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ROTATION AND WINDING NUMBERS FOR PLANAR POLYGONS AND CURVES

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ABSTRACT. The winding and rotation numbers for closed plane polygons and curves appear in various contexts. Here alternative definitions are presented, and relations between these characteristics and several other integer-valued functions are investigated. In particular, a point-dependent "tangent number" is defined, and it is shown that the sum of the winding and tangent numbers is independent of the point with respect to which they are taken, and equals the rotation number.

1. INTRODUCTION

Associated with every planar polygon P (or, more generally, with certain kinds of closed curves in the plane) are several numerical quantities that take only integer values. The best known of these are the *rotation number* of P (also called the "tangent winding number") of P and the *winding number* of P with respect to a point in the plane. The rotation number was introduced for smooth curves by Whitney [1936], but for polygons it had been defined some seventy years earlier by Wiener [1865]. The winding numbers of polygons also have a long history, having been discussed at least since Meister [1769] and, in particular, Möbius [1865]. However, there seems to exist no literature connecting these two concepts. In fact, so far we are aware, there is no instance in which both are mentioned in the same context.

In the present note we shall show that there exist interesting alternative definitions of the rotation number, as well as various relations between the rotation numbers of polygons, winding numbers with respect to given points, and several other integer-valued functions that depend on the embedding of the polygons in the plane. We shall define a "tangent number" of P with respect to a point A, and prove that the sum of the tangent and winding numbers of P with respect to A is independent of A and equal to the rotation number of P. Similar results are valid also for the " normal number" of P with respect to a point.

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With appropriate interpretation, these results remain true for piecewise smooth closed curves. On the other hand, some results concerning polygons seem to lack analogues for curves. For example, for a polygon P, and any point A that does not belong to P, we shall define a "modified polar" of P with respect to A, and show that the rotation number of P is equal to the winding number of this modified polar with respect to A.

2. Basic definitions; the rotation number

A *polygon* P, or, more specifically, an *n-gon* is a family of $n \geq 3$ points $V_1, V_2, V_3, \ldots, V_n$ and n directed line segments $E_1 = [V_1, V_2]$, $E_2 = [V_2, V_3]$, $\ldots, E_{n-1} = [V_{n-1}, V_n]$, $E_n = [V_n, V_1]$ in the Euclidean plane $\mathbf{E}^2$, such that $V_j \neq V_{j+1}$ for $j = 1, 2, \ldots, n$. Here, and in the sequel, it is convenient to take all subscripts modulo n. The n-gon P is denoted by $[V_1, V_2, V_3, \ldots, V_n]$, and we shall call this the *standard presentation* of P. The points V_j are the *vertices* of P, and the directed line segments $[V_j, V_{j+1}]$ are its *edges*. Two polygons are considered identical if the standard presentation of one can be obtained from that of the other by a cyclic permutation of the symbols V_j. All the polygons are "oriented" in the sense that $[V_1, V_2, V_3, \ldots, V_n]$ is to be considered distinct from $[V_n, \ldots, V_3, V_2, V_1]$; two such polygons are said to be of *different orientations*, though their vertices, and the edges (considered as point-sets rather than as directed line segments) coincide. Where appropriate and unambiguous, we shall use the words "clockwise" and "counterclockwise" to describe the orientation of a given polygon.

A polygon is called *ordinary* if no three edges have a common point. It is called *simple* if, in addition to being ordinary, no two edges have a common point which is in the relative interior of each. A simple polygon P is a simple Jordan curve; the complement of P in the plane consists of two disjoint, open, and connected regions, one of which is bounded and simply-connected, and the other unbounded. These are known as the *interior* and the *exterior* of P, respectively. More generally, if P is *any* polygon, its complement $\mathbf{E}^2 \backslash P$ in the plane consist of a finite number of connected and polygonally-connected open sets called *cells* of P, all but one of which are simply-connected and bounded. The exceptional one is unbounded and is called the *exterior* of P. The boundary of each cell consist of a finite number of segments, which can be directed so as to form a simple polygon. A polygon P has a *whisker* at a vertex V if the two edges of P incident with V overlap in a segment of positive length. While we shall admit various coincidences to occur between vertices and edges of the polygons we discuss (see Figure 1), polygons with whiskers will be excluded from all our considerations. Note, however, that the alternative definition of the rotation number given in §5 applies only to *ordinary* polygons. Examples of polygons which illustrate these definitions appear in Figure 1. In each, the orientation of any polygon can be deduced from the labels on the vertices, as well as from the direction of one edge, which is indicated by an arrow.

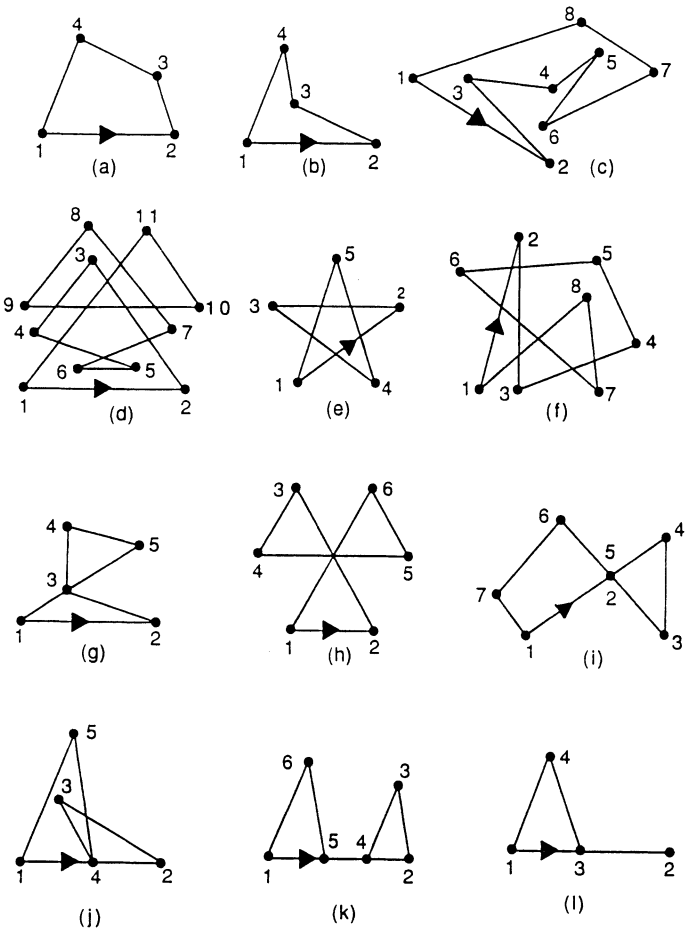


FIGURE 1. Examples of polygons. Vertices are indicated by solid dots labelled by $1, 2, \ldots, n$ (instead of $V_1, V_2, \ldots, V_n$). Polygons (a) to (f) are ordinary, the first three are also simple. The last six polygons are not ordinary. In particular, polygons (i) and (j) are "self-osculatory" in an obvious sense, and polygons (k) and (l) are "overlapping". The polygon (l) has, moreover, a whisker at vertex 2, and is therefore excluded from the considerations of the text. The twelve polygons shown have 2, 2, 2, 11, 7, 10, 3, 4, 3, 4, 3 and 2 cells, respectively, and the rotation numbers of the first eleven are 1, 1, 1, 2, 2, 1, 0, 2, 0, 2 and 1; the rotation number of the last polygon is not defined.

For any two distinct points A, B in the plane we shall use the notation $[A, B\rangle$ for the *ray* (closed halfline) with endpoint A that passes through B. Two rays are said to be *parallel* if one is a translate of the other. Let V_{j-1}, V_j, V_{j+1} be three consecutive vertices of a polygon P and let A_j be any point on the extension of the edge $[V_{j-1}, V_j]$ beyond V_j. The *deflection wedge* $D(V_j)$

of P at V_j is the union of all the rays with endpoint V_j that lie strictly between the rays $[V_j, A_j\rangle$ and $[V_j, V_{j+1}\rangle$ (see Figure 2(a)). Since we are assuming that P is whisker-free, the rays $[V_j, A_j\rangle$ and $[V_j, V_{j+1}\rangle$ do not lie in the same line opposite to each other, hence the meaning of "between" is well-defined. We define $d(V_j)$, the *deflection* of P at V_j, by the signed measure of the angle of $d(V_j)$ at its vertex V_j. If the second ray $[V_j, V_{j+1}\rangle$ is obtained from $[V_j, A_j\rangle$ by a counterclockwise rotation (in $D(V_j)$) then $D(V_j)$ is taken to be positive; if it is obtained by a clockwise rotation, $d(V_j)$ is taken to be negative. Throughout we shall use the absolute system of angle measure, so that a complete counterclockwise turn of 360° or 2π radians has value 1. By definition, the value of $d(V_j)$ satisfies $-\frac{1}{2} < d(V_j) < \frac{1}{2}$.

By translating the deflection wedges so that their vertices all coincide with a point O, we obtain the so-called "second figure" of the polygon (see Figure 2(b)). It is easy to see that these translated wedges fit together so as to cover the plane, and hence, for every polygon P, the quantity $\sum_j d(V_j)$ is an integer; here summation is over all vertices V_j of P. This integer indicates how many times any point (that does not lie on the boundary of a translated wedge) is covered by the wedges. The covering is reckoned in the algebraic sense, that is, a wedge contributes ± 1 to the covering according to the sign of $d(V_j)$. The integer $\sum_j d(V_j)$ is called the *rotation number* of P and is denoted by $r(P)$. It follows that if a simple n-gon P is oriented in a counterclockwise direction then $r(P) = 1$, and if it is oriented in a clockwise direction then $r(P) = -1$. In general, reversing the orientation of a polygon has the effect of changing the sign of its rotation number. The rotation numbers of the polygons shown in Figure 1 are indicated in the caption to the figure.

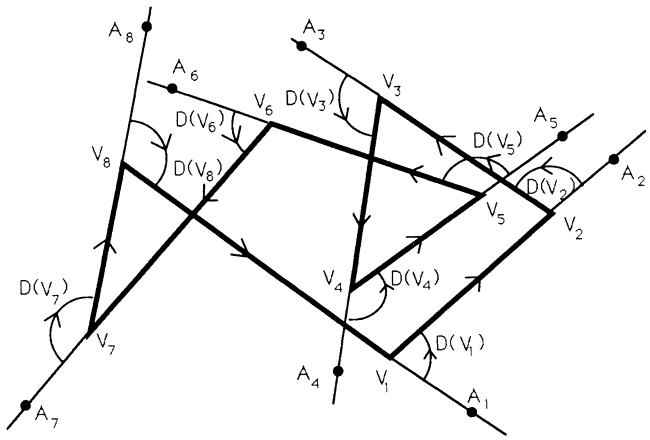
Let $R_j = [V_j, C\rangle$ be any ray with endpoint V_j. We say that R_j is *concordantly tangent* to P at V_j if it is contained in the interior of the deflection wedge $D(V_j)$. The concordant tangency of the ray R_j is *positive* or *negative* according to the sign of $d(V_j)$, see the caption to Figure 2(a). We have the following result:

Theorem 1. *Let P be a whisker-free polygon and R be any ray not parallel to an edge of P. For each vertex V_j of P let R_j be a ray parallel to R with endpoint V_j. Define $\tau(R_j) = 1$ if R_j is positively concordantly tangent to P at V_j; $\tau(R_j) = -1$ if R_j is negatively concordantly tangent to P at V_j; and $\tau(R_j) = 0$ otherwise. Then*

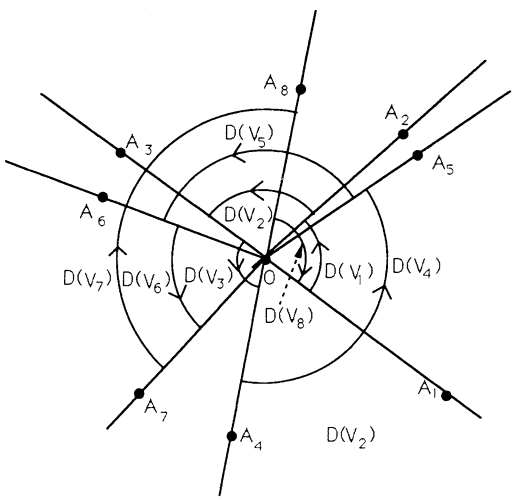
$$r(P) = \sum_j \tau(R_j)$$

where summation is over all the vertices of P.

Proof. Consider the second figure of P, and let R_O be the ray parallel to R with endpoint O. Then it is clear that $\sum_j \tau(R_j)$ is precisely the multiplicity with which the ray R_O is covered by the translated wedges, and so is equal to the rotation number $r(P)$. □



(a)



(b)

FIGURE 2. The definitions of "deflection wedge" and "rotation number". The construction of the "second figure" illustrated in (b) for the polygon in (a) goes back to Wiener [1865]; it shows that the polygon P in (a) satisfies $r(P) = 1$. Rays directed vertically up are concordantly tangent at V_2, V_5 and V_7 (these rays are not shown); the concordant tangency is positive at V_2 and V_5, negative at V_7.

The restriction that R is not parallel to an edge of P can be removed if one counts, instead of vertices, the *connected subsets* of P at which rays parallel to R are concordantly tangent, and the definition of the sign of concordancy is modified in the appropriate manner.

3. Tangent and normal numbers

For any polygon P let $W(P)$ be the *web* of P, that is, the union of all the lines determined by the edges of P (see Figure 3(a)). For any point $A \in \mathbf{E}^2 \backslash W(P)$ and for each $j = 1, 2, \ldots, n$, let B_j be any point on the extension

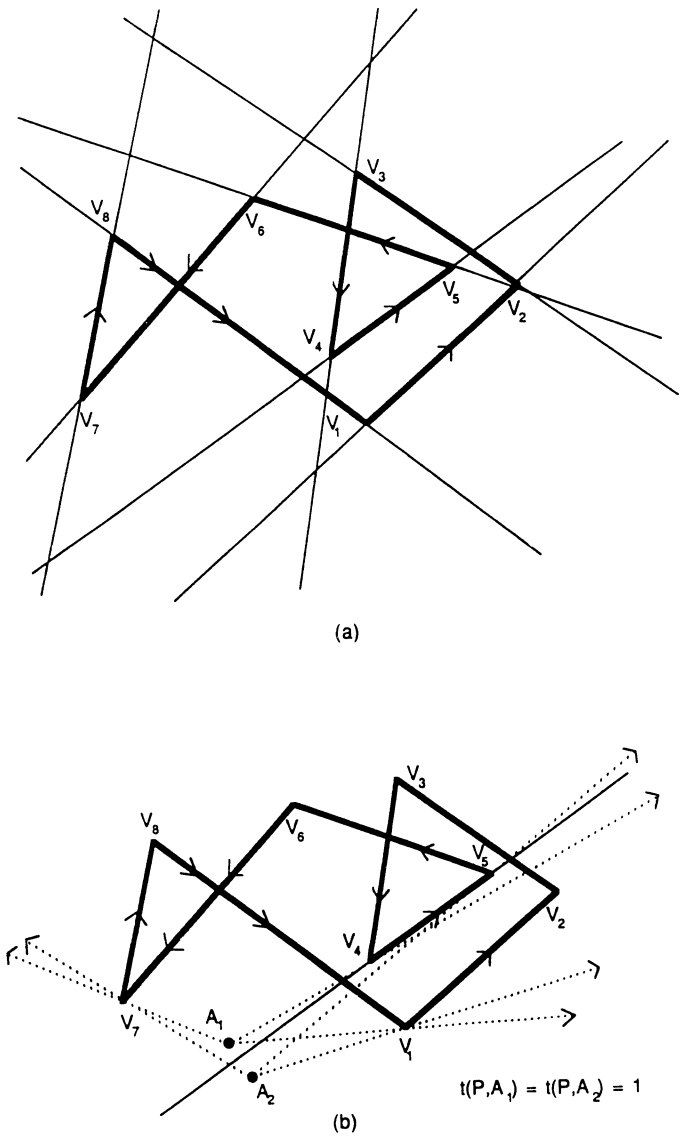


FIGURE 3. (a) The web $W(P)$ of the polygon P is indicated by thin lines. (b) to (e) Determination of the tangent number of P with respect to various points. Concordant tangents are indicated by dotted lines. In (b), the points A_1 and A_2 are on opposite sides of a line (indicated by a thin solid line) of the web $W(A)$.

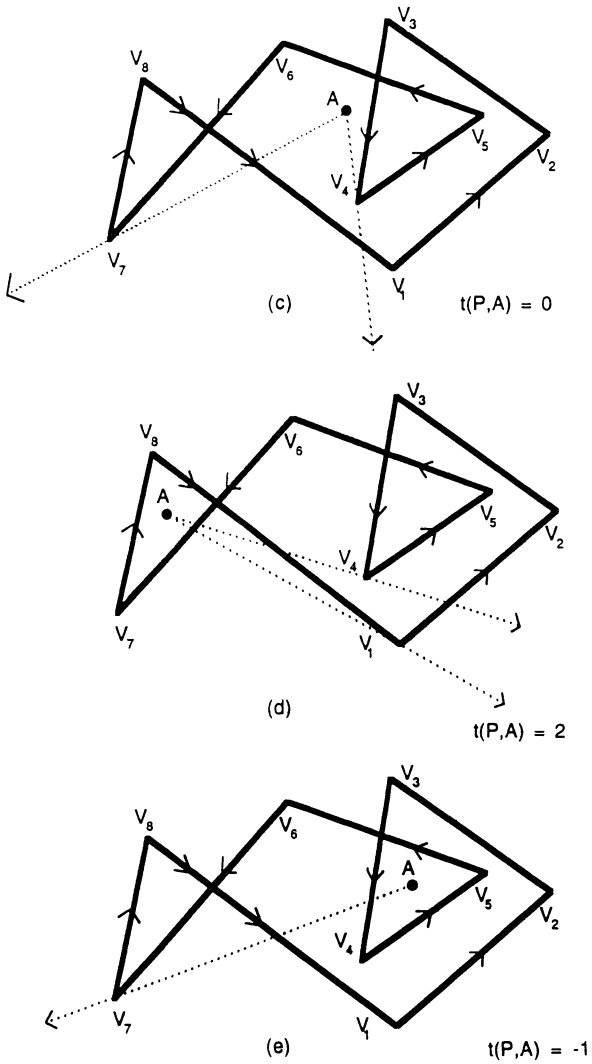


FIGURE 3 (continued).

of the line segment $[A, V_j]$ beyond V_j, and $M_j = [V_j, B_j\rangle$. We define the *tangent number* $t(P, A)$ of P with respect to $A \in \mathbf{E}^2 \backslash W(P)$ by $t(P, A) = \sum_j \tau(M_j)$. Here $\tau(M_j)$ is defined as in Theorem 1, that is, as ± 1 depending on whether M_j is positively or negatively concordantly tangent at V_j, and the summation is over all the vertices of P; see Figure 3(b)–(e) for illustrative examples.

It is clear that $t(P, A)$ is a continuous function of A so long as A does not belong to $W(P)$, and since it takes only integer values it is constant in any connected component of $\mathbf{E}^2 \backslash W(P)$. Further, $t(P, A)$ can be extended by continuity to each cell of P. This follows (see Figure 3(b)) since, as is easily

verified, for two nearby positions of A (say A_1 and A_2) separated by a line of $W(P)$ but not by an edge of P, $t(P, A_1) = t(P, A_2)$. With this extended definition $t(P, A)$ is continuous and therefore constant *on each cell of* P. By taking A very far from P (that is, situated sufficiently far compared to the size of P) in a direction which is not parallel to any edge of P, we see that $t(P, A)$ can be made arbitrarily close to $r(P)$. Since both functions are integer-valued, this implies the following:

Theorem 2. *For every whisker-free polygon P and for each A in the exterior of P we have $r(P) = t(P, A)$.* □

For each vertex V_j of a polygon P let points B_j and C_j be chosen so that each of the triangles $V_{j-1}B_jV_j$ and $V_jC_jV_{j+1}$ is right-angled at V_j and both are positively oriented. Let R_j be the ray $[V_j, B_j\rangle$ such that the triangle $V_{j-1}B_jV_j$ is positively oriented, S_j the ray $[V_j, C_j\rangle$. The *normal wedge* $N(V_j)$ of P at V_j is the union of all the rays with endpoint V_j that lie strictly between R_j and S_j (see Figure 4(a)). It is clear that $N(V_j)$ results from the deflection wedge $D(V_j)$ by a clockwise rotation through $\frac{1}{4}$ (that is, $\pi/2$ radians). We denote by $n(V_j)$ is the signed measure of the angle of $N(V_j)$ at V_j; hence $n(V_j) = d(V_j)$ and so $\sum_j n(V_j) = r(P)$. Let A be any point of $\mathbf{E}^2 \setminus P$ such that $[V_j, A\rangle$ is not in the boundary of any normal wedge $N(V_j)$. We define indices $\nu(V_j, A)$ and $\nu(E_j, A)$ for all vertices V_j and edges E_j of P in the following manner:

$$\nu(V_j, A) = \begin{cases} 0 & \text{if } [V_j, A\rangle \text{ does not belong to the normal wedge } N(V_j) \\ & \text{of } P \text{ at } V_j; \\ 1 & \text{if } [V_j, A\rangle \text{ belongs to } N(V_j) \text{ and } n(V_j) > 0; \\ -1 & \text{if } [V_j, A\rangle \text{ belongs to } N(V_j) \text{ and } n(V_j) < 0. \end{cases}$$

$$\nu(E_j, A) = \begin{cases} 1 & \text{if the foot of the normal from } A \text{ to the line } L_j \text{ determined} \\ & \text{by } E_j \text{ is a point of } E_j, \text{ and the direction of } E_j \text{ as seen} \\ & \text{from } A \text{ is clockwise;} \\ 0 & \text{otherwise.} \end{cases}$$

Then the *normal number* of P with respect to A is the integer defined by

$$n(P, A) = \sum_j \nu(V_j, A) + \sum_j \nu(E_j, A)$$

where the sums are over all the vertices and all the edges of P, see Figure 4(b), (c). In fact, the restriction that A must not be in the boundary of any normal wedge is unnecessary, and the domain of definition of $n(P, A)$ can be extended by continuity so that A may lie anywhere in $\mathbf{E}^2 \setminus P$.

Clearly $n(P, A)$ is a continuous function, and as it takes only integer values it must be constant on each cell of $\mathbf{E}^2 \setminus P$.

Theorem 3. *For each whisker-free polygon P and for each point $A \in \mathbf{E}^2 \setminus P$ we have $n(P, A) = t(P, A)$.*

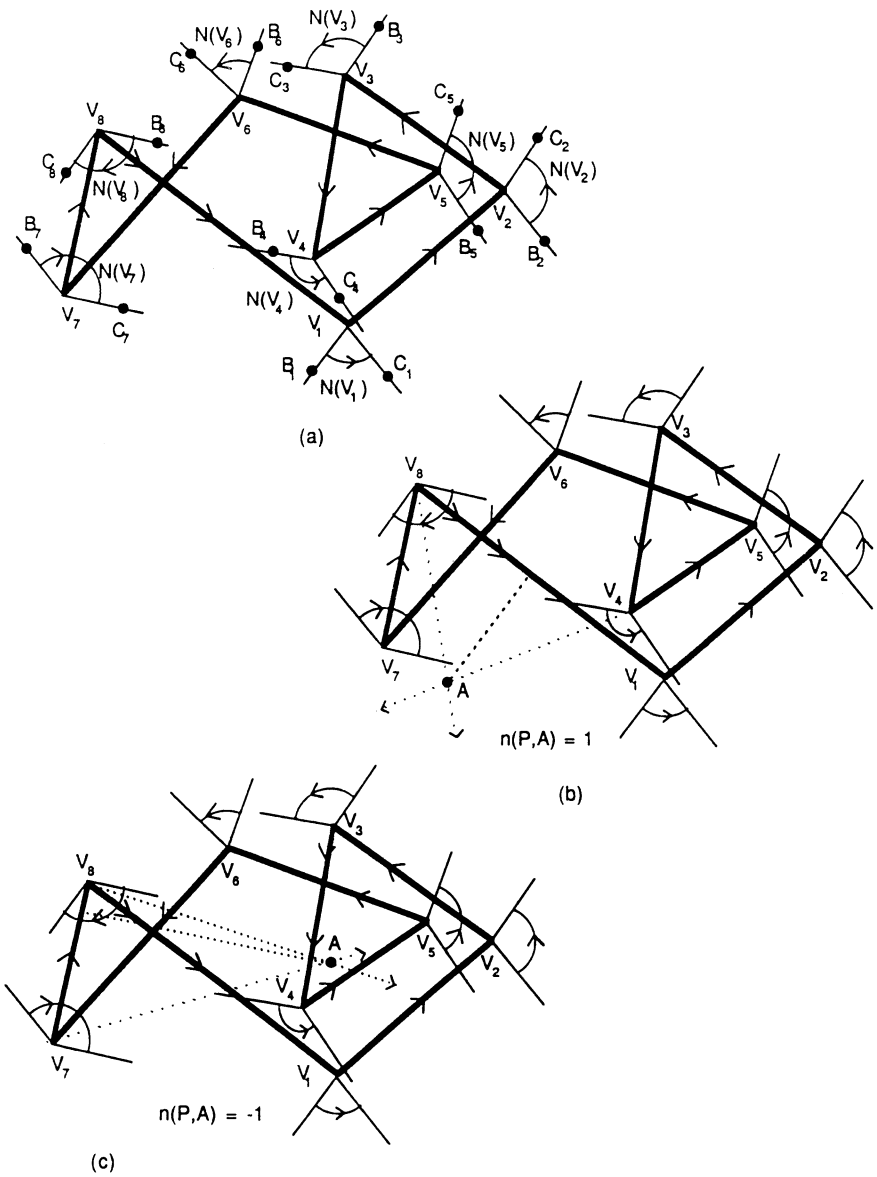


FIGURE 4. (a) The normal wedges $N(V_j)$ of a polygon P. (b) and (c) The determination of the normal numbers $n(P, A)$ for various points. Rays $[V_j, A\rangle$ and normals to edges oriented clockwise as seen from A are indicated by dotted lines.

In particular, this means that for points A belonging to the exterior of P we have $n(P, A) = r(P) = t(P, A)$.

We know of no direct proof of Theorem 3; an indirect proof will follow from the results concerning winding numbers which we shall establish in the next section.

4. The winding number

Let P be a polygon and A any point of $\mathbf{E}^2 \setminus W(P)$. Let $R = [A, B\rangle$ be a ray which contains no vertex of P. For each edge E_j of P we define an index $\psi(R, E_j)$ as follows:

$$\psi(R, E_j) = \begin{cases} 0 & \text{if } R \text{ does not intersect } E_j; \\ 1 & \text{if } E_j \text{ crosses } R \text{ in a counterclockwise direction as viewed from } A; \\ -1 & \text{if } E_j \text{ crosses } R \text{ in a clockwise direction as viewed from } A. \end{cases}$$

Then the *winding number* $w(P, R)$ of P with respect to R is defined by

$$w(P, R) = \sum_j \psi(R, E_j)$$

where summation is over all the edges of P, see Figure 5. Thus $w(P, R)$ is the (signed) difference between the number of edges that cross R in a counterclockwise direction and the number that cross R in a clockwise direction. It is easy to show that $w(P, R)$ takes the same value of all rays R with endpoint A that do not contain a vertex of P. Therefore the value of $w(P, R)$ can more fittingly be denoted by $w(P, A)$. This function of A is continuous and takes only integer values; hence it is constant as A ranges over any cell C of P. Denote its value for A in C by $w(P, C)$. The value of $w(P, C)$ is often called the *density* of the cell C with respect to P. Clearly, if A lies in the exterior of P then $w(P, A) = 0$. The function $w(P, A)$ is useful in the calculation of the areas and other properties of polygons. An alternative definition of $w(P, A)$ equivalent to the one above is given by the equation $w(P, A) = \sum_j a(E_j)$, where $a(E_j)$ is the angle subtended by the edge E_j of P at A, taken with the appropriate sign. The following basic result connects the rotation number of a polygon P, and the tangent and winding numbers of P with respect to a point A.

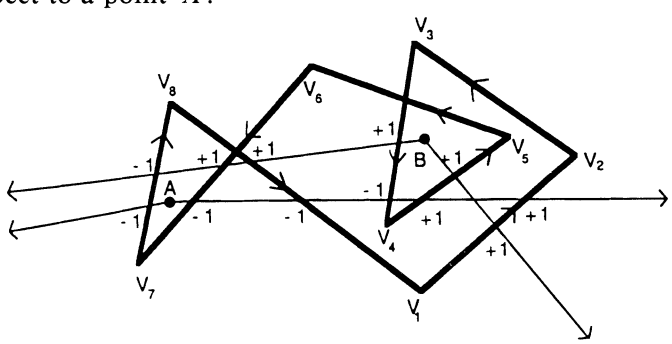


FIGURE 5. For several rays issuing from A and B, the values of ψ are indicated near each intersection with the polygon P. In this example we have $w(P, A) = -1$, $w(P, B) = 2$.

Theorem 4. *For every whisker-free polygon P and every point A not belonging to P we have*

$$w(P, A) + t(P, A) = r(P). \tag{1}$$

Proof. We shall first show the left side of (1) is constant for A not in P. To do this, refer to Figure 6 for the difference between the values of $w(P, A_1)$ and $w(P, A_2)$ on the one hand, and $t(P, A_1)$ and $t(P, A_2)$ on the other, when A_1 and A_2 lie in adjacent cells of P. We have to consider the four cases shown, and in each it will be observed that $w(P, A) + t(P, A)$ takes the same value in all cases. Hence this quantity is constant in $\mathbf{E}^2 \setminus P$. Since, for sufficiently distant points, $w(P, A) = 0$ and $t(P, A) = r(P)$, the relation (1) is established. □

Theorem 5. *For every whisker-free polygon P and every point A not belonging to P we have*

$$w(P, A) + n(P, A) = r(P). \quad \square$$

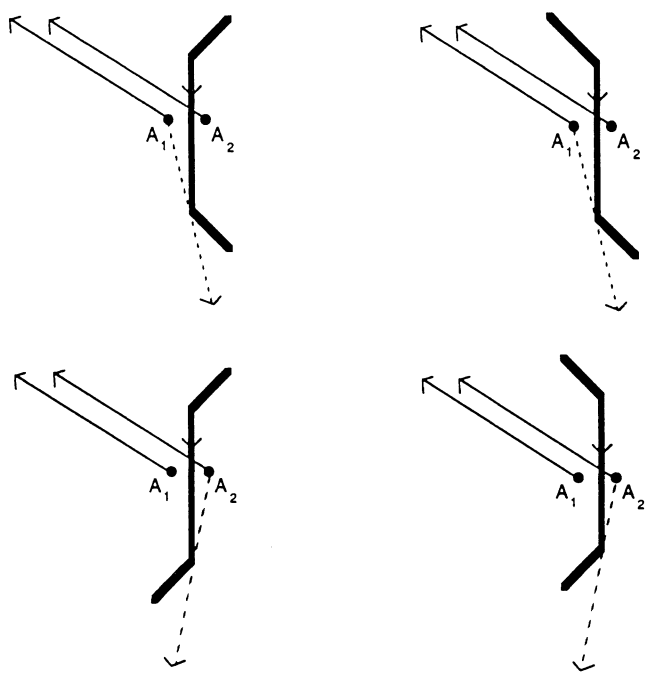


FIGURE 6. The main step in the proof that $t(P, A) + w(P, A)$ is constant. The heavily drawn segments represent part of the polygon P. The points A_1 and A_2 are close enough to the edge that separates them to have analogous positions relative to all the vertices and edges of P that are not shown. In all the cases illustrated we have $w(P, A_2) - w(P, A_1) = 1$, $t(P, A_2) = t(P, A_1) = -1$.

This is proved in an exactly analogous manner to Theorem 4. For two positions of A in adjacent cells of P we calculate $w(P, A) + n(P, A)$ and notice that it takes the same value in each case. Hence this quantity is constant in $\mathbf{E}^2 \setminus P$. Moreover, if A is sufficiently distant from P, and is such that no line through A that meets an edge of P is perpendicular to that edge, then clearly $w(P, A) = 0$ and $n(P, A) = r(P)$. □

From Theorems 4 and 5 immediately follows the validity of Theorem 3.

5. An alternative definition of the rotation number

In the case of ordinary polygons, there is an alternative definition of the rotation number. Let P now denote an ordinary polygon, and A be any point of P which belongs to a single edge. Starting from A, we follow P and note, each time we reach *for the first time* any self-intersection point D of P, whether the crossing edge traverses from left to right, or from right to left. In the former case we define the *crossing index* $c(D, A)$ relative to A as $+1$, in the latter as -1. See the example in Figure 7(a). Then we have the following theorem.

Theorem 6. *For every ordinary polygon* P

$$r(P) = w(C_1, P) + w(C_2, P) + \sum_{D \in P} c(D, A), \tag{2}$$

where $w(C_1, P)$ *and* $w(C_2, P)$ *are the winding numbers of the two cells of* P *adjacent to* A, *and the summation extends over all the self-intersection points* D *of* P.

Proof. The theorem is clearly true if P has no self-intersections. Next, we note that the right-hand side of (2) is independent of A, since on crossing a self-intersection point of P the changes in the winding numbers of the adjacent cells are exactly compensated by the changes in the sum of the indices $c(D, A)$. Now let A be any self-intersection point of P with the property that the part of P from one crossing at M to the revisit of M is a simple polygon P^*; denote the remaining part of P by P^{**}. Clearly, P^{**} has fewer self-intersections than P and we can use induction. Let A be a point of P^{**} just preceding M, so that the part of P from A to M contains no self-intersections of P. Since the contributions to $\sum_D c(D, A)$ which arise from self-intersection points D of P which belong to P^* cancel each other out, using relation (2) for P^{**} (which is permissible by induction) we have

$$\begin{aligned} &w(C_1, P) + w(C_2, P) + \sum_{D \in P} c(D, A) \\ &\quad = w(C_1, P^{**}) + w(C_2, P^{**}) + r(P^*) + \sum_{D \in P^{**}} c(D, A) \\ &\quad = r(P^*) + r(P^{**}) = r(P). \end{aligned}$$

See Figure 7(b), (c), (d), in which this proof is illustrated for several possible positions of A and P^*. □

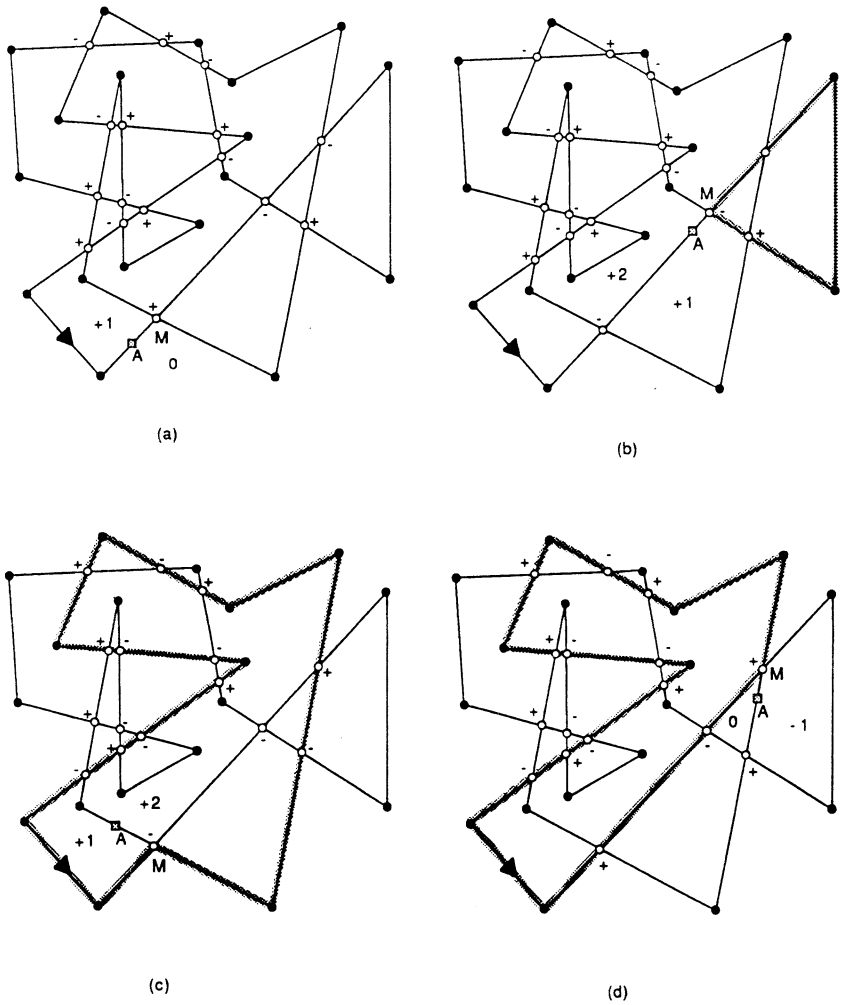


FIGURE 7. (a) The determination of the rotation number $r(P)$ by the densities of the cells adjacent to A, and the crossing indexes relative to A. (To avoid clutter, + and − are used for $+1$ and -1.) (b), (c), (d) Illustrations of the construction used in the proof of Theorem 6. In each case, the polygon P^* is indicated by wide, shaded lines.

6. THE MODIFIED POLARITY

First let us recall the usual concept of polarity of a polygon P with respect to a circle C. Without loss of generality (by a change of scale and applying a suitable translation) we may suppose C is the circle of unit radius centered at the origin O, which is the initial point for all the position vectors. *Polarity* is the correspondence Π which maps a point T (other than O) with position vector t, onto the line (not through O) $\langle x, t\rangle = 1$ which we denote by $\Pi(T)$.

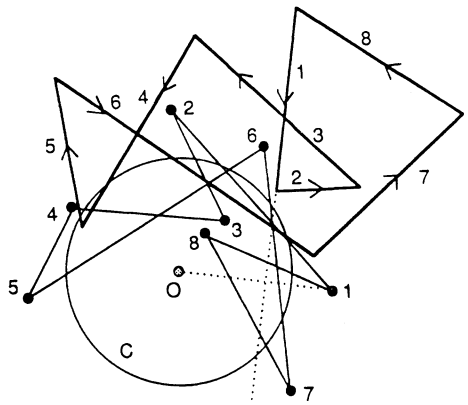


FIGURE 8. An illustration of the construction of the polygon $\Pi(P)$ (thin solid lines) dual of the polygon P (heavy lines) by the polarity Π determined by the circle C. To avoid clutter, we write j instead of L_j and $\Pi(L_j)$.

Geometrically $\Pi(T)$ is perpendicular to the vector t, and its distance from O is the reciprocal of the length of t. The images under polarity of all the points on a line L that does not pass through O is a pencil of lines, that is, the set of all (except one) lines passing through a point, which we denote by $\Pi(L)$; the one exception is the line joining $\Pi(L)$ to O. We may say that $\Pi(L)$ corresponds to L under the polarity. Thus polarity maps points onto lines and lines onto points, preserves incidences of points and lines, and it is involutory in that $\Pi(\Pi(L)) = L$ for any line L not through O, and $\Pi(\Pi(T)) = T$ for any point $T \neq O$.

We shall denote by L_j the line containing the edge E_j of P, oriented in the same direction as E_j. It is well known that polarity can be used to establish a *duality* between pairs of polygons which are such that no edge of either lies on a line that passes through O. To be precise, the polygon $\Pi(P)$ polar to P is determined by its vertices, namely $\Pi(L_1)$, $\Pi(L_2)$, ..., $\Pi(L_n)$, and these, in turn, determine the edges of $\Pi(P)$; see Figure 8 for an illustrative example. The same result is obtained by considering $\Pi(P)$ to be determined by its lines $\Pi(V_1)$, $\Pi(V_2)$, ..., $\Pi(V_n)$, which are the polars of the vertices of P.

This polarity is not completely satisfactory for our purposes, so we define a *modified polarity* as follows. Let P be a polygon such that no line L_j passes through O. For each edge E of P define a point $\Gamma(L)$ (where L is the line containing E) in the following way:

$$\Gamma(L) = \begin{cases} \Pi(L) \text{ if } O \text{ is on the } \textit{left} \text{ of } L \text{ (when looking along } L \text{ in the direction} \\ \text{of its orientation);} \\ -\Pi(L) \text{ if } O \text{ is on the } \textit{right} \text{ of } L. \end{cases}$$

The edges of $\Gamma(P)$ are naturally determined by its vertices $\Gamma(L_1)$, $\Gamma(L_2)$, ..., $\Gamma(L_n)$, and $\Gamma(P)$ is called the *modified polar* of P (see Figure 9). If O

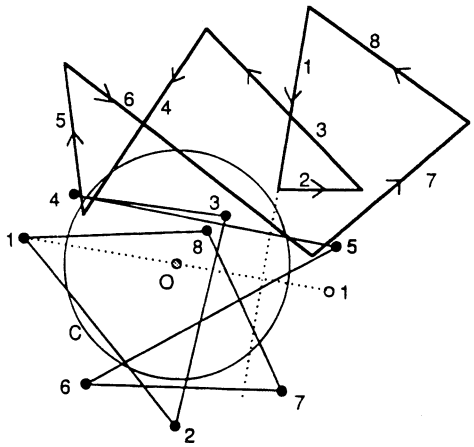


FIGURE 9. An illustration of the construction of the modified polar $\Gamma(P)$ for the polygon P shown in Figure 8.

lies on the left of all the edges of P (as will happen, for example, if O lies in the interior of a convex polygon oriented in a counterclockwise direction), then clearly $\Gamma(P) = \Pi(P)$. In this case $\Gamma(\Gamma(P)) = P$, but in general Γ *is not involutory* — that is $\Gamma(\Gamma(P)) \neq P$. However $\Gamma(P)$ has the following interesting property:

Theorem 7. *If P is a whisker-free polygon such that no line determined by an edge of P passes through the origin O, then $r(P) = w(\Gamma(P), O)$.*

Proof. We recall from §3 that $r(P) = \sum_j n(V_j)$, where $n(V_j)$ is the measure of the normal cone at the vertex V_j of P. As can be seen from the construction in Figure 9, in each possible situation the value (including the sign) of $n(V_j)$ equals to the measure of the angle $a([\Gamma(L_{j-1}), \Gamma(L_j)])$ subtended at O by the edge $[\Gamma(L_{j-1}), \Gamma(L_j)]$ of $\Gamma(P)$. Therefore, summing up for all j, we get the required equality. □

7. SIMPLE CLOSED CURVES

Most of the results stated above for polygons have analogues or generalizations to closed, piecewise smooth curves. The changes needed are obvious. For example, in computing the rotation number of such a curve one has to take into account not only the contributions from the vertices (points at which the left and right tangents fail to coincide) but also those arising from the curvature of the smooth arcs. However, the curve must not have cusps; for such curves various complications arise and in some cases the interpretation is not clear. In fact it was to avoid such complications that, in our treatment of polygons, we restricted attention to those that are whisker-free; whiskers are just the polygonal version of cusps. For curves Theorem 1 is valid in the following form. It can be proved by straightforward approximation arguments.

Theorem 8. *Given any ray L, the rotation number of a piecewise smooth, cusp-free closed curve C is the number of rays parallel to L that are positively concordantly tangent to C less the number of those that are negatively concordantly tangent.*

The other results of §§3, 4 and 5 have similar analogues. However, we know of no result analogous to Theorem 7 which holds for curves other than polygons.

8. Remarks

If P is a regular polygon with center O, then it is easily seen that $r(P) = w(P, O)$; equality also holds if the symmetries of P act transitively on its vertices, or on its edges, with O as the center of symmetry, and if all the edges are directed clockwise (or all counterclockwise) as seen from O. Since these kinds of polygons were the ones most frequently investigated, and served as models for the classical theory, the equality for them of the winding number (with respect to O) and the rotation number probably explains why the latter concept has been neglected. (A vertex-transitive octagon P for which $r(P) \neq w(P, O)$ is shown in Figure 10.) In fact, Wiener [1865] explicitly defined what we call $r(P)$; he used the phrase "die Art von P", which may be translated "the kind of P", for our $r(P)$. However, in what seems to have been a misguided effort to belittle Wiener, Hess [1874] appropriated the word "Art" for a different concept (which, though related to $r(P)$, is needlessly complicated, besides being useless and misleading; it is rather revealing that in all situations in which Hess's concept can be used, its value coincides with the one given by Wiener). Brückner [1900] adopted Hess' approach and since Brückner's book became the basic reference for the theory of polygons, it is not surprising that Wiener's work

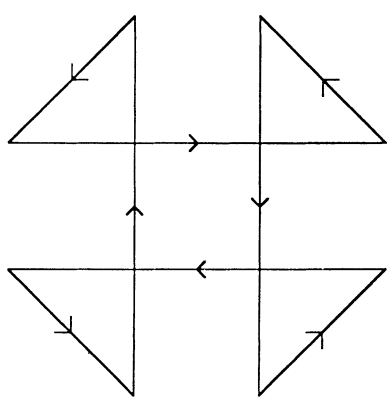


FIGURE 10. A vertex-transitive octagon P for which the rotation number $r(P) = 3$ differs from the winding number with respect to the center, $w(P, O) = 1$. Some edges of P are directed clockwise, and some counterclockwise, as seen from the center O.

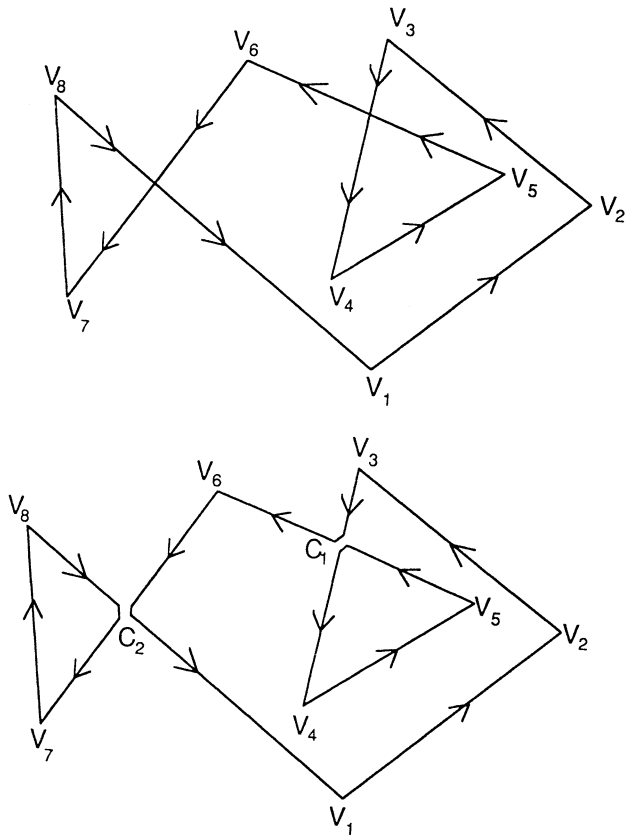


FIGURE 11. The simultaneous "switching" at all self-intersection points of an ordinary polygon P (shown in (a)) yields a noncrossing family of simple polygons (shown in (b)).

was almost forgotten. As a result, many of the useful ideas Wiener introduced were not developed. There are several indications that Brückner considered himself to be the intellectual heir of Hess, and, as such, took a similar attitude to Wiener's work. From the standpoint of the present, we venture to suggest that Wiener's treatment of polygons was superior to that of the others, and it is regrettable that Hess and Brückner tried (and to some extent succeeded in) suppressing his work. There are only a few later references to Wiener's definition of the rotation number of a polygon; the most recent one known to us is in Steinitz [1922], which was actually written in 1916. However, even this mention did not prevent Wiener's work from sinking into long oblivion; in fact, many of Steinitz's own remarkable ideas remained unnoticed for many years.

One of the techniques first used by Wiener [1865] consists in splitting off a simple loop from a self-intersecting ordinary polygon by "switching" the two outgoing parts of edges at a suitable self-intersection; in the proof of Theorem 6 we employed this "switching" at the point M. Remarkably, Wiener seems not

to have had the idea of "switching" *simultaneously* at all self-intersection points of P (see Figure 11). If this is done, we immediately obtain the following reduction theorem which yields alternative definitions of the winding, tangent and rotation numbers of a polygon:

Theorem 9. (i) *The "switching" at all self-intersection points of an ordinary polygon P yields a uniquely defined family $\mathbf{S}(P)$ of simple and mutually noncrossing polygons.*

(ii) *For each point $A \notin P$, the winding number $w(P, A)$ equals the difference between the numbers of positively and of negatively oriented polygons of $\mathbf{S}(P)$ that contain A in their interior.*

(iii) *For each point $A \notin P$, the tangent number $t(P, A)$ equals the difference between the numbers of positively and of negatively oriented polygons of $\mathbf{S}(P)$ that contain A in their exterior.*

(iv) *The rotation number $r(P)$ equals the difference between the numbers of positively and of negatively oriented polygons of $\mathbf{S}(P)$.*

Parts (i) and (ii) of this result are due (in rather cryptic form) to C. G. J. Jacobi (1804–1851). Their exact date is uncertain, and they were published only posthumously, by O. Hermes (see Jacobi [1866]). This, and a short note by Hermes [1866] which extends parts (i) and (ii) (in a noncanonical way) to certain nonordinary polygons, seem to have been largely forgotten; the only mention we were able to find is in Steinitz [1922]. Part (iv) of Theorem 9 appears in Steinitz [1922, p. 8], in a slightly stronger formulation. Clearly, the results of Theorem 9 extend to piecewise smooth closed cusp-free curves without triple points or self-osculations.

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