ON GELFAND PAIRS ASSOCIATED WITH SOLVABLE LIE GROUPS

CHAL BENSON, JOE JENKINS AND GAIL RATCLIFF

ABSTRACT. Let $G$ be a locally compact group, and let $K$ be a compact subgroup of $\text{Aut}(G)$, the group of automorphisms of $G$. There is a natural action of $K$ on the convolution algebra $L^1(G)$, and we denote by $L^1_K(G)$ the subalgebra of those elements in $L^1(G)$ that are invariant under this action. The pair $(K, G)$ is called a Gelfand pair if $L^1_K(G)$ is commutative. In this paper we consider the case where $G$ is a connected, simply connected solvable Lie group and $K \subseteq \text{Aut}(G)$ is a compact, connected group. We characterize such Gelfand pairs $(K, G)$, and determine a moduli space for the associated $K$-spherical functions.

INTRODUCTION

Let $G$ be a locally compact group, and let $K$ be a compact subgroup of $\text{Aut}(G)$, the group of automorphisms of $G$. There is a natural action of $K$ on the convolution algebra $L^1(G)$, and we denote by $L^1_K(G)$ the subalgebra of those elements in $L^1(G)$ that are invariant under this action. The pair $(K, G)$ is called a Gelfand pair if $L^1_K(G)$ is commutative. A more general and more usual definition of Gelfand pairs assumes that $K$ is a compact subgroup of $G$. One then defines $(K, G)$ to be a Gelfand pair if the subalgebra of $K$-bi-invariant elements in $L^1(G)$ is commutative. This is the case, for example, if $(G, K)$ is a Riemannian symmetric pair, as was shown by Gelfand in 1950, [Ge]. In this paper we consider the case where $G$ is a connected, simply connected solvable Lie group and $K \subseteq \text{Aut}(G)$ is a compact, connected group.

For the remainder of the paper, unless otherwise stated, $S$ will denote a connected, simply connected solvable Lie group and $N$ will denote a connected, simply connected nilpotent Lie group, with corresponding Lie algebras $\mathfrak{S}$, $\mathfrak{N}$, and $K$ will denote a compact, connected subgroup of the appropriate automorphism group.

The classification of Gelfand pairs involving solvable groups presupposes a classification for such pairs involving nilpotent groups, which is the subject we

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Theorem A. If \((K, N)\) is a Gelfand pair then \(N\) is at most two step.

The proof is based on the observation that \((K, G)\) is a Gelfand pair if, and only if, products (as sets) of \(K\)-orbits in \(G\) commute, i.e. for each \(x, y \in G\),
\[
(K \cdot x)(K \cdot y) = (K \cdot y)(K \cdot x).
\]

The criterion that we generally use to determine if \((K, N)\) is a Gelfand pair is contained in a theorem due to Carcano, [Ca], which we now recall. Let \(\pi \in \hat{N}\), and denote by \(K_\pi\) the set of all elements \(k \in K\) such that \(\pi_k \simeq \pi\)
where \(\pi_k\) is the element of \(\hat{N}\) defined by \(\pi_k(x) = \pi(k \cdot x)\) for all \(x \in N\). Then there is a projective representation \(W_\pi\) of \(K_\pi\) on \(H_\pi\), the representation space of \(\pi\). \(W_\pi\) is called the intertwining representation for \(\pi\). If \(\sigma\) is the cocycle of \(W_\pi\) there is a decomposition
\[
W_\pi = \sum_{T \in K_\pi} c(T, W_\pi)T,
\]
where \(c(T, W_\pi)\) denotes the multiplicity of \(T\) in \(W_\pi\). Carcano’s theorem states that \((K, N)\) is a Gelfand pair if \(c(T, W_\pi) \leq 1\) for all \(\pi\) in a set of full Plancherel measure, and that, conversely, if \((K, N)\) is a Gelfand pair then \(c(T, W_\pi) \leq 1\) for every \(\pi \in \hat{N}\).

Since the representations of 2-step nilpotent groups factor through tensor products of representations of Heisenberg \(\times\) abelian groups, the classification of Gelfand pairs \((K, N)\) reduces to classification of Gelfand pairs \((K, H_n)\), where \(H_n\) is the \(2n+1\)-dimensional Heisenberg group. We realize \(H_n\) as \(\mathbb{C}^n \times \mathbb{R}\) with multiplication given by \((z, t)(z', t') = (z + z', t + t' + 2zz')\). If \(K \subseteq \text{Aut}(H_n)\), then, after conjugating by an element of \(\text{Aut}(H_n)\) if necessary, we may assume that \(K \subseteq U(n)\), the group of \(n \times n\) unitary matrices acting on \(\mathbb{C}^n\) in the usual fashion. Given such a \(K\), we denote by \(K_C\) its complexification, which may be considered as a subgroup of \(Gl(n, \mathbb{C})\). We denote by \(\mathbb{C}[\mathbb{C}^n]\) the polynomial ring over \(\mathbb{C}^n\). There is a natural action of \(K_C\) on \(\mathbb{C}[\mathbb{C}^n]\).

Theorem B. Suppose that \(K\) acts irreducibly on \(\mathbb{C}^n\). \((K, H_n)\) is a Gelfand pair if, and only if, \(K_C\) acts without multiplicity on \(\mathbb{C}[\mathbb{C}^n]\).

Victor Kac, [Ka], has given a complete list of all such groups \(K_C\) acting without multiplicity on \(\mathbb{C}[\mathbb{C}^n]\). If the action of \(K\) on \(\mathbb{C}^n\) is not irreducible, consider the irreducible decomposition \(\mathbb{C}^n = \bigoplus_{j=1}^{p_j} V_j\), and let \(K_j\) denote the subgroup of \(U(V_j)\) given by the (irreducible) action of \(K\) on \(V_j\). The subset of \(H_n\) given by \(V_j \times \mathbb{R}\) is isomorphic to \(H_{m_j}\), where \(m_j = \dim(V_j)\). For \(n_1, \ldots, n_p \in \mathbb{Z}^+\) let \(\mathbb{P}^{n_1, \ldots, n_p} = \bigotimes_{j=1}^{p} P_{j, n_j}\), where \(P_{j, n_j}\) is a \(K_j\)-irreducible subspace of \(\mathbb{C}[V_j]\).

Theorem C. \((K, N)\) is a Gelfand pair if, and only if, the subrepresentations of \(K\) on the various \(\mathbb{P}^{n_1, \ldots, n_p}\) are all distinct.
We next consider the free, two-step nilpotent Lie group on \( n \)-generators, \( F(n) \). We identify its Lie algebra \( \mathcal{F}(n) \) with \( \mathbb{R}^n \oplus \Sigma_n \), where \( \mathbb{R}^n \) is viewed as \( 1 \times n \) real matrices, \( \Sigma_n \) is the set of \( n \times n \) skew symmetric matrices, and the bracket is defined by \([ (u, U), (v, V) ] = (0, u^t v - v^t u)\). The automorphism group of \( \mathcal{F}(n) \) is identified with \( \text{Gl}(n, \mathbb{R}) \times \text{Hom}(\Sigma_n, \Sigma_n) \) with the action of \((A, \nu)\) on \((u, U)\) given by \((A, \nu) \cdot (u, U) = (uA, A^t UA + \nu(u))\). Thus, \( \text{O}(n) \), the group of \( n \times n \) orthogonal matrices is a maximal compact subgroup of \( \text{Aut}(\mathcal{F}(n)) \). We denote by \( SO(n) \) the subgroup of matrices of determinant one.

**Theorem D.** Let \( K \) be a closed (not necessarily connected) subgroup of \( \text{SO}(n) \). \((K, F(n))\) is a Gelfand pair if, and only if \( K = SO(n) \).

Suppose now that a two-step \( N \) is given with \([N, N] = \mathcal{Z} \), where \( \mathcal{Z} \) is the center of \( N \). (If this condition is not satisfied, then \( N \) has an abelian direct product factor that does not play a role in the current considerations.) Given a compact, connected \( K \subseteq \text{Aut}(N) \), we fix a \( K \)-invariant inner product, \( \langle \cdot, \cdot \rangle \), on \( \mathcal{N} \), and denote by \( \mathcal{N}' \), the orthogonal complement of \( \mathcal{N} \) in \( \mathcal{N} \).

Let \( X_1, \ldots, X_n \) be an orthonormal basis for \( \mathcal{N}_1 \). Define the homomorphism \( \lambda : \mathcal{N} \rightarrow \mathcal{N}_1 \) by setting \( \lambda(e_i) = X_i \) (where \( e_1, \ldots, e_n \) is the standard basis for \( \mathbb{R}^n \)), and \( \lambda(E_{i,j}) = [X_i, X_j] \), (where \( E_{i,j} = [(e_i, 0), (e_j, 0)] \in \mathcal{F}(n) \)). Let \( \mathcal{H} \) denote the kernel of \( \lambda \), which is identified with \( \Sigma_n \). Note that \( \lambda : \mathbb{R}^n \rightarrow \mathcal{N}'_1 \) is an isometry (where \( \mathcal{F}(n) \) is equipped with the (standard) inner product \( \langle (u, U), (v, V) \rangle = uv^t + \frac{1}{2} \text{tr}(UV^t) \)). Given \( k \in K \), we define \( \tilde{k} \in \text{Aut}(\mathcal{F}(n)) \) by \( \tilde{k}(e_i) = \lambda^{-1}(k \cdot (\lambda(e_i))) \) and \( \tilde{k}(E_{i,j}) = [\tilde{k} \cdot e_i, \tilde{k} \cdot e_j] \), and set \( \tilde{K} = \{ \tilde{k} | k \in K \} \). Then \( \tilde{K} \subseteq \text{O}(n) \), and one has that \( K \) is maximal compact if, and only if, \( \tilde{K} = \text{O}_\mathcal{Z}(n) := \{ A \in \text{O}(n) | A \cdot \mathcal{H} := \{ A \cdot \mathcal{H} : = A \cdot \mathcal{H} \cdot A \} = \mathcal{H} \} \).

Let \( \mathcal{L} \) denote the orthogonal complement in \( \Sigma_n \) of \( \mathcal{H} \), and set \( \mathcal{N}_\mathcal{L} = \mathbb{R}^n \oplus \mathcal{L} \) with Lie bracket defined by \([ (u, U), (v, V) ]_\mathcal{L} = \tilde{P}_\mathcal{L}(u^t v - v^t u) \), where \( \tilde{P}_\mathcal{L} \) is the orthogonal projection of \( \Sigma_n \) onto \( \mathcal{L} \). Then \( \mathcal{N}_\mathcal{L} \simeq \mathcal{N} \) and \( \tilde{K} \subseteq \text{Aut}(\mathcal{N}_\mathcal{L}) \).

For nonzero \( B \in \mathcal{L} \), let \( \mathcal{H}_B \) denote the subset of \( \mathcal{N}_\mathcal{L} \) given by \( \mathbb{R}^n B \oplus \mathbb{R} B \), i.e. the range of \( B \) in \( \mathbb{R}^n \) plus the line through \( B \), and define a Lie bracket similar to the above by following the bracket in \( \mathcal{F}(n) \) with the orthogonal projection onto \( \mathbb{R} B \). The quotient Lie algebra \( \mathcal{N}_\mathcal{L}/\mathcal{L}_0 \), where \( \mathcal{L}_0 \) is the orthogonal complement in \( \mathcal{L} \) of \( \mathbb{R} B \) is isomorphic to the direct sum of ideals \( \mathcal{N}_B \) and \( \mathbb{R}^n B \), the latter being commutative. Let \( H_B \) denote the simply connected Lie group corresponding to \( \mathcal{N}_B \), and given \( b \in (\mathbb{R}^n B) \), let \( \tilde{K}_{(b, B)} = \{ \tilde{k} | \tilde{k} \cdot (b, B) = (b, B) \} \).

**Theorem E.** \((K, N)\) is a Gelfand pair if \((\tilde{K}_{(b, B)}, H_B)\) is a Gelfand pair for all \((b, B)\) in a set of full Plancherel measure, and conversely, if \((K, N)\) is a Gelfand pair, then \((\tilde{K}_{(b, B)}, H_B)\) is a Gelfand pair for all \( B \in \mathcal{L}, b \in (\mathbb{R}^n B) \).
We demonstrate the use of Theorem E in two examples. In the first, let $N$ be the group whose Lie algebra has a basis $X, Y_1, Y_2, Z_1, Z_2$, and with all non-trivial commutators determined by $[X, Y_1] = Z_1$ and $[X, Y_2] = Z_2$. We show that there is no compact subgroup $K \subseteq \text{Aut}(N)$ for which $(K, N)$ is a Gelfand pair.

In the second example, we give a short proof of a theorem due to H. Leptin [Le] which states that if $K$ is the $n$-dimensional torus (and $N$ is a two-step group with $[\mathcal{N}, \mathcal{N}] = \mathbb{Z}$, the center of $\mathcal{N}$) then $(K, N)$ is a Gelfand pair if, and only if, $N$ is the quotient of the direct product of $n$-copies of $H_1$, with $K$ lifting to a $U(1)$ action on each factor $H_1$.

We turn now to solvable groups. The essential new ingredient is another theorem due to H. Leptin, which was privately communicated to the authors. Since a proof has not appeared in the literature, we include his proof here.

**Theorem (Leptin).** Let $\mathcal{S}$ be a solvable Lie algebra with nilradical $\mathcal{N}$. Let $K$ be a compact, connected subgroup of $\text{Aut}(\mathcal{S})$, and let $\mathcal{S}_0 = \{X \in \mathcal{S}| k \cdot X = X, \forall k \in K\}$. Then $\mathcal{S} = \mathcal{S}_0 + \mathcal{N}$.

For $X \in \mathcal{S}$, let $i_X$ denote the inner-automorphism of $S$ determined by exp $X$, and denote by rad($S$) the simply connected nilpotent Lie group whose Lie algebra is the nilradical of $\mathcal{S}$. Using Leptin's theorem we can prove

**Theorem F.** $(K, S)$ is a Gelfand pair if, and only if, $(K, \text{rad}(S))$ is a Gelfand pair, and for each $X \in \mathcal{S}_0$, $y \in S$ there is a $k \in K$ such that $i_X(y) = k \cdot y$.

Finally, we consider the $K$-spherical functions associated to a Gelfand pair $(K, S)$. Recall that a $K$-spherical function $\phi$ is a continuous, complex valued function defined on $S$ satisfying $\phi(e) = 1$ and $\int_K \phi(xk \cdot y) dk = \phi(x)\phi(y)$ for each $x, y \in S$. It is well known that integration against a $K$-spherical function, $\phi$, defines a complex homomorphism on $L^1_K(S)$, that this homomorphism is continuous if $\phi$ is bounded, and that each continuous homomorphism of $L^1_K(S)$ is obtained in this manner. We denote by $\Delta(K, S)$ the set of continuous homomorphisms on $L^1_K(S)$. It follows from Theorem F, that if $(K, S)$ is a Gelfand pair then $S$ has polynomial growth, [Je], and hence that $L^1(S)$ is a symmetric Banach *-algebra, [Lu]. From this one can show that the bounded $K$-spherical functions are positive definite, in sharp contrast to the case when $(G, K)$ is a Riemannian symmetric pair (cf. [He]).

We first consider Gelfand pairs $(K, N)$. One shows that if $\pi \in \hat{N}$ and $\pi' = \pi_k$, then the intertwining representations $W_{\pi}$ and $W_{\pi'}$ have the same irreducible subspaces.

**Theorem G.** Let $(K, N)$ be a Gelfand pair. Then $\phi$ is a bounded $K$-spherical function if, and only if, there is a $\pi \in \hat{N}$ and a $\xi \in V_\alpha \subseteq H_\pi$, $\|\xi\| = 1$, such that for each $x \in N$, $\phi(x) = \phi_{\pi, \xi}(x) := \int_K \langle \pi(k \cdot x)\xi, \xi \rangle dk$. 


where $V_\alpha$ is an irreducible subspace for the intertwining representation $W^n$. Furthermore, bounded $K$-spherical functions $\phi_\pi, \xi = \phi_{\pi'}, \xi'$ if, and only if, $\pi' = \pi_k$ for some $k \in K$ and $\xi, \xi'$ belong to the same $V_\alpha$.

Theorem G states that there is a 1-1 correspondence between $\Delta(K, N)$ and the fibered product $\widetilde{N}/K \times \sigma(W^n, \mathbf{H}_\pi)$, where $\widetilde{N}/K$ denotes the $K$-orbits in $\widetilde{N}$, and $\sigma(W^n, \mathbf{H}_\pi)$ denotes the irreducible components of $W^n$ in $\mathbf{H}_\pi$.

Suppose now that $(K, S)$ is a Gelfand pair. Let $X_1, \ldots, X_p$ be a basis for a complement of $\mathcal{N}$, the nilradical of $\mathcal{P}$, in $\mathcal{P}_0$. For each $y \in S$, there exist unique $n(y) \in N (= \exp(\mathcal{N}))$ and $t(y) \in \mathbf{R}^p$ such that $y = n(y)\Pi_i \exp(t_i(y)X_i)$.

**Theorem H.** $\phi$ is a bounded $K$-spherical function on $S$ if, and only if, $\phi|_N$ is a bounded $K$-spherical function on $N$ and there exists $a \in \mathbf{R}^p$ such that $\phi(y) = \phi(n(y))e^{i(a, t(y))}$. Thus,

$$\Delta(K, S) = \Delta(K, N) \times \mathbf{R}^p.$$  

**Remarks.** A number of authors, in addition to those already mentioned, have considered Gelfand pairs of the form $(K, N)$, and the associated $K$-spherical functions. In [HR] it is shown that the usual action of a maximal torus in $U(n)$ on $H_n$ provides an example of a Gelfand pair, and the $K$-spherical functions are expressed in terms of Laguerre polynomials. The paper [KR] exhibits examples $(K, N)$, where $N$ is an irreducible group of Heisenberg type and $K$ is either $\text{Spin}(n)$ or a maximal connected compact subgroup of $\text{Aut}(n)$. In [Ca], examples are presented where $N$ arises as the Šilov boundary of a Siegel domain of type II and $K = SU(p) \times U(q)$. The generalized Laguerre polynomials introduced in [Hz] are shown in [Di] to be associated to certain Gelfand pairs $(U(n), H_n)$.

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**Preliminaries**

Consider a unimodular group $G$ with $K \subseteq G$ a compact subgroup. We denote the $L^1$-functions that are invariant under both the left and right actions of $K$ on $G$ by $L^1(G//K)$. These form a subalgebra of the group algebra $L^1(G)$ with respect to the convolution product

$$(1.1) \quad f \ast g(x) = \int_G f(y)g(y^{-1}x)\, dy = \int_G f(xy^{-1})g(y)\, dy.$$  

According to the traditional definition, one says that $K \subseteq G$ is a Gelfand pair if $L^1(G//K)$ is commutative.
Suppose now that $K$ is a compact group acting on $G$ by automorphisms via some homomorphism $\phi: K \rightarrow \text{Aut}(G)$. One can form the semidirect product $K \ltimes G$, with group law

$$(k_1, x_1)(k_2, x_2) = (k_1 k_2, x_1 k_1 \cdot x_2),$$

where we write $k \cdot x$ for $\phi(k)(x)$. Right $K$-invariance of a function $f: K \alpha G \rightarrow \mathbb{C}$ means that $f(k, x)$ depends only on $x$. Accordingly, if one defines $f_G: G \rightarrow \mathbb{C}$ by $f_G(x) = f(e, x)$, then one obtains a bijection $L^1(K \alpha G/\!/K) \simeq L^1_K(G)$ given by $f \mapsto f_G$. Here $L^1_K(G)$ denotes the $K$-invariant functions on $G$, i.e. those $f \in L^1(G)$ such that $f(k \cdot x) = f(x)$ for all $x \in G$ and $k \in K$. One verifies easily that this map respects the convolution product and we see that $K \subseteq K \alpha G$ is a Gelfand pair if, and only if, the convolution algebra $L^1_K(G)$ is commutative. Thus, the definition given in the introduction agrees with the more standard one.

Note that if $(K_1, G)$ is a Gelfand pair and $K_1 \subseteq K_2$, then $(K_2, G)$ is also a Gelfand pair. Also note that we can assume that $K$ acts faithfully on $G$ since we can always replace $K$ by $K/\ker(\phi)$. In this way we can regard $K$ as a compact subgroup of $\text{Aut}(G)$. It is a useful fact that the Gelfand pair property depends only on the conjugacy class of $K$ in $\text{Aut}(G)$.

**Lemma 1.3.** Let $K, L$ be compact groups acting on $G$ which are conjugate inside $\text{Aut}(G)$. Then $(K, G)$ is a Gelfand pair if, and only if, $(L, G)$ is a Gelfand pair.

**Proof.** For $f \in L^1(G)$, define $f^L \in L^1_L(G)$ by

$$f^L(x) = \int_L f(l \cdot x) \, dl.$$ 

The map $f \mapsto f^L$ is onto $L^1_L(G)$. Suppose that $L = uKu^{-1}$ for some $u \in \text{Aut}(G)$. Then

$$f^L(x) = \int_K f((uku^{-1}) \cdot x) \, dk$$

$$= \int_K (f \circ u)(k \cdot (u^{-1}(x))) \, dk$$

$$= (f \circ u)^K(u^{-1}(x)).$$

It follows that $f^L(u(x)) = (f \circ u)^K(x)$ and that $L^1_L(G) \rightarrow L^1_K(G): f \mapsto f \circ u := \Phi(f)$ is a vector space isomorphism.

Let $dx$ denote Haar measure on $G$. Then $u^*(dx) = \Delta(u) \, dx$ for some nonzero real number $\Delta(u)$. We will show that $\Phi(f) * \Phi(g) = \Delta(u) \Phi(f * g)$. It
follows that $f * g = g * f \Leftrightarrow \Phi(f) * \Phi(g) = \Phi(g) * \Phi(f)$. We compute 
\[
(\Phi(f) * \Phi(g))(x) = \int_G \Phi(f)(y)\Phi(g)(y^{-1}x) \, dy \\
= \int_G (f \circ u)(y)(g \circ u)(y^{-1}x) \, dy \\
= \int_G f(u(y))g(u^{-1}u(x)) \, du(y) \\
= \Delta(u) \int_G f(y)g(y^{-1}u(x)) \, du(y) \\
= \Delta(u)(f * g)(u(x)) \\
= \Delta(u)\Phi(f * g)(x).
\]

Suppose now that $G$ is a Lie group. For $D \in \mathcal{E}'(G)$, the space of compactly supported distributions, define the $K$-average $D^K$ by
\[
\langle D^K, f \rangle = \langle D, f^K \rangle,
\]
for each $f \in C^\infty_c(G)$, where $f^K$ is defined by (1.4). The space of $K$-invariant, compactly supported distributions is
\[
\mathcal{E}'_K(G) = \{D \in \mathcal{E}'|D^K = D\} = \{D^K|D \in \mathcal{E}'(G)\}.
\]
If $\delta_x$ is the delta function at $x \in G$ then $\delta^K_x \in \mathcal{E}'_K(G)$ has compact support $K \cdot x$. One has
\[
\langle \delta^K_x, f \rangle = \int_K f(k \cdot x) \, dk.
\]

**Lemma 1.8.** The $K$-invariant test functions are dense in $\mathcal{E}'_K(G)$.

**Proof.** Merely note that if $\{u_n\} \subseteq \mathcal{E}(G)$, and $u_n \to D \in \mathcal{E}'(G)$, then $u^K_n \to D^K = D$, for each $D \in \mathcal{E}'_K(G)$. □

The convolution of distributions $D_1, D_2 \in \mathcal{E}'(G)$ is defined by
\[
\langle D_1 * D_2, f \rangle = \langle D_1(x), D_2, l_x^{-1}f \rangle,
\]
where $l_x f(y) = f(x^{-1}y)$. In particular, one has
\[
\langle \delta^K_x * \delta^K_y, f \rangle = \int_K \int_K f((k_1 \cdot x)(k_2 \cdot y)) \, dk_1 \, dk_2.
\]

**Lemma 1.11.** If $(K, G)$ is a Gelfand pair then convolution in $\mathcal{E}'_K(G)$ is commutative.

**Proof.** This follows immediately from commutativity of $L^1_K(G)$ and Lemma 1.8. □
Theorem 1.12. \((K, G)\) is a Gelfand pair if, and only if, for all \(x, y \in G\), \(xy \in (K \cdot y)(K \cdot x)\).

Proof. Suppose that \(xy \notin (K \cdot y)(K \cdot x)\). We will show that \(\delta^K_x \ast \delta^K_y \neq \delta^K_y \ast \delta^K_x\), so \((K, G)\) fails to be a Gelfand pair by Lemma 1.11. Indeed, one can find a non-negative test function \(f: G \to \mathbb{R}\) with \(f(xy) = 1\) and \(f((K \cdot y)(K \cdot x)) = 0\) by compactness of \((K \cdot y)(K \cdot x)\). But then (1.10) shows that \(\langle \delta^K_x \ast \delta^K_y, f \rangle\) is positive, whereas \(\langle \delta^K_y \ast \delta^K_x, f \rangle = 0\).

Conversely, suppose \(xy \in (K \cdot y)(K \cdot x)\) for all \(x, y \in G\), and let \(f, g \in L^1_K(G)\). Then
\[
f \ast g(x) = \int_G f(xy)g(y^{-1}) dy = \int_G f((k_3 \cdot y)x)g(y^{-1}) dy,
\]
where \(xy = (k_1 \cdot y)(k_2 \cdot x) = k_2((k_3 \cdot y)x)\). Note that \(k_1, k_2, \) and \(k_3\) depend on the integration variable \(y\). Using \(K\)-invariance of \(f\) we write
\[
f \ast g(x) = \int_G \int_K f(k \cdot ((k_3 \cdot y)x))g(y^{-1}) dk \, dy \,
\]
via \(k \mapsto kk_3^{-1}\)
\[
= \int_K \int_G f(y(kk_3^{-1} \cdot x))g(k^{-1} \cdot y^{-1}) dy \, dk
\]
via \(y \mapsto k^{-1} \cdot y\)
\[
= \int_G \int_K f(y(kk_3^{-1} \cdot x))g(y^{-1}) dk \, dx
\]
using \(K\)-invariance
\[
= \int_G \int_K f(y(k \cdot x))g(y^{-1}) dk \, dy
\]
via \(k \mapsto kk_3\)
\[
= \int_K g \ast f(k \cdot x) dk
\]
changing the order of integration
\[
= g \ast f(x)
\]
using \(K\)-invariance. \(\Box\)

It is not difficult to check that the condition in Theorem 1.12 is equivalent to the more symmetrical condition that \((K \cdot x)(K \cdot y) = (K \cdot y)(K \cdot x)\).

Three-step groups

We now begin our consideration of Gelfand pairs that involve nilpotent groups. Let \(N\) be a connected, simply connected nilpotent Lie group with Lie algebra \(\mathcal{N}\). Recall the descending central series for \(\mathcal{N}\),
\[
\mathcal{N} = \mathcal{N}^{(1)} \supset \mathcal{N}^{(2)} \supset \cdots \supset \mathcal{N}^{(n)} \supset \mathcal{N}^{(n+1)} = \{0\},
\]
Lemma 2.3. Let $N$ be an $n$-step group with $n \geq 3$. Then

$[N, N^{(n-1)}] \neq \{0\}.$

Proof. Suppose $[N_1, N^{(n-1)}] = \{0\}$, and choose any $n$ elements $X_1, X_1, \ldots, X_{n-1}, Y \in N$. Then $W = [X_1, [X_2, \ldots, [X_{n-2}, X_{n-1}] \ldots]]$ is an element of $N^{(n-1)}$, and writing $Y = U + V$ where $U \in N_1$, $V \in N^{(2)}$, we see that

$[Y, W] = [U, W] + [V, W] = [V, W] = 0$

since $[N_1, N^{(n-1)}] = 0$ and any $n$-fold bracket of terms in $N^{(2)}$ must vanish. However, this shows that $N$ cannot be $n$-step since all $n$-fold brackets in $N$ are zero. □

The main result of this section is

Theorem 2.4. If $N$ is an $n$-step group with $n \geq 3$ then there are no Gelfand pairs $(K, N)$.

Proof. Since $K$ is compact, there is a $K$-invariant inner product $\langle \cdot, \cdot \rangle$ on $N$. Indeed, such an inner product can be obtained by averaging an arbitrary one with respect to the $K$-action. Form the decomposition (2.2) using this inner product and choose any $X \in N_1$, $Y \in N_{n-1}$ with $[X, Y] \neq 0$. This is possible by Lemma 2.3, and the observations that $N^{(n-1)} = N_{n-1} \oplus N_n$ and $N_n$ is contained in the center.

Let exp denote the exponential map from $N$ to $N$. We will show that for $x = \exp(X)$, $y = \exp(Y)$ one has $xy \notin (K \cdot y)(K \cdot x)$. Suppose otherwise, and pick $k_1, k_2 \in K$ so that $xy = (k_1 \cdot y)(k_2 \cdot x)$. By the Baker-Campbell-Hausdorff formula one has

$X + Y + \frac{1}{2}[X, Y] = k_2 \cdot X + k_1 \cdot Y + \frac{1}{2}[k_1 \cdot Y, k_2 \cdot X],$

where $(k, X) \mapsto k \cdot X$ is the derived action of $K$ on $N$.

Since any automorphism of $N$ must preserve each $N^{(k)}$, we have $k_1 \cdot Y \in N^{(n-1)}$. Thus $X$ and $k_2 \cdot X$ differ by an element $W \in N^{(n-1)}$, so that $k_2 \cdot X = X + W$. As $N_1$ and $N^{(n-1)}$ are orthogonal subspaces in $N$ and the $K$-action preserves orthogonality, we see that $W = 0$. That is $k_2 \cdot X = X$, and (2.5) becomes

$Y + \frac{1}{2}[X, Y] = k_1 \cdot Y + \frac{1}{2}[k_1 \cdot Y, X].$
The same trick now shows that \( k_1 \cdot Y = Y \), since the two differ by an element of \( \mathcal{M}_\pi^* \). Finally, (2.6) becomes \([X, Y] = [Y, X]\), which is impossible since \([X, Y] \neq 0\). \( \square \)

### Some representation theory

This section will serve to introduce some notation and to describe a result due to G. Carcano. Since this result is of primary importance to our analysis, we will include a sketch of the proof.

If \( \pi \) and \( \pi' \) are irreducible unitary representations of \( N \), we write \( \pi \simeq \pi' \) to indicate that \( \pi \) and \( \pi' \) are unitarily equivalent. We denote by \( \tilde{\mathcal{N}} \) the equivalence classes of irreducible unitary representations of \( N \). Given \( k \in K \) and \( \pi \in \tilde{\mathcal{N}} \) we denote by \( \pi_k \) the representation defined by

\[
(3.1) \quad \pi_k(x) = \pi(k \cdot x).
\]

The stabilizer of \( \pi \) under this action is

\[
(3.2) \quad K_\pi = \{ k \in K : \pi_k \simeq \pi \}.
\]

We denote by \( \mathcal{O}_\pi \) the coadjoint orbit in \( \mathcal{M}_\pi^* \) corresponding to \( \pi \) according to the Kirillov theory, and note that \( K_\pi \) is also the stabilizer of \( \mathcal{O}_\pi \) under the dual action of \( K \) on \( \mathcal{M}_\pi^* \).

For each \( k \in K_\pi \), one can choose an intertwining operator \( W_\pi(k) \) with

\[
\pi_k(x) = W_\pi(k) \pi(x) W_\pi(k)^{-1}
\]

for each \( x \in N \). The map \( k \mapsto W_\pi(k) \) need not be a representation of \( K_\pi \). Indeed, the \( W_\pi(k) \)'s are only characterized up to multiplicative constants in the circle \( T \) by the intertwining condition. In fact, there will be a map

\[
(3.3) \quad \sigma (= \sigma_\pi) : K_\pi \times K_\pi \to T
\]

for which \( W_\pi(k_1 k_2) = \sigma(k_1, k_2) W_\pi(k_1) W_\pi(k_2) \). The map \( \sigma \) can be made measurable and is called the multiplier for the projective representation \( W_\pi \). We call \( W_\pi \) the *intertwining representation* for the representation \( \pi \).

Many aspects of representation theory can be extended to projective representations as well (cf. [Ma]). In particular, compactness of \( K_\pi \) implies that \( W_\pi \) decomposes as a direct sum of irreducible (projective) representations. Writing \( c(T, W_\pi) \) for the multiplicity of \( T \) in \( W_\pi \), one has

\[
(3.4) \quad W_\pi = \sum_{T \in \hat{K}_\pi^\sigma} c(T, W_\pi) T.
\]

Here, \( \hat{K}_\pi^\sigma \) denotes the set of unitary equivalence classes of projective representations of \( K_\pi \) with multiplier \( \sigma (= \sigma_\pi) \). The following theorem is from [Ca].

**Theorem 3.5.** If \( (K, N) \) is a Gelfand pair, then \( c(T, W_\pi) \leq 1 \) for all \( \pi \in \tilde{\mathcal{N}} \), and conversely, if \( c(T, W_\pi) \leq 1 \) for almost all (with respect to Plancherel measure) \( \pi \in \tilde{\mathcal{N}} \) then \( (K, N) \) is a Gelfand pair.

**Proof.** For completeness we sketch what is essentially Carcano's proof.
Let \( \pi \in \hat{N} \) and let \( W_\pi \) be the intertwining representation of \( K_\pi \) with multiplier \( \sigma \). If \( T \) is any irreducible projective representation of \( K_\pi \) with multiplier \( \overline{\sigma} \), then

\[
R(k, x) = T(k) \otimes \pi(x) W_\pi(k)
\]

is an irreducible representation of \( K_\pi \rtimes N \) whose restriction to \( N \) is a multiple of \( \pi \), and the induced representation \( \text{Ind}_{K_\pi \rtimes N}^G(R) \) is irreducible for \( K \rtimes N \). By considering all \( \pi \) and \( T \), one obtains all equivalence classes of irreducible representations of \( K \rtimes N \) in this manner (cf. [Ma]).

It is well known that if \( K \subset G \) is a Gelfand pair, then for each irreducible representation \( \pi \) of \( G \), the space of \( K \)-fixed vectors has dimension \( c(1_K, \pi|_K) \in \{0, 1\} \) (cf. [He]). For the representation \( R \) given by (3.6), one has

\[
\text{Ind}_{K_\pi \rtimes N}^G(R)|_K \cong \text{Ind}_{K_\pi}^K(R|_K) = \text{Ind}_{K_\pi}^K(T \otimes W_\pi),
\]

and by Frobenius reciprocity for compact groups,

\[
c(1_K, \text{Ind}_{K_\pi}^K(T \otimes W_\pi)) = c(1_K|_{K_\pi}, T \otimes W_\pi) = c(1_{K_\pi}, T \otimes W_\pi).
\]

This last value can be written as \( c(T, W_\pi) \) since \( 1_{K_\pi} \) has multiplicity 1 in \( T \) and multiplicity 0 in \( T \otimes S \) for \( S \) not equivalent to \( T \). This shows the necessity of the condition.

Now suppose \( \pi \in \hat{N} \) satisfies the multiplicity condition. Denote the Hilbert space on which it acts by \( H_\pi \), and form the decomposition

\[
(3.7) \quad H_\pi = \sum_{T \in \hat{K_\pi}} H_{\pi \cdot T}
\]

into \( K_\pi \)-irreducible subspaces. (If \( T \) is not a subrepresentation of \( W_\pi \), then \( H_{\pi \cdot T} = \{0\} \).) If \( f \in L^1_K(N) \) then one shows that the operator \( \pi(f) \) commutes with every \( W_\pi(k) \). Since each factor \( H_{\pi \cdot T} \) in (3.7) occurs only once, \( \pi(f) \) must preserve these factors and thus, acts as a scalar in each by Schur's Lemma. It follows that if \( f, g \in L^1_K(N) \) then the operators \( \pi(f) \) and \( \pi(g) \) commute and hence \( \pi(f \ast g) = \pi(g \ast f) \). When this equality holds for almost all \( \pi \in \hat{N} \), one concludes that \( f \ast g = g \ast f \) by appealing to the Plancherel Theorem. \( \Box \)

We remark that the result holds more generally for compact actions on separable locally compact groups.

**Heisenberg groups**

The \((2n + 1)\)-dimensional Heisenberg group \( H_n \) has Lie algebra \( \mathfrak{h}_n \) with basis \( X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z \) and structure equations given by \([X_i, Y_i] = Z\). The group \( \text{Sp}(n, \mathbb{R}) \) of real \( 2n \times 2n \) symplectic matrices acts on \( \text{Span}(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \) by automorphisms of \( \mathfrak{h}_n \). It is well known that \( U(n) = \text{Sp}(n, \mathbb{R}) \cap O(2n) = \text{Sp}(n, \mathbb{R}) \cap SO(2n) \) is a maximal compact
connected subgroup of $\text{Aut}(H_n)$ (cf. [Ho]). (The full automorphism group contains inner automorphisms, dilations and an involution that sends $Z$ to $-Z$ in addition to these symplectic automorphisms.) If one models $H_n$ as $\mathbb{C}^n \times \mathbb{R}$, as we generally will, then $U(n)$ becomes the group of $n \times n$ unitary matrices acting on $\mathbb{C}^n$ in the usual fashion.

We recall the representation theory of $H_n$. A generic set of coadjoint orbits in $\mathcal{N}_n^*$ is parametrized by nonzero $\lambda \in \mathbb{R}$, where the orbit $\mathcal{O}_\lambda$ is the hyperplane in $\mathcal{N}_n^*$ of all functionals taking the value $\lambda$ at $Z$. The action of $U(n)$ on $\mathcal{N}_n^*$ preserves each $\mathcal{O}_\lambda$. Hence, if $\pi_\lambda$ is the element of $\hat{H}_n$ corresponding to $\mathcal{O}_\lambda$, then $U(n)$ also preserves the equivalence class of $\pi_\lambda$.

One can realize $\pi_\lambda$ in the Fock space

\begin{equation}
H_\lambda(n) = \left\{ \text{entire } f: \mathbb{C}^n \to \mathbb{C} | \int_{\mathbb{C}^n} e^{-2|\lambda||w|^2} |f(w)|^2 \, dw < \infty \right\}
\end{equation}

as

\begin{equation}
\pi_\lambda(z, t)f(w) = e^{-i\lambda t + i\lambda(2(w, z) - |z|^2)}f(w - z)
\end{equation}

for $\lambda > 0$ and

\begin{equation}
\pi_\lambda(z, t)f(w) = e^{-i\lambda t - i\lambda(2(w, z) - |z|^2)}f(w - z)
\end{equation}

for $\lambda < 0$. Here $(w, z)$ denotes the Hermitian inner product on $\mathbb{C}^n$. We refer the reader to [Ho or Ta] for a discussion of the Fock model.

Define $W_\lambda(k): H_\lambda(n) \to H_\lambda(n)$ by

\begin{equation}
W_\lambda(k) f(z) = f(k^{-1} z).
\end{equation}

Then $W_\lambda(k)$ intertwines $\pi_\lambda(z, t)$ and $(\pi_\lambda)_k(z, t) = \pi_\lambda(k z, t)$. We verify this for $\lambda > 0$. Indeed,

\begin{equation}
W_\lambda(k)(\pi_\lambda(k^{-1} z, t)f)(w) = \pi_\lambda(k^{-1} z, t)f(k^{-1} w)
\begin{align*}
&= e^{-i\lambda t + i\lambda(2(k^{-1} w, k^{-1} z) - |k^{-1} z|^2)}f(k^{-1} w - k^{-1} z) \\
&= e^{-i\lambda t + i\lambda(2(w, z) - |z|^2)}W_\lambda(k)f(w - z) \\
&= (\pi_\lambda(z, t)W_\lambda(k)f)(w),
\end{align*}
\end{equation}

and hence

\begin{equation}
W_\lambda(k)\pi_\lambda(z, t)W_\lambda(k)^{-1} = \pi_\lambda(k z, t)
\end{equation}

as claimed. That is, $U(n)$ is the stabilizer of the equivalence class of $\pi_\lambda \in \hat{H}_n$ under the action of $U(n)$ and $W_\lambda: H_\lambda(n) \to H_\lambda(n)$ is the intertwining representation as in (3.4). (We remark that up to a factor of $\det(k)^\frac{1}{2}$, $W_\lambda$ lifts to the oscillator representation on the double cover $MU(n)$ of $U(n)$ (cf. [Ta]).) It follows that for any compact subgroup $K \subseteq U(n)$, $K_{\pi_\lambda} = K$, and the intertwining representation of $K$ is given by the restriction of $W_\lambda$ to $K$.

Given a compact, connected subgroup $K \subseteq U(n)$, we denote its complexification by $K_C$. The action of $K$ on $\mathbb{C}^n$ yields a representation of $K_C$ on
Theorem 4.6. Let $K$ be a compact, connected subgroup of $U(n)$ acting irreducibly on $\mathbb{C}^n$. The following are equivalent: (i) $(K, H_n)$ is a Gelfand pair. (ii) The representation of $K_C$ on $\mathbb{C}^n$ is multiplicity free. (iii) The representation of $K_C$ on $\mathbb{C}^n$ is equivalent to one of the representations in the following table:

<table>
<thead>
<tr>
<th>Group</th>
<th>Acting On</th>
<th>Subject To</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Sl(n, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n \geq 2$</td>
</tr>
<tr>
<td>$Gl(n, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n \geq 1$</td>
</tr>
<tr>
<td>$Sp(k, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 2k$</td>
</tr>
<tr>
<td>$\mathbb{C}^* \times Sp(k, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 2k$</td>
</tr>
<tr>
<td>$\mathbb{C}^* \times SO(n, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n \geq 2$</td>
</tr>
<tr>
<td>$Gl(k, \mathbb{C})$</td>
<td>$S^2(\mathbb{C}^k) \simeq \mathbb{C}^n$</td>
<td>$n = k(k+1)/2, \ k \geq 2$</td>
</tr>
<tr>
<td>$Sl(k, \mathbb{C})$</td>
<td>$\Lambda^3(\mathbb{C}^k) \simeq \mathbb{C}^n$</td>
<td>$n = (\binom{k}{2})$ and $k$ is odd</td>
</tr>
<tr>
<td>$Gl(k, \mathbb{C})$</td>
<td>$\Lambda^2(\mathbb{C}^k) \simeq \mathbb{C}^n$</td>
<td>$n = (\binom{k}{2})$</td>
</tr>
<tr>
<td>$Sl(k, \mathbb{C}) \times Sl(l, \mathbb{C})$</td>
<td>$\mathbb{C}^k \otimes \mathbb{C}^l \simeq \mathbb{C}^n$</td>
<td>$n = kl, \ k \neq l$</td>
</tr>
<tr>
<td>$Gl(k, \mathbb{C}) \times Sl(l, \mathbb{C})$</td>
<td>$\mathbb{C}^k \otimes \mathbb{C}^l \simeq \mathbb{C}^n$</td>
<td>$n = kl$</td>
</tr>
<tr>
<td>$Gl(2, \mathbb{C}) \times Sp(k, \mathbb{C})$</td>
<td>$\mathbb{C}^2 \otimes \mathbb{C}^{2k} \simeq \mathbb{C}^n$</td>
<td>$n = 4k$</td>
</tr>
<tr>
<td>$Sl(3, \mathbb{C}) \times Sp(k, \mathbb{C})$</td>
<td>$\mathbb{C}^3 \otimes \mathbb{C}^{2k} \simeq \mathbb{C}^n$</td>
<td>$n = 6k$</td>
</tr>
<tr>
<td>$Gl(3, \mathbb{C}) \times Sp(k, \mathbb{C})$</td>
<td>$\mathbb{C}^3 \otimes \mathbb{C}^{2k} \simeq \mathbb{C}^n$</td>
<td>$n = 6k$</td>
</tr>
<tr>
<td>$Gl(4, \mathbb{C}) \times Sp(4, \mathbb{C})$</td>
<td>$\mathbb{C}^4 \otimes \mathbb{C}^8 \simeq \mathbb{C}^n$</td>
<td>$n = 32$</td>
</tr>
<tr>
<td>$Sl(k, \mathbb{C}) \times Sp(4, \mathbb{C})$</td>
<td>$\mathbb{C}^k \otimes \mathbb{C}^8 \simeq \mathbb{C}^n$</td>
<td>$n = 8k, \ k &gt; 4$</td>
</tr>
<tr>
<td>$Gl(k, \mathbb{C}) \times Sp(4, \mathbb{C})$</td>
<td>$\mathbb{C}^k \otimes \mathbb{C}^8 \simeq \mathbb{C}^n$</td>
<td>$n = 8k, \ k &gt; 4$</td>
</tr>
<tr>
<td>$\mathbb{C}^* \times Spin(7, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 8$</td>
</tr>
<tr>
<td>$\mathbb{C}^* \times Spin(9, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 16$</td>
</tr>
<tr>
<td>$Spin(10, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 16$</td>
</tr>
<tr>
<td>$\mathbb{C}^* \times Spin(10, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 16$</td>
</tr>
<tr>
<td>$\mathbb{C}^* \times G_2$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 7$</td>
</tr>
<tr>
<td>$\mathbb{C}^* \times E_6$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 27$</td>
</tr>
</tbody>
</table>
Proof. The complexification $K_C$ of $K$ is connected, reductive, algebraic (cf. [BtD]) and acts irreducibly on $C^n$. Moreover, the representation of $K$ on $C^n$ is multiplicity free if, and only if, the complexified representation of $K_C$ on $C^n$ is multiplicity free. The multiplicity free irreducible linear representations of connected, reductive, algebraic groups have been classified by V. Kac. The table given here is taken from Theorem 3 of [Ka]. This gives the equivalence of (ii) and (iii).

The equivalence of (i) and (ii) is an immediate consequence of Theorem 3.5 once one observes that for each $\lambda \neq 0$, $W_\lambda$ is the completion of the associated representation of $K$ on $C[C^n]$. □

Remarks. Some comments are in order regarding the table. $C^*$ denotes the nonzero complex numbers, $S^2$ the symmetric 2-tensors and $\Lambda^2$ the alternating 2-tensors. The group $C^* \times Sp(k, C)$ acts on $C^{2k}$ via $(\lambda, A) \cdot v = \lambda v A$. We can view $C^* \times Sp(k, C)$ as the group of $n \times n$ complex matrices that transform the standard symplectic structure on $C^n$ into a scalar multiple of itself. There are similar interpretations for the other groups $C^* \times G$. $Spin(n, C) = Spin(n, R)_C$ is a double cover of $SO(n, C)$ and acts by the complexified half-spin representation. $Spin(7, C)$ and $Spin(9, C)$ are simply connected and $\pi_1(Spin(10, C)) = Z_2$.

Suppose now that the action of $K$ on $C^n$ is reducible, and let

$$C^n = \sum_{j=1}^{p} V_j$$

be a decomposition of $C^n$ into $K$-irreducible (not necessarily complex) subspaces. If $(K, H_n)$ is a Gelfand pair, then the $V_\alpha$'s are orthogonal with respect to the skew-symmetric form on $C^n$ given by $\Lambda: (z, w) \mapsto \Im(z \cdot \bar{w})$. Indeed, if $z_i \in V_\alpha_i$ for $i = 1, 2$ then by Theorem 1.12 there exist $k_1, k_2 \in K$ such that $(z_1, 0)(z_2, 0) = (k_2 \cdot z_2, 0)(k_1 \cdot z_1, 0)$. It follows that

$$\sum_i z_i = \sum_i k_i \cdot z_i$$

and that

$$\Lambda(z_1, z_2) = \Lambda(k_2 \cdot z_2, k_1 \cdot z_1).$$

Since the $V_\alpha$'s are orthogonal with respect to the usual Hermitian inner product $\langle \cdot, \cdot \rangle$ on $C^n$ and are $K$-invariant, one concludes from (4.8) that $k_i \cdot z_i = z_i$, for $i = 1, 2$, and hence from (4.9) that $\Lambda(z_1, z_2) = 0$. It now follows that the $V_\alpha$'s have complex structure, i.e. $iV_\alpha = V_\alpha$. Suppose not, and let $z \in V_\alpha$ such that $iz \notin V_\alpha$. Then $iz = \sum \beta z_\beta$, and $z_\beta \neq 0$ for some $\beta \neq \alpha$. Thus,

$$|z|^2 = -\Lambda(z, iz) = \sum \beta -\Lambda(z, z_\beta) = -\Lambda(z, z_\alpha) < |z|^2.$$

Finally, since the $V_\alpha$'s are invariant under multiplication by $i$, the skew-symmetric form $\Lambda$ is nondegenerate on each $V_\alpha$. Therefore, if $m_j = \dim(V_j)$,
$H_{m_j} \simeq V_j \times \mathbb{R}$. (This isomorphism is made explicit in the proof of Theorem 5.12.)

Let $K_j$ denote the subgroup of $U(V_j)$, the group of unitary transformations on $V_j$ obtained by the restriction of $K$ to $V_j$, and let

\begin{equation}
C[V_j] = \sum_{n=0}^{\infty} P_{j,n}
\end{equation}

be the decomposition of the polynomial ring over $V_j$ into $K_j$-irreducible subspaces, with the convention that $P_{1,0} = \{0\}$. For each $p$-tuple $(n_1, \ldots, n_p) \in (\mathbb{Z}^+)^p$, let $P_{n_1, \ldots, n_p} = P_{1,n_1} \otimes \cdots \otimes P_{p,n_p}$. If $W_{\lambda,j}$ denotes the intertwining representation associated to the pair $(K_j, H_{m_j})$ as above, then for each $k \in K$, the restriction of $W_{\lambda}$ to $P_{n_1, \ldots, n_p}$ is given by $W_{\lambda,1} \otimes \cdots \otimes W_{\lambda,p}$. Thus, if $(K, H_n)$ is a Gelfand pair, Theorem 4.6 implies that $(K_j, H_{m_j})$ is a Gelfand pair for each $j = 1, \ldots, p$. But it also implies the stronger condition that the subrepresentations of $K$ on $P_{n_1, \ldots, n_p}$, as $(n_1, \ldots, n_p)$ ranges over $(\mathbb{Z}^+)^p$, are distinct. This establishes the necessity of the condition in the following theorem. The sufficiency is an immediate consequence of Theorem 4.6 and the observation that

\[ C[C^n] = \sum_{(n_1, \ldots, n_p) \in (\mathbb{Z}^+)^p} P_{n_1, \ldots, n_p}. \]

**Theorem 4.11.** $(K, H_n)$ is a Gelfand pair if, and only if, the subrepresentations of $W_{\lambda}$ on $P_{n_1, \ldots, n_p}$ are distinct as $(n_1, \ldots, n_p)$ ranges over $(\mathbb{Z}^+)^p$.

We consider two examples. For the first, let $K$ be the subgroup of matrices of determinant one in $U(2) \times U(1) \subseteq U(3)$, i.e. $K = \{(A, \det(A)) \mid A \in U(2)\}$. The decomposition of $C^3$ corresponding to (4.7) is $C^3 = C^2 \oplus C$, in the obvious sense, and corresponding to (4.10) one has that $C[C^2] = \sum_{n=1}^{\infty} P_{1,n}$, where $P_{1,n}$ is the space of homogeneous polynomials in $z_1, z_2$ of degree $n$, and $C[C^1] = \sum_{n=1}^{\infty} P_{2,n}$, where $P_{2,n} = Cz_3^n$. The intertwining representation of $K$ on $P_{n_1,n_2}$ is equivalent to the representation $A \mapsto (\det(A))^{n_2}W_{\lambda}(A)$ of $U(2)$ on $P_{1,n}$. These representations are clearly irreducible and inequivalent for distinct $(n_1, n_2)$. Thus $(K, H_3)$ is a Gelfand pair.

For the second example, let $K$ be the subgroup of $U(1) \times U(1)$ consisting of all matrices of determinant one. In this case, both $(K_1, H_1)$ and $(K_2, H_1)$ are Gelfand pairs, and in fact, the subrepresentations of the intertwining representations of $K_1$ and $K_2$ on $C[C^1]$ are distinct (corresponding to $Z^+$ for $K_1$, and $Z^-$ for $K_2$). However, the intertwining representation on $P_{n,n}$ is the identity for each $n$, and thus $(K, H_2)$ is not a Gelfand pair.

We conclude this section with an immediate corollary to Theorem 4.11.

**Corollary 4.12.** Let $K_j$ be a compact subgroup of $U(n_j)$ for $1 \leq j \leq p$, $K = \prod K_j$, and let $n = \sum n_j$. Then $(K, H_n)$ is a Gelfand pair if, and only if $(K_j, H_{n_j})$ is a Gelfand pair for $1 \leq j \leq p$.
**Free Groups**

In this section we turn our attention to the free, two-step nilpotent Lie group on $n$-generators, $F(n)$. We realize its Lie algebra, $\mathcal{F}(n)$, as $\mathbb{R}^n \oplus \Sigma_n$, where $\mathbb{R}^n$ is viewed as $1 \times n$ real matrices, $\Sigma_n$ is the space of real $n \times n$ skew symmetric matrices, and the Lie bracket is given by

$$[(u, U), (v, V)] = (0, u^t v - v^t u). \tag{5.1}$$

The group law is thus

$$(u, U) (v, V) = (u + v, U + V + \frac{1}{2}(u^t v - v^t u)). \tag{5.2}$$

**Lemma 5.3.** There is a bijection between $\text{Aut}(F(n)) \simeq \text{Aut}(\mathcal{F}(n))$ and the set $\text{Gl}(n, \mathbb{R}) \times \text{Hom}(\mathbb{R}^n, \Sigma_n)$.

**Proof.** The exponential map establishes the isomorphism

$$\text{Aut}(F(n)) \simeq \text{Aut}(\mathcal{F}(n)).$$

For $(A, \nu) \in \text{Gl}(n, \mathbb{R}) \times \text{Hom}(\mathbb{R}^n, \Sigma_n)$, define $\phi_{(A, \nu)} : \mathcal{F}(n) \to \mathcal{F}(n)$ by

$$\phi_{(A, \nu)}(u, U) = (uA, A^t U A + \nu(u)). \tag{5.4}$$

It is easy to check that $\phi_{(A, \nu)}$ is a Lie algebra automorphism. On the other hand, if $\phi : \mathcal{F}(n) \to \mathcal{F}(n)$ is any given automorphism, then $\phi = \phi_{(A, \nu)}$, where $A$ and $\nu$ are the composites

$$\mathbb{R}^n \hookrightarrow \mathcal{F}(n) \xrightarrow{\phi} \mathcal{F}(n) \to \mathbb{R}^n$$

and

$$\mathbb{R}^n \hookrightarrow \mathcal{F}(n) \xrightarrow{\phi} \mathcal{F}(n) \to \Sigma_n$$

respectively. \(\square\)

Note that the correspondence in Lemma 5.3 becomes a group isomorphism if the set $\text{Gl}(n, \mathbb{R}) \times \text{Hom}(\mathbb{R}^n, \Sigma_n)$ is given the group structure

$$\langle (A, \nu)(B, \mu) = (AB, A \cdot \mu + \nu B), \rangle \tag{5.4}$$

with $\text{Gl}(n, \mathbb{R})$ acting on $\Sigma_n$ by $A \cdot V = A^t V A$. In particular, we see that a maximal compact subgroup of $\text{Aut}(F(n))$ can be identified with $O(n)$, the group of real orthogonal matrices. This acts on $\mathcal{F}(n)$ by

$$A \cdot (u, U) = (uA, A \cdot U) = (uA, A^t U A), \tag{5.5}$$

and preserves the inner product

$$\langle (u, U), (v, V) \rangle = uv^t + \frac{1}{2} \text{tr}(UV^t). \tag{5.6}$$

Suppose that $\mathcal{Z}$ is a subspace of $\Sigma_n$. We define a Lie algebra $\mathcal{N}_\mathcal{Z} := \mathbb{R}^n \times \mathcal{Z}$ with bracket

$$[(u, U), (v, V)]_\mathcal{Z} = (0, P_\mathcal{Z}(u^t v - v^t u)), \tag{5.7}$$

where $P_\mathcal{Z}$ is the orthogonal projection of $\Sigma_n$ onto $\mathcal{Z}$.
We now describe the coadjoint orbits in $\mathcal{F}(n)^*$ and $\mathcal{N}_Z^*$. First, using the inner product (5.6) we identify $\mathcal{F}(n)^*$ with $\mathcal{F}(n)$ and $\mathcal{N}_Z^*$ with $\mathcal{N}_Z$. This gives an inclusion $\mathcal{N}_Z^* \rightarrow \mathcal{F}(n)^*$ dual to the projection $P_Z$. For $B \in \Sigma_n$, define a map

\begin{equation}
J_B : \mathbb{R}^n \rightarrow \mathbb{R}^n
\end{equation}

by $\langle J_B(u), v \rangle = \langle B, u^t v - v^t u \rangle$. Similarly, if $B \in \mathcal{Z}$ define a map

\begin{equation}
J_B^\mathcal{Z} : \mathbb{R}^n \rightarrow \mathbb{R}^n
\end{equation}

by $\langle J_B^\mathcal{Z}(u), v \rangle = \langle B, [(u, 0), (v, 0)]_\mathcal{Z} \rangle$. In fact, though, for $B \in \mathcal{Z}$, $J_B = J_B^\mathcal{Z}$ since

$$\langle J_B(u), v \rangle = \langle B, [(u, 0), (v, 0)] \rangle = \langle B, [(u, 0), (v, 0)] \rangle = \langle J_B(u), v \rangle.$$

Accordingly, we denote both maps by $J_B$. One computes

$$\langle J_B(u), v \rangle = \langle B, [(u, 0), (v, 0)] \rangle = \frac{1}{2} \text{tr}(B(u^t v - v^t u)^t) = \langle uB, v \rangle$$

to conclude that

\begin{equation}
J_B(u) = uB.
\end{equation}

The coadjoint orbit through $(b, B) \in \mathcal{F}(n)^*$ ($\cong \mathcal{F}(n)$) is

$$\mathcal{O}_{(b, B)} = \text{Ad}^*(F(n))(b, B).$$

For $(u, U), (v, V) \in \mathcal{F}(n)$ one has

$$\langle \text{Ad}^* \exp(u, U)(b, B), (v, V) \rangle = \langle (b, B), (v, V) + [(u, U), (v, V)] \rangle = bv^t + \frac{1}{2} \text{tr}(BV^t) + \frac{1}{2} \text{tr}(Bu^t v - v^t u)^t) = \langle (b, B), (v, V) \rangle + \langle J_B(u), v \rangle = \langle (b + J_B(u), B), (v, V) \rangle.$$

Thus,

\begin{equation}
\mathcal{O}_{(b, B)} = (b, B) + \text{Image}(J_B), 0) = (b + R^n B, B).
\end{equation}

The same reasoning shows that when $B \in \mathcal{Z}$ the orbit $\mathcal{O}_{(b, B)}$ through $(b, B) \in \mathcal{N}_Z^*$ is also given by $(b + R^n B, B)$, i.e. the inclusion $\mathcal{N}_Z^* \rightarrow \mathcal{F}(n)^*$ maps $\mathcal{O}_{(b, B)}$ diffeomorphically to $\mathcal{O}_{(b, B)}$. Accordingly, we denote both of these orbits by $\mathcal{O}_{(b, B)}$, and will write $\mathcal{O}_{B}$ for $\mathcal{O}_{(0, B)}$.

For $n$ even, the orbits $\mathcal{O}_{B} := \mathcal{O}_{(0, B)} = R^n \times \{B\}$ with $B$ nondegenerate provide a generic set of orbits in $\mathcal{F}(n)^*$, while for $n$ odd, the orbits $\mathcal{O}_{(b, B)}$ with $b \in R^n$ and $B$ of rank $(n - 1)$ form a generic set. (Note that these orbits are not distinct since $\mathcal{O}_{(b_1, B)} = \mathcal{O}_{(b_2, B)}$, provided $b_1 - b_2 \in R^n B$.)
Theorem 5.12. \((SO(n), F(n))\) is a Gelfand pair for all \(n \geq 2\).

Proof. The proof is an application of Theorem 3.5. Since the generic orbits in \(\mathcal{F}(n)^*\) depend on the parity of \(n\), we consider the cases separately.

Suppose first that \(n = 2k\) and let \(B \in \Sigma \) be nondegenerate. We may also assume that \(B\) has distinct eigenvalues which we denote \(\pm i\lambda_1, \ldots, \pm i\lambda_k\), with \(\lambda_j > 0\). The orbits \(\mathcal{O}_B = \mathbb{R}^n \times \{B\}\) for such \(B\) form a generic set in \(\mathcal{F}(n)^*\).

Let \(\mathcal{H}_B\) denote the Lie algebra defined in (5.7) with \(\mathcal{Z} = \mathbb{R}B\). \(B\) is central in \(\mathcal{H}_B\) and for \(u, v \in \mathbb{R}^n\) one has
\[
[u, v] = (J_B(u), v)B = \omega_B(u, v)B,
\]
where \(\omega_B(u, v) = uBv^t\) is the skew symmetric bilinear form on \(\mathbb{R}^n\) with matrix \(B\). Nondegeneracy of \(B\) implies that \(\mathcal{H}_B\) is isomorphic to the Heisenberg algebra \(\mathcal{H}_k\). We can make this isomorphism explicit by changing the basis on \(\mathbb{R}^n\). Suppose \(B\) has eigenvectors \(\alpha_1, \ldots, \alpha_k\) in \(\mathbb{C}^k\) corresponding to the eigenvalues \(i\lambda_1, \ldots, i\lambda_k\). Writing \(\alpha_j = v_j + iu_j\), one has \(u_jB = \lambda_jv_j\) and \(v_jB = -\lambda_ju_j\). The matrix of \(B\) in the basis \(\{u_1, v_1, \ldots, u_k, v_k\}\) is
\[
B = \begin{pmatrix}
\lambda_1J & 0 & \cdots & 0 \\
0 & \lambda_2J & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_kJ
\end{pmatrix}
\]
where
\[
J = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]
By scaling the \(\alpha_j\)'s we can ensure that \(\{u_1, v_1, \ldots, u_k, v_k\}\) is an orthonormal basis. Writing \(X'_j = (u_j, 0), Y'_j = (v_j, 0)\), and \(Z = (0, B)\) in \(\mathcal{H}_B\) we obtain a basis in which the Lie bracket in (5.7) becomes \([X'_j, Y'_j] = \lambda_jZ\) with other brackets vanishing. Replacing \(X'_j\) by \(X'_j = (1/\sqrt{\lambda_j})X'_j\), and \(Y'_j\) by \(Y'_j = (1/\sqrt{\lambda_j})Y'_j\) one obtains a basis \(\{X_1, Y_1, \ldots, X_k, Y_k, Z\}\) for \(\mathcal{H}_B\) in which the nonzero brackets are determined by \([X_j, Y_j] = Z\).

Let \(Sp(\omega_B) = \{A \in GL(n, \mathbb{R})|ABA^t = B\}\). This is the group of linear transformations preserving the symplectic form \(\omega_B\). The stabilizer of \(\mathcal{O}_B\) under the action of \(SO(n)\) is
\[
K_B = SO(n) \cap Sp(\omega_B) = \{A \in SO(n)|AB = BA\}.
\]
\(K_B\) also acts on \(\mathcal{H}_B\) and stabilizes \(\mathcal{O}_B\) regarded as an orbit in \(\mathcal{H}_B^*\). In view of (5.14), \(K_B\) acts on \(\mathcal{H}_B\) as \(U(1)^k\) on \(\text{Span}(X_1, Y_1, \ldots, X_k, Y_k)\). Here each factor \(U(1) = SO(2) \cap Sp(1, \mathbb{R}) = \{A \in SO(2)|AJ = JA\}\) acts on \(\text{Span}(X_j, Y_j)\) in the usual fashion. The representations of \(H_B = \exp(\mathcal{H}_B)\) and \(F(n)\) given by \(\mathcal{O}_B\) coincide under the orthogonal projection \(\mathcal{F}(n) \to \mathcal{H}_B\) and hence have the same intertwining representations. In view of Corollary 4.12, this must satisfy the conditions of Theorem 3.5, and we conclude that \((SO(n), F(n))\) is a Gelfand pair.
Theorem 5.17. Theorem 5.17.

viewed as a compact subgroup

then (K, F(n)) is not a Gelfand pair.

Proof. Choose a basis for C

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Theorem 3.5. 0

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obtain a generic set of orbits \( \mathcal{O}_{(b, B)} \) in \( \mathcal{F}(n)^* \) from such pairs \((b, B)\).

Let \( \mathcal{N}_B \) be defined as in (5.7) with \( \mathcal{L} = RB \), and let \( X \) be any nonzero vector in \( \ker(B) \). From (5.10) one concludes that the center of \( \mathcal{N}_B \) is given by \( \text{Span}(B, X) \) and that \( \mathcal{N}_B = \mathcal{H}_B \times \mathbb{R} \) (as Lie algebras) where \( \mathcal{H}_B = \mathcal{N}_B / \mathbb{R}X \cong \mathcal{H}_k \).

In view of (5.5), the stabilizer of \( \mathcal{O}_{(b, B)} \) under the action of \( SO(n) \) is given by

\[
K_{(b, B)} = \{ A \in SO(n) | bA = b \text{ and } AB = BA \} = \{ A \in SO(2k) | AB = BA \},
\]

where we are regarding \( SO(2k) \) as the stabilizer of \( b \in \mathbb{R}^n \) under the action of \( SO(n) \).

\( \mathcal{O}_{(b, B)} \) can be viewed as an orbit in \( \mathcal{N}_B \) and also as an orbit in \( \mathcal{H}_B \). The action of \( K_{(b, B)} \) on \( \mathcal{N}_B \) descends to \( \mathcal{H}_B \) since each \( A \in K_{(b, B)} \) preserves \( \ker(B) \). Just as in the case where \( n \) is even, one shows that this corresponds to the action of \( U(1)^k \) on \( \mathcal{H}_k \) and completes the proof using Corollary 4.12 and Theorem 3.5. □

Theorem 5.17. If \( K \) is a proper, closed (not necessarily connected) subgroup of \( SO(n) \) then \((K, F(n))\) is not a Gelfand pair.

Proof. As in the proof of Theorem 5.12, one must consider separately the cases \( n \) even and \( n \) odd. Here we present the argument for the case \( n = 2k \). We assume at first that \( K \) is connected. The stabilizer of a generic orbit \( \mathcal{O}_B \) can be viewed as a compact subgroup \( A_B \) of \( K_B \cong U(1)^k \) (see equation (5.15)). We regard \( A_B \) as acting on a Heisenberg group \( H_k \) and conclude that if \((K, F(n))\) is a Gelfand pair then so is \((A_B, H_k)\), as in the proof of Theorem 5.12.

For a suitable choice of \( B \), \( A_B \) is a proper subgroup of \( K_B \). Indeed, let \( C \in SO(n) \setminus K \) and let \( T \) be a maximal torus in \( SO(n) \) that contains \( C \). Choose a basis for \( \mathbb{C}^k \cong \mathbb{R}^n \) which transforms \( T \) into the usual \( U(1)^k \) and let \( B \) be given in this basis by

\[
( J \ 0 \ \ldots \ 0 )
0 \ 2J \ \ldots \ 0 \\
\vdots \ \vdots \ \ddots \ \vdots \\
0 \ 0 \ \ldots \ kJ
\]

One has \( K_B = T \) so that \( A_B = K \cap K_B \) is a proper subgroup of \( K_B \).

\( A = A_B \) is a proper connected subgroup of \( U(1)^k \) and hence is a torus. One can decompose \( \mathbb{C}^k \) into a sum of weight spaces for the action of \( A \),

\[
\mathbb{C}^k = \sum_{\alpha \in \mathcal{A}^*} V_\alpha.
\]

Here \( \alpha \in \mathcal{A}^* \), where \( \mathcal{A} \) is the Lie algebra of \( A \),

\[
V_\alpha = \{ v \in \mathbb{C}^k | \exp(X) \cdot v = e^{2\pi i \alpha(X)} v \text{ for all } X \in \mathcal{A} \},
\]
and $P$ denotes the set of weights: $P = \{ \alpha \in \mathcal{A}^* \mid V_\alpha \neq \{0\} \}$. Each $\alpha \in P$ is an integral form, that is $\alpha(L) \subseteq \mathbb{Z}$, where $L = \ker(\exp: \mathcal{A} \to A)$.

There is a corresponding decomposition of the polynomial functions on $\mathbb{C}^k$:

\[(5.21) \quad \mathbb{C}[\mathbb{C}^k] = \bigotimes \mathbb{C}[V_\alpha]. \]

The $A$-action on $\mathbb{C}[\mathbb{C}^k]$ preserves each $\mathbb{C}[V_\alpha]$ and acts via the character

\[(5.22) \quad \chi_\alpha(\exp(X)) = e^{2\pi ig(X)}.
\]

There are two cases to consider:

(i) Some weight space $V_\alpha$ has $\dim_{\mathbb{C}}(V_\alpha) > 1$.

(ii) $\dim_{\mathbb{C}}(V_\alpha) = 1$ for all $\alpha \in P$.

Suppose (i). Any decomposition $V_\alpha = U \oplus W$ into nontrivial subspaces $U$ and $W$ will be preserved by the $A$-action. Moreover, $A$ will act on the invariant subspaces $\mathbb{C}[U]$ and $\mathbb{C}[W]$ of $\mathbb{C}[\mathbb{C}^k]$ via the character $\chi_\alpha$. This shows that the action of $A$ on $\mathbb{C}^k$ is not multiplicity free and hence that $(K, F(n))$ is not a Gelfand pair.

Next assume that $\dim_{\mathbb{C}}(V_\alpha) = 1$ for all $\alpha \in P$. In this case, $P$ consists of $k$ weights $\{\alpha_1, \ldots, \alpha_k\}$ and we obtain a basis $\{v_1, \ldots, v_k\}$ of $\mathbb{C}^k$ by choosing $v_j \in V_{\alpha_j}$ with $v_j \neq 0$. Note that any monomial $v_1^{i_1}v_2^{i_2} \cdots v_k^{i_k}$ generates an $A$-invariant subspace in $\mathbb{C}[\mathbb{C}^k]$.

As $\dim(\mathcal{A}) < k$, the weights $\alpha_1, \ldots, \alpha_k$ must satisfy some nontrivial linear dependence relation:

\[(5.23) \quad c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k = 0. \]

In fact, one can find an integer solution $(c_1, c_2, \ldots, c_k)$ to this equation, since the forms $\alpha_j$ are integral. Suppose $c_1, \ldots, c_i$ are nonnegative and that $c_{i+1}, \ldots, c_k$ are negative (after rearranging the weights). Consider the monomials

\[(5.24) \quad p = v_1^{c_1} \cdots v_i^{c_i} \quad \text{and} \quad q = v_{i+1}^{-c_{i+1}} \cdots v_k^{-c_k}. \]

One has

$$\exp(X)p = e^{2\pi ig(c_1\alpha_1 + \cdots + c_i\alpha_i)(X)}p \quad \text{and} \quad \exp(X)q = e^{-2\pi ig(c_{i+1}\alpha_{i+1} + \cdots + c_k\alpha_k)(X)}q$$

for $X \in \mathcal{A}$. One concludes that the $A$-irreducible subspaces of $\mathbb{C}[\mathbb{C}^k]$ spanned by $p$ and $q$ are equivalent. As in case (i), the action of $A$ on $\mathbb{C}^k$ is not multiplicity free and $(K, F(n))$ fails to be a Gelfand pair.

Finally, consider a nonconnected, proper subgroup $K \subseteq SO(n)$. The stabilizer $A' = A_B'$ of a generic orbit $G_B$ now has the form $A' = A \times F$, where $A$ is a torus with $\dim(A) < k$ and $F$ is a finite abelian group. As before, we decompose $\mathbb{C}^k$ into weight spaces $V_\alpha$ for the action of $A$. Note that the action of $F$ and $A$ commute so that each $V_\alpha$ is $F$-invariant. As before, we consider two cases:
(i) Suppose \( \dim(V_\alpha) > 1 \). Choose two linearly independent vectors \( u, v \in V_\alpha \). The actions of \( A' \) on the monomials \( u^{[F]} \) and \( v^{[F]} \) agree and hence the representation of \( A' \) on \( C^K \) is not multiplicity free.

(ii) Suppose \( \dim(V_\alpha) = 1 \) for all \( \alpha \). In this case, the actions of \( A' \) on \( p^{[F]} \) and \( q^{[F]} \) agree, where \( p \) and \( q \) are given by (5.24). \( \Box \)

**Two-step groups**

In this section we do not assume that \( K \) is a connected group. Suppose now that a two-step \( N \) is given with \([N', N] = \mathcal{Z}\), where \( \mathcal{Z} \) is the center of \( N' \).

If this condition is not satisfied, then \( N = N_1 \oplus \mathcal{G} \) where \( N_1 \) is a \( K \)-invariant, nilpotent Lie algebra with \([N_1, N_1]\) spanning the center of \( N_1 \), and \( \mathcal{G} \) is commutative. Thus, \( N = N_1 \times A \) and \( L^1(N) = L^1(N_1) \otimes L^1(A) \). It is now easy to show that \( L^1_k(N) \) is commutative if, and only if, \( L^1_k(N_1) \) is commutative.

Thus there is no loss in assuming that \([N', N] = \mathcal{Z}\).

Given a compact subgroup \( K \subseteq \text{Aut}(N) \), we fix a \( K \)-invariant inner product \( \langle \cdot, \cdot \rangle \) on \( N' \), and denote by \( N'_1 \) the orthogonal complement to \( \mathcal{Z} \) in \( N' \).

Let \( X_1, \ldots, X_n \) be an orthonormal basis for \( N'_1 \). Define the homomorphism \( \lambda: \mathcal{F}(n) \rightarrow N' \) by setting \( \lambda(e_i) = X_i \) (where \( e_1, \ldots, e_n \) is the standard basis for \( \mathbb{R}^n \)), and \( \lambda(E_{i,j}) = [X_i, X_j] \), (where \( E_{i,j} = [(e_i, 0), (e_j, 0)] \in \mathcal{F}(n) \)).

Let \( \mathcal{H} \) denote the kernel of \( \lambda \) \((\subseteq \Sigma_n)\). Note that \( \lambda: \mathbb{R}^n \rightarrow N'_1 \) is an isometry (where \( \mathcal{F}(n) \) is equipped with the inner product \( \langle (u, U), (v, V) \rangle = \langle (0, uV^t + \frac{1}{2}\text{tr}(UV^t)) \rangle \)). Given \( k \in K \), we define \( \tilde{k} \in \text{Aut}(\mathcal{F}(n)) \) by \( \tilde{k}(e_i) = \lambda^{-1}(k \cdot \lambda(e_i)) \) and \( \tilde{k}(E_{i,j}) = [\tilde{k} \cdot e_i, \tilde{k} \cdot e_j] \), and set \( \tilde{K} = \{\tilde{k} | k \in K\} \). Note that \( \tilde{K} \cong K \).

**Lemma 6.1.** Let \( K \) be a compact subgroup of \( \text{Aut}(N) \). For any choice of orthonormal basis of \( N'_1 \), \( \tilde{K} \) is a compact subgroup of \( O(n) \). If \( \tilde{K}, \tilde{K}' \) are constructed using different orthonormal bases of \( N'_1 \) then \( \tilde{K} = A' \tilde{K}' A \) for some \( A \in O(n) \). \( K \) is a maximal compact subgroup of \( \text{Aut}(N) \) if, and only if, \( \tilde{K} = O_{\mathcal{H}}(n) \) := \{\( A \in O(n) | A \cdot \mathcal{H} (:= A' \mathcal{H} A) = \mathcal{H} \}\}.

**Proof.** Given \( \tilde{k} \in \tilde{K} \), \( \tilde{k}(\mathbb{R}^n) \subseteq \mathbb{R}^n \). Thus, there is an \( A_k \in GL(n, \mathbb{R}) \) such that \( \tilde{k} \cdot (u, U) = (uA_k, A_k \cdot U) \). Since \( \lambda: \mathbb{R}^n \rightarrow N'_1 \) is an isometry and the inner product on \( N \) is \( K \)-invariant, \( A_k \in O(n) \). Finally note that \( \lambda \tilde{k} = k \lambda \).

It follows that \( \mathcal{H} = \ker(\lambda) \) is \( \tilde{k} \)-invariant, and hence that \( \tilde{K} \subseteq O_{\mathcal{H}}(n) \).

Suppose that \( A \in O_{\mathcal{H}}(n) \). Define \( k_A \in \text{Aut}(N) \) by requiring that \( k_A \cdot \lambda((u, U)) = \lambda(A \cdot (u, U)) \). It is clear that \( A \mapsto k_A : O_{\mathcal{H}}(n) \rightarrow \text{Aut}(N) \) is a 1-1 homomorphism, and hence, since \( O(n) \) is a maximal compact subgroup of \( GL(n, \mathbb{R}) \), that \( K \) is a maximal compact subgroup of \( \text{Aut}(N) \) if, and only if, \( \tilde{K} = O_{\mathcal{H}}(n) \). \( \Box \)

Let \( \mathcal{Z} \) denote the orthogonal complement in \( \Sigma_n \) of \( \mathcal{H} \), and let \( N'_\mathcal{Z} = \mathbb{R}^n \times \mathcal{Z} \) be the Lie algebra defined as in (5.7), i.e. with Lie bracket defined by...
[(u, U), (v, V)]_Z = P_Z(u'v - v'u), where \( P_Z \) is the orthogonal projection of \( \Sigma_n \) onto \( Z \). Let \( \lambda: \mathcal{T}(n)/\mathcal{K} \to \mathcal{N} \) be the canonical isomorphism, define \( i: \mathcal{N}_Z \to \mathcal{T}(n)/\mathcal{K} \) by \( i(X) = X + \mathcal{K} \), and let \( \lambda = \lambda \circ i \). Then \( \lambda \) is a Lie algebra isomorphism. Since \( \tilde{\mathcal{K}} \subseteq \mathcal{O}_Z(n) \), by restriction we may consider \( \tilde{\mathcal{K}} \subseteq \text{Aut}(N_Z) \), where \( N_Z = \exp(\mathcal{N}_Z) \). One can easily check that \( k \cdot \lambda(X) = \lambda(\tilde{k} \cdot X) \) and thus prove

**Lemma 6.2.** \((K, N)\) is a Gelfand pair if, and only if, \((\tilde{K}, N_Z)\) is a Gelfand pair.

Pick a nonzero \( B \in Z \). Let \( \mathcal{N}_B \) denote the Lie algebra defined as in (5.7) with \( Z = RB \). \( \mathcal{N}_B \) is a concrete realization of the quotient Lie algebra \( \mathcal{N}_Z/\mathcal{I}_0 \), where \( \mathcal{I}_0 \) is the orthogonal complement in \( Z \) of \( RB \). Let \( \mathcal{K}_B \) denote the subset of \( \mathcal{N}_B \) given by \( R^B \times RB \), and define a Lie bracket as in (5.7). Let \( N_B \) and \( H_B \) denote the corresponding simply connected Lie groups. Since the bilinear form defined on \( R^B \) by \( B \) is nondegenerate on its range, one has as in the proof of Theorem 5.12 (see equation (5.13)) that \( H_B \) is isomorphic to a Heisenberg group.

Given \( b \in (R^B)^\perp \), the orthogonal complement in \( R^B \) of the range of \( B \), set

\[
(6.3) \quad \tilde{K}_{(b, B)} = \{ \tilde{k} \in \tilde{K} \mid \tilde{k} \cdot B = B, \text{ and } \tilde{k} \cdot b = b \}.
\]

By restriction, we may consider \( \tilde{K}_{(b, B)} \) as a subgroup of \( \text{Aut}(H_B) \).

**Theorem 6.4.** If \((K, N)\) is a Gelfand pair then \((\tilde{K}_{(b, B)}, H_B)\) is a Gelfand pair for all \( B \) in \( Z \), and all \( b \in (R^B)^\perp \). Conversely, if \((\tilde{K}_{(b, B)}, H_B)\) is a Gelfand pair for \((b, B)\) in a set of full Plancherel measure, then \((K, N)\) is a Gelfand pair.

**Proof.** Recall that we identify Lie algebras and their duals using the selected inner products. Given \( B \in Z \) and \( b \in (R^B)^\perp \) we let \( \mathcal{O}_{(b, B)} \) denote the orbit in \( N_Z \) (\( \cong N_Z^* \)) through \((b, B)\). By (5.11), \( \mathcal{O}_{(b, B)} = (b + R^B, B) \). Thus, \( \tilde{K}_{(b, B)} \) is the subgroup of \( \tilde{K} \) that preserves the equivalence class of \( \pi_{(b, B)} \), the representation of \( N_Z \) corresponding to \( \mathcal{O}_{(b, B)} \).

As above, let \( Z_0 \) be the orthogonal complement in \( Z \) of \( RB \). Then \( Z_0 \) is the subset of \( Z \) on which the functional \( B \) vanishes. Thus, \( \pi_{(b, B)} \) factors through a representation of \( N_B = N_Z/\exp(Z_0) \).

Note that for \( u \in R^n \) and \( v \in (R^B)^\perp \), equation (5.10) implies that

\[
[(u, 0), (v, 0)]_{RB} = P_{RB}([(u, 0), (v, 0)]) = (B, [(u, 0), (v, 0)])B
= (J_B(u), v)B = (uB, v)B = 0.
\]
Thus, \( \mathcal{N}_B \) is the direct sum of the Heisenberg Lie algebra \( \mathfrak{H}_B = \mathbb{R}^n B \times \mathbb{R}B \) and the commutative algebra \( (\mathbb{R}^n B)^\perp \) (= \( (\mathbb{R}^n B)^\perp \times \{0\} \)). Writing \( N_B = H_B \times (\mathbb{R}^n B)^\perp \), \( \pi_{(b, B)} \) factors as \( \pi_B \otimes \chi_b \), where \( \pi_B \) is the element of \( \mathcal{H}_B \) corresponding to \( B \) and \( \chi_b \) is the unitary character defined on \( (\mathbb{R}^n B)^\perp \) by \( \chi_b(v) = e^{2\pi i(b,v)} \).

The intertwining representation of \( \tilde{K}_{(b, B)} \) fixes the factor \( \chi_b \), and thus is multiplicity free if, and only if, the representation of \( \tilde{K}_{(b, B)} \) on the space of \( \pi_B \) is multiplicity free. This proves the theorem. \( \square \)

Remark. If \( K \) is a maximal compact, connected subgroup of \( \text{Aut}(N) \) then \( \tilde{K}_{(b, B)} = O(\mathbb{R}^n B) \times O_b((\mathbb{R}^n B)^\perp) \), where \( O_b(V) \) denotes the group of all orthogonal transformations of \( V \) that fix \( v \in V \). We consider two applications of Theorem 6.4. In the first, let \( \mathcal{N} \) be the Lie algebra with basis \( X, Y_1, Y_2, Z_1, Z_2 \), and with all nonzero brackets determined by \( [X, Y_j] = Z_j \) for \( j = 1, 2 \).

Let \( K \) be a maximal compact subgroup of \( \text{Aut}(\mathcal{N}) \), and fix a \( K \)-invariant inner product on \( \mathcal{N} \). Pick an orthonormal basis \( X_i, i = 1, 2, 3 \), for \( \mathcal{Z}^\perp \), and define \( \lambda : \mathcal{F}(3) \to \mathcal{N} \) by requiring that \( \lambda(e_i) = X_i \), \( i = 1, 2, 3 \). Then, \( \dim(\mathcal{H} = \ker \lambda) = 1 \). Thus, if \( \mathcal{Z} \) is the orthogonal complement to \( \mathcal{H} \) in \( \Sigma_3 \), \( \dim(\mathcal{Z}) = 2 \). Hence, if \( B \in \mathcal{Z} \), \( B \neq 0 \), and \( b \in \mathbb{R}^3 \), one easily sees that \( \tilde{K}_{(b, B)} = \{e\} \). Thus there are no compact subgroups \( K' \) of \( \text{Aut}(\mathcal{N}) \) such that \( (K', N) \) is a Gelfand pair.

The next application of Theorem 6.4 will be to offer a short proof of a theorem due to H. Leptin, [Le]. We assume, as always, that \( \mathcal{N} \) is the nilpotent Lie algebra of a simply connected group \( N \) with \( [\mathcal{N}, \mathcal{N}] = \mathcal{Z} \), the center of \( \mathcal{N} \).

Theorem (Leptin). Suppose that \( K \) is the \( k \)-torus contained in \( \text{Aut}(N) \). Then \( (K, N) \) is a Gelfand pair if, and only if, \( N \) is the quotient of the direct product of \( k \)-copies of the 3-dimensional Heisenberg group \( H_1 \), with \( K \) acting trivially on the center of \( N \) and lifting to the product of the usual \( U(1) \) action on each factor \( H_1 \).

Proof. Let \( \lambda : \mathcal{F}(n) \to \mathcal{N} \), and \( \tilde{K} \subseteq \text{Aut}(F(n)) \) be defined as above. Let

\[
\mathbb{R}^n = \sum_{i=1}^{k} V_{\alpha_i}
\]

be the decomposition into \( \tilde{K} \)-root spaces. First note that if \( X_{\alpha_i} \in V_{\alpha_i}, i = 1, 2 \), and \( \alpha_1 \neq \alpha_2 \), then \( [X_{\alpha_1}, X_{\alpha_2}] = 0 \). Indeed, since \( (\tilde{K}, N_Z) \) is a Gelfand pair, there exist \( k_i \in \tilde{K}, i = 1, 2 \), such that

\[
X_{\alpha_1} + X_{\alpha_2} + \frac{1}{2}[X_{\alpha_1}, X_{\alpha_2}] = k_1 \cdot X_{\alpha_1} + k_2 \cdot X_{\alpha_2} + \frac{1}{2}[k_2 \cdot X_{\alpha_2}, k_1 \cdot X_{\alpha_1}].
\]

From the \( \tilde{K} \)-invariance of each \( V_{\alpha} \), one concludes that \( k_i \cdot X_{\alpha} = X_{\alpha} \), and thus that \( [X_{\alpha_1}, X_{\alpha_2}] = 0 \).
Next observe that for \( \alpha \in \{ \alpha_i \mid 1 \leq i \leq k \} \), \( \dim(V_{\alpha}) = 2 \). For this note that if \( \tilde{K}_\alpha \) is the action of \( \tilde{K} \) on \( \mathcal{M}_\alpha := V_\alpha \oplus \mathcal{Z} \), considered as a subalgebra of \( \mathcal{M}_\mathcal{Z} \), then \( (\tilde{K}_\alpha, \exp(\mathcal{M}_\alpha)) \) is a Gelfand pair. \( \dim(V_{\alpha}) > 1 \), since for each nonzero \( X \in V_\alpha \) there is a \( Y \in V_\alpha \) such that \( [X, Y] \neq 0 \), and since \( \tilde{K}_\alpha \) acts as a subgroup of \( T \) on \( \mathcal{M}_\alpha \), one concludes as in the proof of Theorem 5.17 that \( \dim(V_{\alpha}) = 2 \), and so \( n = 2k \).

Let \( \{e_{2i-1}, e_{2i}\} \) be an orthonormal basis for \( V_\alpha \), and let
\[
\Omega = \text{span}\{E_{2i-1,2i} \mid 1 \leq i \leq k\}.
\]
We will show that if \( B \in \mathcal{Z} \), the orthogonal complement to \( \mathcal{H} := \ker(\lambda) \) in \( \Sigma_{2k} \), then \( B \in \Omega \). Given such a \( B \), let \( R^B = \sum_{i=1}^k V_i \) be the decomposition corresponding to the standard form of the skew-symmetric \( B \). Since \( B \) is nondegenerate on its range, for each nonzero \( X \in R^B \) there is a \( Y_X \in R^B \) such that \( [X, Y_X] \neq 0 \). Since \( (\bar{K}_B, H_B) \) is a Gelfand pair, one concludes as before, that if \( X \in V_i \), then \( Y_X \in V_i \). It then follows that \( V_i = \text{span}\{ \tilde{K}_B \cdot X \} \) for any nonzero \( X \in V_i \). This amounts to showing that if \( \bar{K}_B \cdot X = X \) for some \( X \in V_i \), then \( X = 0 \). But this is clear, for otherwise, by Theorem 1.12, there exist \( k \in \bar{K}_B \) such that
\[
X + Y_X + \frac{1}{2}[X, Y_X] = X + k \cdot Y_X + \frac{1}{2}[k \cdot Y_X, X].
\]
This forces the contradiction that \( [X, Y_X] = 0 \). It now follows that each \( V_i \) equals some \( V_{\alpha_j} \), and hence that \( B \in \Omega \). Therefore, \( \mathcal{H} \) contains the orthogonal complement to \( \Omega \) in \( \Sigma_{2k} \), and \( F(n)/\exp(\mathcal{H}) \) is the quotient of the direct product of \( k \)-copies of \( H_1 \). Finally, since \( \tilde{K} \) fixes each element of \( \Omega \), \( \tilde{K} \) acts trivially on the center of \( \mathcal{N} \). \( \square \)

**SOLVABLE GROUPS**

We now consider a simply connected solvable Lie group \( S \) with Lie algebra \( \mathfrak{S} \). We denote by \( \mathcal{N}_\mathfrak{S} \), or more simply by \( \mathcal{N} \), the nilradical of \( \mathfrak{S} \). Given a compact subgroup \( \tilde{K} \subseteq \text{Aut}(\mathfrak{S}) \), we set
\[
\mathfrak{S}_0 = \{ X \in \mathfrak{S} \mid k \cdot X = X, \ \forall k \in \tilde{K} \}.
\]
The following theorem and proof was communicated to the authors by H. Leptin.

**Theorem (Leptin).** If \( K \) is connected, then \( \mathfrak{S} = \mathfrak{S}_0 + \mathcal{N} \).

**Proof.** Let \( \mathfrak{S}_C = \mathfrak{S} \otimes_R C \) be the complexification of \( \mathfrak{S} \). Then \( K \subseteq \text{Aut}(\mathfrak{S}_C) \), \( (\mathfrak{S}_0)_C = (\mathfrak{S}_C)_0 \), and \( \mathcal{N}_\mathfrak{S}_C = (\mathcal{N}_\mathfrak{S})_C \). Thus, we may assume that \( \mathfrak{S} \) is complex.

Now, if \( K \) is abelian and
\[
\mathfrak{S}_\chi = \{ X \in \mathfrak{S} \mid k \cdot X = \chi(k) X, \ \forall k \in K \},
\]
then
\[
(7.1) \quad \mathfrak{S} = \sum_{\chi \in \hat{K}} \mathfrak{S}_\chi.
\]
If \( X \in \mathcal{S}_x \), \( X \neq 0 \), and \( \lambda \) is an eigenvalue of \( \text{ad}X \), then there is a nonzero \( Y \in \mathcal{S} \) such that \( [X, Y] = \lambda Y \). For \( k \in K \),

\[
k \cdot (\lambda Y) = [k \cdot X, k \cdot Y] = \chi(k)[X, k \cdot Y].
\]

Thus, \( \chi(k) \lambda \) is also an eigenvalue of \( \text{ad}X \) for all \( k \in K \). But if \( \chi \neq \varepsilon \), the identity, \( \chi(K) = T \), and thus, \( \lambda t \) is an eigenvalue of \( \text{ad}X \) for all \( t \in T \). It follows that \( \lambda = 0 \), and so \( \text{ad}X \) is nilpotent. Therefore, \( \mathcal{S}_x \subseteq N \) for all \( \chi \neq \varepsilon \), i.e. \( \mathcal{S} = \mathcal{S}_0 + N \).

We turn now to the general case. Let \( t \in T \subseteq K \), and \( X \in \mathcal{S} \). Since \( \mathcal{S} = \mathcal{S}_0 + N \), where \( \mathcal{S}_0 = \{ X \in \mathcal{S} \mid t \cdot X = X, \ \forall t \in T \} \), by the argument above, \( t \cdot X \equiv X \pmod{N} \). But every element of \( K \) is in a torus, and so for all \( k \in K \), \( k \cdot X \equiv X \pmod{N} \). It follows that

\[
X_0 := \int_K k \cdot X dk \equiv X \pmod{N}.
\]

Since \( X_0 \in \mathcal{S}_0 \), the theorem is proven. \( \Box \)

Given \( X \in \mathcal{S} \), we define \( i_x \in \text{Aut}(S) \) by \( i_x(y) = \exp(X)y \exp(-X) \). Consider the following condition:

\[
(7.2) \quad \text{For each } X \in \mathcal{S}_0, \ y \in S, \ \exists k \in K \ \exists i_x(y) = k \cdot y.
\]

**Theorem 7.3.** Suppose \( K \) is connected. Then \( (K, S) \) is a Gelfand pair if, and only if, \( (K, N) \) is a Gelfand pair, and condition (7.2) is satisfied.

**Proof.** Suppose \( (K, S) \) is a Gelfand pair. By Theorem 1.12, for all \( x, y \in N \), \( xy \in (K \cdot y)(K \cdot x) \), which implies that \( (K, N) \) is a Gelfand pair. Furthermore, if \( X \in \mathcal{S}_0 \) and \( y \in S \), then \( \exp(X)y \in (K \cdot y)(K \cdot \exp(X)) = (K \cdot y) \exp(X) \). This proves the necessity of the conditions.

Suppose now the converse. Note that \( S = \exp(\mathcal{S}_0)N \). Given \( X, Y \in \mathcal{S}_0 \), and \( x, y \in N \) we compute

\[
(K \cdot \exp(X)x)(K \cdot \exp(Y)y) = \exp(X)(K \cdot x)\exp(Y)(K \cdot y)
\]

\[
= \exp(X)\exp(Y)(\exp(-Y)(K \cdot x)\exp(Y))(K \cdot y)
\]

\[
= \exp(X)\exp(Y)(K \cdot x)(K \cdot y)
\]

\[
= \exp(X)\exp(Y)(K \cdot y)(K \cdot x)
\]

\[
= (\exp(X)(K \cdot \exp(Y)y)\exp(-X))(K \cdot (\exp(X)x)
\]

\[
= (K \cdot \exp(Y)y)(K \cdot \exp(X)x).
\]

Theorem 1.12 implies that \( (K, S) \) is a Gelfand pair. \( \Box \)

Recall that a connected Lie group \( G \) is said to be \text{type-R} if the eigenvalues of \( \text{ad}X \), as a linear operator on \( \mathcal{S} \), are pure imaginary. Note that \( i_x(\exp(Y)) = \exp(\text{ad}(\exp(X)) \cdot Y) = \exp(\exp(\text{ad}X) \cdot Y) \). Thus, if (7.2) is satisfied, and \( \| \cdot \| \) is a \( K \) invariant norm on \( \mathcal{S} \), then for all \( X \in \mathcal{S}_0 \), \( \| \exp(\text{ad}X) \cdot Y \| = \| i_x \cdot Y \| = \| Y \| \). This implies that the eigenvalues of \( \text{ad}X \) are pure imaginary for all \( X \in \mathcal{S}_0 \). The same holds true for \( X \in N \), since \( \text{ad}X \) is nilpotent as a
linear operator on \( S \). Thus

**Corollary 7.4.** If \((K, S)\) is a Gelfand pair, then \( S \) is type-R.

A very simple example of a Gelfand pair \((K, S)\) involving a non-nilpotent group is given by letting \( S = \mathbb{R} \ltimes \mathbb{C} \), with \( \mathbb{R} \) acting on \( \mathbb{C} \) by \( t: z \mapsto e^{it}z \), and \( K = U(1) \) acting as usual on \( \mathbb{C} \).

**Spherical functions**

In this section we identify a moduli space for the \( K \)-spherical functions associated to a Gelfand pair \((K, S)\). Recall that a \( K \)-spherical function associated to such a pair is a continuous, complex-valued function, \( \phi \), defined on \( S \), satisfying

\[
\phi(e) = 1 \quad \text{and} \quad \int_K \phi(xk \cdot y) \, dk = \phi(x)\phi(y)
\]

for all \( x, y \in S \). It easily follows that a \( K \)-spherical function is \( K \)-invariant. One also has that integration against a \( K \)-spherical function, \( \phi \), defines a complex-valued homomorphism on \( L^1_K(N) \), that this homomorphism is continuous if \( \phi \) is bounded, and that all continuous homomorphisms of \( L^1_K(N) \) are given in this manner (cf. [He]). We first consider \( K \)-spherical functions associated to a Gelfand pair \((K, N)\).

**Lemma 8.2.** Suppose \( \phi \) is a bounded \( K \)-spherical function on \( N \). Then there is a \( \pi \in \hat{N} \) and a unit vector \( \xi \in H_{\pi} \) such that

\[
\phi(x) = \int_K \langle \pi(k \cdot x) \xi, \xi \rangle \, dk
\]

for each \( x \in N \).

**Proof.** Let \( \lambda_{\phi}: L^1_K(N) \to \mathbb{C} \) be given by integration against \( \phi \).

Since \( L^1(N) \) is a symmetric Banach \(*\)-algebra, [Le2], there is a representation \( \pi \) of \( L^1(N) \) and a one-dimensional subspace \( H_{\phi} \) of \( H_{\pi} \) such that \( (\pi|_{L^1_K(N)}, H_{\phi}) \) is equivalent to \( (\lambda_{\phi}, \mathbb{C}) \). As \( \lambda_{\phi} \) is irreducible, the extension \( \pi \) is also irreducible (cf. [Na]). Using approximate identities at each point of \( N \), one can show that \( \pi \) is the integrated version of some \( \pi \in \hat{N} \), with \( H_{\pi} = H_{\pi} \).

Choose \( \xi \in H_{\phi} \) with \( \|\xi\| = 1 \). Then for each \( f \in L^1_K(N) \), \( \pi(f)\xi = \lambda_{\phi}(f)\xi \), so that

\[
\langle \phi, f \rangle = \lambda_{\phi}(f) = \langle \pi(f)\xi, \xi \rangle = \int_N f(x)\langle \pi(x)\xi, \xi \rangle \, dx = \int_K \int_N f(k^{-1} \cdot x)\langle \pi(x)\xi, \xi \rangle \, dx \, dk
\]

since \( f \) is \( K \)-invariant.
Theorem 8.7. (i) \( \phi \) is \( K \)-invariant, we change the order of integration and obtain

\[
\int_{N} \frac{1}{K} \int_{N} f(x) \langle \pi(k \cdot x), \xi \rangle \, dx \, dk.
\]

Since \( \phi \) is \( K \)-invariant, we change the order of integration and obtain

\[
\phi(x) = \int_{K} \langle \pi(k \cdot x), \xi \rangle \, dk. \quad \square
\]

Notation. We denote the function defined by (8.3) as \( \phi_{\pi, \xi} \).

Corollary 8.4. If \( \phi \) is a bounded \( K \)-spherical function on \( N \), then \( \phi \) is positive definite.

Recall from §3 that for \( \pi \in \hat{\pi} \) we denote by \( K_{\pi} \) the subgroup of \( K \) that preserves the equivalence class of \( \pi \), and that \( W_{\pi} \) denotes the intertwining representation of \( K_{\pi} \).

Let \( H_{\pi} = \sum_{\alpha} V_{\alpha} \) be the decomposition of \( H_{\pi} \) into irreducible subspaces invariant under the action of \( W_{\pi} \). The assumption that \( (K, N) \) is a Gelfand pair implies that as \( K_{\pi} \)-modules, the \( V_{\alpha} \)’s are inequivalent for different \( \alpha \)’s.

Lemma 8.5. If \( \pi' = \pi_{k_{0}} \), then \( K_{\pi'} = k_{0}^{-1} K_{\pi} k_{0} \).

Proof. If \( k' \in K_{\pi'} \), then \( \pi_{k'}(x) \simeq \pi' \). That is, \( \pi_{k_{0}'}(x) = W_{\pi'}(k') \pi(x) W_{\pi'}^{*}(k') \) for each \( x \in N \). Thus

\[
\pi_{k_{0}k_{0}'k_{0}^{-1}}(x) = \pi_{k_{0}k_{0}'}(k_{0}^{-1} \cdot x) = \pi_{k_{0}'}(k_{0}^{-1} \cdot x) = W_{\pi'}(k') \pi_{k_{0}'}(x) W_{\pi'}^{*}(k') = W_{\pi_{k_{0}k_{0}'k_{0}^{-1}}}(x).
\]

Thus, \( \pi_{k_{0}k_{0}'k_{0}^{-1}} \simeq \pi \), so \( k_{0}k_{0}'k_{0}^{-1} \in K_{\pi} \). \( \square \)

Note that for \( k' \in K_{\pi'} \), the above calculation shows that we could choose \( W_{\pi_{k_{0}k_{0}'k_{0}^{-1}}} \), so that \( W_{\pi_{k_{0}k_{0}'k_{0}^{-1}}}(k') = W_{\pi_{k_{0}k_{0}'k_{0}^{-1}}} \).

Corollary 8.6. For \( \pi' = \pi_{k_{0}} \), \( H_{\pi} \) and \( H_{\pi'} \) have the same decomposition into \( W_{\pi_{k_{0}}} \)- and \( W_{\pi_{k_{0}'}k_{0}^{-1}} \)-irreducible subspaces respectively.

Theorem 8.7. (i) \( \phi_{\pi, \xi} \) is a \( K \)-spherical function if, and only if, \( \xi \in V_{\alpha} \) for some \( \alpha \), and \( \| \xi \| = 1 \). (ii) \( \phi_{\pi, \xi} = \phi_{\pi', \eta} \) if, and only if, there is a \( k \in K \) such that \( \pi' = \pi_{k} \) and \( \xi, \eta \) belong to the same \( V_{\alpha} \).

Proof. Let \( f \in L^{1}_{K}(N) \). Since \( f \) is \( K_{\pi} \)-invariant, \( \pi(f) \) commutes with the action of \( W_{\pi} \) on \( H_{\pi} \). Since \( W_{\pi} \) is multiplicity free, \( \pi(f) \) preserves each \( V_{\alpha} \). Now by Schur’s lemma, the irreducibility of \( W_{\pi} \) on \( V_{\alpha} \) implies that \( \pi(f) \) acts as a scalar multiple of the identity on each \( V_{\alpha} \). Note that this scalar is computed by the formula \( \langle \pi(f)\xi, \xi \rangle \) for any \( \xi \in V_{\alpha} \) with \( \| \xi \| = 1 \).

For \( \xi \in V_{\alpha} \) with \( \| \xi \| = 1 \), \( \phi_{\pi, \xi} \) is clearly a continuous function on \( N \). We only need to show that \( \lambda_{\phi} \) (with \( \phi = \phi_{\pi, \xi} \)) is a homomorphism on \( L^{1}_{K}(N) \).
Note that for $f \in L^1_K(N)$,

$$
\langle \phi_{\pi, \xi}, f \rangle = \int_N \int_K \langle \pi(k \cdot x)\xi, \xi \rangle f(x) \, dk \, dx
$$

(8.8)

$$
= \int_K \int_N \langle \pi(x)\xi, \xi \rangle f(k^{-1} \cdot x) \, dx \, dk
= \langle \pi(f)\xi, \xi \rangle.
$$

Thus, if $f, \ g \in L^1_K(N)$,

$$
\lambda_{\phi}(f \ast g) = \langle \pi(f \ast g)\xi, \xi \rangle = \langle \pi(f)\pi(g)\xi, \xi \rangle
= \langle \pi(g)\xi, \xi \rangle\langle \pi(f)\xi, \xi \rangle = \lambda_{\phi}(f)\lambda_{\phi}(g).
$$

Conversely, suppose $\xi \in H_\pi$, $\|\xi\| = 1$. Write $\xi = \sum t_\alpha \xi_\alpha$ with $\xi_\alpha \in V_\alpha$, $\|\xi_\alpha\| = 1$, $t_\alpha \geq 0$, and $\sum t_\alpha^2 = \|\xi\|^2 = 1$. Then

$$
\langle \phi_{\pi, \xi}, f \rangle = \langle \pi(f)\xi, \xi \rangle = \sum_{\alpha, \beta} t_\alpha t_\beta \langle \pi(f)\xi_\alpha, \xi_\beta \rangle = \sum_{\alpha} t_\alpha^2 \langle \pi(f)\xi_\alpha, \xi_\alpha \rangle
$$

since $\pi(f)$ preserves the mutually orthogonal $V_\alpha$'s.

Thus, for $\xi = \sum t_\alpha \xi_\alpha$, $t_\alpha \geq 0$, $\phi_{\pi, \xi} = \sum t_\alpha^2 \phi_{\pi, \xi_\alpha}$, and $\|\xi\|^2 = 1$ implies that $\sum t_\alpha^2 = 1$. Note that positive definite homomorphisms are extreme points in the Gelfand space of $L^1_K(N)$, so if $\phi_{\pi, \xi}$ is a positive definite $K$-spherical function, it cannot be a convex sum of positive definite $K$-spherical functions. Thus $\xi = \xi_\alpha$ for some $\alpha$.

Now suppose $\pi' = \pi_{k_0}$ and $\xi, \eta$ belong to $V_\alpha \subseteq H_\pi$. Then

$$
\langle \phi_{\pi, \xi}, f \rangle = \langle \pi(f)\xi, \xi \rangle = \langle \pi(f)\eta, \eta \rangle
$$

since $\pi(f)$ is constant on $V_\alpha$

$$
= \int_N \int_K \langle \pi(k \cdot x)\eta, \eta \rangle f(x) \, dk \, dx
= \int_K \int_N \langle \pi(k_0k \cdot x)\eta, \eta \rangle f(x) \, dx \, dk
= \langle \phi_{\pi', \eta}, f \rangle.
$$

Thus, $\phi_{\pi, \xi} = \phi_{\pi', \eta}$.

For the converse of (ii), we need to understand $K\bar\otimes N$ via the Mackey machine. Let $\pi \in \hat{N}$, and suppose the intertwining representation $W_\pi$ of $K\pi$ is a $\sigma$-representation, as described in §3. Let $T$ be any $\sigma$-representation of $K\pi$. Then $\rho = T \otimes \pi W_\pi$ is an irreducible representation of $K\pi \propto N$. Let $\hat{\rho}$ be the representation of $K \propto N$ induced from $\rho$. Then $\hat{\rho} \in K\propto N$, and any irreducible representation of $K \propto N$ is obtained in this manner. More precisely,
$K \ltimes N$ is given by pairs $(\pi, T)$, where $\pi \in \hat{N}$, and $T \in \hat{K}$. Another pair $(\pi', T')$ yields an equivalent representation if, and only if, $\pi' \cong \pi_{k_0}$ for some $k_0$ and $T' \cong T \circ i_{k_0}$, where $i_{k_0} : K_{\pi'} \to K_{\pi} = k_0 K_{\pi} k_0^{-1}$.

As a function on $G = K \ltimes N$, any positive definite $K$-spherical function is given as follows: Let $\rho \in \hat{G}$. If there is a $K$-fixed vector $v \in H_{\rho}$ (the space of $K$-fixed vectors has dimension at most one), then $\phi(x) = \langle \rho(x)v, v \rangle$. This yields a 1-1 correspondence between the representations in $\hat{G}$ with $K$-fixed vectors and positive definite $K$-spherical functions on $G$ (cf. [He]).

By Frobenius reciprocity, we see that the dimension of the space of $K$-fixed vectors in $H_{\rho}$ equals the dimension of the space of $K_{\pi}$-fixed vectors in $H_{\rho}$. Note that $T \otimes W_{\pi}$ has $K_{\pi}$-fixed vectors if, and only if, $\overline{T}$ is a subrepresentation of $W_{\pi}$, i.e. $H_{T} = V_{\alpha}$ for some $W_{\pi}$-irreducible component of $H_{\pi}$, and $T = \overline{W_{\pi}}|_{V_{\alpha}}$. Thus there is a 1-1 correspondence between positive definite $K$-spherical functions and pairs $(\pi, V_{\alpha})$, where $\pi \in \hat{N}$ and $V_{\alpha} \subseteq H_{\pi}$ is a $W_{\pi}$-irreducible component. We will see that these $K$-spherical functions coincide with the formulas in the statement of the theorem. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for $V_{\alpha}$, and set

$$v = \frac{1}{\sqrt{m}} \sum v_i \otimes v_i,$$

regarded as an element of $H_{\rho} = V_{\alpha} \otimes H_{\pi}$. For $k \in K_{\pi}$,

$$\rho(k)v = \frac{1}{\sqrt{m}} \sum_i \overline{W_{\pi}}(k)v_i \otimes W_{\pi}(k)v_i$$

$$= \frac{1}{\sqrt{m}} \sum_{i,j,k} \overline{a}_{i,j} v_j \otimes a_{i,k} v_k,$$

where $A = (a_{i,j})$ is the matrix corresponding to $W_{\pi}(k)|_{V_{\alpha}}$. But

$$\sum_i a_{i,j} a_{i,k} = (A^{*}A)_{j,k} = \delta_{j,k}.$$}

Thus

$$\rho(k)v = \frac{1}{\sqrt{m}} \sum_j v_j \otimes v_j,$$

so $v$ is a $K_{\pi}$-fixed vector in $H_{\rho}$.

To construct a corresponding $K$-fixed vector in $H_{\rho}$, define $f : K \ltimes N \to V_{\alpha} \otimes H_{\pi}$ by $f(k, n) = (1 \otimes \pi(n))v$. To ensure that $f \in H_{\rho}$, we need $f(hg) = \rho(h)f(g)$, for $h \in K_{\pi} \ltimes N$, $g \in K \ltimes N$. (Actually it is sufficient to take $g = (k, e)$ with $k \in K$.) We have

$$f((k, n)(k, e)) = f(k, n) = (1 \otimes \pi(n))v.$$

On the other hand,

$$\rho(k, n)f(k, e) = \overline{W_{\pi}}(k) \otimes \pi(n)W_{\pi}(k)v$$

$$= (1 \otimes \pi(n))\rho(k, n)v = (1 \otimes \pi(n))v,$$

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as required. Thus \( f \in H_{\hat{\rho}} \), and for \( k \in K \),
\[
\hat{\rho}(k)f(k', n) = f((k', n)(k, e)) = f(k'k, n) = (1 \otimes \pi(n))v = f(k', n),
\]
so \( f \) is a \( K \)-fixed vector.

We check that \( f \) is a unit vector.

We have
\[
\|f\|^2 = \int_{(K \times N)/(K, \times N)} \|f(k, n)\|^2 \, dk \, dn
\]
\[
= \int_{(K \times N)/(K, \times N)} \|(1 \otimes \pi(n))v\|^2 \, dk \, dn
\]
\[
= \int_{K/K} \|v\|^2 \, dk = 1,
\]
since
\[
\|v\|^2 = \frac{1}{m} \sum_{i=1}^{m} \|v_i \otimes v_i\|^2 = 1.
\]

The \( K \)-spherical function \( \hat{\phi} \) on \( G \) associated with \( f \) is given by \( \hat{\phi}(g) = \langle \hat{\rho}(g)f, f \rangle \). The restriction \( \phi \) of \( \hat{\phi} \) to \( N \) is given by
\[
\phi(n) = \langle \hat{\rho}(n)f, f \rangle
\]
\[
= \int_{K/K} \langle \hat{\rho}(n)f(k), f(k) \rangle \, dk
\]
\[
= \int_{K/K} \langle f((k, e)(e, n)), f(k) \rangle \, dk
\]
\[
= \int_{K/K} \langle f(k, k \cdot n), f(k) \rangle \, dk
\]
\[
= \int_{K/K} \langle (1 \otimes \pi(k \cdot n))v, v \rangle \, dk.
\]

For \( k \in K \),
\[
\langle (1 \otimes \pi(k \cdot n))v, v \rangle = \frac{1}{m} \sum_{i,j} \langle v_j \otimes \pi(k \cdot n)v_j, v_i \otimes v_i \rangle
\]
\[
= \frac{1}{m} \sum_i \langle \pi(k \cdot n)v_i, v_i \rangle
\]

For \( k \in K \_e \),
\[
\sum_i \langle \pi(k \cdot n)v_i, v_i \rangle = \sum_i \langle W_\pi(k)\pi(n)W_\pi(k)^{-1}v_i, v_i \rangle
\]
\[
= \sum_i \langle \pi(n)W_\pi(k)^{-1}v_i, W_\pi(k)^{-1}v_i \rangle
\]
\[
= \sum_i \langle \pi(n)v_i, v_i \rangle.
\]
by an easy trace argument. Thus,
\[ \phi(n) = \frac{1}{m} \int_{\mathcal{N}} \sum_{i} \langle \pi(k\cdot n)v_i, v_i \rangle \, dk \]
\[ = \frac{1}{m} \sum_{i} \phi_{\pi, v_i}(n) = \phi_{\pi, 1/2} \sum_{i} v_i(n). \]

Thus, \( \phi = \phi_{\pi, \xi} \), where \( \xi \) is any element of \( V_\alpha \) (since any unit vector in \( V_\alpha \) can be written as \( 1/\sqrt{m} \sum v_i \) for some orthonormal basis \( \{v_1, \ldots, v_n\} \)). □

Suppose now that \((K, S)\) is a Gelfand pair. Note that if \( \phi \) is a \( K \)-spherical function, \( X, Y \in \mathcal{S}_0 \), and \( y \in S \), then by (8.1)
\[ \phi(y \exp X \exp Y) = \phi(y)\phi(\exp X)\phi(\exp Y). \]

One also sees from (8.1) that the restriction of \( \phi \) to \( N := \exp(\mathcal{N}) \), where \( \mathcal{N} \) is the nilradical of \( \mathcal{S} \), is a \( K \)-spherical function. This indicates how one constructs \( K \)-spherical functions on \( S \).

Let \( X_1, \ldots, X_p \) be a basis for a complement of \( \mathcal{N} \), the nilradical of \( \mathcal{S} \), in \( \mathcal{S}_0 \). Since \( S \) is simply connected, for each \( y \in S \), there exist unique \( n(y) \in N (= \exp(\mathcal{N})) \) and \( t(y) \in \mathbb{R}^p \) such that \( y = n(y)\Pi_i \exp(t_i(y)X_i) \). Thus, if \( \phi \) is a bounded \( K \)-spherical function on \( S \) then
\[ \phi(y) = \phi(n(y))\Pi_i \phi(\exp(t_i(y))) \]
for each \( y \in S \). Again by (8.1), for any \( X \in \mathcal{S}_0 \), the mapping \( t \mapsto \phi(\exp(tX)) \) is a homomorphism of \( \mathbb{R} \) into \( C \). Thus, there exist an \( a \in \mathbb{R}^p \) such that \( \phi(y) = \phi(n(y))e^{i(a \cdot t(y))} \). Thus one has

**Theorem 8.11.** \( \phi \) is a bounded \( K \)-spherical function on \( S \) if, and only if, there is a bounded \( K \)-spherical function \( \psi \) on \( N \) and an \( a \in \mathbb{R}^p \) such that \( \phi(y) = \psi(n(y))e^{i(a \cdot t(y))} \). Thus \( \Delta(K, S) = \Delta(K, N) \times \mathbb{R}^p \).

**References**


DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF MISSOURI-ST. LOUIS, ST. LOUIS, MISSOURI 63121 (C1792@UMSLVAXA)

DEPARTMENT OF MATHEMATICS AND STATISTICS, THE UNIVERSITY AT ALBANY/SUNY, ALBANY, NEW YORK 12222 (JWJ71@ALBNY1VX)

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF MISSOURI-ST. LOUIS, ST. LOUIS, MISSOURI 63121