by

\[
H_i = \pi_F \circ \tilde{h}_2^{-1} \circ i_{2\pi} \circ \pi_F \circ \tilde{h}_1 \circ i_0,
\]
where \( j = 1 \) if \( t \in [0, \frac{1}{2}] \) and \( j = 2 \) if \( t \in [\frac{1}{2}, 1] \). Hence, \( H_0 = \text{id}_{\tilde{F}} \) and \( H_1 = k \) are isotopic, which is a contradiction. Therefore, \( \tilde{f}(1) = \tilde{y}_0 \) and \((h \circ \rho)_\#(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \rho_\#(\pi_1(\tilde{X}, \tilde{y}_0)) \). \( \square \)

**Lemma 15.** Let \( U \) and \( V \) be open connected subsets of \( B \). Let \( X = U \times F \), \( Y = V \times F \), \( \tilde{X} = U \times \tilde{F} \), and \( \tilde{Y} = V \times \tilde{F} \). Let \( \rho_X: \tilde{X} \to X \) and \( \rho_Y: \tilde{Y} \to Y \) be covering maps defined by \( \rho_X = \text{id}_U \times \tau \) and \( \rho_Y = \text{id}_V \times \tau \) respectively. If \( h: X \to Y \) is a homeomorphism, then there exists a map \( \tilde{h}: \tilde{X} \to \tilde{Y} \) such that the diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{h} & \tilde{Y} \\
\rho_X \downarrow & & \downarrow \rho_Y \\
X & \xrightarrow{h} & Y 
\end{array}
\]

commutes.

**Proof.** By Lemma 2, for every \( u \in U \) there exists a \( v \in V \) such that \( h(\{u\} \times F) = \{v\} \times F \).

Let \( x_0 \in X \), \( y_0 \in Y \), \( \tilde{x}_0 \in \tilde{X} \), and \( \tilde{y}_0 \in \tilde{Y} \) be points such that \( \rho_X(\tilde{x}_0) = x_0 \), \( \rho_Y(\tilde{y}_0) = y_0 \), and \( h(x_0) = y_0 \). It is enough to show that if \( f: [0, 1] \to \tilde{X} \) is a loop with \( f(0) = f(1) = \tilde{x}_0 \), then there exists a loop \( \tilde{f}: [0, 1] \to \tilde{Y} \) with \( \tilde{f}(0) = \tilde{f}(1) = \tilde{y}_0 \) such that \( h \circ \rho_X \circ f = \rho_Y \circ \tilde{f} \). Without loss of generality, we may assume that \( f([0, 1]) \subset U \times \{\tilde{s}_0\} \), where \( \tilde{x}_0 = (u_0, \tilde{s}_0) \).

Let \( t_0 = 0 < t_1 < \cdots < t_m = 1 \) be a sequence of points in \([0, 1]\), and let \( f': [0, 1] \to \tilde{X} \) be a loop such that \( f'([0, 1]) \subset U \times \{\tilde{s}_0\} \), \( f'(t_i) = f(t_i) \) for \( i = 0, \ldots, m \), \( f'([0, 1] - \{t_0, \ldots, t_m\}; [0, 1] - \{t_0, \ldots, t_m\}) \to f'([0, 1] - \{t_0, \ldots, t_m\}) \)
is one-to-one, and for each \(i = 1, \ldots, m\), there is a \(j = 0, 1, 2\) such that 
\[ h \circ \rho_X \circ f'([t_{i-1}, t_i]) \cup h \circ \rho_X \circ f([t_{i-1}, t_i]) \subset V \times (G^j \cup G^{(j+1) \mod 3}). \]

If \(P\) is a simple closed curve in \(f'([0, 1])\), then \(h \circ \rho_X(P)\) is a simple closed curve in \(h \circ \rho_X \circ f'([0, 1])\). Let \(\pi_V : V \times F \to V\) be the projection. For each \(u \in U\), \(h(\{u\}) \times F = \{v\} \times F\) for some \(v \in V\), then \(\pi_V \circ h \circ \rho_X(P)\) is a simple closed curve in \(V\), and \(h \circ \rho_X(P) \times F = (\pi_V \circ h \circ \rho_X(P)) \times F\). By Lemma 14, for every \(z \in \rho_Y^{-1} \circ h \circ \rho_X(P)\) there exists a simple closed curve \(\tilde{P} \subset \tilde{Y}\) containing \(z\) such that \(\rho_Y(\tilde{P}) = h \circ \rho_X(P)\). Also, for \(i = 0, \ldots, m\), every loop whose image is the set \(h \circ \rho_X \circ f'([t_{i-1}, t_i]) \cup h \circ \rho_X \circ f([t_{i-1}, t_i])\) lifts to a loop in \(\tilde{F}\). Hence, there exists a loop \(\tilde{f} : [0, 1] \to \tilde{X}\) such that \(h \circ \rho_X \circ f = \rho_Y \circ \tilde{f}\) and \(\tilde{f}(0) = \tilde{f}(1) = \tilde{f}_0\). □

Consider \(C\) to be a subset of \(D\), \(C = \{(b, s) \in D : s \in L\}\). Let \(s_0\) be the point of \(L \cap G_0\). Consider \(B\) to be a subset of \(D\), \(B = \{(b, s) \in D : s = s_0\}\). Let \(R : D \to C\) be a retraction defined by \(R(b, s) = (b, r(s))\), where \(s \in G - G_1\). In this section, the notation \(N_p\) is used for fibers of \(D\) (homeomorphic to \(F\)). Hence, if \(p = (b_1, s_1)\), then \(N_p = \{(b, s) \in D : b = b_1, b = f_2^{(6)}(b_1), \text{ or } b = f_2^{(6)}(b_1)\}\), where \(f_2\) is the homeomorphism defined in §2. If \(p \in C\), then the fiber of \(C\) containing \(p\), homeomorphic to \(S^1\), is denoted by \(K_p\), i.e., \(K_p = N_p \cap C\).

**Lemma 16.** If \(h : D \to D\) is a homeomorphism, then there exists an orientation preserving homeomorphism \(\overline{h} : C \to C\) such that \(\overline{h}(B) = B\) and such that for every \(p\) and \(q\) in \(C\), if \(h(N_p) = N_q\) then \(\overline{h}(K_p) = K_q\).

**Proof.** Define a homeomorphism \(\phi : D \to D\) by \(\phi(b, s) = (f_2^{(3)}(b), s)\). Let \(X = \{(r, \theta, z, \bar{z}, \bar{y}, \bar{z}, s) \in D : 0 < \theta < \frac{2\pi}{3}, \frac{2\pi}{3} < \theta < \frac{4\pi}{3}, \text{ or } \frac{4\pi}{3} < \theta < 2\pi\}\). Let \(U\) be a component of \(X \cap B\). Notice that \(X\) is homeomorphic to \(U \times F\) and the set \(h(X) \cap B\) is not empty. Let \(V\) be a component of \(h(X) \cap B\).

We claim that \(h(X)\) is homeomorphic to \(V \times F\). Since \(V\) is connected, if \(V \cap \phi(V) \neq \emptyset\), then \(V\) is invariant under \(\phi\). Then, there exist a point \(p_0 \in V\) and an arc \(P_0 \subset V\) joining \(p_0\) and \(\phi(p_0)\) such that \(P_0 \cap \phi(P_0) = \{p_0\}\). Hence \(P_0 \cup \phi(P_0) \cup \phi^2(P_0)\) is a simple closed curve contained in \(V\). Consider the Cartesian product \(P_0 \times F\). Let \(\pi_F : P_0 \times F \to F\) be the projection, and let \(i_p : F \to P_0 \times F\) be the inclusion defined by \(i_p(s) = (p, s)\) for \(p \in P_0\). The set \(Y = \bigcup_{p \in P_0} N_p\) is homeomorphic to \((P_0 \times F) / k\), where \(k : \{p_0\} \times F \to \{\phi(p_0)\} \times F\) is a homeomorphism such that \(\pi_F \circ k \circ i_{p_0}\) is a 1-shift homeomorphism.

The embedding \(h^{-1}_Y : Y \to X\) preserves fibers, i.e., for every \(p \in Y\), we have \((h^{-1}_Y)(N_p) = N_q\) for some \(q \in X\). By an argument similar to that of the proof of Lemma 14, \(\pi_F \circ k \circ i_{p_0}\) is isotopic to the identity, which is a contradiction. Hence, \(V \cap \phi(V) = \emptyset\), and \(\bigcup_{p \in V} N_p\) is homeomorphic to \(V \times F\). Since \(h(X)\) is connected, \(h(X) = \bigcup_{p \in V} N_p\). Notice that for every \(p \in V\), \(V\) intersects each fiber \(N_p\) at exactly one point, and there is a homeomorphism of \(\bigcup_{p \in V} N_p\)
onto $V \times F$ which takes each fiber $N_p$ onto some fiber $\{q\} \times F$.

Let $\omega: B \times \tilde{F} \to D$ be a covering map such that $\omega(b, s) = (b, \tau(s))$ for $(b, s) \in B \times (F_0 - F_1)$. By Lemma 15, there exists a map $\hat{h}_U: U \times \tilde{F} \to B \times \tilde{F}$ with $\hat{h}(U \times \tilde{F}) = V \times \tilde{F}$ and such that the diagram

\[
\begin{array}{ccc}
U \times \tilde{F} & \xrightarrow{h_U} & B \times \tilde{F} \\
\downarrow \omega|_{U \times \tilde{F}} & & \downarrow \omega \\
X & \xrightarrow{h|_X} & D
\end{array}
\]

commutes.

For every $p \in U$, the closure of $U$, there exists a neighborhood $W_p$ of $p$ in $B$ such that the set $\bigcup_{q \in W_p} N_q$ is homeomorphic to the Cartesian product $W_p \times F$. Furthermore, for every $p \in U$, there exists a unique map $\hat{h}_p: W_p \times \tilde{F} \to B \times \tilde{F}$ such that the diagram

\[
\begin{array}{ccc}
W_p \times \tilde{F} & \xrightarrow{\hat{h}_p} & B \times \tilde{F} \\
\downarrow \omega|_{W_p \times \tilde{F}} & & \downarrow \omega \\
\bigcup_{q \in W_p} N_q & \xrightarrow{\hat{h}|_{\bigcup_{q \in W_p} N_q}} & D
\end{array}
\]

commutes, and $\hat{h}_p|_{(U \cap W_p) \times \tilde{F}} = \hat{h}_U|_{(U \cap W_p) \times \tilde{F}}$. Therefore there exists a map $\bar{h}: \bar{U} \times \tilde{F} \to B \times \tilde{F}$ such that $\hat{h}|_{

The set $\omega^{-1}(C)$ is homeomorphic to $B \times E^1$. Assume that $\omega^{-1}(C) = B \times E^1$ and $\omega^{-1}(C) \cap (\bar{U} \times \tilde{F}) = \bar{U} \times E^1$. Let $(b, s) \in B \times E^1$, where $b = (r, \theta, z, \bar{x}, \bar{y}, \bar{z})$. Let $i = 0, 1, 2$. Assume that if $i \leq s \mod 3 < i + 1$, then $\omega(b, s) = (b_i, s_i)$, where $b_i = (r, \theta + \frac{2\pi i}{3}, z, \bar{x}, \bar{y}, \bar{z})$, and $s_i = (s \mod 3) - i$.

Let $\tilde{R}: B \times \tilde{F} \to B \times E^1$ be a retraction such that the diagram

\[
\begin{array}{ccc}
B \times \tilde{F} & \xrightarrow{\tilde{R}} & B \times E^1 \\
\downarrow \omega & & \downarrow \omega|_{B \times E^1} \\
D & \xrightarrow{R} & C
\end{array}
\]

commutes. Note that if $p, \varphi(p) \in \bar{U}$ and $s \in E^1$, then

$\omega(p, s) = \omega(\varphi(p), s - 1)$.

Hence, if $p, \varphi(p) \in \bar{U}$, $s \in E^1$, and $\tilde{R} \circ \tilde{h}(p, s) = (q, r)$, then $\tilde{R} \circ \tilde{h}(\varphi(p), s - 1) = (\varphi(q), r - 1)$ or $\tilde{R} \circ \tilde{h}(\varphi(p), s - 1) = (\varphi(q), r + 1)$. By Lemma 13, if $\tilde{R} \circ \tilde{h}(p, s) = (q, r)$, then $\tilde{R} \circ \tilde{h}(\varphi(p), s - 1) = (\varphi(q), r - 1)$ for $p, \varphi(p) \in \bar{U}$ and $s \in E^1$. Finally, if $(p, s) \in \bar{U} \times E^1$ is a point and $\tilde{R} \circ \tilde{h}(p, s) = (q, r)$, then define $h': \bar{U} \times E^1 \to \bar{V} \times E^1$ by $h'(p, s) = (q, s)$. Let $\tilde{h}: C \to C$ be such that the diagram

\[
\begin{array}{ccc}
\bar{U} \times E^1 & \xrightarrow{h'} & \bar{V} \times E^1 \\
\downarrow \omega|_{\bar{U} \times E^1} & & \downarrow \omega|_{\bar{V} \times E^1} \\
C & \xrightarrow{\tilde{h}} & C
\end{array}
\]
commutes. \( \overline{h} \) is an orientation preserving homeomorphism, \( \overline{h}(B) = B \), and if \( h(N_p) = N_q \), then \( \overline{h}(K_p) = K_q \) for \( p, q \in C \).

**Theorem.** The continuum \( D \) is homogeneous but not bihomogeneous.

**Proof.** By Lemma 4, \( D \) is homogeneous. Let \( p_0 = (1, 0, 0, 0, 0, 0, s_0) \) and \( p_1 = (1, \frac{2\pi}{9}, 0, 0, 0, 0, s_0) \). Suppose that there exists a homeomorphism \( h: D \rightarrow D \) such that \( h(p_0) = p_1 \) and \( h(p_1) = p_0 \). Then, by Lemma 16, there exists an orientation preserving homeomorphism \( \overline{h}: C \rightarrow C \) such that \( \overline{h}(B) = B \), \( \overline{h}(K_{p_0}) = K_{p_1} \), and \( \overline{h}(K_{p_1}) = K_{p_0} \). However, by Lemma 12, \( \overline{h} \) is orientation reversing. Therefore \( D \) is not bihomogeneous.

4. **PROBLEMS**

**Definition.** A space \( X \) is said to be *semilocaly bihomogeneous* if for every \( p \) there exists a neighborhood \( U \) of \( p \) such that for every \( q \in U \) there exists a homeomorphism \( h: X \rightarrow X \) with \( h(p) = q \) and \( h(q) = p \).

**Remark 1.** The continuum \( D \) constructed in this paper is semilocally bihomogeneous.

**Problem 1.** Does there exist a homogeneous continuum (locally connected continuum) which is not semilocally bihomogeneous?

**Problem 2.** Does there exist a homogenous, locally compact metric space (continuum) \( X \) such that for no two points \( p \) and \( q \) in \( X \) there exists a homeomorphism \( h: X \rightarrow X \) with \( h(p) = q \) and \( h(q) = p \)?

**Remark 2.** Cook's example (see [3]) of a homogeneous, nonbihomogeneous, locally compact metric space is of dimension 2. The example constructed in this paper is of dimension 7.

**Problem 3.** What is the lowest dimension of a homogeneous, nonbihomogeneous, locally compact metric space (continuum)?

**Remark 3.** W. R. R. Transue points out that by his result of [8], Cook's example is embeddable in \( E^3 \).

**Problem 4.** Does there exist a homogeneous, nonbihomogeneous continuum embeddable in \( E^3 \)?

**Problem 5.** Is every homogeneous, metric, absolute neighborhood retract bihomogeneous?

**References**


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