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A SPANNING SET FOR $\mathscr{C}(I^n)$

THOMAS BLOOM

ABSTRACT. $\mathscr{C}(I^n)$ denotes the Banach space of continuous functions on the unit n-cube, I^n, in $\mathbf{R}^n$. Let $\{a^i\}$, $i = 0, 1, 2, \ldots$, be a countable collection of n-tuples of positive real numbers satisfying $\lim_i a_j^i = +\infty$ for $j = 1, \ldots, n$. We canonically enlarge the family of monomials $\{x^{a^i}\}$ to a family of functions $\mathscr{F}(A)$.

Conjecture. The linear span of $\mathscr{F}(A)$ is dense in $\mathscr{C}(I^n)$ if and only if $\sum_{i=0}^{\infty} 1/|a^i| = +\infty$. For $n = 1$ this is equivalent to the Müntz-Szasz theorem. For $n > 1$ we prove the necessity in general and the sufficiency under the additional hypothesis that there exist constants $G, N > 1$ such that $|a^i| \leq G\exp(i^N)$ for all i.

INTRODUCTION

Let $0 < b_0 < b_1 < \cdots$. The Müntz-Szasz theorem gives a sufficient condition for the linear span o the function 1 and the monomials $t^{b_0}, t^{b_1}, \ldots$ to be dense in the continuous function on $I = [0, 1]$. The condition is

(1) $\lim b_i = +\infty$.
(2) $\sum_{i=0}^{\infty} 1/b_i = +\infty$.

Furthermore, given (1) then (2) is necessary.

The sufficient condition may be deduced from results on zero sets of functions bounded and analytic in a half plane [10, 13], although other proofs are known [4, 5].

In several variables the corresponding Müntz-Szasz problem would be to find necessary and sufficient conditions that the linear span of a family of monomials $\{t^{a^i} = (t_1^{a_1^i}) \cdots (t_n^{a_n^i})\}_{i=0,1,2,\ldots}$ (where each $a_j^i \geq 0$) be dense in the space of continuous functions on the unit n-cube $I^n = [0, 1]^n$. An essentially equivalent problem is the question of characterizing discrete sets of uniqueness for functions of several variables bounded in a cone. Interesting results on this problem have been found by Korevaar-Hellerstein [7], Ronkin [11, 12], and Berndtsson [1]. For example, in [1] the following result is proved (see also [11, 12]).

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Theorem. [1] *Let K be an open cone in $\mathbf{R}^n$ with vertex the origin and $T(K) = \{z \in \mathbf{C}^n \mid \mathrm{Re}(z) \in K\}$. Suppose f is bounded and analytic on $T(K)$. Let E be a discrete subset K such that for some constant $h > 0$, $e_1, e_2 \in E \Rightarrow |e_1 - e_2| \geq h$. Let $n(r) = \#E \cap B(0, r)$. Assume f vanishes on E. Then f is identically zero if*

$$\overline{\lim}\, n(r) r^{-n} > 0.$$

This sufficient condition is not necessary, and finding neessary and sufficient conditions would seem to be very difficult. In the case $n = 1$ (the one-variable case) the above result is not sharp enough to yield the classical Müntz-Szasz theorem.

In this paper we will take a somewhat different approach to the Müntz-Szasz problem in several variables. Given a family of monomials $\{t^a\}_{a \in A}$, we will canonically associate a family of functions $\mathscr{F}(A)$ (see §1). The functions of $\mathscr{F}(A)$ will include sums of monomials multiplied by rational functions in the logarithms of the coordinates. We will give sufficient conditions for the linear span of the family $\mathscr{F}(A)$ to be dense in continuous functions on the n-cube (Theorem 2.2).

Conjecture. Let $A = \{a^i\}_{i=0,1,\ldots}$ and assume $\lim_i a_j^i = +\infty$ for $j = 1, \ldots, n$. Then the linear span of $\mathscr{F}(A)$ is dense in $\mathscr{C}(I^n)$ if and only if $\sum_i 1/|a^i| < +\infty$.

For $n = 1$ this conjecture is equivalent to the Müntz-Szasz theorem (see §1.7) and hence is correct. For $n > 1$ we prove the necessity of the divergence of the series $1/|a^i|$ (§5) and the sufficiency under an additional condition on the sequence $|a^i|$ (see Theorem 2.2 and condition (2.2.3)).

The outline of the paper is as follows: The family of functions $\mathscr{F}(A)$ is defined in §1 and its basic algebraic properties are established. The main result of the paper is Theorem 2.2. The proof of Theorem 2.2 is outlined in §§2.4 and 2.5. The estimates used to prove Theorem 2.2 are Theorems 3.2 and 3.9.

In §4 we show that Kergin interpolation—a canonical polynomial interpolation procedure in several variables may be used to define the family $\mathscr{F}(A)$. Our result (i.e., Theorem 2.2) is related to a uniqueness result on bounded functions analytic in products of half planes, where the conditions on the bounded analytic functions involve vanishing on a discrete set and also the vanishing of the Kergin interpolants at subsets of the discrete set (see §4.7).

1. A FAMILY OF FUNCTIONS ON I^n

1.1. Let $x = (x_1, \ldots, x_n)$ be coordinates for $\mathbf{R}^n$. If $x_j \geq 0$ for $j = 1, \ldots, n$ we set $\log x = (\log x_1, \ldots, \log x_n)$. Let $a = (a_1, \ldots, a_n)$ be an n-tuple of positive real numbers (i.e., $a_j > 0$ for $j = 1, \ldots, n$). We will use the notation x^a for the product $(x_1^{a_1}) \cdots (x_n^{a_n})$ and $(\log x)^a$ for $(\log x_1)^{a_1} \cdots (\log x_n)^{a_n}$ in case $x_j \geq 0$ for $j = 1, \ldots, n$.

We let $I^n = \{x \in \mathbf{R}^n \mid 0 \leq x_j \leq 1 \text{ for } j = 1, \ldots, n\}$. $\mathscr{C}(I^n)$ will denote the Banach space of continuous functions on I^n with the sup norm.

1.2. Let A be a countable collection of n-tuples of positive real numbers. We will describe a family of functions $\mathscr{F}(A)$ which is determined by A. $\mathscr{F}(A)$ will include the monomials $\{x^a\}_{a\in A}$ and certain other functions canonically determined by A.

The functions of $\mathscr{F}(A)$ will be continuous on I^n. That is, $\mathscr{F}(A) \subset \mathscr{C}(I^n)$. The main result of the paper (Theorem 2.2) will give sufficient conditions on A so that the linear span of $\mathscr{F}(A)$ is dense in $\mathscr{C}(I^n)$. That is, denoting by $L\mathscr{F}(A)$ the linear span of $\mathscr{F}(A)$ and by $\overline{L\mathscr{F}(A)}$ its closure in the sup norm on I^n, Theorem 2.2 gives sufficient conditions on A so that $\overline{L\mathscr{F}(A)} = \mathscr{C}(I^n)$.

1.3. The family $\mathscr{F}(A)$ will be a union of a countable collection of subfamilies

$$\mathscr{F}(A) = \left(\bigcup_{j=0}^{\infty} \mathscr{F}_j(A)\right) \cup \mathscr{H}. \tag{1.3.1}$$

$\mathscr{H}$ consists of those functions on $\mathscr{C}(I^n)$ which, for some $j = 1, \dots, n$, are constant on the slices $x_j = c$ for $0 \le c \le 1$. In particular, $\mathscr{H}$ includes the constant functions.

$\mathscr{F}_0(A)$ consists of the monomials $\{x^a\}_{a\in A}$.

The functions of $\mathscr{F}_s(A)$ $(s \ge 1)$ may be described as follows:

Each collection of $(s+1)$ elements of A, say $a^0, \dots, a^s$, together with an n-multi-index ν such that $|\nu| = s$, determines a function of $\mathscr{F}_s(A)$ given by

$$(\log x)^\nu \sum_{k=0}^{s} \frac{x^{a^k}}{\prod_{j\neq k,\, j=0}^{s}(a^j - a^k)\cdot \log x}. \tag{1.3.2}$$

Here $(a^j - a^k)\cdot \log x$ denotes the usual inner product $\mathbf{R}^n$; namely,

$$a^j - a^k \cdot \log x = \sum_{i=1}^{n} (a_i^j - a_i^k)\log(x_i). \tag{1.3.3}$$

$\mathscr{F}_s(A)$ consists of all functions arising in this way.

There may be repeated n-tuples in the collection A. That is, we may have $a^j = a^k$ for $j \neq k$. However, we will see that formula (1.3.2) will make sense in this case also.

1.4. We will give an alternate description of the functions in $\mathscr{F}_s(A)$ for $s \ge 1$.

First we note that

$$x^\sigma = \exp(\sigma \cdot \log x) \tag{1.4.1}$$

for $x \in I^n$ and $\sigma \in (\mathbf{R}^+)^n$ where

$$\mathbf{R}^+ = \{\tau \in \mathbf{R} \mid \tau > 0\}.$$

The function x^σ is defined and continuous on $I^n \times (\mathbf{R}^+)^n$ and is, in fact, in $\mathscr{C}^\infty(I^n \times (\mathbf{R}^+)^n)$ where x^σ and all its partial derivatives are zero if one coordinate of x is zero.

The functions of $\mathscr{F}_s(A)$ are divided differences of order s of x^σ multiplied by $(\log x)^\nu$ for ν an n-multi-index of length s.

To be precise, we recall some properties of divided differences (see [4] or [5]).

Let $g(\tau)$ be a real-valued function of the real variable τ defined on an interval $J \subset \mathbf{R}$. Let $b_0, \dots, b_s$ be $(s+1)$ distinct points of J.

We use the notation

$$[b_j]g = g(b_j) \quad \text{for } j = 0, \dots, s \tag{1.4.2}$$

and consider the values of g as 0th order divided differences. kth order divided differences may be defined inductively. For example, the kth order differences of g at $b_0, \dots, b_k$ denoted $[b_0 \cdots b_k]g$ is defined in terms of the $(k-1)$th order difference by

$$\frac{[b_0 \cdots b_{k-1}]g - [b_1 \cdots b_k]g}{b_0 - b_k} = [b_0 \cdots b_k]g. \tag{1.4.3}$$

In fact, it is easily seen that

$$[b_0 \cdots b_s]g = \sum_{j=0}^{s} \frac{g(b_j)}{\prod_{i \neq j,\, i=0}^{s} (b_j - b_i)}. \tag{1.4.4}$$

If the function g is s times continuously differentiable, the sth order divided difference is well defined even if some (or all) of the points b_j coincide. This is readily seen by using the Hermite-Genocchi formula [5]

$$[b_0 \cdots b_s]g = \int_{\Delta^s} g^{(s)}(v_0 b_0 + \cdots + v_s b_s)\, dv_1 \cdots dv_s \tag{1.4.5}$$

where Δ^s is the s-dimensional simplex

$$\Delta^s = \left\{ v \in \mathbf{R}^{s+1} \,\middle|\, \sum_{j=0}^{s} v_j = 1 \text{ and } v_j \geq 0 \text{ for } j = 0, \dots, s \right\}. \tag{1.4.6}$$

Thus if g is in $\mathscr{C}^s(J)$ then $[b_0, \dots, b_s]g$ is continuous on J^{s+1} (as a function of $b_0, \dots, b_s$).

The function of $\mathscr{F}_s(A)$ given by (1.3.2) may be written in the form

$$(\log x)^\nu [a^0 \cdot \log x, \dots, a^s \log x] \exp \tag{1.4.7}$$

where $|\nu| = s$ and $a^0, \dots, a^s$ are elements of A. Here, exp is the one-variable exponential function.

1.5. **Lemma.** *Let $\phi(x) \in \mathscr{F}_s(A)$. Then*

(i) *ϕ is continuous on I^n;*

(ii) *$\phi(x) = 0$ if one of the coordinates $x_j = 0$.*

Proof. The proof is immediate using (1.4.7) and (1.4.5).

1.6. We will use the notation of §1.4. The Lagrange interpolating polynomial to g at $b_0, \dots, b_s$ can be written in Newton form [4, 5]

$$L_s(\tau) = g(b_0) + \sum_{k=1}^{s} [b_0 \cdots b_k] g(\tau - b_0) \cdots (\tau - b_{k-1}). \tag{1.6.1}$$

$L_s(\tau)$ is the unique polynomial of $\deg \le s$ whose values coincide with g at the points $b_0, \dots, b_s$.

To be precise concerning the interpolating points and function, we will sometimes use the notation $L_s(b_0, \dots, b_s; g)(\tau)$ for the interpolating polynomial to g at $b_0, \dots, b_s$.

If some (or all) of the points $b_0, \dots, b_s$ coincide, formula (1.6.1) still makes sense if g is of class $\mathscr{C}^s$. In this case it is referred to as the hermite interpolating polynomial [4, 5] and it has the following property: If the number b is of multiplicity m in $b_0, \dots, b_s$, then $L_s(\tau)$ and $g(\tau)$ must coincide to order $m-1$ at b. That is,

$$L_s^{(i)}(b) = g^{(i)}(b) \quad \text{for } i = 0, 1, \dots, m-1. \tag{1.6.2}$$

We also remark (see [4, 5]) that if g is analytic on an open subset U of $\mathbf{C}$ and $b_0, \dots, b_s \in U$, then there is a unique complex analytic polynomial L_s of $\deg \le s$ which interpolates g at $b_0, \dots, b_s$. We will also denote this polynomial by $L_s(b_0, \dots, b_s; g)(\tau)$.

The functions of $\mathscr{F}_k(A)$ for $k = 0, \dots, s$ arise as follows. In the Newton form (1.6.1), if we substitute

$$\begin{gathered} \tau = \sigma \cdot \log x, \qquad g = \exp, \\ b_j = a^j \cdot \log x \quad \text{for } j = 0, \dots, s, \end{gathered} \tag{1.6.3}$$

we have

$$\begin{aligned} &L_s(a^0 \cdot \log x, \dots, a^s \cdot \log x; \exp)(\sigma \cdot \log x) \\ &\quad = x^{a^0} + \sum_{k=1}^{s} [a^0 \cdot \log x, \dots, a^k \cdot \log x] \exp \prod_{j=0}^{k-1} (\sigma - a^j) \cdot \log x \\ &\quad = x^{a^0} + \sum_{k=1}^{s} \sum_{|\nu|=k} \phi_\nu(x) Q_\nu(\sigma, a^0, \dots, a^k) \end{aligned} \tag{1.6.4}$$

where $\phi_\nu \in \mathscr{F}_k(A)$ for $k = |\nu|$ and Q_ν is a polynomial in $\sigma, a^0, \dots, a^k$.

1.7. We will examine the family $\mathscr{F}(A)$ in the case $n = 1$ (the one-variable case). In this case A consists of a countable collection of positive real numbers. $\mathscr{H}$ consists of the constants and $\mathscr{F}_0 = \{x^a\}_{a \in A}$.

First, suppose all elements of A are distinct. Then, using (1.4.4) one finds that the functions of $\mathscr{F}_s(A)$ for $s \ge 1$ are linear combinations of the monomials $\{x^a\}_{a \in A}$. Thus $L\mathscr{F}(A)$ (the linear span of $\mathscr{F}(A)$) is the same as that of the functions

$$1, \ \{x^a\}_{a \in A}. \tag{1.7.1}$$

Next, suppose A has repeated elements. Suppose that $\alpha \in A$ and is of multiplicity m. Then (1.4.5) shows that $x^\alpha(\log x)^j$ is in $\mathscr{F}_j(A)$ for $j = 1, \dots, m-1$.

The formulas (1.4.3), (1.4.4), and (1.4.5) show that the function of $\mathscr{F}_{m+1}(A)$ determined by the $(m+1)$ elements $a^0, \dots, a^m$ of A is a linear combination of functions in $\bigcup_{j=0}^m \mathscr{F}_j(A)$ unless all a^j are equal for $j = 0, \dots, m$. Thus, in case all elements of A are of finite multiplicity, $L\mathscr{F}(A)$ is the same as the linear span of the functions

$$(1.7.2) \qquad 1, \ \{x^a\}_{a \in A}, \ \{x^\alpha(\log x)^j\} \quad \text{for } j = 1, \dots, m-1 \text{ and } \alpha \in m(A)$$

where $m(A)$ consists of those elements of A of multiplicity m.

Now let $A = \{a^i\}_{i=0,1,2,\dots}$ and suppose

$$(1.7.3) \qquad \lim_i a^i = +\infty.$$

Then the classical theorem of Müntz-Szasz [4, 5, 10, 13] states that the linear span of the functions of (1.7.1) or (1.7.2) is dense in $\mathscr{C}(I)$ if

$$(1.7.4) \qquad \sum_i \frac{1}{a^i} = +\infty.$$

In case A has multiple elements (of at most finite multiplicity), say $A = \{\alpha^i\}$ where the α^i are distinct and each is of multiplicity m_i, then (1.7.4) may be written

$$(1.7.5) \qquad \sum_i \frac{m_i}{\alpha^i} = +\infty.$$

Furthermore, given (1.7.3), then (1.7.4) is a necessary condition for $L\mathscr{F}(A)$ to be dense in $\mathscr{C}(I)$ [13].

2. Main result

2.1. Let $A = \{a^i\}$ be a countable collection of n-tuples of positive real numbers. That is $a^i = (a_1^i, \dots, a_n^i)$ where $a_j^i > 0$ for $j = 1, \dots, n$ and all i. We will give conditions on A so that the linear span of $\mathscr{F}(A)$ is dense in $\mathscr{C}(I^n)$ in Theorem 2.2 below.

The statement of the theorem imposes three conditions on the collection A. Condition (2.2.1) is a generalization of (1.7.3), and (2.2.2) is a generalization of (1.7.4) or (1.7.5).

In the case $n = 1$, condition (2.2.1) and (2.2.2) are sufficient to ensure that $L\mathscr{F}(A)$ is dense in $\mathscr{C}(I)$. This is, in fact, the sufficient condition in the classical Müntz-Szasz theorem.

Condition (2.2.3) is, of course, not needed in case $n = 1$. Whether (2.2.3) (or some similar restriction) is needed in case $n > 1$ remains open. In the introduction it is conjectured that no such restriction is needed.

2.2. **Theorem.** *Let A denote a countable collection of n-tuples of positive real numbers. Then $\overline{L\mathscr{F}(A)} = \mathscr{C}(I^n)$ if A satisfies*

$$\lim_{i\to\infty} a_j^i = +\infty \quad \textit{for } j = 1, \dots, n; \tag{2.2.1}$$

$$\sum_i \frac{1}{|a^i|} = +\infty; \tag{2.2.2}$$

(2.2.3) *there exist constants $G > 0$, $N > 0$ such that*

$$|a^i| \le G\exp(i^N) \quad \textit{for } i = 0, 1, 2, \dots.$$

2.3. The approach to the proof of the above result will be outlined in §§2.4 and 2.5 and will be carried out in §3.

First we make some remarks:

(i) $L\mathscr{F}(A)$ does not depend on the ordering of the elements of A. Hence we may assume

$$|a^0| \le |a^1| \le |a^2| \le \cdots. \tag{2.3.1}$$

(ii) The map $(x_1, \dots, x_n) \to (x_1^\gamma, \dots, x_n^\gamma)$ for γ a real number > 0 is a homeomorphism of I^n and so induces an isometry of $\mathscr{C}(I^n)$. Functions of $\mathscr{F}_s(A)$ are mapped to constant multiples of functions in $\mathscr{F}_s(\gamma A)$ where

$$\gamma A = \{\gamma a\}_{a\in A}. \tag{2.3.2}$$

Thus $L\mathscr{F}(A)$ is dense in $\mathscr{C}(I^n)$ if and only if $L\mathscr{F}(\gamma A)$ is dense (for some $\gamma > 0$). Thus under hypothesis (2.2.1) we may further assume that

$$a_j^i \ge 3 \quad \text{for } j = 1, \dots, n \text{ and all } i. \tag{2.3.3}$$

2.4. *Proof of Theorem* 2.2 (outline). Let $d\mu$ be a finite Borel measure on I^n. Consider the function of the complex variables $z = (z_1, \dots, z_n)$ defined by

$$F_\mu(z) = \int_{I^n} t^z \, d\mu. \tag{2.4.1}$$

F_μ is holomorphic on the cone

$$\Gamma = \{z \in \mathbf{C}^n \mid \operatorname{Re}(z_j) > 0 \text{ for } j = 1, \dots, n\}. \tag{2.4.2}$$

For $z \in \Gamma$ we have the estimate

$$|F_\mu(z)| \le \int_{I^n} t^{\operatorname{Re}(z)} \, d|\mu| \le \int_{I^n} d|\mu| \tag{2.4.3}$$

where $d|\mu|$ is the total variation of $d\mu$. Hence we may conclude that F_μ is bounded on Γ.

Now, by the Riesz representation theorem, every bounded linear functional on $\mathscr{C}(I^n)$ is of the form $f \to S_\mu(f)$ where

$$S_\mu(f) = \int_{I^n} f \, d\mu \tag{2.4.4}$$

for some finite Borel measure $d\mu$ on I^n.

Suppose that S_μ annihilates $L\mathscr{F}(A)$. Then we will show that under the hypothesis of Theorem 2.2, the function F defined by (2.4.1) is identically zero. This implies that $F_\mu(p_1, \dots, p_n) = 0$ for all n-tuples of positive integers $(p_1, \dots, p_n)$, and hence the monomials $(x_1^{p_1}) \cdots (x_n^{p_n})$ are in $\overline{L\mathscr{F}(A)}$ for any n-tuples of positive integers $(p_1, \dots, p_n)$. Since $\mathscr{H}$ contains all the monomials $(x_1^{p_1}) \cdots (x_n^{p_n})$ where $p_i \geq 0$ for $i = 1, \dots, n$ and $p_j = 0$ for some $j = 1, \dots, n$, we can conclude that $\overline{L\mathscr{F}(A)} = \mathscr{C}(I^n)$. The above is the approach to the Müntz-Szasz theorem used in [13].

2.5. First we will examine what information we obtain concerning the function F_μ from the hypothesis that S_μ annihilates $L\mathscr{F}(A)$. We have, from the fact that S_μ annihilates $\mathscr{F}_0(A)$,

$$F_\mu(a) = 0 \quad \text{for } a \in A. \tag{2.5.1}$$

We have, using (1.6.4), $L_s(a^0 \cdot \log x, \dots, a^s \cdot \log x; \exp)(\sigma \cdot \log x)$ is a linear combination of functions in $\bigcup_{j=1}^s \mathscr{F}_j$. Hence

$$\int_{I^n} L_s(a^0 \cdot \log x, \dots, a^s \cdot \log x; \exp)(\sigma \cdot \log x)\, d\mu \equiv 0 \tag{2.5.2}$$

for all $\sigma \in (\mathbf{R}^+)^n$. But then

(2.5.3)

$$\begin{aligned} F(\sigma) &= \int_{I^n} x^\sigma \, d\mu \\ &= \int_{I^n} \{\exp(\sigma \cdot \log x) - L_s(a^0, \log x, \dots, a^s \cdot \log x; \exp)(\sigma \cdot \log x)\}\, d\mu. \end{aligned}$$

Since F is analytic on Γ, to show that $F \equiv 0$ it suffices to show that $F(\sigma) = 0$ for σ in a subset of $(\mathbf{R}^+)^n$ with nonempty interior.

Theorem 3.9 shows that there is a set $E \subset (\mathbf{R}^+)^n$ with nonempty interior such that

$$\lim_{s \to \infty} L_s(a^0 \cdot \log x, \dots, a^s \cdot \log x; \exp)(\sigma \cdot \log x) = \exp(\sigma \cdot \log x)$$

uniformly for all $\sigma \in E$ and all $x \in (0, 1]^n \subset I^n$. One may assume

$$\{x \in I^n \mid \text{some } x_j = 0\}$$

is of measure zero for $d\mu$ since in the definition of F the integrand vanishes on this set. This fact, Theorem 3.9, and (2.5.3) complete the proof that $F \equiv 0$.

2.6. *Remark.* Let $Y = \{x \in I^n \mid \text{some } x_j = 0\}$. Theoem 2.2 gives sufficient conditions on A so that $L(\bigcup_{j=0}^\infty \mathscr{F}_j(A))$ be dense in $\{f \in \mathscr{C}(I^n) \mid f = 0 \text{ on } Y\}$.

3. Convergence of Lagrange-Hermite interpolants

3.1. We will prove (Theorem 3.2) a one-variable result on the convergence of Lagrange-Hermite interpolants of an analytic function bounded on the right

half plane $H = \{\xi \mid \mathrm{Re}(\xi) \geq 0\}$. We will use the estimate only in the case of the funtion $e^{-\xi}$. Theorem 3.9 will result from a version of Theorem 3.2 with parameters. (Results on the convergence of Lagrange interpolants at the integers to functions analytic in the right half plane may be found in [5, Chapter 2].)

3.2. **Theorem.** *Let $g(\xi)$ be analytic and bounded on H. Say $|g(\xi)| \leq 1$ for all $\xi \in H$. Let $\{b_i\}_{i=0,1,2\ldots}$ be a sequence of real numbers such that*

$$\lim_i b_i = +\infty; \tag{3.2.1}$$

$$b_i \geq 3 \quad \textit{for all } i; \tag{3.2.2}$$

(3.2.3) *for some constants $G > 1$ and $N > 1$ we have*

$$b_i \leq G\exp(i^N) \quad \textit{for } i = 0, 1, 2, \ldots.$$

Let α satisfy $0 < \alpha \leq 1$. Then for all $\lambda \in [\alpha, 2]$ there exists a constant $C > 0$ independent of m such that

$$\log|g(\lambda) - L_m(b_0, \ldots, b_m; g)(\lambda)| \leq C - \alpha\sum_{j=0}^{m}\frac{1}{b_j}.$$

The proof of Theorem 3.2 will be concluded in §3.8. It is based on Lemmas 3.6 and 3.7, which involve estimating a contour integral.

3.3. **Corollary.** *If $\sum_{j=0}^{\infty} 1/b_j = +\infty$ then*

$$\lim_m L_m(\lambda) = g(\lambda) \quad \textit{uniformly for } \lambda \in [\alpha, 2].$$

3.4. **Corollary** (to the proof). *C depends only on G, N, α.*

3.5. Let γ be a simple closed curve in H which contains points $b_0, \ldots, b_m, z$ in its interior. Then the difference between $g(z)$ and its Lagrange-Hermite interpolant $L_m(z)$ may be expressed in terms of an integral [4, 5]

$$\begin{aligned} &g(z) - L_m(b_0, \ldots, b_m; g)(z) \\ &\quad = \frac{1}{2\pi i}\prod_{j=0}^{m}(z - b_j)\int_\gamma \frac{g(\xi)\,d\xi}{(\xi - b_0)\cdots(\xi - b_m)(\xi - z)}. \end{aligned} \tag{3.5.1}$$

Let B be a large positive real number (to be specified later). For γ take

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3 \tag{3.5.2}$$

where γ_1 is the straight line from 0 to iB, γ_2 is the semicircle center 0, radius B from iB to $-iB$, and γ_3 is the straight line from $-iB$ to 0.

For $z = \lambda \in [\alpha, 2]$ we have from (3.5.1)

$$|g(\lambda) - L_m(\lambda)| \leq \frac{1}{2\pi\alpha}(b_0 - \lambda)\cdots(b_m - \lambda)\int_\gamma \frac{|d\xi|}{|\xi - b_0|\cdots|\xi - b_m|} \tag{3.5.3}$$

since $|g(\xi)| \leq 1$ for $\xi \in H$ and $|\xi - \lambda| \geq \alpha$ for $\xi \in \gamma$ and $\lambda \in [\alpha, 2]$.

We must estimate

$$\int_\gamma \frac{|d\xi|}{|\xi - b_0| \cdots |\xi - b_m|}. \tag{3.5.4}$$

3.6. **Lemma.** *Let* $1 < T_0 < T_1 < \cdots < T_m$ *be positive reals (to be specified). Let* $d_0 \le d_1 \le \cdots \le d_m$ *be the numbers* $b_0, \dots, b_m$ *in nondecreasing order. Let* $B = d_m T_m$. *Then*

$$\int_\gamma \frac{|d\xi|}{|\xi - b_0| \cdots |\xi - b_m|} \le \frac{1}{b_0 \cdots b_m} \left\{ 2T_0 d_0 + 2\sum_{j=1}^{m} \frac{d_j T_j}{(T_{j-1})^{j-1}} + \frac{\pi d_m T_m}{(T_m - 1)^m} \right\}.$$

Proof. First we note that

$$|\xi - b_0| \cdots |\xi - b_m| = |\xi - d_0| \cdots |\xi - d_m| \tag{3.6.1}$$

so it is equivalent to consider the integral

$$\int_\gamma \frac{|d\xi|}{|\xi - d_0| \cdots |\xi - d_m|}. \tag{3.6.2}$$

We consider the integral over γ_1 first. Subdivide γ_1 as follows:

$$\gamma_1 = \sum_{s=0}^{m} I_s \tag{3.6.3}$$

where each I_s is a directed segment of the imaginary axis. I_s is the straight line from $iT_{s-1} d_{s-1}$ to $iT_s d_s$ for $s = 1, \dots, m$ and I_0 is the straight line from 0 to $iT_0 d_0$.

For $\xi \in I_0$ we have

$$|\xi - d_j| \ge d_j \quad \text{for } j = 0, \dots, m. \tag{3.6.4}$$

For $\xi \in I_s$ $(s > 0)$ we have

$$|\xi - d_j| \ge ((T_{s-1} d_{s-1})^2 + d_j^2)^{1/2} \ge T_{s-1} d_{s-1} \ge T_{s-1} d_j \quad \text{for } j = 0, \dots, s-1 \tag{3.6.5}$$

and

$$|\xi - d_j| \ge d_j \quad \text{for } j = s, \dots, m. \tag{3.6.6}$$

Hence

$$\int_{I_0} \frac{|d\xi|}{|\xi - d_0| \cdots |\xi - d_m|} \le \frac{T_0 d_0}{d_0 \cdots d_m} \tag{3.6.7}$$

and for $s \ge 1$

$$\int_{I_s} \frac{|d\xi|}{|\xi - d_0| \cdots |\xi - d_m|} \le \frac{T_s d_s - T_{s-1} d_{s-1}}{T_{s-1}^{s-1} d_0 \cdots d_m} \le \frac{T_s d_s}{T_{s-1}^{s-1} d_0 \cdots d_m}. \tag{3.6.8}$$

Combining (3.6.3), (3.6.7), and (3.6.8) we have

$$\int_{\gamma_1} \frac{|d\xi|}{|\xi - d_0| \cdots |\xi - d_m|} \le \frac{1}{d_0 \cdots d_m} \left\{ T_0 d_0 + \sum_{j=1}^{m} \frac{T_j d_j}{(T_{j-1})^{j-1}} \right\}. \tag{3.6.9}$$

The integral over γ_3 satisfies the same estimate.

As for the integral over γ_2, we have specified $B = d_m T_m$. Hence, for $\xi \in \gamma_2$ we have

$$|\xi - d_j| \geq d_m T_m - d_j \geq d_m(T_m - 1). \tag{3.6.10}$$

Thus

$$\int_{\gamma_2} \frac{|d\xi|}{|\xi - d_0| \cdots |\xi - d_m|} \leq \frac{\pi\, d_m T_m}{(T_m - 1)^m d_m^m} \leq \frac{\pi\, d_m T_m}{d_0 \cdots d_m (T_m - 1)^m}. \tag{3.6.11}$$

Combining (3.6.9) and (3.6.11), the result follows on noting that $d_0 \cdots d_m = b_0 \cdots b_m$.

3.7. **Lemma.** *Suppose there exist constants $G > 1$ and $N > 1$ such that $b_j \leq G\exp(j^N)$ for $j = 0, 1, 2, \ldots$. That is, condition (3.2.3) is satisfied. Then there is a constant $C = C(G, N) > 0$ independent of m such that*

$$\int_{\gamma} \frac{|d\xi|}{|\xi - b_0| \cdots |\xi - b_m|} \leq \frac{C(G, N)}{b_0 \cdots b_m}.$$

Proof. Let

$$T_j = G\exp(j^N). \tag{3.7.1}$$

Now $b_r \leq T_s$ for $r = 0, 1, \ldots, s$, so $d_s \leq T_s$ for $s = 1, \ldots, m$. Hence

$$\lim_{m \to \infty} (d_m T_m / (T_m - 1)^m) = 0. \tag{3.7.2}$$

Also,

$$\frac{d_s T_s}{T_{s-1}^{s-1}} \leq \frac{G^2 \exp(2s^N)}{G^{s-1} \exp((s-1)^{N+1})} \quad \text{for } s \geq 1 \tag{3.7.3}$$

and the series

$$\sum_{s=1}^{\infty} \frac{G^2 \exp(2s^N)}{G^{s-1} \exp((s-1)^{N+1})} < +\infty. \tag{3.7.4}$$

Hence

$$2T_0 d_0 + 2\sum_{j=1}^{m} \frac{d_j T_j}{(T_{j-1})^{j-1}} + \frac{\pi\, d_m T_m}{(T_m - 1)^m} \leq C(G, N) \tag{3.7.5}$$

where the bound $C(G, N)$ may be chosen independent of m.

The result now follows from Lemma 3.6.

3.8. *Proof of Theorem* 3.2. Using (3.5.3) and Lemma 3.7, we have, for $\lambda \in [\alpha, 2]$,

$$|g(\lambda) - L_m(b_0, \ldots, b_m; g)(\lambda)| \leq \frac{C(G, N)}{2\pi\alpha} \left(1 - \frac{\lambda}{b_0}\right) \cdots \left(1 - \frac{\lambda}{b_m}\right). \tag{3.8.1}$$

Thus

$$\log|g(\lambda) - L_m(b_0, \dots, b_m; g)(\lambda)| \le \log C(G, N) - \log 2\pi - \log\alpha$$

$$+ \sum_{j=0}^{m} \log\left(1 - \frac{\lambda}{b_j}\right) \tag{3.8.2}$$

$$\le C_1 - \alpha \sum_{j=0}^{m} \frac{1}{b_j} \tag{3.8.3}$$

where $C_1 = C_1(G, N, \alpha)$ and we have used the fact that $\log(1-x) < -x$ for $0 < x < 1$.

3.9. **Theorem.** *Let $\{a^i\}_{i=0,1,\dots}$ be a countable collection of n-tuples of positive real numbers satisfying*

(3.9.1) $\lim_i a_j^i = +\infty$ *for* $j = 1, \dots, n$;

(3.9.2) $a_j^i \ge 3$ *for* $j = 1, \dots, n$ *and all* i;

(3.9.3) *there exist constants* $G, N > 1$ *such that* $|a^i| \le G\exp(i^N)$.

Let α satisfy

$$0 < \alpha < \sqrt{2/n}. \tag{3.9.4}$$

Let $E = \{\sigma \in (\mathbf{R}^+)^n \mid \alpha \le \sigma_j \le \sqrt{2/n}$ for $j = 1, \dots, n\}$. Then there exists a constant $C = C(G, N, \alpha)$ such that for all $x \in (0, 1]^n$ we have

$$\log|\exp(\sigma\cdot\log x) - L_m(a^0\cdot\log x, \dots, a^m\cdot\log x; e^\xi)(\sigma\cdot\log x)|$$
$$\le C - \alpha\left(\sum_{i=0}^{m} \frac{1}{|a^i|}\right).$$

Theorem 3.9 will be based on Lemmas 3.10 and 3.11.

Of course, if $\sum_i 1/|a^i| = +\infty$ then Theorem 3.9 implies that

$$L_m(a^0\cdot\log x, \dots, a^m\cdot\log x; e^\xi)(\sigma\cdot\log x)$$

converges uniformly to $\exp(\sigma\cdot\log x)$ for all $\sigma \in E$ and all $x \in (0, 1]^n$. As noted in §2.5, this suffices to prove Theorem 2.2.

3.10. **Lemma.** *Let g be holomorphic in the closed right half plane H and bounded by 1. Let $\beta_0, \dots, \beta_m$ be points of H with $\mathrm{Re}(\beta_j) > 0$. Let γ be a simple closed curve which includes the points $\beta_0, \dots, \beta_m, z$ in its interior. Let $\rho > 0$. Then*

$$|g(\rho z) - L_m(\rho\beta_0, \dots, \rho\beta; g)(\rho z)|$$
$$\le \frac{1}{2\pi}\prod_{j=0}^{m}|z - \beta_j| \int_\gamma \frac{|d\xi|}{|\xi - \beta_0|\cdots|\xi - \beta_m|\,|\xi - z|}.$$

Proof. Let $\psi : H \to H$ be the map given by $\psi(w) = \rho w$. Let

$$\beta_j' = \psi(\beta_j) = \rho\beta_j \quad \text{for } j = 0, \dots, m, \qquad z' = \psi(z) = \rho z. \tag{3.10.1}$$

Let γ' be the image of γ under the map ψ. Then γ' is a simple closed curve in H with $\beta_0', \dots, \beta_m'$, z' in its interior. Thus

$$\begin{aligned} &g(z') - L_m(\beta_0', \dots, \beta_m'; g)(z') \\ &= \frac{1}{2\pi i}\prod_{j=0}^{m}(z' - \beta_j')\int_{\gamma'} \frac{g(\xi')\,d\xi'}{(\xi' - \beta_0')\cdots(\xi' - \beta_m')(\xi' - z')}. \end{aligned} \tag{3.10.2}$$

Let $\xi = \xi'/\rho$ and transform the line integral in (3.10.2) to one over γ. Then, using the substitutions of (3.10.1), the lemma follows.

3.11. **Lemma.** *Let $\rho > 0$, α a real number $0 < \alpha \le 1$, and $\{b_i\}_{i=0,1,\dots}$ a sequence satisfying* (3.2.1), (3.2.2), *and* (3.2.3). *then there exists a constant $C = C(G, N, \alpha)$ such that, for all $\lambda \in [\alpha, 2]$ and all m, we have*

$$\log|e^{-\lambda\rho} - L_m(\rho b_0, \dots, \rho b_m; e^{-\xi})(\rho\lambda)| \le C - \alpha\left(\sum_{i=0}^{m}\frac{1}{b_i}\right).$$

Proof. The proof follows from Lemma 3.10 and the estimate used in the proof of Theorem 3.2 when applied to $g(\xi) = e^{-\xi}$.

3.12. *Proof of Theorem* 3.9. For $x \in (0, 1]^n$ and $\sigma \in E$ we set

$$w = -\log x \in (\mathbf{R}^+)^n. \tag{3.12.1}$$

Also, let

$$\rho = |w|, \qquad v = w/|w|, \quad \text{and} \quad \lambda = \sigma\cdot v. \tag{3.12.2}$$

Hence

$$\rho(\sigma\cdot v) = -\sigma\cdot\log x. \tag{3.12.3}$$

Consider the sequence of positive real numbers $\{a^i\cdot v\}_{i=0,1,\dots}$. We have $0 < a^i\cdot v < |a^i|$ for all i and

$$a^i\cdot v \ge \operatorname*{Min}_{j=1}^{n} a_j^i(v_1 + \cdots + v_n) \ge \operatorname*{Min}_{j=1}^{n} a_j^i \ge 3. \tag{3.12.4}$$

In a similar way we have

$$\alpha < \lambda < 2. \tag{3.12.5}$$

Thus the sequence of positive real numbers $a^i\cdot v$ satisfies conditions (3.2.1), (3.2.2), and (3.2.3).

Applying Lemma 3.11 for $\lambda = \sigma\cdot v$, $b_i = a^i\cdot v$, we have

$$\log|e^{-\rho(\sigma\cdot v)} - L_m(\rho(a^0\cdot v)\cdots\rho(a^m\cdot v); e^{-\xi})(\rho(\sigma\cdot v))| \le C - \alpha\sum_{i=0}^{m}\frac{1}{a^i\cdot v} \tag{3.12.6}$$

where C is independent of m or v.

But $0 < a^i \cdot v \leq |a^i|$ for all i, hence

$$-\alpha \sum_{i=0}^{m} \frac{1}{a^i \cdot v} \leq -\alpha \sum_{i=0}^{m} \frac{1}{|a^i|}. \tag{3.12.7}$$

On substituting $\rho(\sigma \cdot v) = -\sigma \cdot \log x$ and $\rho(a^j \cdot v) = -a^j \cdot \log x$ in (3.12.6) and noting that $L_m(b_0 \cdots b_m; e^{-\xi})(\lambda) = L_m(-b_0, \ldots, -b_m; e^{\xi})(-\lambda)$, the theorem follows.

4. Kergin interpolation

4.1. It is possible to give an intrinsic interpretation to the family of functions $\mathscr{F}(A)$ and to the interpolant $L_m(a^0 \cdot \log x, \ldots, a^m \cdot \log x; e^{\xi})(\sigma \cdot \log x)$ (see Proposition 4.5).

This interpretation relies on a natural polynomial interpolation procedure, Kergin interpolation, for functions of several variables. The basic properties may be found in [6, 8, 9]. I will list some of the properties of this interpolation procedure which are needed.

4.2. Let $P_0, P_1, \ldots, P_m$ be points in $\mathbf{R}^n$ contained in a convex open set V. There is a natural projection $\chi_m : \mathscr{C}^m(V) \to \mathscr{P}^m(\mathbf{R}^n)$, where $\mathscr{C}^m(V)$ denote the space of m-times continuously differentiable functions on V and $\mathscr{P}^m(\mathbf{R}^n)$ denotes the space of polynomials of total degree $\leq m$ in $x = (x_1, \ldots, x_n)$.

This projection satisfies

(4.2.1) $\quad (\chi_m f)(P_j) = f(P_j) \quad$ for $j = 0, \ldots, m$.

(4.2.2) $\quad$ If $P_{j_0}, \ldots, P_{j_s}$ are $(s+1)$ elements $\subset \{P_0, \ldots, P_m\}$ and χ_J denotes the Kergin interpolation for $P_{j_0}, \ldots, P_{j_s}$, then $\chi_J \chi_m = \chi_J$.

(4.2.3) $\quad$ If $f = g \circ \lambda$ where $\lambda : \mathbf{R}^n \to \mathbf{R}$ is linear and $g \in \mathscr{C}^m(\mathbf{R})$, then

$$\chi_m f = L_m(\lambda(P_0), \ldots, \lambda(P_m); g)(\lambda(x)).$$

4.3. Next we list some properties of this interpolation procedure when applied to holomorphic functions. Suppose that $P_0, P_1, \ldots, P_m$ are points in $\mathbf{C}^n$ contained in a convex open set V. Identifying $\mathbf{C}^n$ with $\mathbf{R}^{2n}$, we may apply the Kergin interpolation procedure. Suppose f is holomorphic on V and let

$$\chi_m(f) = \chi_m(\mathrm{Re}(f)) + i\chi_m(\mathrm{Im}(f)). \tag{4.3.1}$$

(4.3.2) $\quad$ Then $\chi_m(f)$ is an analytic polynomial on $\mathbf{C}^n$.

(4.3.3) $\quad$ If $f = g \circ \lambda$ where $\lambda : \mathbf{C}^n \to \mathbf{C}$ is linear and g is analytic on $\mathbf{C}$, then

$$\chi_m f = L_m(\lambda(P_0), \ldots, \lambda(P_m); g)(\lambda(z)).$$

(4.3.4) Suppose f and V are invariant under conjugation, that is, if $z \in V \Rightarrow \bar{z} \in V$ and $f(z) = \overline{f(\bar{z})}$ for all $z \in V$. Suppose $P_0, P_1, \cdots \in V \cap \mathbf{R}^n$; then

$$(\chi_m f)(z)/_{\mathbf{R}^n} = \chi_m(f/_{\mathbf{R}^n}).$$

4.4. **Proposition.** *Let X be a compact subset of $\mathbf{R}^N$ and $d\mu$ a finite Borel measure on X. Let V be open and convex in $\mathbf{C}^n$ and let $P_0, P_1, \ldots, P_m$ be points in V. Let $H(z, t)$ be continuous on $V \times X$, holomorphic in z for t fixed, and bounded on sets of the form $K \times X$ with K compact $\subset V$. Let*

$$h(z) = \int_X H(z, t)\, d\mu(t). \tag{4.4.1}$$

Then

$$\chi_m(h) = \int_X \chi_m H(z, t)\, d\mu(t) \tag{4.4.2}$$

where the interpolant $\chi_m(H(z, t))$ is considered with respect to z for t fixed.

Proof. First we note that using the Cauchy formula for derivatives it follows that for any multi-index ν, $\partial^\nu H(z, t)/\partial z^\nu$ is bounded on sets of the form $K \times X$ and continuous on $V \times X$. Also

$$\frac{\partial^\nu h(z)}{\partial z^\nu} = \int_X \frac{\partial^\nu H(z, t)}{\partial z^\nu}\, d\mu(t). \tag{4.4.3}$$

There is an integral formula for the difference of a function and its interpolant χ_m (see [3, 9]). It is

$$h(z) - \chi_m h = \int_{\Delta^{m+1}} d^{m+1} h\left(v_{m+1} z + \sum_{j=0}^{m} v_j P_j\right) dv_1 \cdots dv_{m+1} \times (z - P_0, \ldots, z - P_m) \tag{4.4.4}$$

where d^{m+1} is the total $(m+1)$th derivative and Δ^{m+1} is the $(m+1)$ simplex in $\mathbf{R}^{m+1}$. Applying the same formula for $H(z, t)$ and using (4.4.3) and an interchange of order of integration, we obtain (4.4.2).

4.5. **Proposition.** *Let $F(z) = \int_{I^n} t^z\, d\mu$ (as in 2.4) and let $A = \{a^i\}_{i=0,1,2}$ be a sequence in $(\mathbf{R}^+)^n$. Then $\chi_m F \equiv 0$ for all m if and only if the linear functional S_μ given by $f \to \int_{I^n} f\, d\mu$ annihilates $\bigcup_{j=0}^{\infty} \mathscr{F}_j(A)$.*

Proof. By Proposition 4.4

$$\chi_m F(z) = \int_{I^n} \chi_m(e^{z \log t})\, d\mu(t) \tag{4.5.1}$$

and using (4.3.3) this equals

$$\int_{I^n} L_m(a^0 \cdot \log t, \ldots, a^m \cdot \log t; \exp)(z \cdot \log t)\, d\mu(t). \tag{4.5.2}$$

For $z = \sigma \in (\mathbf{R}^+)^n$

$$\chi_m F(\sigma) = \int_{I^n} L_m(a^0 \cdot \log t, \dots, a^m \cdot \log t; \exp)(\sigma \cdot \log t)\, d\mu(t). \tag{4.5.3}$$

Thus, if S_μ annihilates $\bigcup_{j=0}^{\infty} \mathscr{F}_j(A)$, using (1.6.4) we have $\chi_m F(\sigma) \equiv 0$ for all m.

Conversely, suppose $\chi_m F \equiv 0$ for all m. Let $a^{j_0}, \dots, a^{j_s}$ be any $(s+1)$ elements of A and let χ_J denote the Kergin interpolation at $a^{j_0}, \dots, a^{j_s}$. If $m = \operatorname{Max}(j_0, \dots, j_s)$, then by (4.2.2)

$$\chi_J \chi_m F = \chi_J F \equiv 0. \tag{4.5.4}$$

This implies

$$\int_{I^n} L_{s+1}(a^{j_0} \cdot \log t, \dots, a^{j_s} \cdot \log t; \exp)(\sigma \cdot \log t)\, d\mu(t) = 0. \tag{4.5.5}$$

In particular, since this is a polynomial in σ of degree $\leq s$, for every multi-index ν satisfying $|\nu| = s$ the coefficient of σ^ν must be zero. But, from (1.6.4) the coefficient of σ^ν in the above expression is

$$\int_{I^n} [a^{j_0} \cdot \log t, \dots, a^{j_s} \cdot \log t] \exp(\log t)^\nu\, d\mu(t). \tag{4.5.6}$$

Thus, S_μ annihilates all elements of $\bigcup_j \mathscr{F}_j(A)$.

4.6. *Remark.* Let $A = \{a^i\}_{i=0,1,\dots}$. For each n multi-index ν such that $|\nu| = s$ let (see (1.3.2))

$$\phi_\nu = (\log x)^\nu \sum_{k=0}^{s} \frac{x^{a^k}}{\prod_{j=0, j\neq k}^{s} (a^j - a^k) \cdot \log x}.$$

Then the linear span of the family ϕ_ν is the same as the linear span of $\bigcup_{j=0}^{\infty} \mathscr{F}_j(A)$. $\mathscr{F}_s(A)$ was defined in (1.3) as consisting of functions of the above form as $a^0, \dots, a^s$ ranged over all collections of $(s+1)$ elements of A (rather than just the first $(s+1)$ elements in a fixed ordering of A).

4.7. A natural question is the following: Let F be a bounded analytic function on the cone Γ (see (2.4.2)). Let $A = \{a^i\}_{i=0,1,2,\dots}$ be a countable sequence of points such that $\lim_i a_j^i = +\infty$ for $j = 1, \dots, n$ and

$$\sum_{i=0}^{\infty} \frac{1}{|a^i|} = +\infty.$$

Suppose that $\chi_m F \equiv 0$ for $m = 0, 1, \dots$. Is $F \equiv 0$? A positive answer to this question would imply that Theorem 2.2 holds without condition (2.2.3).

5. Necessity

5.1. We will examine the necessity of conditions (2.2.1), (2.2.2), and (2.2.3) in Theorem 2.2.

For $n = 1$ (the one-variable case), under (2.2.1) condition (2.2.2) is necessary and sufficient. Condition (2.2.3) is not needed in the one-variable case. Whether or not (2.2.3) (or some other condition) is necessary for $n > 1$ is not known. We will show that, for $n > 1$ under condition (2.2.1), condition (2.2.2) is necessary.

5.2. **Proposition.** *Let $A = \{a^i\}_{i=0,1,\dots}$ be a sequence of n-tuples of positive real numbers satisfying* (2.2.1). *That is,* $\lim_i a_j^i = +\infty$ *for* $j = 1, \dots, n$. *Let* $b_i = \sum_{j=1}^n a_j^i$ *for* $i = 0, 1, \dots$. *Let* b *be a positive real number distinct from* $b_0, b_1, \dots$ *and let* $a_1, \dots, a_n$ *be positive real numbers satisfying* $a_1 + \cdots + a_n = b$. *If* $\sum_{i=0}^\infty \frac{1}{|a^i|} < +\infty$ *then* $t^a \notin \overline{L\mathscr{F}(A)}$.

In particular, $L\mathscr{F}(A)$ *is not dense in* $\mathscr{C}(I^n)$.

Proof. $b_i \geq |a^i|$ for $i = 0, 1, \dots$. Hence

$$\sum_{i=0}^\infty \frac{1}{b_i} < +\infty. \tag{5.2.1}$$

Also

$$\lim_i b_i = +\infty. \tag{5.2.2}$$

Following a construction in [13, p. 338], there exists an analytic function g, bounded on the right half plane vanishing only at $b_0, b_1, \dots$. (The construction in [13] is done in the case the b_i's are distinct but clearly it is valid in general.) g is of the form $g(z) = \int_0^1 t^z \, d\mu$ where $d\mu$ is a finite Borel measure on $[0, 1]$. Also, $g(b) \neq 0$ and t^b is not in the closure of the span of $1, t^{b_0}, \dots$.

Consider

$$G(z_1, \dots, z_n) = \int_{I^n} t^z \, d\nu(t) \tag{5.2.3}$$

where $d\nu$ is the measure on I^n defined by

$$\nu(Z) = \mu(\delta^{-1}(Z)) \tag{5.2.4}$$

where $Z \subset I^n$ and $\delta : I \to I^n$ is given by

$$\delta(t) = (t, t, \dots, t). \tag{5.2.5}$$

Then

$$G(z_1, \dots, z_n) = g(z_1 + \cdots + z_n). \tag{5.2.6}$$

But

$$\begin{aligned} \chi_m G &= \int_{I^n} L_m(a^0 \cdot \log t, \dots, a^m \cdot \log t; \exp)(z \cdot \log t) \, d\nu(t) \\ &= L_m g(z_1 + \cdots + z_n) \\ &= L_m(b_0, \dots, b_m; g)(z_1 + \cdots + z_n) = 0 \quad \text{for all } m \end{aligned} \tag{5.2.7}$$

since g vanishes at $b_0, \dots, b_m$. Hence $d\nu$ annihilates $\bigcup_{j=0}^{\infty} \mathscr{F}_j(A)$. But $g(b) \neq 0$ so $G(a_1, \dots, a_n) \neq 0$ and thus

$$t^a \notin \overline{L\left(\bigcup_{j=0}^{\infty} \mathscr{F}_j(A)\right)}. \tag{5.2.8}$$

We must show that $t^a \notin \overline{L\mathscr{F}(A)}$. We will proceed by contradiction. Suppose

$$t^a = \lim_k (\phi_k(t) + \psi_1^k(t) + \cdots + \psi_n^k(t)) \tag{5.2.9}$$

where $\phi_k(t) \in L(\bigcup_{j=0}^{\infty} \mathscr{F}_j(A))$ and $\psi_j^k(t)$ is constant on the slices $t_j = c$. Thus, we may write $\psi_j^k(t) = \psi_j^k(t_1, \dots, \hat{t}_j, \dots, t_n)$. Let $Y = \{t \in I^n \mid t_j = 0$ for some $j = 1, \dots, n\}$. For $t \in Y$, $\phi_k(t) = 0$ (by Lemma 1.5) and so

$$\lim_k \left(\sum_{j=1}^{n} \psi_j^k(t)\right) = 0 \quad \text{for } t \in Y. \tag{5.2.10}$$

Fix i an integer, $1 \leq i \leq n$, and consider points of the form

$$(t_1, \dots, 0 \swarrow^{i\text{th}}, \dots, t_n) \in Y.$$

If we consider (5.2.10) at points of this form we see there are functions (for $j \neq i$)

$$U_{ij}^k(t_i, \dots, \hat{t}_i, \dots, \hat{t}_j, \dots, t_n) = \psi_j^k(t_1, \dots, 0 \swarrow^{i\text{th}}, \dots, \hat{t}_j, \dots, t_n)$$

such that

$$\lim_k \left(\psi_j^k(t) + \sum_{\substack{j \neq i \\ j=1}}^{n} U_{ij}^k(t_1, \dots, \hat{t}_i, \dots, \hat{t}_j, \dots, t_n\right) = 0 \tag{5.2.11}$$

for all $t \in I^n$. We may deduce that

$$\lim_k \left(\sum_{j=1}^{n} \psi_j^k(t) + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} U_{ij}^k(t)\right) = 0 \quad \text{for all } t \in I^n \tag{5.2.12}$$

and

$$\lim_k \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} U_{ij}^k(t)\right) = 0 \quad \text{for } t \in Y. \tag{5.2.13}$$

Repeating the above argument sufficiently often, we can conclude that there are constants c^k such that

$$\lim_k \left(\sum_{j=1}^{n} \psi_j^k(t) + c^k\right) = 0 \quad \text{for all } t \in I^n \tag{5.2.14}$$

and $\lim_k c^k = 0$. Hence

$$\lim_k \left(\sum_{j=1}^{n} \psi_j^k(t) \right) = 0 \quad \text{for all } t \in I^n. \tag{5.2.15}$$

Thus $t^a = \lim_k \phi_k(t)$, but this contradicts (5.2.8). Hence $t^a \notin \overline{L\mathscr{F}(A)}$.

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Department of Mathematics, University of Toronto, Toronto, Ontario, Canada