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N-BODY SCHRÖDINGER OPERATORS WITH FINITELY MANY BOUND STATES

W. D. EVANS AND ROGER T. LEWIS

ABSTRACT. In this paper we consider a class of second-order elliptic operators which includes atomic-type N-body operators for $N > 2$. Our concern is the problem of predicting the existence of only a finite number of bound states corresponding to eigenvalues below the essential spectrum. We obtain a criterion which is natural for the problem and easy to apply as is demonstrated with various examples. While the criterion applies to general second-order elliptic operators, sharp results are obtained when the Hamiltonian of an atom with an infinitely heavy nucleus of charge Z and N electrons of charge 1 and mass $\frac{1}{2}$ is considered.

1. INTRODUCTION

The eigenvalues below the essential spectrum of a Schrödinger operator have eigenfunctions which are called bound states in nonrelativistic quantum mechanics since a particle in such a state cannot leave the system without additional energy. The operators considered here will be bounded below. Hence, below the least point Σ of the essential spectrum, the spectrum consists of eigenvalues of finite multiplicity and there is either a finite number of them or a sequence which clusters at Σ.

Consider the formal Schrödinger operator $P = -\Delta + q$ on $\mathbb{R}^n$ where q is a real-valued function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $q_- := \max(-q, 0) \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for some $p > n/2$. If

$$\inf\{(P\varphi, \varphi) : \varphi \in C_0^\infty(\mathbb{R}^n),\ \|\varphi\| = 1\} > -\infty,$$

where $(\cdot, \cdot)$, $\|\cdot\|$ are the $L^2(\mathbb{R}^n)$ inner product and norm, then P admits a unique selfadjoint realization H in $L^2(\mathbb{R}^n)$ which is bounded below: see Kato [14] for this result and Agmon [1, Chapter 3] for an up-to-date treatment. If q is bounded below in a neighborhood of infinity it is known that the behavior of

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$q(x)$ for large values of x determines whether the number of bound states of H is finite or infinite. Most of the work done on this problem has dealt with this special case: the literature is extensive—see [10, 11, 24] and the references cited therein. The assumption that q is bounded below near ∞ is too severe as regards atomic type Schrödinger operators since it allows only for 2-body potentials. To illustrate, consider the Hamiltonian of an atom with an infinitely heavy nucleus of charge Z and N electrons of charge 1 and mass $\frac{1}{2}$.

$$P_N := \sum_{i=1}^{N} \left[-\Delta_i - \frac{Z}{|x^i|} \right] + \sum_{1 \le i < j \le N} \frac{1}{|x^i - x^j|}.$$

Here $x^i \in \mathbb{R}^\nu$ is the coordinate of the ith electron and Δ_i denotes the Laplacian in $\mathbb{R}^\nu$ with respect to the variable x^i. The $(N+1)$-body Schrödinger operator P_N thus acts in $\mathbb{R}^{\nu N}$ and has potential

$$q(x) = -Z \sum_{i=1}^{N} \frac{1}{|x^i|} + \sum_{1 \le i < j \le N} \frac{1}{|x^i - x^j|}$$

which is bounded below at infinity only if $N = 1$ (when $q(x) = -Z/|x|$). However, for $N = 2$ and $Z < 1$, Uchiyama [30] proved that H_2, the selfadjoint realization of P_2, has only a finite number of bound states. For $N \ge 2$ Zhislin proved in [37] that H_N has a finite number of bound states if $Z < N - 1$, after proving earlier in [36] that the number is infinite if $Z > N - 1$. Subsequently (see [31, 33, 35]) the case $Z = N - 1$ was settled, establishing that H_N has a finite number of bound states if and only if $Z \le N - 1$.

In this paper we consider a class of second-order elliptic operators which includes multiparticle Schrödinger operators like P_N. We deal specifically with the problem of predicting the existence of only a finite number of bound states and obtain a criterion which is natural for the problem and easy to apply, as we demonstrate in a variety of examples. When restricted to the the operator P_N our result easily recovers the criterion of Uchiyama [30] and Zhislin [36]; namely, that $Z < N-1$. Also the result, due to Zhislin and colleagues, that there is only a finite number of bound states if the potential is short range is a simple consequence of our result. We follow the general spirit of Agmon's excellent lecture notes [1] where the main interset is on bounds for eigenfunctions of N-body Schrödinger operators. Our methods are geometric in nature as were those of Uchiyama and Zhislin. The use of geometric methods in the study of N-body systems dates back to early work of Zhislin [35] but their systematic use is attributable to Enss [9], Deift and Simon [5], Morgan [17] and Simon [28]. Furthermore, the techniques of Sigal in [25, 26, 27] have also influenced our approach. We refer the reader to Chapter 3 of Cycon, Froese, Kirsh and Simon [4] for extra details and references: we express our gratitude to Professor Simon for making available to us an early preprint of this chapter. Also we wish to thank our colleague Yoshimi Saitō for many helpful discussions concerning this problem.

We shall stick closely to the notation used by Agmon in [1]. For any open set Ω in $\mathbb{R}^n$, $H^1(\Omega)$ will denote the Sobolev space of all functions $u \in L^2(\Omega)$ such that the first distributional derivatives are in $L^2(\Omega)$; $H^1(\Omega)$ is a Hilbert space with norm defined by

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega (|\nabla u|^2 + |u|^2)\, dx .$$

We shall also need the Kato space $M(\mathbb{R}^n)$ defined as follows. Let

$$g(x, y) = \begin{cases} |x-y|^{2-n} & \text{if } n \geq 3, \\ |\ln|x-y|| & \text{if } n = 2, \\ 1 & \text{if } n = 1, \end{cases}$$

for $x, y \in \mathbb{R}^n$. Then, $u \in M(\mathbb{R}^n)$ if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\lim_{r \to 0} \int_{B(x,r)} |u(y)| g(x, y)\, dy = 0$$

uniformly in $x \in \mathbb{R}^n$, where $B(x, r)$ is the ball with center x, radius r, in $\mathbb{R}^n$. We shall denote by $M_{\mathrm{loc}}(\mathbb{R}^n)$ the space of functions u which are such that for every $x \in \mathbb{R}^n$ there exists a neighborhood U of x such that $\chi_U u \in M(\mathbb{R}^n)$, where χ_U is the characteristic function of U.

2. The general case

Let

$$P := -\sum_{i,j=1}^{n} \partial_i a^{ij} \partial_j + q \tag{1}$$

in $\mathbb{R}^n$, where ∂_i denotes $\frac{\partial}{\partial x_i}$ and

(i) each a^{ij} is a bounded, continuous, real-valued function on $\mathbb{R}^n$;
(ii) the matrix $A(x) = (a^{ij}(x))$ is symmetric and its smallest eigenvalue $\mu(x)$ is a positive continuous function on $\mathbb{R}^n$;
(iii) $q \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$;
(iv) $q_- := \max(-q, 0) \in M_{\mathrm{loc}}(\mathbb{R}^n)$.

The assumption $q_- \in M_{\mathrm{loc}}(\mathbb{R}^n)$ in (iv) ensures that the following result of Schechter [23, Theorem 7.3, p. 138] holds: for any $\varepsilon > 0$ and any compact subset K of $\mathbb{R}^n$ there exists a constant $C(\varepsilon, K)$ such that

$$\int_K q_- |\varphi|^2\, dx \leq \varepsilon \int_K |\nabla \varphi|^2\, dx + C(\varepsilon, K) \int_K |\varphi|^2\, dx \tag{iv$'$}$$

for all $\varphi \in C_0^\infty(K)$. This in fact is all we require and it is not necessary that $q_- \in M_{\mathrm{loc}}(\mathbb{R}^n)$. Indeed, in §4, Example 6, we deal with short range potentials (when $q_- \in L^{n/2}(\mathbb{R}^n)$) which satisfy the above inequality but are not in $M_{\mathrm{loc}}(\mathbb{R}^n)$. We assume (iv) in order that we may readily use some results of Agmon [1] when we discuss the N-body problem in §3.

Following Agmon [1], we shall use the notation

$$\langle \nabla_A \varphi , \nabla_A \psi \rangle := \sum_{i,j=1}^{n} a^{ij} \partial_j \varphi \partial_i \overline{\psi}$$

and

$$|\nabla_A \varphi|^2 := \langle \nabla_A \varphi , \nabla_A \varphi \rangle .$$

Associated with P is the sesquilinear form ρ defined by

$$\rho[\varphi , \psi] := \int_{\mathbb{R}^n} \{ \langle \nabla_A \varphi , \nabla_A \psi \rangle + q \varphi \overline{\psi} \} \, dx$$

for $\varphi , \psi \in C_0^\infty(\mathbb{R}^n)$, and we write $\rho[\varphi , \varphi]$ as $\rho[\varphi]$. We shall denote the usual norm and inner product on $L^2(\mathbb{R}^n)$ by $\| \cdot \|$ and $(\cdot , \cdot)$ respectively. The Euclidean norm and inner product on $\mathbb{R}^n$ will be denoted by $| \cdot |$ and $\langle \cdot , \cdot \rangle$ respectively.

Proposition 1. *Let* (i)–(iv) *be satisfied and suppose that*

$$\Sigma(P) := \sup_{K \text{ compact}} [\inf\{\rho[\varphi] : \varphi \in C_0^\infty(\mathbb{R}^n \setminus K) \text{ and } \|\varphi\| = 1\}] > -\infty . \tag{2}$$

Then ρ is a densely defined symmetric form which is bounded below and closable in $L^2(\mathbb{R}^n)$.

The proof of Proposition 1 is given in Appendix 1 but it is essentially Theorem 3.2 of Agmon [1]. If α is a positive constant such that

$$\rho[\varphi] + \alpha \|\varphi\|^2 \geq \|\varphi\|^2 , \qquad \varphi \in C_0^\infty(\mathbb{R}^n) , \tag{3}$$

then $(\rho[\cdot] + \alpha \| \cdot \|^2)^{1/2}$ is a norm on $C_0^\infty(\mathbb{R}^n)$ and these norms are equivalent for all such α. If V denotes the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to one of these norms then it is continuously embedded in $L^2(\mathbb{R}^n)$ and when identified with a subspace of $L^2(\mathbb{R}^n)$ it is the domain of the closure $\tilde{\rho}$ of ρ in $L^2(\mathbb{R}^n)$. By the First Representation Theorem [15, p. 322] there is a selfadjoint operator H which is bounded below and is uniquely determined by the properties

(4) (a) $\mathscr{D}(H) = \{u : u \in V , Pu \in L^2(\mathbb{R}^n)\}$; $Hu = Pu$;

(b) $\tilde{\rho}[u , v] = (Hu , v)$, for every $u \in \mathscr{D}(H)$ and $v \in V$.

This operator H is the same as that determined by Agmon in [1,Theorem 3.2], where it is also proved that

$$\mathscr{D}(H) = \{u : u \in L^2(\mathbb{R}^n) \cap H^1_{\text{loc}}(\mathbb{R}^n) , \ qu \in L^1_{\text{loc}}(\mathbb{R}^n) , \ Pu \in L^2(\mathbb{R}^n)\} , \tag{5}$$

$$\inf \sigma(H) = \Lambda(P) := \inf\{\rho[\varphi] : \varphi \in C_0^\infty(\mathbb{R}^n) , \ \|\varphi\| = 1\} , \tag{6}$$

$$\inf \sigma_e(H) = \Sigma(P) , \tag{7}$$

where $\sigma(H)$ and $\sigma_e(H)$ denote the spectrum and essential spectrum of H respectively and $\Sigma(P)$ is defined in (2). Note that in (4)(a) and (5), Pu is to

be interpreted in the distributional sense. The original version of (7), under a stronger hypothesis, is due to Persson [20].

To proceed, we need other characterizations of $\Sigma(P)$ obtained by Agmon in [1]. First we require the following definitions.

Definition 2. For $\omega \in S^{n-1} = \{\omega \in \mathbb{R}^n : |\omega| = 1\}$, $\delta \in (0, \pi)$ and $R > 0$ define the truncated cone

$$\Gamma(\omega : \delta, R) := \{x \in \mathbb{R}^n : \langle x, \omega \rangle > |x| \cos\delta, \ |x| > R\}.$$

Let

$$\Sigma(\omega : \delta, R) := \inf\{\rho[\varphi] : \varphi \in C_0^\infty(\Gamma(\omega : \delta, R)), \ \|\varphi\| = 1\},$$

and

$$K(\omega : P) := \lim_{\delta \to 0} \lim_{R \to \infty} \Sigma(\omega : \delta, R).$$

Definition 3. For $x \in \mathbb{R}^n$, $R > 0$, and $B(x, R) := \{y \in \mathbb{R}^n : |y - x| < R\}$, let

$$\Lambda_R(x : \rho) := \inf\{\rho[\varphi] : \varphi \in C_0^\infty(B(x, R)), \ \|\varphi\| = 1\}.$$

The results of Agmon [1] that we shall need are contained in

Proposition 4.

(i) *$K(\cdot : P)$ is lower semicontinuous on S^{n-1} and*

$$\Sigma(P) = \min\{K(\omega : P) : \omega \in S^{n-1}\}. \tag{8}$$

(ii) *$\Lambda_R(x : P)$ is continuous in (x, R) on $\mathbb{R}^n \times \mathbb{R}^1_+$ and*

$$\Sigma(P) = \lim_{R \to \infty} \liminf_{|x| \to \infty} \Lambda_R(x : P). \tag{9}$$

Part (i) of Proposition 4 and (7) motivate the first of the four parts of our basic hypothesis $\mathscr{H}$. Actually this part will not be needed until Corollary 11 and our main Theorem 12.

$\mathscr{H}(1)$ Suppose that $K(\omega : P)$ assumes its minimum at only a finite number of points $\{\omega_k : k = 1, 2, \ldots, m\} \subset S^{n-1}$, i.e. for every $\omega \in S^{n-1} \setminus \{\omega_k : k = 1, 2, \ldots, m\}$ and $k = 1, 2, \ldots, m$

$$\Sigma(P) = K(\omega_k : P) < K(\omega : P).$$

A form of the main theorem, Theorem 12 below, does hold when $K(\omega : P)$ assumes its minimum on a proper subset of S^{n-1} with a sufficiently smooth boundary. The authors intend to study the ramifications of that fact in later work. For N-body atomic-type systems, it is reasonable to require that $K(\omega : P_N)$ assume its minimum at only a finite number of points ω_k on S^{n-1}; see Agmon [1] and the discussion in §3 below. In this case the equivalent hypothesis for a finite number of bound states is that $\Sigma_3 > \Sigma(P_N)$, where Σ_3 denotes the infimum of the spectra of all possible "three-cluster decompositions" of the $(N+1)$-body Hamiltonians, the latter being $(N-1)$-body Hamiltonians. The

case in which $\Sigma_3 = \Sigma(P_N)$ can be somewhat pathological resulting in the Efimov effect (see [7, 8]). We shall elaborate on these notions in §3 but for a comprehensive account, including full definitions and an up-to-date list of references we refer the reader to [4, 21, 22].

Geometric methods have proved to be fundamental in the study of qualitative spectral analysis of many-body Hamiltonians. A key ingredient is the so-called IMS localization formula associated with a partition of unity defined in

Definition 5. A family of functions $\{J_\beta\}_{\beta\in B}$ indexed by a set B is called a partition of unity if

(i) $0 \le J_\beta(x) \le 1$, $x \in \mathbb{R}^n$,

(ii) $\sum_{\beta\in B} J_\beta^2(x) = 1$, $x \in \mathbb{R}^n$,

(iii) $\{J_\beta\}_{\beta\in B}$ is locally finite, i.e. for any compact set K in $\mathbb{R}^n$ we have $J_\beta|_K \equiv 0$ for all but a finite number of $\beta \in B$,

(iv) $J_\beta \in C^\infty(\mathbb{R}^n)$ for all $\beta \in B$,

(v) $\sup_{x\in\mathbb{R}^n}\left(\sum_{\beta\in B} |\nabla J_\beta(x)|^2\right) < \infty$.

Lemma 6 (The IMS localization formula). *If $\{J_\beta\}_{\beta\in B}$ is a partition of unity and Ω is an open subset of $\mathbb{R}^n$, then for any $\varphi \in C_0^\infty(\mathbb{R}^n)$,*

$$\int_\Omega \left[|\nabla_A\varphi|^2 + q|\varphi|^2\right] dx = \sum_{\beta\in B}\int_\Omega \left[|\nabla_A(J_\beta\varphi)|^2 + q|J_\beta\varphi|^2 - |\nabla_A J_\beta|^2|\varphi|^2\right] dx.$$

Proof. An elementary calculation gives

$$|\nabla_A(J_\beta\varphi)|^2 = J_\beta^2|\nabla_A\varphi|^2 + |\nabla_A J_\beta|^2|\varphi|^2 + \tfrac{1}{2}\langle\nabla_A J_\beta^2, \nabla_A|\varphi|^2\rangle$$

on using the fact that the matrix $A(x)$ is symmetric. On substituting in the right-hand side of the asserted identity, and recalling that $\sum_\beta J_\beta^2 = 1$ the result follows.

The IMS localization formula is credited to Ismagilov [13], Morgan [17] and Morgan and Simon [18]; Sigal [25] recognized its importance to the problem studied here. It is usually stated in terms of operators (see [4, p. 28]) but we need it in the above form.

Next, we introduce a simple partition of unity which plays a leading role in what follows. Although the cluster decompositions used in partitions of unity in related papers are not mentioned specifically here or in the applications to the $(N+1)$-body Hamiltonians in §3, their existence does underlie our construction. This will be made clear in §3.

Lemma 7. *Let $\{\Gamma(\omega_k : \delta_k, \frac{1}{2})\}_{k=1}^m$ be mutually disjoint truncated cones associated with points $\{\omega_k\}_{k=1}^m$ on S^{n-1}. Then there exists a partition of unity $\{J_0, J_1, J_2\}$ which satisfies the following conditions:*

(i) $\operatorname{supp} J_0 \subset B(0, 1)$;

(ii) $\operatorname{supp} J_1 \subset \bigcup_{k=1}^m \Gamma(\omega_k : \delta, \frac{1}{2})$;

(iii) $\operatorname{supp} J_2 \subset \mathbb{R}^n \setminus \bigcup_{k=1}^m \Gamma(\omega_k : \delta_k/2, \frac{1}{2}) \cup B(0, \frac{1}{2})$;

(iv) *J_1 and J_2 are homogeneous of degree zero in* $\mathbb{R}^n \setminus B(0, 1)$;

(v) *for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$|\nabla J_1(x)|^2 + |\nabla J_2(x)|^2 \le \{\varepsilon J_1(x)^2 + C_\varepsilon J_2(x)^2\}|x|^{-2}$$

for all $x \in \mathbb{R}^n \setminus B(0, 1)$.

The proof is given in Appendix 2.

Hereafter we shall denote by D a bounded open subset of $\mathbb{R}^n$ which contains the unit ball $B(0, 1)$ and is such that

(10) the embedding $H^1(D) \hookrightarrow L^2(D)$ is compact.

This is the well-known Rellich property and is satisfied if D has a continuous boundary: see Edmunds and Evans [6, Chapter V] for this and other results.

As a consequence of the IMS localization formula we conclude that

$$(11)\quad \rho[\varphi] = \int_D [|\nabla_A \varphi|^2 + q|\varphi|^2]\,dx + \sum_{i=1}^{2} \int_{D^c} [|\nabla_A (J_i\varphi)|^2 + q|J_i\varphi|^2 - |\nabla_A J_i|^2 |\varphi|]\,dx$$

where $D^c = \mathbb{R}^n \setminus D$. It was recognized by Uchiyama and Zhislin that for P_N the property of having a finite number of bound states depends on the behavior of the potential in truncated cones: for P these are the $\Gamma(\omega_k : \delta_k, \frac{1}{2})$, $k = 1, \dots, m$, in which J_1 is supported. This fundamental fact is reflected in the next two parts of our basic hypothesis $\mathscr{H}$. There is an implicit assumption that D has a Lipschitz boundary.

$\mathscr{H}(2)$ There exists $\varepsilon_1 > 0$ and a function σ defined on ∂D such that

$$\int_{D^c} [|\nabla_A (J_1\varphi)|^2 + (q - \varepsilon_1 |x|^{-2})|J_1\varphi|^2]\,dx \ge \Sigma(P) \int_{D^c} |J_1\varphi|^2\,dx + \int_{\partial D} \sigma |J_1\varphi| 2\,ds$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$.

$\mathscr{H}(3)$ For some $\varepsilon_2 \in (0, 1)$ and $C_{\varepsilon_2} > 0$

$$\int_{\partial D} \sigma |J_1\varphi|^2\,dx \ge -\varepsilon_2 \int_D |\nabla_A \varphi|^2\,dx - C_{\varepsilon_2} \int_D |\varphi|^2\,dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$.

Hypothesis $\mathscr{H}(3)$ holds if $\sigma_- \in L^\gamma(\partial D)$ for $\gamma = n - 1$ when $n > 2$ and $\gamma \in (1, \infty]$ when $n = 2$ (see [10, Lemma 1]).

The hypotheses $\mathscr{H}(2)$ and $\mathscr{H}(3)$ can be replaced by the single hypothesis obtained by combining the two, namely

$\mathscr{H}(2')$ There exist $\varepsilon_1, \varepsilon_2 \in (0, 1)$ and $C_{\varepsilon_2} > 0$ such that

$$\int_{D^c} [|\nabla_A (J_1\varphi)|^2 + (q - \varepsilon_1 |x|^{-2})|J_1\varphi|^2]\,dx \ge \Sigma(P) \int_{D^c} |J_1\varphi|^2\,dx - \varepsilon_2 \int_D |\nabla_A \varphi|^2\,dx - C_{\varepsilon_2} \int_D |\varphi|^2\,dx.$$

However, we felt that it is helpful to express them separately, although we shall revert to $\mathscr{H}(2')$ in our main Theorem 17 in §3 for technical reasons. We ask the reader to bear in mind that in the rest of this section $\mathscr{H}(2')$ can replace $\mathscr{H}(2)$ and $\mathscr{H}(3)$.

Lemma 8. *If $\mathscr{H}(2)$ and $\mathscr{H}(3)$ hold then, for all $\varphi \in C_0^\infty(\mathbb{R}^n)$,*

$$\rho[\varphi] \geq \Sigma(P) \int_{D^c} |J_1\varphi|^2\, dx + h[\varphi], \tag{12}$$

where $h[\varphi] = h[\varphi, \varphi]$ and

$$\begin{aligned} h[\varphi, \psi] := &\int_D [(1-\varepsilon_2)\langle \nabla_A\varphi, \nabla_A\psi\rangle + (q - C_{\varepsilon_2})\varphi\overline{\psi}]\, dx \\ &+ \int_{D^c} [\langle \nabla_A(J_2\varphi), \nabla_A(J_2\psi)\rangle + (q - C_{\varepsilon_1}|x|^{-2})J_2^2\varphi\overline{\psi}]\, dx. \end{aligned} \tag{13}$$

Proof. Since $A(x)$ is bounded, then by Lemma 7(v)

$$\sum_{i=1}^{2} |\nabla_A J_i(x)|^2 \leq \{\varepsilon_1 J_1(x)^2 + C_{\varepsilon_1} J_2(x)^2\}|x|^{-2}$$

for $|x| \geq 1$. On substituting this in (11) we obtain

$$\begin{aligned} \rho[\varphi] \geq &\int_D [|\nabla_A\varphi|^2 + q|\varphi|^2]\, dx \\ &+ \int_{D^c} [|\nabla_A(J_1\varphi)|^2 + (q - \varepsilon_1|x|^{-2})|J_1\varphi|^2]\, dx \\ &+ \int_{D^c} [|\nabla_A(J_2\varphi)|^2 + (q - C_{\varepsilon_1}|x|^{-2})|J_2\varphi|^2]\, dx \end{aligned}$$

and (12) follows from $\mathscr{H}(2)$ and $\mathscr{H}(3)$ (or $\mathscr{H}(2')$).

We shall regard the form h as acting in the weighted space $L^2(\mathbb{R}^n; w^2\, dx)$, where

$$w(x) = \begin{cases} 1 & \text{for } x \in D, \\ J_2(x) & \text{for } x \notin D. \end{cases} \tag{14}$$

Also, we shall use the notation

$$\begin{aligned} &\int_{\mathbb{R}} \langle \tilde{\nabla}_A(w\varphi), \tilde{\nabla}_A(w\varphi)\rangle\, dx \\ &\qquad := \int_D \langle \nabla_A\varphi, \nabla_A\psi\rangle\, dx + \int_{D^c} \langle \nabla_A(J_2\varphi), \nabla_A(J_2\psi)\rangle\, dx \end{aligned}$$

for $\varphi \in C_0^\infty(\mathbb{R}^n)$. Note that $\nabla_A(w\varphi)$ is not defined on ∂D.

The final part of our hypothesis is

$\mathscr{H}(4)$ Given $\varepsilon_3 > 0$ there exists $C_{\varepsilon_3} > 0$ such that

$$\int_{\mathbb{R}^n} q_-|w\varphi|^2\,dx \le \varepsilon_3 \int_{\mathbb{R}^n} |\tilde{\nabla}_A(w\varphi)|^2\,dx + C_{\varepsilon_3}\int_{\mathbb{R}^n} |w\varphi|^2\,dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$.

This part of the hypothesis compares with condition $(1)(\mathrm{iv})'$.

Proposition 9. *If $\mathscr{H}(2)$–(4) are satisfied then the form h in (13) defined for φ, $\psi \in C_0^\infty(\mathbb{R}^n)$ is densely-defined, symmetric, bounded below, and closable in $L^2(\mathbb{R}^n; w^2\,dx)$.*

Proof. It is clear that h is densely defined and symmetric. Also, by $\mathscr{H}(4)$, for all $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned} h[\varphi] \ge (1-\varepsilon_3)\int_{D^c} |\nabla_A(J_2\varphi)|^2\,dx + \int_{D^c}(q_+ - C_{\varepsilon_1}|x|^{-2} - C_{\varepsilon_3})|J_2\varphi|^2\,dx \\ + (1-\varepsilon_2-\varepsilon_3)\int_D |\nabla_A\varphi|^2\,dx + \int_D (q_+ - C_{\varepsilon_2} - C_{\varepsilon_3})|\varphi|^2\,dx, \end{aligned}$$

where $\varepsilon_3 > 0$ is arbitrary. Thus, if $\varepsilon = \varepsilon_2 + \varepsilon_3 < 1$ we conclude that there exists $c_\varepsilon > 0$ such that

$$h[\varphi] \ge (1-\varepsilon)\int_{\mathbb{R}^n} |\tilde{\nabla}_A(w\varphi)|^2\,dx + \int_{\mathbb{R}^n}(q_+ - c_\varepsilon)|w\varphi|^2\,dx.$$

Hence there exists $\alpha = \alpha(\varepsilon) > 0$ such that

$$h[\varphi] + \alpha\|\varphi\|_w^2 \ge (1-\varepsilon)\int_{\mathbb{R}^n} |\tilde{\nabla}_A(w\varphi)|^2\,dx + \int_{\mathbb{R}^n} q_+|w\varphi|^2\,dx + \|\varphi\|_w^2, \tag{16}$$

where $\|\cdot\|_w$ denotes the norm in $L^2(\mathbb{R}^n; w^2\,dx)$. It follows that h is bounded below.

To prove that h is closable, we consider a sequence of functions $\varphi_n \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$h[\varphi_n - \varphi_m] \to 0, \qquad \|\varphi_n\|_w \to 0 \tag{17}$$

as n, $m \to \infty$; we are required to show that $h[\varphi_n] \to 0$. From (16) we see that (17) implies

$$\int_{\mathbb{R}^n} [|\tilde{\nabla}_A(w[\varphi_n - \varphi_m])|^2 + |w[\varphi_n - \varphi_m]|^2]\,dx \to 0 \tag{17a}$$

$$\int_{\mathbb{R}^n} (q_+ + 1)|w[\varphi_n - \varphi_m]|^2\,dx \to 0 \tag{17b}$$

$$\int_{\mathbb{R}^n} |w\varphi_n|^2 \to 0. \tag{17c}$$

Since A is locally bounded below by a positive number, (17a) implies that for any compact set Ω, $\{w\varphi_n\}$ is Cauchy in $H^1(\Omega)$, i.e., in accordance with

our notation in (15), $\{\varphi_n\}$ is Cauchy in $H^1(D \cap \Omega)$ and $\{J_2\varphi_n\}$ is Cauchy in $H^1(D^c \cap \Omega)$. Since $H^1(\Omega)$ is continuously embedded in $L^2(\Omega) \subset L^2(\mathbb{R}^n)$ it follows from (17c) that $w\varphi_n \to 0$ in $H^1(\Omega)$.

Since A is bounded above on $\mathbb{R}^n$, then for some $C > 0$,

$$\int_\Omega [|\tilde{\nabla}_A(w\varphi_n)|^2] + |w\varphi_n|^2] \leq C\|w\varphi_n\|^2_{H^1(\Omega)} \to 0.$$

Consequently, $\{w\varphi_n\}$ converges weakly to zero in the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\int_{\mathbb{R}^n} [|\tilde{\nabla}_A(w\varphi)|^2 + |w\varphi|^2]\, dx$$

and from (17a) we obtain

$$\int_{\mathbb{R}^n} [|\tilde{\nabla}_A(w\varphi_n)|^2 + |w\varphi_n|^2]\, dx \to 0. \tag{18}$$

From (17b), $\{w\varphi_n\}$ is Cauchy in $L^2(\mathbb{R}^n; (q_+ + 1)\, dx)$ and in view of (17c) its limit must be zero:

$$\int_{\mathbb{R}^n} (q_+ + 1)|w\varphi_n|^2\, dx \to 0. \tag{19}$$

Since

$$h[\varphi_n] \leq \int_{\mathbb{R}^n} [|\tilde{\nabla}_A(w\varphi_n)|^2 + q_+|w\varphi_n|^2]\, dx$$

it follows from (18) and (19) that $h[\varphi_n] \to 0$ and so h is closable. The proof is therefore complete.

Let $\tilde{h}$ denote the closure of h in $L^2(\mathbb{R}^n; w^2\, dx)$. Its domain $\mathscr{D}(\tilde{h})$ is identified with the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_h$ defined by

$$\|\varphi\|_h^2 := h[\varphi] + \alpha\|\varphi\|_w^2, \tag{20}$$

where α is any positive number for which (16) is satisfied. Let $\tilde{H}$ denote the lower semibounded, selfadjoint operator in $L^2(\mathbb{R}^n; w^2\, dx)$ associated with $\tilde{h}$ by the First Representation Theorem, i.e.

$$(\tilde{H}\psi, \varphi)_w = \tilde{h}[\psi, \varphi]$$

for all $\psi \in D(\tilde{H}) \subset D(\tilde{h})$ and $\varphi \in D(\tilde{h})$, where $(\cdot, \cdot)_w$ is the inner product on $L^2(\mathbb{R}^n; w^2\, dx)$.

Next, we establish a result of the type obtained by Persson in [20] for the least point of the essential spectrum $\sigma_e(\tilde{H})$ of the above operator $\tilde{H}$.

Theorem 10. *Let $\mathscr{H}$(2)–(4) be satisfied and with $B_k := B(0, k)$, $k = 1, 2, \ldots$, define*

$$l_k := \inf\{h[\varphi]\colon \varphi \in C_0^\infty(B_k^c),\ \|\varphi\|_w = 1\}$$

and

$$l := \lim_{k\to\infty} l_k = \sup_{k\geq 1} l_k .$$

Then

$$\inf \sigma_e(\widetilde{H}) = l .$$

Proof. Let $\lambda \in \sigma_e(\widetilde{H})$ with a singular sequence $\{\psi_n\}$, i.e.

$$\psi_n \in D(\widetilde{H}), \qquad \|\psi_n\|_w = 1, \qquad \psi_n \rightharpoonup 0, \qquad (\widetilde{H} - \lambda I)\psi_n \to 0$$

in $L^2(\mathbb{R}^n; w^2\,dx)$. For all $\varphi \in D(\tilde{h})$ we have

$$\tilde{h}[\psi_n, \varphi] = ([\widetilde{H} - \lambda I]\psi_n, \varphi)_w + \lambda(\psi_n, \varphi)_w \to 0$$

and

$$(\psi_n, \varphi)_h := \tilde{h}[\psi_n, \varphi] + \alpha(\psi_n, \varphi)_w \to 0 .$$

Thus $\psi_n \rightharpoonup 0$ in $D(\tilde{h})$. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $D(\tilde{h})$, there exists a sequence $\{\varphi_n\}$ in $C_0^\infty(\mathbb{R}^n)$ such that $\|\psi_n - \varphi_n\|_h \to 0$ as $n \to \infty$. This implies that $\varphi_n \rightharpoonup 0$ in $D(\tilde{h})$, $\|\psi_n - \varphi_n\|_w \to 0$ and $\|\varphi_n\|_w \to 1$ since $\|\psi_n\|_w = 1$. Also from

$$\tilde{h}[\psi_n] = ([\widetilde{H} - \lambda I]\psi_n, \psi_n)_w + \lambda\|\psi_n\|_w^2 \to \lambda$$

it follows that $h[\varphi_n] \to \lambda$. Hence, in summary we have

$$\text{(21)} \qquad \begin{cases} \text{(a)} & \varphi_n \in C_0^\infty(\mathbb{R}^n), \\ \text{(b)} & \varphi_n \rightharpoonup 0 \text{ in } D(\tilde{h}), \\ \text{(c)} & \|\varphi_n\|_w \to 1, \\ \text{(d)} & h[\varphi_n] \to \lambda . \end{cases}$$

Let k be a sufficiently large positive integer in order that $D \subset B_k$ and choose $\theta \in C_0^\infty(\mathbb{R}^n)$ to satisfy

$$\theta(x) = \begin{cases} 1 & \text{for } x \in B_k, \\ 0 & \text{for } x \notin B_{k+1}, \end{cases}$$

$\theta(x) \in [0, 1]$, and $|\nabla\theta(x)|$ is bounded on $\mathbb{R}^n$. For all $\varphi \in C_0^\infty(\mathbb{R}^n)$, $(1-\theta)\varphi \in C_0^\infty(B_k^c)$ and so

$$\begin{aligned} l_k\|(1-\theta)\varphi\|_w^2 &\leq h[(1-\theta)\varphi] \\ &= \int_{B_k^c} [|\nabla_A([1-\theta]J_2\varphi)|^2 + (q - C_{\varepsilon_1}|x|^{-2})|[1-\theta]J_2\varphi|^2]\,dx \\ &\leq (1+\delta_0)\int_{B_k^c} (1-\theta)^2|\nabla_A(J_2\varphi)|^2\,dx + C(\delta_0)\int_{B_{k+1}} |J_2\varphi|^2\,dx, \\ &\quad + \int_{B_k^c} \{(1-\theta)^2[q_+ - C_{\varepsilon_1}|x|^{-2}] - q_- + \eta q_-\}|J_2\varphi|^2\,dx, \end{aligned}$$

where $\delta_0 > 0$ is arbitrary and $\eta = 1 - (1-\theta)^2 \in C_0^\infty(\overline{B}_{k+1})$,

$$\begin{aligned}
&\le (1+\delta_0)\int_{B_k^c} |\nabla_A(J_2\varphi)|^2\,dx + \int_{B_k^c}(q - C_{\varepsilon_1}|x|^{-2})|J_2\varphi|^2\,dx \\
&\quad + \int_{B_k^c}\eta(q_- + C_{\varepsilon_1}|x|^{-2})|J_2\varphi|^2\,dx + C(\delta_0)\int_{B_{k+1}}|J_2\varphi|^2\,dx \\
&\le h[\varphi] + \delta_0\int_{B_k^c}|\nabla_A(J_2\varphi)|^2\,dx + \int_{B_k^c}\eta(q_- + C_{\varepsilon_1}|x|^{-2})|J_2\varphi|^2\,dx \\
&\quad + \int_{B_k}(q_- + C_{\varepsilon_2})|\varphi|^2\,dx + C(\delta_0)\int_{B_{k+1}}|J_2\varphi|^2\,dx
\end{aligned}$$

from (13). Hence, from (16) and (20), for any $\delta_1 > 0$ there exists $C(\delta_1) > 0$ such that

$$l_k\|(1-\theta)\varphi\|_w^2 \le h[\varphi] + \delta_1\|\varphi\|_h^2 + \int_{\mathbb{R}^n}\eta q_-|w\varphi|^2\,dx + C(\delta_1)\int_{B_{k+1}}|w\varphi|^2\,dx.$$

By $\mathscr{H}(4)$ and (16), given any $\delta_2 > 0$ there is $C(\delta_2) > 0$ and an absolute constant $C > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^n}\eta q_-|w\varphi|^2\,dx &\le \left\{\int_{\mathbb{R}^n} q_-|w\varphi|^2\,dx\int_{\mathbb{R}^n} q_-|w\eta\varphi|^2\,dx\right\}^{1/2} \\
&\le C\|\varphi\|_h\left\{\delta_2^2\|\varphi\|_h^2 + C(\delta_2)\int_{B_{k+i}}|w\varphi|^2\,dx\right\}^{1/2}
\end{aligned}$$

since $\eta = 0$ outside B_{k+1}. On substituting in (22) with $\varphi = \varphi_n$, we obtain if $l_k \ge 0$

$$\begin{aligned}
l_k\{\|\varphi_n\|_w - \|\theta\varphi_n\|_w\}^2 &\le l_k\|(1-\theta)\varphi_n\|_w^2 \\
&\le h[\varphi_n] + \delta_1\|\varphi_n\|_h^2 + C(\delta_1)\int_{B_{k+1}}|w\varphi_n|^2\,dx \\
&\quad + C\|\varphi_n\|_h\left\{\delta_2^2\|\varphi_n\|_h^2 + C(\delta_2)\int_{B_{k+1}}|w\varphi_n|^2\,dx\right\}^{1/2};
\end{aligned} \tag{23}$$

this remains true for $l_k < 0$ if the sign is reverse in the first bracket.

The next step is to prove that $\int_{B_{k+1}}|w\varphi_n|^2\,dx \to 0$. Since $\varphi_n \rightharpoonup 0$ in $D(\tilde{h})$ we have that $\|\varphi_n\|_h \le C$. Hence from (16) and the fact that A is positive, we get

$$\int_{B_{k+1}}[|\tilde{\nabla}(w\varphi_n)|^2 + |w\varphi_n|^2]\,dx \le C(k).$$

Hence $\{w\varphi_n\}$ is precompact in $L^2(B_{k+1})$. Thus there is a subsequence (which we continue to denote by $\{w\varphi_n\}$ for simplicity of notation) such that $w\varphi_n \to \Phi$, say, in $L^2(B_{k+1})$. Let χ_{k+1} denote the characteristic function of B_{k+1}.

Then $\Phi\chi_{k+1}/w \in L^2(\mathbb{R}^n; w^2\,dx)$ and

$$\begin{aligned}\int_{B_{k+1}} |\Phi|^2\,dx &= \lim_{n\to\infty}\int_{B_{k+1}} (w\varphi)\overline{\Phi}\,dx\\ &= \lim_{n\to\infty}(\varphi_n, \Phi\chi_{k+1}w^{-1})_w\\ &= 0\end{aligned}$$

since $\varphi_n \rightharpoonup 0$ in $D(\tilde{h})$ implies that $\varphi_n \rightharpoonup 0$ in $L^2(\mathbb{R}^n; w^2\,dx)$. This last assertion follows from the fact that $\tilde{h}$ is closed and hence $D(\tilde{h})$ is continuously embedded in $L^2(\mathbb{R}^n; w^2\,dx)$. Hence $\Phi = 0$ and

$$\int_{B_{k+1}} |w\varphi_n|^2\,dx \to 0$$

as asserted. On allowing $n \to \infty$ in (23) and noting that

$$\|\theta\varphi_n\|_w^2 \le \int_{B_{k+1}} |w\varphi_n|^2\,dx,$$

we obtain from (21) that

$$l_k \le \lambda + C[\delta_1 + \delta_2].$$

Since δ_1 and δ_2 are arbitrary and C is independent of k we conclude that $l \le \lambda$ and hence $l \le \inf\sigma_e(\widetilde{H})$.

We now prove that $l \ge \inf\sigma_e(\widetilde{H})$. Let $\mu < \inf\sigma_e(\widetilde{H})$. Then in $(-\infty, \mu)$, $\sigma(\widetilde{H})$ consists of a finite number (M say) of eigenvalues λ_n, repeated according to multiplicity, with corresponding eigenfunctions ψ_n say. Then, for all $\varphi \in D(\widetilde{H})$,

$$\tilde{h}[\varphi] = (\widetilde{H}\varphi, \varphi) = \sum_{n=1}^{M}\lambda_n|(\varphi, \psi_n)_w|^2 + \int_\mu^\infty \lambda\,d(E_\lambda\varphi, \varphi)_w,$$

where $\{E_\lambda\}$ is the spectral resolution of $\widetilde{H}$,

$$\begin{aligned}&\ge \sum_{n=1}^{M}\lambda_N|(\varphi, \psi_n)_w|^2 + \mu\left\{\int_{-\infty}^{\infty} - \int_{-\infty}^{-\mu}\right\} d(E_\lambda\varphi, \varphi)_w,\\ &= \sum_{n=1}^{M}\lambda_n|(\varphi, \psi_n)_w|^2 + \mu\|\varphi\|_w^2 - \mu\sum_{n=1}^{M}|(\varphi, \psi_n)_w|^2 \qquad (24)\\ &= \sum_{n=1}^{M}(\lambda_n - \mu)|(\varphi, \psi_n)_w|^2 + \mu\|\varphi\|_w^2.\end{aligned}$$

Since $D(\widetilde{H})$ is dense in $D(\tilde{h})$ it follows that (24) also holds for all $\varphi \in D(\tilde{h})$. Now choose $\varphi_j \in C_0^\infty(\mathbb{R}^n)$, $j = 1, 2, \ldots$, with disjoint supports and such that

$$h[\varphi_j] \to l, \qquad \|\varphi_j\|_w = 1. \tag{25}$$

Since $\{\varphi_j\}$ is orthonormal, $\varphi_j \rightharpoonup 0$ in $L^2(\mathbb{R}^n; w^2\,dx)$ and hence on allowing $j \to \infty$ in (24) with $\varphi = \varphi_j$ we get

$$l \geq \mu .$$

Since $\mu < \inf \sigma_e(\widetilde{H})$ is arbitrary, we conclude that $l \geq \inf \sigma_e(\widetilde{H})$ and the proof is complete.

Corollary 11. *Under the hypothesis of Theorem* 10

$$\min\{K(\omega\colon P)\colon\ \omega \in S^{n-1} \setminus U\} \leq l$$

for an open set U containing the unit vectors $\{\omega_k\}_1^m$ in Lemma 7. *If, in addition $\mathscr{H}(1)$ is satisfied relative to $\{\omega_k\}_1^m$, then $\Sigma(P) < l$ and hence $\widetilde{H}$ has only a finite number of eigenvalues* (*counting multiplicity*) *below $\Sigma(P)$.*

Proof. By Lemma 2.3 of Agmon [1], for all $\varphi \in C_0^\infty(D^c)$ and any $\delta > 0$ there is R_δ such that

$$\begin{aligned} h[\varphi] &= \int_{D^c} [|\nabla_A(J_2\varphi)|^2 + (q - C_{\varepsilon_1}|x|^{-2})|J_2\varphi|^2]dx \\ &\geq \int_{\mathbb{R}^n} [\Lambda_R(x\colon P) - \delta - C_{\varepsilon_1}|x|^{-2}]|J_2\varphi|^2\,dx \end{aligned}$$

whenever $R \geq R_\delta$, $\Lambda_R(x\colon P)$ being defined in Definition 3. Hence, for k sufficiently large and $R \geq R_\delta$

$$l_k \geq \inf\{\Lambda_R(x\colon P)\colon x \in (\operatorname{supp} J_2) \cap B_k^c\} - 2\delta$$

whence

$$l \geq \liminf_{k\to\infty}\{\Lambda_R(x\colon P)\colon x \in (\operatorname{supp} J_2) \cap B_k^c\} - 2\delta$$

and, since δ is arbitrary and $R \geq R_\delta$ is the only restriction on R, we have

$$l \geq \lim_{R\to\infty} \liminf_{k\to\infty}\{\Lambda_R(x\colon P)\colon x \in (\operatorname{supp} J_2) \cap B_k^c\}. \tag{26}$$

It follows from (26) that given $\delta > 0$ there exist R_1 and a sequence $\{x_k\} \subset (\operatorname{supp} J_2) \cap B_k^c$ such that for k large

$$l \geq \Lambda_{R_1}(x_k\colon P) - \delta .$$

From Lemma 7 it is clear that there is an open set U containing $\{\omega_k\}_1^m$ such that $x_k/|x_k| \in S^{n-1} \setminus U$. Since $\{x_k/|x_k|\}$ is bounded it contains a subsequence converging to some $\omega_0 \in S^{n-1} \setminus U$. Therefore, for every $\varepsilon > 0$ there exist N_ε, $k_\varepsilon > 0$ such that

$$B(x_{k_\varepsilon}, R_1) \subset \Gamma(w_0\colon \varepsilon, N_\varepsilon)$$

with $k_\varepsilon \to \infty$ and $N_\varepsilon \to \infty$ as $\varepsilon \to 0$. This implies that

$$\Lambda_{R_1}(x_{k_\varepsilon}\colon P) \geq \Sigma(\omega_0\colon \varepsilon, N_\varepsilon)$$

by Definitions 2 and 3. Thus

$$l \geq \Sigma(\omega_0\colon \varepsilon, N_\varepsilon) - \delta$$

whence

$$l \geq K(\omega_0 : P)$$

and the first part of the corollary is proved. The remainder of the proof follows since $l = \inf \sigma_e(\widetilde{H})$ by Theorem 10.

Our main result is

Theorem 12. *Let* (1)(i)–(iv) *and* $\mathscr{H}$(1)–(4) *be satisfied. Then* H *has only a finite number of bound states.*

Proof. We first show that (12) has an extension to $\varphi \in \mathscr{D}(\tilde{\rho})$. By definition, if $\varphi \in \mathscr{D}(\tilde{\rho})$ then there exists a sequence $\{\varphi_n\}$ in $C_0^\infty(\mathbb{R}^n)$ such that

$$\tilde{\rho}[\varphi - \varphi_n] \to 0, \qquad \|\varphi - \varphi_n\| \to 0.$$

If α is such that (16) holds, we deduce from (12) that

$$\begin{aligned} 0 &\leq h[\varphi_n - \varphi_m] + \alpha\|\varphi_n - \varphi_m\|_w^2 \\ &\leq \rho[\varphi_n - \varphi_m] + (\alpha + |\Sigma(P)|)\|\varphi_n - \varphi_m\|^2 \to 0 \end{aligned}$$

since $w(x)$, $J_1(x) \in [0, 1]$. Hence $h[\varphi_n - \varphi_m] \to 0$ and

$$\|\varphi_n - \varphi_m\|_w \leq 2\|\varphi_n - \varphi\| \to 0.$$

This implies that φ viewed as an element of $L^2(\mathbb{R}^n : w^2\,dx)$ is in $\mathscr{D}(\tilde{h})$ and $\tilde{h}[\varphi] = \lim_{n\to\infty} h[\varphi_n]$. On putting $\varphi = \varphi_n$ in (12) and allowing $n \to \infty$ we obtain

$$\tilde{\rho}[\varphi] \geq \Sigma(P) \int_{D^c} |J_1\varphi|^2\,dx + \tilde{h}[\varphi] \tag{27}$$

for all $\varphi \in \mathscr{D}(\tilde{\rho}) \subset \mathscr{D}(\tilde{h})$.

We know from Corollary 11 that $\widetilde{H}$ has only a finite set of eigenfunctions $E := \{\psi_1, \dots, \psi_k\}$ corresponding to eigenvalues below $\Sigma(P)$. Let $\varphi \in \mathscr{D}(\tilde{\rho})$ be orthogonal to $\{w^2\psi_1, \dots, w^2\psi_k\}$ in $L^2(\mathbb{R}^n)$. Then, from above, we have that $\varphi \in \mathscr{D}(\tilde{h})$ and is orthogonal to E in $L^2(\mathbb{R}^n; w^2\,dx)$. Hence

$$\tilde{h}[\varphi] \geq \Sigma(P)\|\varphi\|_w^2$$

and from (27)

$$\tilde{\rho}[\varphi] \geq \Sigma(P)\left\{\int_{D^c} |J_1\varphi|^2\,dx + \int_{D^c} |J_2\varphi|^2\,dx + \int_D |\varphi|^2\,dx\right\} = \Sigma(P)\|\varphi\|^2$$

since $J_1^2 + J_2^2 = 1$ in D^c. It follows that H has at most k bound states.

In [29] Simon gives a result (Theorem C.8.2, p. 517), largely due to Allegretto and Piepenbrink, which is related to Theorem 12, but with conditions on q which are too severe to include applications to N-body Schrödinger operators for $N > 2$. Simon conjectures that the implication (a) $\Rightarrow$ (c) in his theorem is true under a much weaker hypothesis. If his conjecture is true then our conditions $\mathscr{H}$(2) and $\mathscr{H}$(3) could be simplified.

3. $(N+1)$-BODY SCHRÖDINGER OPERATORS

Here we apply the results of §1 to an important special case of the formal operator P in (1). We refer the reader to Agmon [1] for elaboration of many of the facts which we use.

Consider an atomic type system of N particles $(N \geq 2)$ each moving in ν-dimensional space $\mathbb{R}^\nu$ relative to a fixed nucleus that is assumed to have infinite mass. The coordinates of the particles are denoted by $x^i \in \mathbb{R}^\nu$, $i = 1, 2, \ldots, N$, with the nucleus at the origin of $\mathbb{R}^\nu$. The formal Schrödinger operator for such an $(N+1)$-body system is of the form

$$P = -\sum_{i=1}^{N}(2m_i)^{-1}\Delta_i + \sum_{i=1}^{N} v_{oi}(x^i) + \sum_{1\leq i<j\leq N} v_{ij}(x^i - x^j), \tag{28}$$

where Δ_i denotes the ν-dimensional Laplacian with respect to the x^i variable and m_i is the mass of the ith particle. When the particles are electrons moving with respect to a fixed nucleus of infinite mass, each v_{oi} represents the binding force of the nucleus while each v_{ij} represents the repulsive force between the ith and jth electron.

The configuration space X of the system consists of the product of N copies of $\mathbb{R}^\nu$, i.e. $X = \bigotimes_{i=1}^{N}\mathbb{R}^\nu$. Points $x \in X$ are of the form $x = (x^1, x^2, \ldots, x^N)$ where $x^i \in \mathbb{R}^\nu$ and if $x^i = (x_1^i, x_2^i, \ldots, x_\nu^i)$, then x is identified with the point $(x_1^1, x_2^1, \ldots, x_\nu^1, x_1^2, \ldots, x_\nu^N)$ in $\mathbb{R}^{\nu N}$; let I denote the identification map. We define on X the new inner product

$$\langle x, y\rangle_X := \sum_{i=1}^{N} 2m_i\langle x^i, y^i\rangle, \tag{29}$$

where $x = (x^1, \ldots, x^1)$, $y = (y^1, \ldots, y^N)$ and $\langle\cdot,\cdot\rangle$ is the Euclidean inner product on $\mathbb{R}^\nu$. This is related to the Euclidean inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{\nu N}$ by

$$\langle x, y\rangle_X = \langle GIx, IY\rangle,$$

where G is the $(\nu N)\times(\nu N)$ diagonal matrix with diagonal blocks $(2m_1 I_\nu, 2m_2 I_\nu, \ldots, 2m_N I_\nu)$, I_ν being the identity matrix on $\mathbb{R}^\nu$. The identification map $I\colon x \to \mathbb{R}^{\nu N}$ is viewed as the coordinate map of the νN-dimensional Riemannian manifold X with the inner product $\langle\cdot,\cdot,\rangle_X$. The gradient $\nabla_X\varphi$ of $\varphi \in C^1(X)$ is the function

$$\nabla_x\varphi\colon x \mapsto ((2m_i)^{-1}\nabla_1\varphi(x), \ldots, (2m_N)^{-1}\nabla_N\varphi(x)),$$

where ∇_i is the gradient with respect to x^i. The Laplace-Beltrami operator on X is

$$\Delta_X := \sum_{i=1}^{N}(2m_i)^{-1}\Delta_i.$$

With this notation the formal Schrödinger operator (28) can now be written as

$$P = -\Delta_X + V,$$

where V is the potential of the system; namely,

$$V(x) = \sum_{i=1}^{N} v_{oi}(x^i) + \sum_{1\leq i<j\leq N} v_{ij}(x^i - x^j).$$

The sesquilinear form ρ_X associated with P in $L^2(X)$ is given by

$$\rho_X[\varphi, \psi] = \int_X \overline{\psi} P\varphi \, d_X x = \int_X \{\langle \nabla_X \varphi, \nabla_X \overline{\psi}\rangle_X + V\varphi\overline{\psi}\} \, d_X x$$

for $\varphi, \psi \in C_0^\infty(X)$, where the measure $d_X x$ induced by $\langle \cdot, \cdot \rangle_X$ on X is given by

$$d_X x = \sqrt{g}\, dx^1 \cdots dx^N$$

and $g = \det G = 2^{\nu N} \prod_{i=1}^N m_i^\nu$. Thus

$$(30) \quad \rho_X[\varphi, \psi] = \sqrt{g} \int_{\bigotimes_{i=1}^N \mathbb{R}^\nu} \left\{ \sum_{i=1}^N (2m_i)^{-1} \langle \nabla_i \varphi, \nabla_i \overline{\psi}\rangle + V\varphi\overline{\psi} \right\} dx^1 \cdots dx^N .$$

The map

$$U : \varphi \mapsto g^{-1/4} \varphi \circ I^{-1}$$

is unitary from $L^2(\mathbb{R}^{\nu N})$ onto $L^2(X)$ and furthermore, for $\varphi, \psi \in C_0^\infty(\mathbb{R}^{\nu N})$

$$(31) \quad \rho_X[U\varphi, U\psi] = \int_{\mathbb{R}^{\nu N}} \left\{ \sum_{i=1}^N (2m_i)^{-1} \langle \nabla_i \varphi, \nabla_i \overline{\psi}\rangle + V\varphi\overline{\psi} \right\} dx =: \rho[\varphi, \psi],$$

where ρ is the form associated with P in $L^2(\mathbb{R}^{\nu N})$.

We shall assume that the functions v_{ij} in the potential V are real-valued and satisfy

$$(32) \quad \begin{array}{ll} \text{(i)} & v_{ij} \in L^2_{\text{loc}}(\mathbb{R}^\nu) \text{ and } \lim_{|y| \to \infty} v_{ij}(y) = 0 \quad \text{for } 0 \le i < j \le N, \\ \text{(ii)} & v_{ij} \ge 0 \quad \text{for } 1 \le i < j \le N, \\ \text{(iii)} & (v_{oi})_- \in M_{\text{loc}}(\mathbb{R}^\nu) \quad \text{for } 1 \le i \le N. \end{array}$$

As noted in §2, we may replace 32(iii) by the analogue of $1(\text{iv})'$. We shall use this fact in §4, Example 6.

In [1, Lemma 4.7] and the discussion [1, pp. 67, 71] Agmon proves that the conditions (32) ensure that P satisfies the hypothesis of Proposition 1(with $q = V$) as an operator in $\mathbb{R}^{\nu N}$ and hence ρ is bounded below and closable in $L^2(\mathbb{R}^{\nu N})$. From (31) and the fact that $U : L^2(\mathbb{R}^{\nu N}) \to L^2(X)$ is unitary, it follows that ρ_X is also bounded below and closable in $L^2(X)$. Furthermore (31) extends to the domains of the closures $\tilde{\rho}_X$, $\tilde{\rho}$ of ρ_X, ρ respectively, i.e. $\mathscr{D}(\tilde{\rho}_X) = U\mathscr{D}(\tilde{\rho})$ and

$$\tilde{\rho}_X[U\varphi, U\psi] = \tilde{\rho}[\varphi, \psi]$$

for all $\varphi, \psi \in \mathscr{D}(\tilde{\rho})$. If H_X, H are respectively the selfadjoint operators associated with $\tilde{\rho}_X$, $\tilde{\rho}$ by the First Representation Theorem then

$$H_X = UHU^{-1}.$$

Therefore H_X and H are unitarily equivalent and thus share the same spectral properties. We refer to H_X as the selfadjoint realization of the formal Schrödinger operator P for the $(N+1)$-body system in $L^2(X)$.

For each unit vector $\omega \in X$ (i.e. $|\omega|_X = \langle \omega, \omega \rangle_X^{1/2} = 1$) set

$$J(\omega) := \{i \in \{1, 2, \dots, N\} : \omega^i = 0\}$$

and

$$X_\omega := \{x \in X : x^i = 0 \text{ if } i \notin J(\omega)\}.$$

If $M = \#J(\omega)$, then X_ω is the configuration space of a system of $M+1$ particles with formal Schrödinger operator

$$P_\omega := \sum_{i \in J(\omega)} [-(2m_i)^{-1}\Delta_i + v_{oi}(x^i)] + \sum_{\substack{i<j \\ i,j \in J(\omega)}} v_{ij}(x^i - x^j). \tag{33}$$

If I_ω is the identification map $X_\omega \to \mathbb{R}^{\nu M}$ and $g_\omega = 2^{\nu M} \prod_{i \in J(\omega)} m_i^\nu$, then $U_\omega : \varphi \mapsto g_\omega^{1/4} \varphi \circ I_\omega^{-1}$ is unitary map from $L^2(\mathbb{R}^{\nu M})$ onto $L^2(X_\omega)$. The form associated with P_ω in $L^2(X_\omega)$ is given by

(34)

$$\rho_{X_\omega}[\varphi, \psi] = \sqrt{g}_\omega \int_{\bigotimes_{i \in J(\omega)} \mathbb{R}^\nu} \left\{ \sum_{i \in J(\omega)} [(2m_i)^{-1} \langle \nabla_i \varphi, \nabla_i \psi \rangle + v_{oi} \varphi \overline{\psi}] + \sum_{\substack{i<j \\ i,j \in J(\omega)}} v_{ij} \varphi \overline{\psi} \right\} \bigotimes_{i \in J(\omega)} dx^i$$

for $\varphi, \psi \in C_0^\infty(X_\omega)$. It is bounded below and closable in $L^2(X_\omega)$ and we denote the associated selfadjoint operator by H_ω.

In [1, Lemma 4.11 and the subsequent discussion] Agmon proves the following characterization of the function $K(\omega : P)$ in Definition 2 for ω a unit vector in X. The truncated cones are now

$$\Gamma_X(\omega : \delta, R) := \{x \in X : \langle x, \omega \rangle_X > |x|_X \cos\delta, |x|_X > R\}$$

and

$$\Sigma(\omega : \delta, R) := \inf\{\rho_X[\varphi] : \varphi \in C_0^\infty(\Gamma_X(\omega : \delta, R)), \|\varphi\| = 1\}$$

where $\|\cdot\|$ is the norm on $L^2(X)$.

Lemma 13. *If* (32) *holds then*

(i) $K(\omega : P) = \inf\{\rho_{X_\omega}[\varphi] : \varphi \in C_0^\infty(X_\omega), \|\varphi\|_{L^2(X_\omega)} = 1\} = \inf \sigma(H_\omega)$ *if* $J(\omega) \neq \varnothing$ *and* $K(\omega : P) = 0$ *otherwise*;

(ii) $\inf \sigma_e(H_X) = \min_{|\omega|=1} K(\omega : P) = \Sigma(P)$;

(iii) $\max_{|\omega|=1} K(\omega : P) = 0$.

An important consequence of Lemma 13(i) is that $K(\omega : P)$ takes only a finite number of values as ω ranges over the unit sphere in X since there are only a finite number of possible forms ρ_{X_ω}. Also from Lemma 13(i) and (ii)

$$\inf \sigma_e(H_X) = \min_{|\omega|=1} [\inf \sigma(H_\omega)] \tag{35}$$

and this constitutes the main part of the celebrated HVZ theorem (see [12, 25, 32, 35]). Following Agmon in [1] we classify this alternatively as follows.

For each proper, nonempty subset J of $\{1, 2, \dots, N\}$ define

$$X_J := \{x \in X : x^i = 0 \text{ if } i \notin J\}$$

and let H_J denote the selfadjoint realization in $L^2(X_J)$ of the formal operator P_J defined as in (33) with $J(\omega)$ replaced by J. Then

$$\inf \sigma_e(H_X) = \min_J [\inf \sigma(H_J)], \tag{36}$$

where the minimum is taken over all proper nonempty subsets of J.

The first part of our main hypothesis in this section is

$$\mathscr{A}(1) \qquad \inf \sigma_e(H_X) < \min_{\#J \le N-2} [\inf \sigma(H_J)].$$

This assumption corresponds to Sigal's hypothesis in [25, Theorem 4.3] that $\inf \sigma_e(H_X)$ is determined only by two-cluster breakups, i.e. $\inf \sigma_e(H_X)$ is determined only by N-body Hamiltonians. In the event that equality occurs in $\mathscr{A}(1)$ a phenomenon known as the Efimov effect may produce infinitely many bound states even if V has compact support.

If $\mathscr{A}(1)$ is assumed, then in Lemma 13(ii)

$$\inf \sigma_e(H_X) = K(\omega : P)$$

only if, for some $i \in \{1, \dots, N\}$, $\omega^i \neq 0$ and $\omega^j = 0$ if $j \neq i$. Since each $\omega^i \in \mathbb{R}^\nu$, there are ν mutually orthogonal choices for ω^i and by Lemma 13(i), $K(\omega : P)$ is the same for each.

Let $\{e(1), \dots, e(\nu)\}$ be the canonical basis for $\mathbb{R}^\nu$, i.e. $e(i)$ has 1 in the ith position and 0 elsewhere, and set

$$\omega(i, j) = (0, \dots, 0, \omega^i(j), 0, \dots, 0) \in X$$

where $\omega^i(j) = (2m_i)^{-1/2} e(j)$. Then $\omega(i, j)$, $i = 1, \dots, N$, $j = 1, \dots, \nu$, are unit vectors in X and from above $K(\omega(i, j) : P)$ does not depend on j. For $R > 0$, define

$$\begin{aligned} D_R &:= \{x \in X : |\langle x, \omega(i, j)\rangle_X| < R,\ i = 1, \dots, N,\ j = 1, \dots, \nu\} \\ &= \{x \in X : |x_j^i| < R/\sqrt{2m_i},\ i = 1, \dots N,\ j = 1, \dots, \nu\} \\ &= \bigotimes_{i=1}^{N} (-R/\sqrt{2m_i}, R/\sqrt{2m_i})^\nu \end{aligned} \tag{37}$$

and set

$$\Gamma_j^i := \Gamma_X(\omega(i, j) : \delta(i, j), \tfrac{1}{2}) \tag{38}$$

where the $\delta(i, j)$ are small enough to ensure that the truncated cones Γ_j^i are disjoint in X. In what follows the partition of unity $\{J_0, J_1, J_2\}$ will be that constructed in Lemma 7 but now relative to the truncated cones Γ_j^i in X, and the set D in hypothesis $\mathscr{H}$ will be D_R in (37) with R sufficiently large.

Lemma 14. *Let* (32) *be satisfied. Then for* R *sufficiently large* V *satisfies* $\mathscr{H}(4)$ *on* q *with* $D = D_R$.

Proof. Since $V_- \le \sum_{i=1}^N (v_{oi})_-$, it suffices to prove that each v_{oi} satisfies $\mathscr{H}(4)$.

Let $D_R^i = (-R/\sqrt{2m_i}, R/\sqrt{2m_i})^\nu$, the ν-dimensional cube in $\mathbb{R}^\nu$, and choose R to be sufficiently large that $D_R \supset B_X(0, 1)$, the unit ball in X, and $(v_{oi})_- \in L^\infty(\mathbb{R}^\nu \backslash B_\nu(0, (R-1)/\sqrt{2m_i}))$ in accordance with (32)(i), where $B_\nu(0, r)$ denotes the ball center 0 and radius r in $\mathbb{R}^\nu$. Since $(v_{oi})_- \in M_{\text{loc}}(\mathbb{R}^\nu)$ by (32)(iii) we obtain from Schechter [23, Theorem 7.3 on p. 138] (see also Lemma 0.3 in Agmon [1, p. 10]) that for any $\varepsilon > 0$, there exists a constant $K(\varepsilon, R)$ such that

$$\int_{D_R^i} (v_{oi})_- |w\psi|^2 \, dx^i \le \varepsilon \int_{D_R^i} |\nabla_i w\psi|^2 \, dx^i + K(\varepsilon, R) \int_{D_R^i} |w\psi|^2 \, dx^i \tag{39}$$

for all $\psi \in C_0^\infty(X)$ which have supports in D_R^i as functions of x^i (i.e. when x^j, $j \ne i$, are fixed). Note that w is now defined with respect to $D = D_R$ and hence is 1 when $x \in D_R$ and J_2, a C^∞ function, otherwise.

The next step is to construct a partition of unity $\{h_1, h_2\}$ in $\mathbb{R}^\nu$ as in Lemma 7 of Appendix 2. These functions lie in $C^\infty(\mathbb{R}^\nu)$ and satisfy

$$\begin{aligned}
&\text{(i)} && h_1^2 + h_2^2 = 1 \quad \text{in } \mathbb{R}^\nu, \\
&\text{(ii)} && \operatorname{supp} h_1 \subset B_\nu(0, R/\sqrt{2m_i}), \\
&\text{(iii)} && h_1 = 1 \quad \text{in } B_\nu\left(0, \frac{R-1}{\sqrt{2m_i}}\right), \\
&\text{(iv)} && |\nabla_i h_1|^2 + |\nabla_i h_2|^2 \quad \text{is bounded in } \mathbb{R}^\nu.
\end{aligned} \tag{40}$$

It follows that

$$\sum_{l=1}^2 |\nabla_i (h_l \varphi)|^2 = |\nabla_i \varphi|^2 + |\varphi|^2 \left(\sum_{l=1}^2 |\nabla_i h_l|^2 \right). \tag{41}$$

Thus, for $\varphi \in C_0^\infty(X)$, $h_1\varphi$ has support in $B_\nu(0, R/\sqrt{2m_i}) \subset D_R^i$ as a function of x^i, and for any $\varepsilon > 0$,

$$\begin{aligned}
\int_{\mathbb{R}^\nu} (v_{oi})_- |w\varphi|^2 \, dx^i &= \int_{B_\nu(0, R/\sqrt{2m_i})} (v_{oi})_- |h_1 w\varphi|^2 \, dx^i \\
&\quad + \int_{\mathbb{R}^\nu \backslash B_\nu(0, (R-1)/\sqrt{2m_i})} (v_{oi})_- |h_2 w\varphi|^2 \, dx_i \\
&\le \frac{\varepsilon}{(2m_i)} \int_{D_R^i} |\nabla_i (h_1 w\varphi)|^2 \, dx^i + K(\varepsilon, R) \int_{D_R^i} |w h_1 \varphi|^2 \, dx^i \\
&\quad + K(\varepsilon, R) \int_{\mathbb{R}^\nu} |h_2 w\varphi|^2 \, dx^i
\end{aligned}$$

since $(v_{oi})_-$ is bounded outside $B_\nu(0, (R-1)/\sqrt{2m_i})$ in $\mathbb{R}^\nu$,

$$\leq \frac{\varepsilon}{(2m_i)} \int_{D_R^i} |\nabla_i(w\varphi)|^2 \, dx^i + K(\varepsilon, R) \int_{\mathbb{R}^\nu} |w\varphi|^2 \, dx^i$$
$$\leq \varepsilon \int_{D_R^i} |\nabla_X(w\varphi)|_X^2 \, dx^i + K(\varepsilon, R) \int_{\mathbb{R}^\nu} |w\varphi|^2 \, dx^i$$

on using (40)(i), (iv) and (41). On integrating with respect to x^j, $j \neq i$, over $\mathbb{R}^\nu$ we obtain

$$\int_X (v_{oi})_- |w\varphi|^2 \, d_X x \leq \varepsilon \int_X |\tilde{\nabla}_X(w\varphi)|_X^2 \, d_X x + K(\varepsilon, R) \int_X |w\varphi|^2 \, d_X x$$

and the lemma is proved.

We denote by $\rho_X[\varphi, \psi : \Omega]$ the form (30) but with the integration over a set $\Omega \subset X$, and for $k = 1, 2, \ldots, N$ we define

$$(42) \quad \tau_k[\varphi, \psi : \Omega] := \int_\Omega \left\{ (2m_k)^{-1} \langle \nabla_k \varphi, \nabla_k \psi \rangle + \left[v_{ok}(x^k) + \sum_{\substack{i<j \\ i=k \text{ or } j=k}} v_{ij}(x^i - x^j) \right] \varphi \overline{\psi} \right\} d_X x$$

for $\varphi, \psi \in C_0^\infty(X)$.

Lemma 15. *Let* (32) *be satisfied. Then for* $\delta(k, j)$ *sufficiently small*

$$\rho_X[\varphi : \Gamma_j^k \setminus D_R] \geq \Sigma(P) \int_{X \setminus D_R} |\varphi|^2 \, d_X x + \tau_k[\varphi : \Gamma_j^k \setminus D_R]$$

for all $\varphi \in C_0^\infty(\Gamma_j^k)$.

Proof. First we show that for $\delta(k, j)$ sufficiently small

$$(43) \qquad \Gamma_j^k \setminus D_R = \Gamma_j^k \setminus \{x \in X : |x_j^k| < R/\sqrt{2m_k}\}.$$

The set on the right in (43) is obviously contained in that on the left. Let $x \in \Gamma_j^k \setminus D_R$. Then

$$\sqrt{2m_k}|x_j^k| = |\langle x, \omega(k, j)\rangle_X| > |x|_X \cos\delta(k, j)$$

and since $x \notin D_R$,

$$|x_l^i| \geq R/\sqrt{2m_i}$$

for some i and l. Thus

$$|x_j^k|^2 > (m_i|x_l^i|^2 + m_k|x_j^k|^2) \frac{\cos^2 \delta(k, j)}{m_k}$$

whence

$$|x_j^k|^2 > \left\{ \frac{m_i}{m_k} \frac{\cos^2 \delta(k, j)}{(1 - \cos^2 \delta(k, j))} \right\} |x_l^i|^2 \geq R^2/2m_k$$

for $\delta(k, j)$ small enough. Thus (43) is proved.

Now we set $Y = \bigotimes_{i=1}^{N-1} \mathbb{R}^\nu \cong X_{\omega(k,j)}$. We may view any $x \in X$ as $x = (y, z)$ where $y \in Y$ and $z = x^k \in Z = \mathbb{R}^\nu$. We shall also denote this decomposition as $X = Y \otimes Z$. If Π_Y denotes the projection of X onto Y we then have from (43) that

$$\Gamma_j^k \setminus D_R = (\Pi_Y(\Gamma_j^k)) \otimes \{z \in Z : |z_j| > R/\sqrt{2m_k}\}.$$

Hence, for any $\varphi \in C_0^\infty(\Gamma_j^k)$,

$$\begin{aligned}
&\rho_X[\varphi : \Gamma_j^k \setminus D_R] - \tau_k[\varphi : \Gamma_j^k \setminus D_R] \\
&\quad = (2m_k)^{\nu/2} \int_{|z_j| > R/\sqrt{2m_k}} \rho_{X_{\omega(k,j)}}[\varphi(\cdot, z) : \Pi_Y(\Gamma_j^k)]\, dz \\
&\quad \geq (2m_k)^{\nu/2} \int_{|z_j| > R/\sqrt{2m_k}} K(\omega(k, j) : P) \|\varphi(\cdot, z)\|_{L^2(Y)}^2\, dz
\end{aligned}$$

by Lemma 13(i),

$$\geq \Sigma(P) \int_{X \setminus D_R} |\varphi|^2\, d_X x$$

by Lemma 13(ii). The lemma is therefore proved.

We are now in a position to prove the main theorem in this section which follows as a corollary of Theorem 12. In an heuristic sense the theorem states that, in the absence of an Efimov-type effect, there will be no more than a finite number of bound states provided that (for each $i = 1, \ldots, N$) the repulsive strength of the ith electron with the other electrons is larger than the binding strength of the nucleus when the ith electron is at a much greater distance from the nucleus than any other electron.

Theorem 16. *Suppose that V satisfies* (32) *and that $\mathscr{A}(1)$ holds. For $i = 1, 2, \ldots, N$, and $j = 1, 2, \ldots, \nu$, let Γ_j^i be defined in* (38) *and let D_R be defined in* (37). *Suppose that R is sufficiently large and $\delta(i, j)$ sufficiently small for Lemmas* 14 *and* 15 *to be satisfied. If there exist $\varepsilon_1, \varepsilon_2 \in (0, 1)$ such that*

$$\begin{aligned}
&\tau_i[\varphi : \Gamma_j^i \setminus D_R] - \varepsilon_1 \int_{X \setminus D_R} |x|^{-2} |\varphi(x)|^2\, d_X x \\
&\quad \geq -\varepsilon_2 \int_{D_R} |\nabla_i \varphi|^2\, d_X x - C_{\varepsilon_2} \int_{D_R} |\varphi|^2\, d_X x
\end{aligned} \tag{44}$$

for all $\varphi \in C_0^\infty(\Gamma_j^i)$ and all values of i, j, then H_X has only a finite number of bound states.

Proof. We have already noted that (32) implies that ρ satisfies the hypothesis of Proposition 1. Also $\mathscr{A}(1)$ and Lemma 13 imply that $\mathscr{H}(1)$ is satisfied while Lemma 14 establishes $\mathscr{H}(4)$. Lemma 15 and (44) yield

$$\rho_X[\varphi : \Gamma_j^i \setminus D_R] - \varepsilon_1 \int_{X \setminus D_R} |x|^{-2} |\varphi(x)|^2 \, d_X x$$
$$\geq \Sigma(P) \int_{X \setminus D_R} |\varphi|^2 \, d_X x - \varepsilon_2 \int_{D_R} |\nabla \varphi|^2 \, d_X x - C_{\varepsilon_2} \int_{D_R} |\varphi|^2 \, d_X x .$$

Hence the hypothesis $\mathscr{H}(2')$, which can replace $\mathscr{H}(2)$ and $\mathscr{H}(3)$ in Theorem 12, is satisfied and the theorem is proved.

We express the hypothesis (44) thus, rather than in terms of a function σ on ∂D_R in accordance with $\mathscr{H}(2)$ and $\mathscr{H}(3)$, because it is easier to verify in the examples discussed in the next section.

Theorem 16 is comparable to Theorem 4.3 of Sigal [25] or Theorem 3.2.3 in Cycon, Froese, Kirsch, and Simon [4]. There the hypothesis corresponding to $\mathscr{A}(1)$ is that $\inf \sigma_e(H_X)$ is attained only by two-cluster breakups and he also assumes that a one-body Hamiltonian of the form $-(1-\eta)\Delta + W_a^\delta$ on $L^2(X_a) \simeq L^2(\mathbb{R}^\nu)$ has only a finite number of bound states for all $\delta > 0$ and some $\eta \in (0, 1)$; the potential W_a^δ is given by

$$W_a^\delta := \langle \psi^a, I_a \chi_a \psi^a \rangle + \delta^{-1} \{ \langle \psi^a, (I_a \chi_a)^2 \psi^a \rangle - \langle \psi^a, I_a \chi_a \psi^a \rangle^2 \},$$

where for a given cluster decomposition a, I_a is the intercluster interaction and ψ^a is a certain normalized ground state related to the internal Hamiltonian of the cluster decomposition a: the function χ_a is the characteristic function of the $\operatorname{supp} j_a$ where j_a is an element of a Deift–Agmon–Sigal partition of unity corresponding to the cluster decomposition a. We refer the reader to [4, 22] for explanation of these terms and further details. Sigal notes in [25] that the above result is essentially due to Zhislin and his colleagues [2, 31, 35, 37, 38]. Also it applies to the general N-body problem of which P in (28) is a special case (see [1, pp. 78–80]). Two applications of the result are given in [4, 25], the first to N-body short-range potentials and the other proving Zhislin's result in [37] mentioned in §1. However, as mentioned in [4], the theorem is "abstract" in the sense that applications are very difficult requiring knowledge of the ground states ψ^a, etc. Our alternative hypothesis (44) appears to be much more amenable to applications as we demonstrate in §4.

4. Examples

Let

$$V_i(x) := v_{oi}(x^i) + \sum_{\substack{k<l \\ k=i \text{ or } l=i}} v_{kl}(x^k - x^l) \tag{45}$$

denote the potential in $\tau_i[\cdot]$. We now give examples of V_i for which (44) is satisfied. Of course, Theorem 16 requires that V satisfy condition (32) as well as $\mathscr{A}(1)$.

Example 1. Suppose that

$$V_i(x) = v_{oi}(|x^j|) + \sum_{\substack{k=1\\k\neq i}}^{N} v_k(|x^i - x^k|), \tag{46}$$

where the v_k are nonincreasing and for some $\delta > 0$

$$\liminf_{|x^i|\to\infty} \left\{ |x^i|^2 v_{oi}(|x^i|) + \sum_{\substack{k=1\\k\neq i}}^{N} v_k([1+\delta]|x^i|) \right\} > 0. \tag{47}$$

Then (44) is satisfied for R sufficiently large and $\delta(i, j)$ sufficiently small.

Proof. If $x \in \Gamma_j^i$ we have, for any $k \neq i$,

$$2m_i|x_j^i|^2 > (2m_i|x^i|^2 + 2m_k|x^k|^2)\cos^2\delta(i, j)$$

whence

$$|x^k| < \left[\left(\frac{m_i}{m_k}\right)^{1/2} \tan\delta(i, j)\right] |x^i| < \delta|x^i|$$

for $\delta(i, j)$ small enough. Also, by (43), any $x \in \Gamma_j^i D_R$ satisfies

$$|x^i| \geq |x_j^i| \geq R/\sqrt{2m_i}.$$

From (47) there exist positive constants ε_1, R_0 such that

$$v_{oi}(|x^i|) + \sum_{\substack{k=1\\k\neq i}}^{N} v_k([1+\delta]|x^i|) > \frac{2\varepsilon_1}{|x^i|^2}$$

for $|x^i| > R_0$. Thus, from $|x^i - x^k| < (1+\delta)|x^i|$ and the assumption that the v_k are nonincreasing, we obtain

$$v_i(x) - \frac{\varepsilon_1}{|x|^2} \geq v_{oi}(|x^i|) + \sum_{\substack{k=1\\k\neq i}}^{N} v_k([1+\delta]|x^i|) - \frac{\varepsilon_1}{|x^i|^2} \geq \frac{\varepsilon_1}{|x^i|^2}$$

for all $x \in \Gamma_j^i \setminus D_R$ with R large enough. The assertion is therefore proved.

An important special case of Example 1 is

Example 2. Let $Z \in \mathbb{R}$ and

$$V_i(x) \geq \frac{-Z}{|x^i|^{\alpha_i}} + \sum_{\substack{k=1\\k\neq i}}^{N} \frac{1}{|x^i - x^k|^{\beta_i}}. \tag{48}$$

Then (47) is satisfied in each of the following cases:

(i) $0 \leq \beta_i < \alpha_i$,
(ii) $Z < N - 1$ and $0 \leq \beta_i = \alpha_i$.

(Condition 32(iii) further requires that each $\alpha_i < 2$.)

Proof. For any $\delta > 0$ we have in the notation of (46)

$$\begin{aligned} v_{oi}(|x^i|) + \sum_{\substack{k=1\\k\neq i}}^{N} v_k([1+\delta]|x^i|) \\ \geq \frac{1}{|x^i|^{\alpha_i}}\left\{-Z + \sum_{k\neq i}(1+\delta)^{-\beta_i}|x^i|^{\alpha_i-\beta_i}\right\} \\ = \frac{1}{|x^i|^{\alpha_i}}\{-Z + (N-1)(1+\delta)^{-\beta_i}|x^i|^{\alpha_i-\beta_i}\}. \end{aligned}$$

It now easily follows that (47) is satisfied in both cases for δ small enough.

The case $\alpha_i = \beta_i = 1$ of (ii) is the result of Uchiyama [30] and Zhislin [37] mentioned earlier. To deal with $Z = N-1$ when $\alpha_i = \beta_i = 1$, it appears that properties of the ground state of an N-body operator are needed; see [25]. In order that each $(v_{oi})_- \in M_{\mathrm{loc}}(\mathbb{R}^\nu)$, as required in 32(iii), each α_i must be strictly less than 2 (when $\nu > 2$).

In the remaining examples, Π_i denotes the projection of X onto the ith copy of the $\mathbb{R}^\nu$. Hence, with $\{e(1), \dots, e(\nu)\}$ the canonical basis for $\mathbb{R}^\nu$,

$$\Pi_i(\Gamma^i_j/D_R) = \left\{x^i : \langle x^i, e(j)\rangle > |x^i|\cos\delta(i,j),\ |x^i| > \frac{R}{\sqrt{2m_i}}\right\},$$

$$D^i_R := \Pi_i D_R = \left(-\frac{R}{\sqrt{2m_i}}, \frac{R}{\sqrt{2m_i}}\right)^\nu$$

and $-e(j)$ is the outward normal to $\Pi_i(\Gamma^i_j) \setminus D^i_R$.

Example 3. If $\nu > 2$ and

$$\liminf_{R\to\infty}\ \inf_{x\in\Gamma^i_j\setminus D_R} |x^i|^2 V_i(x) > -\frac{(\nu-2)^2}{8m_i} \tag{49}$$

then (44) is satisfied for R large enough.

Proof. The crucial step is the result obtained from the Friedrichs inequality in [16, Theorem 1] with the choice of $g(x^i) = \ln|x^i|$. It is that for all $\varphi \in C_0^\infty(\Gamma^i_j)$

$$\begin{aligned} \frac{1}{2m_i}\int_{\Pi_i(\Gamma^i_j)\setminus D^i_R} |\nabla_i\varphi|^2\,dx^i \geq \frac{(\nu-2)^2}{8m_i}\int_{\Pi_i(\Gamma^i_j)\setminus D^i_R} |x^i|^{-2}|\varphi(x)|^2\,dx^i \\ + \frac{(\nu-2)}{8m_i}\int_{\partial D^i_R\cap\Pi_i(\Gamma^i_j)} \frac{x^i\cdot e(j)}{|x^i|^2}|\varphi(s)|^2\,ds. \end{aligned}$$

Since $|x^i| = R/\sqrt{2m_i}$ on $\partial D_R^i \cap \Pi_i(\Gamma_j^i)$, we have that

$$\begin{aligned}
&\int_{\Pi_i(\Gamma_j^i)\setminus D_R^i} \left\{ \frac{1}{2m_i}|\nabla_i\varphi|^2 + V_i(x)|\varphi|^2 \right\} dx^i \\
&\quad \geq \int_{\Pi_i(\Gamma_j^i)\setminus D_R^i} \left\{ V_i(x) + \frac{(\nu-2)^2}{8m_i|x^i|^2} \right\} |\varphi|^2\, dx^i - \frac{(\nu-2)}{4\sqrt{2m_i}R} \int_{\partial D_R^i \cap \Pi_i(\Gamma_j^i)} |\varphi|^2\, ds \\
&\quad > \varepsilon_1 \int_{\Pi_i(\Gamma_j^i)\setminus D_R^i} |x|^2|\varphi|^2\, dx^i - \int_{\partial D_R^i \cap \Pi_i(\Gamma_j^i)} |\varphi|^2\, ds
\end{aligned}$$

for some $\varepsilon_1 > 0$ and R large enough, on using the hypothesis. For any $\varepsilon_2 > 0$ there exists $C_{\varepsilon_2} > 0$ such that

$$\int_{\partial D_R^i \cap \Pi_i(\Gamma_j^i)} |\varphi|^2\, ds \leq \int_{\Pi_i(\Gamma_j^i)\cap D_R^i} (\varepsilon_2|\nabla_i\varphi|^2 + C_{\varepsilon_2}|\varphi|^2)\, dx^i .$$

Hence, in all we have

$$\begin{aligned}
&\int_{\Pi_i(\Gamma_j^i)\setminus D_R^i} \left\{ \frac{1}{2m_i}|\nabla_i\varphi|^2 + \left(V_i(x) - \frac{\varepsilon_1}{|x|^2} \right) |\varphi|^2 \right\} dx^i \\
&\qquad \geq -\varepsilon_2 \int_{\Pi_i(\Gamma_j^i)\cap D_R^i} |\nabla_i\varphi|^2\, dx^i - C_{\varepsilon_2} \int_{\Pi_i(\Gamma_j^i)\cap D_R^i} |\varphi|^2\, dx^i .
\end{aligned} \tag{50}$$

From (43), if $x \in \Gamma_j^i$ and $x^i \in D_R^i$ then we must have $x \in D_R$ if $\delta(i, j)$ is sufficiently small. Hence, for $x^i \in D_R^i$ fixed, the support of any $\varphi \in C_0^\infty(\Gamma_j^i)$ lies in D_R. Consequently, on integrating (50) with respect to x^k for $k \neq i$, we obtain (44).

Example 4. Let $\nu > 2$, $f \in C^2(\mathbb{R}^\nu)$ and for $x \in \Gamma_j^i \setminus D_R$ suppose that for some $\varepsilon_1 > 0$

$$(V_i(x))_- + \frac{\varepsilon_1}{|x|^2} \leq \Delta_i f(x^i) \tag{51}$$

and

$$\lim_{R\to\infty} \sup_{x^i \in \Pi_i(\Gamma_j^i)\setminus D_R^i} |x^i||\nabla_i f(x^i)| < \frac{\nu-2}{8m_i} . \tag{52}$$

Then (44) is satisfied for R large enough.

Proof. The proof is based on that of [10, Theorem 9]. Choose R sufficiently large that

$$|x^i|\,|\nabla_i f(x^i)| < \frac{\nu-2}{8m_i}$$

for $x \in \Gamma_j^i \setminus D_R$. Then for $\varphi \in C_0^\infty(\Gamma_j^i)$

$$\begin{aligned}
&\int_{\Pi_i(\Gamma_j^i)\setminus D_R^i} \left[(V_i(x))_- + \frac{\varepsilon}{|x|^2} \right] |\varphi(x)|^2 \, dx^i \\
&\quad \le \int_{\Pi_i(\Gamma_j^i)\setminus D_R^i} \Delta_i f(x^i) |\varphi(x)|^2 \, dx^i \\
&\quad \le -\int_{\Pi_i(\Gamma_j^i)\cap \partial D_R^i} \nabla_i f \cdot e(j) |\varphi|^2 \, ds + 2 \int_{\Pi_i(\Gamma_j^i)\setminus D_R^i} |\nabla_i f| |\nabla_i \varphi| |\varphi| \, dx^i \\
&\quad \le \int_{\Pi_i(\Gamma_j^i)\cap \partial D_R^i} |\nabla_i f| |\varphi|^2 \, dx + \frac{(\nu-2)}{4m_i} \int_{\Pi_i(\Gamma_j^i)\setminus D_R^i} |x^i|^{-1} |\nabla_i \varphi| |\varphi| \, dx^i \\
&\quad \le \int_{\Pi_i(\Gamma_j^i)\cap \partial D_R^i} \left[|\nabla_i f| - \frac{(\nu-2)}{4m_i} (x^i \cdot e(j)) |x^i|^{-2} \right] |\varphi|^2 \, ds \\
&\qquad + \frac{1}{2m_i} \int_{\Pi_i(\Gamma_j^i)\setminus D_R^i} |\nabla_i \varphi|^2 \, dx^i .
\end{aligned}$$

Hence

$$\int_{\Pi_i(\Gamma_j^i)\setminus D_R^i} \left\{ \frac{1}{2m_i} |\nabla_i \varphi|^2 + \left(V_i - \frac{\varepsilon_1}{|x|^2} \right) |\varphi|^2 \right\} dx^i \ge \int_{\Pi_i(\Gamma_j^i)\cap \partial D_R^i} \sigma |\varphi|^2 \, ds ,$$

where σ is bounded. The rest follows as in Example 3.

Our next example is an application of Example 4.

Example 5. Let $\nu > 2$ and suppose that there is a function $Q \in L^1_{\mathrm{loc}}(0, \infty)$ such that for $x \in \Gamma_j^i \setminus D_R$

$$(V_i(x))_- \le Q(|x^i|) \tag{53}$$

and

$$\lim_{R\to\infty} \sup_{|x^i| > R/\sqrt{2m_i}} |x^i|^{2-\nu} \int_R^{|x^i|} t^{\nu-1} Q(t) \, dt < \frac{\nu-2}{8m_i} . \tag{54}$$

Then (51) and (52) are satisfied.

Proof. From (54) there exist positive numbers ε_0, R_0 such that

$$|x^i|^{2-\nu} \int_R^{|x^i|} t^{\nu-1} Q(t) \, dt + 2\varepsilon_0 < \frac{\nu-2}{8m_i}$$

for $x \in \Gamma_j^i \setminus D_R$ and $R \ge R_0$. Let

$$f(t) = \int_{R_0}^t s^{1-\nu} \int_{R_0}^s u^{\nu-1} Q(u) \, du \, ds + \varepsilon_0 \ln t .$$

Then

$$|x^i| |\nabla_i f(|x^i|)| = |x^i|^{2-\nu} \left| \int_{R_0}^{|x^i|} t^{\nu-1} Q(t) \, dt + \varepsilon_0 \right|$$

and

$$\Delta_i f(|x^i|) = Q(|x^i|) + (\nu - 2)\varepsilon_0 |x^i|^{-2}.$$

Thus (51) and (52) are satisfied.

The final example gives the result of Zhislin and colleagues on short-range potentials.

Example 6. Let $\nu \geq 3$ and in (45) suppose that $(v_{oi}) \in L^{\nu/2}(\mathbb{R}^\nu)$ and $v_{kl} \geq 0$ for $k \neq 0$. Then H_X has only a finite number of bound states if $\mathscr{A}(1)$ is satisfied.

Proof. Let $\varphi \in C_0^\infty(\Gamma_j^i)$. By Hölder's inequality and Sobolev's Embedding Theorem (see [6, Theorem III.3.6]) we obtain

$$\begin{aligned}\int_{\mathbb{R}^\nu \setminus D_R^i} (v_{oi})_- |\varphi|^2 \, dx^i &\leq \left(\int_{\mathbb{R}^\nu \setminus D_R^i} (v_{oi})_-^{\nu/2} \, dx^i \right)^{2/\nu} \left\{ \int_{\mathbb{R}^\nu} |\varphi|^{2\nu/(\nu-2)} \, dx^i \right\}^{1-2/\nu} \\ &\leq \gamma \left(\int_{\mathbb{R}^\nu \setminus D_R^i} (v_{oi})_-^{\nu/2} \, dx^i \right)^{2/\nu} \int_{\mathbb{R}^\nu} |\nabla_i \varphi|^2 \, dx^i\end{aligned}$$

for some constant γ. Given $\varepsilon > 0$ we choose $R = R(\varepsilon)$ such that

$$\gamma \left(\int_{\mathbb{R}^\nu \setminus D_R^i} (v_{oi})_-^{\nu/2} \, dx^i \right)^{2/\nu} < \varepsilon / 2m_i \qquad (i = 1, \ldots, N).$$

Then, we have

$$\int_{\mathbb{R}^\nu \setminus D_R^i} (v_{oi})_- |\varphi|^2 \, dx^i < \varepsilon / 2m_i \int_{\mathbb{R}^\nu} |\nabla_i \varphi|^2 \, dx^i. \tag{55}$$

Also, Hardy's inequality gives

$$\int_{\mathbb{R}^\nu} \frac{|\varphi|^2}{|x|^2} \, dx^i \leq \frac{4}{(\nu - 2)^2} \int_{\mathbb{R}^\nu} |\nabla_i \varphi|^2 \, dx^i. \tag{56}$$

Hence, for all $\varphi \in C_0^\infty(\Gamma_j^i)$,

$$\begin{aligned}&\int_{\mathbb{R}^\nu \setminus D_R^i} \left\{ \frac{1}{2m_i} |\nabla_i \varphi|^2 + \left[V_i(x) - \frac{\varepsilon}{|x|^2} \right] |\varphi|^2 \right\} dx^i \\ &\quad \geq \left(1 - \varepsilon - \frac{8m_i}{(\nu - 2)^2} \varepsilon \right) \int_{\mathbb{R}^\nu \setminus D_R^i} \frac{1}{2m_i} |\nabla_i \varphi|^2 \, dx^i \\ &\qquad - \frac{\varepsilon}{2m_i} \int_{D_R^i} |\nabla_i \varphi|^2 \, dx^i - \frac{4}{(\nu - 2)^2} \varepsilon \int_{D_R^i} |\nabla_i \varphi|^2 \, dx^i\end{aligned}$$

which yields (44) on choosing $\varepsilon_1 = \varepsilon < \left(1 + \frac{8m_i}{(\nu-2)^2}\right)^{-1}$ and $\varepsilon_2 = \left(\frac{1}{2m_i} + \frac{4}{(\nu-2)^2}\right)\varepsilon$; recall that in view of (43), if the $\delta(i, j)$ are small enough, any $x \in \Gamma_j^i$ with $x^i \in D_R^i$ must lie in D_R.

We must next show that $\mathscr{H}(4)$ is satisfied under the present hypothesis. First, we write $(v_{oi})_- = h_{1i} + h_{2i}$, where $h_{1i} \in L^\infty(\mathbb{R}^\nu)$ and $\left(\int_{\mathbb{R}^\nu} h_{2i}^{\nu/2}\, dx^i\right)^{2/\nu} < \varepsilon/2m_i$. It then follows that for all $\varphi \in C_0^\infty(X)$

$$\begin{aligned}\int_{\mathbb{R}^\nu} (v_{oi})_- |J_2\varphi|^2\, dx^i &= \int_{\mathbb{R}^\nu} (h_{1i} + h_{2i}) |J_2\varphi|^2\, dx^i \\ &\le \|h_{1i}\|_{L^\infty(\mathbb{R}^\nu)} \|J_2\varphi\|^2_{\mathbb{R}^\nu} + \gamma \left(\int h_{2i}^{\nu/2}\, dx^i\right)^{2/\nu} \|\nabla_i(J_2\varphi)\|^2_{\mathbb{R}^\nu} \\ &\le \frac{\varepsilon}{2m_i}\gamma \|\nabla_1(J_2\varphi)\|^2_{\mathbb{R}^\nu} + C(\varepsilon)\|(J_2\varphi)\|^2_{\mathbb{R}^\nu}.\end{aligned}$$

Also, on using the Sobolev Embedding Theorem for $H^1(D_R^i) \hookrightarrow L^{2\nu/\nu-2}(D_R^i)$ (see [6, Theorem V.4.13])

$$\begin{aligned}\int_{D_R^i} (v_{oi})_- |\varphi|^2\, dx^i &\le \|h_{1i}\|_{L^\infty(\mathbb{R}^\nu)} \|\varphi\|^2_{D_R^i} + \frac{\varepsilon\gamma'}{2m_i}\{\|\nabla_i\varphi\|^2_{D_R^i} + \|\varphi\|^2_{D_R^i}\} \\ &< \frac{\varepsilon\gamma'}{2m_i}\|\nabla_i\varphi\|^2_{D_R^i} + C(\varepsilon)\|\varphi\|^2_{D_R^i}.\end{aligned}$$

On integrating with respect to the other variables x^j, $j \ne i$, we see that $\mathscr{H}(4)$ is satisfied. Note also that $1(\mathrm{iv})'$ has been established. The result is therefore proved.

5. Appendices

Appendix 1. *Proof of Proposition* 1.

Let $\Lambda_R(x : P)$ be the function defined in Definition 3. In [1, Lemma 2.3] Agmon proves that given $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\rho[\varphi] \ge \int_{\mathbb{R}^n} (\Lambda_R(x, P) - \varepsilon)|\varphi(x)|^2]\, dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ and any $R \ge R_\varepsilon$. From this and Proposition 4(ii) it follows easily that ρ is bounded below (see [1, p. 46]). It is clear that ρ is densely defined in $L^2(\mathbb{R}^n)$ and symmetric. Hence we only need to show that ρ is closable.

Without loss of generality we assume for the rest of the proof that

$$\rho[\varphi] \ge \|\varphi\|^2.$$

Then $\rho[\cdot]^{1/2}$ is a norm on $C_0^\infty(\mathbb{R}^n)$; this is the norm denoted by $|||\cdot|||$ by Agmon in the proof of his Theorem 3.2 in [1]. Let V be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to $\rho[\cdot]^{1/2}$. The key fact is that proved by Agmon in [1, Lemma 3.1], namely that there exists a positive continuous function k on $\mathbb{R}^n$ for which

$$\rho[\varphi] + \|\varphi\|^2 \ge \int_{\mathbb{R}^n} k(x)[|\nabla_A\varphi(x)|^2 + |q(x)||\varphi(x)|^2]\, dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. It follows that for any compact set $\Omega \subset \mathbb{R}^n$

$$2\rho[\varphi] \geq \rho[\varphi] + \|\varphi\|^2 \geq K(\Omega) \int_\Omega [|\nabla_A \varphi|^2 + |q||\varphi|^2]\, dx\,; \tag{57}$$

where

$$K(\Omega) = \min_\Omega k(x) > 0\,.$$

Let $\{\varphi_m\}$ be a sequence of $C_0^\infty(\mathbb{R}^n)$-functions which is ρ-convergent to 0, i.e. $\|\varphi_n\| \to 0$ and $\rho[\varphi_n - \varphi_m] \to 0$ as $n, m \to \infty$. To prove that ρ is closable we must show that $\rho[\varphi_m] \to 0$. From (57) and the fact that $A(x) \geq \mu(x)I$, where μ is a positive continuous function, it follows that $\{\varphi_m\}$ is Cauchy in $H^1(\Omega)$ and also in the weighted space $L^2(\Omega; |q|dx)$. Since $\varphi_m \to 0$ in $L^2(\mathbb{R}^n)$ we have that $\varphi_m \to 0$ in both $H^1(\Omega)$ and $L^2(\Omega; |q|dx)$. Hence, as A is bounded on $\mathbb{R}^n$, we conclude that

$$\int_\Omega [\nabla_A \varphi_m|^2 + |q||\varphi_m|^2] \to 0 \tag{58}$$

and this is true for any compact subset Ω of $\mathbb{R}^n$. Let ψ denote the limit of $\{\varphi_m\}$ in V and, for $\varphi \in C_0^\infty(\mathbb{R}^n)$, set $\Omega = \operatorname{supp}\varphi$. Then

$$\rho[\psi, \varphi] = \lim_{m\to\infty} \rho[\varphi_m, \varphi] \to 0$$

from (58). This proves that $\psi = 0$ in V and hence the proof is complete.

Appendix 2. *Proof of Lemma* 7.

Choose $\varphi_1, \varphi_2 \in C^\infty(0, \infty)$ satisfying the conditions

$$\varphi_1(t), \varphi_2(t) \in [0, 1] \quad \text{for } t \in [0, \infty),$$
$$\varphi(t) = \begin{cases} 1 & t \in [(m+l)/2, \infty), \\ 0, & t \in [0, l], \end{cases}$$

and

$$\varphi_2(t) = \begin{cases} 1, & t \in [0, (m+l)/2), \\ 0, & t \in [m, \infty), \end{cases}$$

where $0 < l < m < \infty$. Define

$$h_i(t) = \varphi_i(t)[\varphi_1^2(t) + \varphi_2^2(t)]^{-1/2}, \qquad i = 1, 2.$$

Then

$$h_1 = \begin{cases} 1, & t \geq m, \\ 0, & t < l, \end{cases}$$
$$h_i(t) \in [0, 1] \quad \text{for } t \geq 0,\ i = 1, 2, \text{ and}$$
$$h_1^2(t) + h_2^2(t) \equiv 1 \quad \text{for } t \geq 0.$$

For ω_k and δ_k given in the hypothesis let $m = \cos\delta_k/2$ and $l = \cos\delta_k$. The function $h_1(\langle x, \omega_k\rangle/|x|)$ is a $C^\infty(\mathbb{R}^n \setminus \{0\})$-function whose support lies in the

cone $\Gamma(\omega_k : \delta_k, 0)$. In $\Gamma(\omega_k : \delta_k/2, 0)$ the function is identically equal to 1. Define

$$j_1(x) = \sum_{k=1}^{m} h_1(\langle x, \omega_k \rangle / |x|)$$

and

$$j_2(x) = [1 - j_1^2(x)]^{1/2}$$

for $x \neq 0$. Since the cones $\{\Gamma(\omega_k : \delta_k, 0)\}_{k=1}^{m}$ are mutually disjoint then

$$j_1^2(x) = \sum_{k=1}^{m} h_1^2(\langle x, \omega_k \rangle / |x|)$$

and for $x \neq 0$

$$j_2(x) = \begin{cases} 1, & x \in \mathbb{R}^n \setminus \bigcup_{k=1}^{m} \Gamma(\omega_k : \delta_k, 0), \\ 0, & x \in \bigcup_{k=1}^{m} \Gamma(\omega_k : \delta_k/2, 0), \\ h_2(\langle x, \omega_k \rangle / |x|), & x \in \Gamma(\omega_k : \delta_k, 0), \ k = 1, \dots, m. \end{cases}$$

Hence, $j_1, j_2 \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and $j_1^2 + j_2^2 \equiv 1$ for $x \neq 0$.

Let $\psi(x) \in C_0^\infty(B(0, 3/4))$ with $\psi(x) \equiv 1$ for $x \in B(0, 1/2)$ and $\psi(x) \in [0, 1]$ for all x. Define

$$f_i(x) = (1 - \psi(x)) j_i(x) \quad \text{for } i = 1, 2.$$

Then

$$f_1^2(x) + f_2^2(x) = \begin{cases} 1, & |x| \geq 3/4 \\ 0, & |x| \leq 1/2. \end{cases}$$

Let $f_0(x) \in C_0^\infty(B(0, 1))$ with $f_0(x) \equiv 1$ for $x \in B(0, 3/4)$ and $f_0(x) \in [0, 1]$ for all x. Finally, define

$$J_i(x) = f_i(x)[f_0^2(x) + f_1^2(x) + f_2^2(x)]^{-1/2}$$

for $i = 0, 1, 2$. Conditions (i)–(iv) of the conclusion of Lemma 7 are clearly satisfied. Next, we show that condition (v) is satisfied as well.

By condition (iv) of Lemma 7, ∇J_i is homogeneous of degree -1 for $|x| > 1$ and $i = 1, 2$. Hence, it will suffice to show that (v) holds on $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Let

$$C_k = \Gamma(\omega_k : \delta_k/2, \tfrac{1}{2}) \cap S^{n-1}$$

for $k = 1, \dots, m$. Then, the sets C_k, $k = 1, \dots, m$, are mutually disjoint and

$$\sum_{i=1}^{2} |\nabla J_i(\omega)|^2 \equiv 0 \quad \text{for } \omega \in \bigcup_{k=1}^{m} C_k.$$

For $\delta \in (0, 1/2)$ define

$$C(\delta) = \{\omega \in S^{n-1} : J_1^2(\omega) > 1 - \delta\}.$$

Then $C(\delta) \supset \bigcup_{k=1}^{m} C_k$. Since $J_1(\omega)$ and $\sum_{i=1}^{2} |\nabla J_i(\omega)|^2$ are uniformly continuous on S^{n-1}, there is a $\delta = \delta(\varepsilon)$ sufficiently small in order that

$$\sum_{i=1}^{2} |\nabla J_i(\omega)|^2 < \varepsilon/2 \quad \text{for } \omega \in C(\delta).$$

Since $J_1^2(\omega) > \frac{1}{2}$ on $C(\delta)$, then

$$\sum_{i=1}^{2} |\nabla J_i(\omega)|^2 < \varepsilon J_1^2(\omega), \qquad \omega \in C(\delta).$$

On $S^{n-1} \backslash C(\delta)$, $J_2^2(\omega) = 1 - J_1^2(\omega) \geq \delta$. Let B be the bound for $\sum_{i=1}^{2} |\nabla J_i(\omega)|^2$ on S^{n-1} and set $C_\varepsilon = B/\delta$. Then for $\omega \in S^{n-1} \setminus C(\delta)$

$$\sum_{i=1}^{2} |\nabla J_i(\omega)|^2 \leq C_\varepsilon \delta \leq C_\varepsilon J_2^2(\omega).$$

Hence, (v) holds for any $\omega \in S^{n-1}$. (Related constructions may be found in [25].)

Added after review. The referee has kindly informed us of a recent paper by S. A. Vugal′ter and G. M Zhislin, *On the spectrum of Schrödinger operators of multiparticle systems with short-range potentials*, Trans. Moscow Math Soc. (1987), 97–114. In that paper the authors study the problem of finiteness of the number of bound states of multiparticle systems with "virtual levels" being present in subsystems. The potentials are short-range. (We refer the reader to that paper for the definition of virtual levels. Yafaev [34] gives an intuitive discussion as well.) Conditions $\mathscr{H}(1)$ and $\mathscr{A}(1)$ above exclude the possiblity that $\Sigma(P)$ could be zero and that virtual levels could be present in subsystems. This phenomenon is a key ingredient for the occurrence of the Efimov effect mentioned earlier. Previous work studying the presence of virtual levels in subsystems of N-body operators and their influence on the number of discrete eigenvalues have only considered the case $N = 3$. Vugal′ter and Zhislin obtain results for the case $N \geq 4$ in their latest paper. Among other things, their results show that for $N \geq 4$ the existence of two 2-particle subsystems with virtual levels does not necessarily lead to an infinite number of bound states, in contrast to the case when $N = 3$.

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School of Mathematics, University of Wales College of Cardiff, Senghennydd Road, Cardiff CF2 4AG, Wales, United Kingdom

Department of Mathematics, University of Alabama at Birmingham, Birmingham, Alabama 35294