THE DIFFEOTOPY GROUP
OF THE TWISTED 2-SPHERE BUNDLE OVER THE CIRCLE

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ABSTRACT. The diffeotopy group of the nontrivial 2-sphere bundle over the
circle is shown to be isomorphic to \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). The first generator is induced by
a reflection across the base circle, while a second generator comes from rotating
the 2-sphere fiber as one travels around the base circle. The technique employed
also shows that homotopic diffeomorphisms are diffeotopic.

1. INTRODUCTION

Gluck proved in \([G]\) that the diffeotopy group \( \mathcal{H} \) of \( S^1 \times S^2 \) is isomorphic
to \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). In fact, for a large class of 3-dimensional manifolds, Ru-
binstein, Laudenbach, Waldhausen and others computed the diffeotopy groups.
The methods exploited there do not seem to work for the twisted \( S^2 \)-bundle
over \( S^1 \), \( S^1 \times S^2 \). Therefore, we use different methods in conjunction with
Gluck’s arguments to compute the diffeotopy group \( S^1 \times S^2 \). In this paper, we
shall prove the following:

Theorem. The diffeotopy group \( \mathcal{G} \) of the twisted 2-sphere bundle over \( S^1 \),
\( S^1 \times S^2 \), is \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \).

Corollary. Each diffeomorphism homotopic to the identity is diffeotopic to the
identity.

2. NOTATION AND PROOF OF THE THEOREM

We adopt the following notation. Let \( I \) be the unit interval \([0, 1]\) and \( S^1 \)
the unit circle in the plane, i.e., the set of all complex numbers whose absolute
value is 1. We will use \( \exp 2\pi i \theta \) as a point of \( S^1 \), where \( \theta \) is a real number
and \( i = \sqrt{-1} \). Let \( S^2 \) be the unit sphere in the 3-dimensional Euclidean space.
We will write \( v \) as a point of \( S^2 \). In \( S^1 \times S^2 \), \( \sim \) means every \( (\exp 2\pi i \theta , v) \)
in \( S^1 \times S^2 \) is identified with \( (-\exp 2\pi i \theta , -v) \). \( D^2 \) will be the unit disk in the

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plane which is the set of all complex numbers whose absolute value is less than or equal to 1.

To compute $G$, we need the following crucial lemma.

(2.1) **Lemma.** Let $\text{Map}^1(S^2, S^2)$ be the set of all degree one continuous maps from $S^2$ to $S^2$. We assume that the topology is induced from the compact open topology.

Define a $\mathbb{Z}_2$-action on $\text{Map}^1(S^2, S^2)$ by

$$
\mathbb{Z}_2 \times \text{Map}^1(S^2, S^2) \to \text{Map}^1(S^2, S^2)
$$

$$
\lambda \mapsto A \circ \lambda \circ A
$$

where $A$ is the antipodal mapping of $S^2$. Then the fundamental group of the quotient space is $\mathbb{Z}_2'$. 

**Proof.** Put $E = \{[\alpha] | \alpha: I \to \text{Map}^1(S^2, S^2), \alpha(0) = \text{the identity map}\}$, where $[\alpha] = [\beta]$ means that $\alpha(1) = \beta(1)$ and $\alpha \ast \beta$ is homotopic to a constant path. More precisely, $\alpha \ast \beta$ is the composition of $\alpha$ and $\beta$, i.e.,

$$
\alpha \ast \beta(t) = \begin{cases}
\alpha(2t), & 0 \leq t \leq 1/2, \\
\beta(2t - 1) = \beta(2 - 2t), & 1/2 \leq t \leq 1.
\end{cases}
$$

Define $\pi: E \to \text{Map}^1(S^2, S^2)$ by $\pi([\alpha]) = \alpha(1)$. The space $E$ is simply connected, since $\text{Map}^1(S^2, S^2)$ is path-connected, and has a universal covering space (cf. [M, p. 394]).

We define two commuting $\mathbb{Z}_2$-actions on $E$. The first is given by

$$(\varepsilon, [\alpha]) \to [A \circ \alpha \circ A]$$

where $\varepsilon$ denotes the nontrivial element of $\mathbb{Z}_2$ in the first action. Next, we are going to use the fact, proved in [Hu], that $\Pi_1 \text{Map}^1(S^2, S^2) = \mathbb{Z}_2$. Then, given any path $\alpha$ starting at the identity, we can find a path $\gamma$, also, starting at the identity map with $\alpha(1) = \gamma(1)$ and $\alpha \ast \gamma$ not homotopic to a constant path. We define the second involution

$$(\varepsilon, [\alpha]) \to [\gamma]$$

where $\varepsilon$ is the nontrivial element of $\mathbb{Z}_2$. This involution describes the generator of the group of covering transformations on $E$. This enables us to conclude that

$$
\Pi_1(\text{Map}^1(S^2, S^2)/\mathbb{Z}_2) = \mathbb{Z}_2, \quad \text{i.e.} \quad \Pi_1(E/\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \mathbb{Z}_2,
$$

because of M. Armstrong's result in [A]:

Let $G$ be a discontinuous group of homeomorphism of a simply connected, locally path connected, Hausdorff space $X$. Then the fundamental group of the quotient $X/G$ is $G/N$ where $N$ is the subgroup of $G$ generated by those elements which have fixed points.

Therefore, we have only to show that there exists no $[\alpha]$ in $E$ such that $\varepsilon \varepsilon [\alpha] = [\alpha]$ since $\varepsilon$ has no fixed point and $\varepsilon$ has a fixed point.
Suppose there exists such $[\alpha]$. Then we get
\[ x[\alpha] = x[\gamma] = [A \circ \gamma \circ A] = [\alpha]. \]

Since $\alpha(1) = A \circ \gamma(1) \circ A = \gamma(1)$, $\alpha(1)$ is in the set $\mathcal{F}$ of fixed points of the $\mathbb{Z}_2$-action on $\text{Map}^1(S^2, S^2)$. Since $[\gamma] = [A \circ \alpha \circ A]$, $(A \circ \alpha \circ A) \ast \overline{\gamma} = 0$, and $\alpha \ast \overline{\gamma}$ is not homotopic to a constant path. Since $(A \circ \alpha \circ A) \ast \overline{\gamma} \ast \gamma \ast \overline{\alpha}$ is not homotopic to a constant path, $(A \circ \alpha \circ A) \ast \overline{\alpha}$ is not homotopic to a constant path. This will lead to a contradiction.

We claim that there is a path $\beta$ in $\text{Map}^1(S^2, S^2)$ going from identity to $\alpha(1)$ which is fixed under the involution on $\text{Map}^1(S^2, S^2)$. We use the fact that the set of self-homotopy equivalences of $RP^2$ is path-connected (see [GK]). This set corresponds exactly to the set of maps in $\text{Map}^1(S^2, S^2)$ that are fixed by the given action $\lambda \mapsto A \circ \lambda \circ A$. So there is a map $\beta: I \to \text{Map}^1(S^2, S^2)$ with $\beta(0) = \text{identity}$ and $\beta(1) = \alpha(1)$, such that $\beta(t)$ lies in the fixed points set $\mathcal{F}$. Observe that $(A \circ \alpha \circ A) \ast \overline{\alpha} \ast \beta \ast \overline{\alpha}$ is not homotopic to a constant path.

Let $\delta = \beta \ast \overline{\alpha}$. Then $A \circ \delta \circ A = (A \circ \beta \circ A) \ast (A \circ \overline{\alpha} \circ A) = \beta \ast (A \circ \overline{\alpha} \circ A)$. Since $\lambda \mapsto A \circ \lambda \circ A$ is an involution and $\Pi_1(\text{Map}^1(S^2, S^2)) = \mathbb{Z}_2$, $A \circ \delta \circ A$ is homotopic to $\delta$. Also, each element is its own inverse and $\beta \ast (A \circ \overline{\alpha} \circ A)$ is homotopic to $(A \circ \alpha \circ A) \ast \beta$. Thus $(A \circ \alpha \circ A) \ast \overline{\alpha} \ast \beta \ast \overline{\alpha}$ is homotopic to $\delta \ast \overline{\delta}$, which is trivial. This is a contradiction. We have proved the lemma. □

(2.2) Corollary. Let $g$ be the self-diffeomorphism of $S^1 \times S^2$ defined by
\[ [\exp 2\pi i \theta, v] \mapsto \left[ \begin{array}{ccc} \exp 2\pi i \theta & \cos 4\pi \theta & 0 \\ -\sin 4\pi \theta & \cos 4\pi \theta & 0 \\ 0 & 0 & 1 \end{array} \right] (v). \]

Then $g$ cannot be extended to a map from $(D^2 \times S^2)/\sim$ to itself, where $\sim$ means every point $(r \exp 2\pi i \theta, v)$ in $D^2 \times S^2$ is identified with $(-r \exp 2\pi i \theta, -v)$, $0 \leq r \leq 1$.

Proof. Suppose there exists an extension $k$ of $g$. Since $D^2 \times S^2$ is the universal covering space of $(D^2 \times S^2)/\sim$, we can lift $k$ to $\tilde{k}$ on $D^2 \times S^2$. Let us examine the value of the second coordinate under the mapping $\tilde{k}$, i.e., consider the following commutative diagram
\[ (r \exp 2\pi i \theta, v) \mapsto (\_, K(r \exp 2\pi i \theta, v)) \]

\[ \begin{array}{ccc} D^2 \times S^2 & \overset{k}{\longrightarrow} & D^2 \times S^2 \\ \downarrow & & \downarrow \\ (D^2 \times S^2)/\sim & \overset{k}{\longrightarrow} & (D^2 \times S^2)/\sim \end{array} \]
Without loss of generality, since $\text{Map}^1(S^2, S^2)$ is path-connected, we may assume that $K$ is a map from $D^2 \times S^2$ to $S^2$ such that

$$K(\exp 2\pi i \theta, v) = \begin{pmatrix} \cos 4\pi \theta & \sin 4\pi \theta & 0 \\ -\sin 4\pi \theta & \cos 4\pi \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}(v),$$

and

$$-K(-r \exp 2\pi i \theta, -v) = K(r \exp 2\pi i \theta, v).$$

Define $\tilde{K}$ from $D^2$ to $\text{Map}^1(S^2, S^2)$ by

$$r \exp 2\pi i \theta \rightarrow K(r \exp 2\pi i \theta, \_).$$

From the diagram above, we get $\tilde{K}(-r \exp 2\pi i \theta) = A \circ \tilde{K}(r \exp 2\pi i \theta) \circ A$. Consider $p \circ \tilde{K}$. Then $p \circ \tilde{K}(-r \exp 2\pi i \theta) = p \circ \tilde{K}(r \exp 2\pi i \theta)$ where $p$ is the projection from $\text{Map}^1(S^2, S^2)$ to $\text{Map}^1(S^2, S^2)/\mathbb{Z}_2$.

Define $\tilde{\tilde{K}}$ from $D^2$ to $\text{Map}^1(S^2, S^2)/\mathbb{Z}_2$ by

$$\tilde{\tilde{K}}(r \exp 2\pi i \theta) = p \circ \tilde{K}(r \exp \pi i \theta).$$

This is well defined and

$$\tilde{\tilde{K}}(\exp 2\pi i \theta) = \left\langle \begin{pmatrix} \cos 2\pi \theta & \sin 2\pi \theta & 0 \\ -\sin 2\pi \theta & \cos 2\pi \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

in $\text{Map}^1(S^2, S^2)/\mathbb{Z}_2$ where $\langle \rangle$ means the image under the projection $p$.

$$\tilde{\tilde{K}}(0) = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

This implies $\tilde{\tilde{K}}$ is a homotopy between following two maps

$$\exp 2\pi i \theta \rightarrow \left\langle \begin{pmatrix} \cos 2\pi \theta & \sin 2\pi \theta & 0 \\ -\sin 2\pi \theta & \cos 2\pi \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

and

$$\exp 2\pi i \theta \rightarrow \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Since $p_* : \pi_1(\text{Map}^1(S^2, S^2) \rightarrow \pi_1(\text{Map}^1(S^2, S^2)/\mathbb{Z}_2) = \mathbb{Z}_2$ is onto, and the map $\tilde{T}$

$$\exp 2\pi i \theta \rightarrow \begin{pmatrix} \cos 2\pi \theta & \sin 2\pi \theta & 0 \\ -\sin 2\pi \theta & \cos 2\pi \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
is the nontrivial loop in $\text{Map}^1(S^2, S^2)$ (which we shall show later),

$$\exp 2\pi i \theta \rightarrow \begin{pmatrix} \cos 2\pi \theta & \sin 2\pi \theta & 0 \\ -\sin 2\pi \theta & \cos 2\pi \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

represents the nontrivial loop, we have a contradiction. It remains to show $\tilde{T}$ is nontrivial. Consider the nontrivial $S^2$-fiber bundle over $S^2$ with structure group $SO(3)$, i.e., the space is given as follows:

$$(D^2 \times S^2) \cup_T (D^2 \times S^2)$$

= gluing two copies of $D^2 \times S^2$ along the boundary by $T$.

where $T$ is a map from $S^1 \times S^2$ to itself given by

$$(\exp 2\pi i \theta, v) \rightarrow (\exp 2\pi i \theta, \tilde{T}(\exp 2\pi i \theta)(v)).$$

Suppose $\tilde{T}$ is homotopic to a constant path. Then we can get a homotopy equivalence from $(D^2 \times S^2) \cup_{id} (D^2 \times S^2) (= S^2 \times S^2)$ to $(D^2 \times S^2) \cup_T (D^2 \times S^2)$, and we have a contradiction (see [S]). This completes the proof. □

Remark. If we use the results in [KKR], we can give a shorter proof of the corollary above. More precisely, if $g$ can be extended, then we may construct two spaces which must be homotopy equivalent, but by the homotopy invariant in [KKR], the two spaces can not be homotopy equivalent. So we have a contradiction.

(2.3) Theorem. The diffeotopy group $\mathcal{G}$ of $S^1 \tilde{\times} S^2$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. Our argument is divided into 4 steps.

1. We construct a map $\varphi$ from $\mathcal{G}$ to $\mathcal{H}/\mathbb{Z}_2 (= \mathbb{Z}_2 \oplus \mathbb{Z}_2)$.
2. We show that the image of $\varphi$ is $\mathbb{Z}_2$.
3. Ker $\varphi$ is $\mathbb{Z}_2$.
4. $\mathbb{Z}_2 \rightarrow \mathcal{G} \rightarrow \mathbb{Z}_2$ is split.

Recall that $\mathcal{H} = \text{Diff}(S^1 \times S^2)/\sim$ where $\sim$ is the normal subgroup consisting of those diffeomorphisms which are diffeotopic to identity and $\mathcal{G} = \text{Diff}(S^1 \tilde{\times} S^2)/\sim$, where $\sim$ means the same as above.

According to Gluck,

$$\mathcal{H} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle T \rangle \oplus \langle f \rangle \oplus \langle h \rangle,$$

$$S^1 \times S^2 \rightarrow S^1 \times S^2,$$

$$T: (\exp 2\pi i \theta, v) \rightarrow \left( \begin{pmatrix} \cos 2\pi \theta & \sin 2\pi \theta & 0 \\ -\sin 2\pi \theta & \cos 2\pi \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)(v),$$

$$f: (\exp 2\pi i \theta, v) \rightarrow (\exp(-2\pi i \theta), v),$$

$$h: (\exp 2\pi i \theta, v) \rightarrow (\exp 2\pi i \theta, -v).$$
(i) First step. Define a $\mathbb{Z}_2$-action on $H$ as follows:

$$\mathbb{Z}_2 \times H \to H$$

$$\langle \ell \rangle \to \langle \phi \circ \ell \rangle$$

where $\phi: S^1 \times S^2 \to S^1 \times S^2$ is a diffeomorphism defined by

$$(\exp 2\pi i \theta, v) \to (-\exp 2\pi i \theta, -v).$$

We want to construct a map $\varphi$ from $G$ to $H/\langle \phi \rangle$. To do it, we have to show that, given any diffeomorphism $w$ of $S^1 \tilde{\times} S^2$, we have a lift defined up to a covering transformation, i.e.,

$$\begin{array}{ccc}
S^1 \times S^2 & \xrightarrow{\tilde{w}} & S^1 \times S^2 \\
p\downarrow & & \downarrow p \\
S^1 \tilde{\times} S^2 & \xrightarrow{w} & S^1 \tilde{\times} S^2
\end{array}$$

where $p: S^1 \times S^2 \to S^1 \tilde{\times} S^2$ is the natural projection.

To show the existence of $\tilde{w}$, consider $(w \circ p)_* \Pi_1(S^1 \times S^2)$. $S^1 \times S^2$ is a double covering of $S^1 \tilde{\times} S^2$, so $p_\* \Pi_1(S^1 \times S^2)$ is $2Z \subseteq \mathbb{Z} = \Pi_1(S^1 \tilde{\times} S^2)$. Hence, any automorphism of $\Pi_1(S^1 \tilde{\times} S^2)$ preserves $p_\* \Pi_1(S^1 \times S^2)$. By the lifting lemma, there exist $\tilde{w}$ from $S^1 \times S^2$ to itself such that $p \circ \tilde{w} = w \circ p$. Note that $p \circ \phi \circ \tilde{w} = w \circ p$, since $\phi$ is a regular covering transformation. Furthermore, if $w_1$ is isotopic to $w_2$ by an isotopy $H$ on $S^1 \tilde{\times} S^2$, then, as in the above argument, we have $\tilde{H}: I \times S^1 \times S^2 \to S^1 \times S^2$ such that $p \circ \tilde{H} = H \circ (\text{Id} \times p)$. Now we can define a map $\varphi$ from $G$ to $H/\mathbb{Z}_2$ by $\langle \eta \rangle \to \langle \eta \rangle$.

Since $f \circ \phi = \phi \circ f$, $f$ induces a self-diffeomorphism on $S^1 \tilde{\times} S^2$. Thus $\text{Im} \varphi \supset \mathbb{Z}_2$. Note that since $h$ is isotopic to $\phi$, $\langle h \rangle$ is trivial in $H/\mathbb{Z}_2$.

(ii) Second step. We know that $\text{Im} \varphi \supset \mathbb{Z}_2$, from First step. To demonstrate $\text{Im} \varphi = \mathbb{Z}_2$, we shall show that there exist no $T'$ in $H$ which is isotopic to $T$ and commutes with $\phi$.

Suppose that there exists such $T'$. That means the following:

$$S^1 \times S^2 \times I \to S^1 \times S^2,$$

$$\{(\exp 2\pi i \theta, v), t\} \to (G_t(\exp 2\pi i \theta, v), F_t(\exp 2\pi i \theta, v))$$

where $t = 0$, $G_0 = \exp 2\pi i \theta$, and

$$F_0(\exp 2\pi i \theta, v) = \begin{pmatrix} \cos 2\pi \theta & \sin 2\pi \theta & 0 \\ -\sin 2\pi \theta & \cos 2\pi \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (v).$$
Note that $F_0$ is the second coordinate of $T$. For $t = 1$, we have
\[
(\exp 2\pi i \theta, v) \quad \longrightarrow \quad (G_1(\exp 2\pi i \theta, F_1(\exp 2\pi i \theta, v))
\]
\[
\begin{array}{ccc}
S^1 \times S^2 & \longrightarrow & S^1 \times S^2 \\
\phi \downarrow & & \phi \downarrow \\
S^1 \times S^2 & \longrightarrow & S^1 \times S^2
\end{array}
\]
\[
(- \exp 2\pi i \theta, -v) \longrightarrow (G_1(- \exp 2\pi i \theta, -v), F_1(- \exp 2\pi i \theta, -v)).
\]
From the commutativity of the above diagram (since $T' \circ \phi = \phi \circ T'$)
\[
F_1(\exp 2\pi i \theta, -v) = -F_1(\exp 2\pi i \theta, v)
\]
Define $F'_1$ from $S^1$ to $\text{Map}^1(S^1, S^2)$ by
\[
\exp 2\pi i \theta \rightarrow F_1(\exp 2\pi i \theta, _{-}).
\]
Note that, since $\Pi_1(\text{Map}^1(S^2, S^2)) = \mathbb{Z}_2$ is abelian, we need not worry about choosing a base point.
Recall the $\mathbb{Z}_2$-action on $\text{Map}^1(S^2, S^2)$
\[
\mathbb{Z}_2 \times \text{Map}^1(S^2, S^2) \rightarrow \text{Map}^1(S^2, S^2)
\quad \lambda \rightarrow A \circ \lambda \circ A
\]
For each $t$,
\[
[F'_1] \in \Pi_1(\text{Map}^1(S^2, S^2)) \quad \text{and}
\quad [p \circ F'_1] \in \Pi_1(\text{Map}^1(S^2, S^2)/\mathbb{Z}_2),
\]
where $[\ ]$ means the equivalence class of loops. Then $[p \circ F'_1]$ is trivial, since $p \circ F'_1(\exp 2\pi i \theta) = p \circ F'_1(\exp 2\pi i \theta)$. Therefore $[p \circ F_0]$ is trivial, since $p \circ F'_1$ is homotopic to $p \circ F'_0$.
By (2.1), $\Pi_1(\text{Map}^1(S^2, S^2)/\mathbb{Z}_2) = \mathbb{Z}_2$. Since $\Pi_1(\text{Map}^1(S^2, S^2)) = \mathbb{Z}_2$ and the nontrivial element is represented by
\[
F_0': S^1 \rightarrow \text{Map}^1(S^2, S^2)
\quad \exp 2\pi i \theta \rightarrow \begin{pmatrix}
\cos 2\pi \theta & \sin 2\pi \theta & 0 \\
-\sin 2\pi \theta & \cos 2\pi \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
then, the fact that the nontrivial element in $\Pi_1(\text{Map}^1(S^2, S^2)/\mathbb{Z}_2)$ is lifted as the nontrivial element in $\Pi_1(\text{Map}^1(S^2, S^2))$ implies a contradiction. Thus, we can conclude that $\text{Im} \varphi = \mathbb{Z}_2$. 

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(iii) Third step. Suppose \( q \) in \( \ker \phi \). Then \( \bar{q} \) is isotopic to the identity or \( \phi \circ \bar{q} \) is isotopic to the identity of \( S^1 \times S^2 \). By a straightforward argument of Gluck (see [G, pp. 315–316] and cf. [T]), we deform \( q \) so that the restriction to \( \{1, -1\} \times S^2 \) is the identity. Since \( \bar{q} \) (or \( \phi \circ \bar{q} \)) is isotopic to the identity, \( q \) can be considered as a diffeomorphism \( \bar{q} \) from \( I \times S^2 \) to \( I \times S^2 \) such that the restriction to \( \{1, -1\} \times S^2 \) of \( \bar{q} \) is the identity.

By Gluck (cf. [Ha]), \( \bar{q} \) is isotopic to the identity or to \( \bar{d} \) while fixing the \( \{1, -1\} \times S^2 \), where \( \bar{d} : I \times S^2 \to I \times S^2 \)

\[
(t, v) \to \left(t, \begin{pmatrix}
\cos 2\pi t, & \sin 2\pi t & 0 \\
-\sin 2\pi t, & \cos 2\pi t & 0 \\
0 & 0 & 1
\end{pmatrix} (v) \right).
\]

We claim that \( \bar{d} \) is \( \bar{g} \), where \( g \) is the self-diffeomorphism of \( S^1 \times S^2 \) in (2.2). Obviously, \( g \) is the identity on \( \{1, -1\} \times S^2 \subset S^1 \times S^2 \). Restrict \( g \) to \( S^1 \times S^2 - \{1, -1\} \times S^2 \), and under the following identification,

\[
\begin{array}{c}
(\theta, v) \\
(0, 1/2) \times S^2
\end{array} \longrightarrow \begin{array}{c}
\{\exp 2\pi i\theta, v\} \\
S^1 \times S^2 - \{1, -1\} \times S^2
\end{array}
\]

we get \( g' : (0, 1/2) \times S^2 \to (0, 1/2) \times S^2 \)

\[
(\theta, v) \to \left[\begin{pmatrix}
\cos 4\pi \theta & \sin 4\pi \theta & 0 \\
-\sin 4\pi \theta & \cos 4\pi \theta & 0 \\
0 & 0 & 1
\end{pmatrix} (v) \right] \exp 2\pi i\theta,
\]

Identify \( (0, 1/2) \times S^2 \) with \( (0, 1) \times S^2 \) by \( (\theta, v) \to (2\theta, v) \).

Then, \( g'' : (0, 1) \times S^2 \to (0, 1) \times S^2 \)

\[
(2\theta, v) \to \left\{\begin{pmatrix}
\cos 4\pi \theta & \sin 4\pi \theta & 0 \\
-\sin 4\pi \theta & \cos 4\pi \theta & 0 \\
0 & 0 & 1
\end{pmatrix} (v) \right\}, \quad 0 < \theta < 1/2.
\]
If we replace $2\theta$ by $\theta$, we get

$$g'' = l: (0, 1) \times S^2 \to (0, 1) \times S^2$$

$$(\theta, v) \to \left( \theta, \begin{pmatrix} \cos 2\pi \theta & \sin 2\pi \theta & 0 \\ -\sin 2\pi \theta & \cos 2\pi \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} (v) \right).$$

This proves the claim, i.e., $\bar{g} = \bar{d}$. Hence, $q$ is isotopic to the identity or $g: g$ is not isotopic to the identity, otherwise $g$ could be extended to $(D^2 \times S^2)/\sim$ (see (2.2)). To show $\ker \varphi = \mathbb{Z}_2$, it remains to show that $(g)^2 = \text{id}$, i.e., $g^2$ is isotopic to the identity. An isotopy is constructed as follows:

$$S^1 \sim S^2 \times I \to S^1 \sim S^2$$

$$\{[(\exp 2\pi i \theta, v)], t\} \to \{[(\exp 2\pi i \theta, H(\exp 4\pi i \theta, t)(v))]\}$$

where $H: S^1 \times I \to SO(3)$ is a homotopy between the maps

$$\exp 2\pi i \theta \to \begin{pmatrix} \cos 4\pi \theta & \sin 4\pi \theta & 0 \\ -\sin 4\pi \theta & \cos 4\pi \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \exp 2\pi i \theta \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(iv) Fourth step. The splitting follows from the fact that $\langle f \rangle$ has order 2. Now we have completed the proof. \(\square\)

From (2.2) and (2.3), we get the following result.

**Corollary.** Any self-diffeomorphism of $S^1 \sim S^2$ homotopic to the identity is diffeotopic to the identity.

**References**


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