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MULTIPLIERS, LINEAR FUNCTIONALS AND THE FRÉCHET ENVELOPE OF THE SMIRNOV CLASS $N_*(\mathbb{U}^n)$

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ABSTRACT. Linear topological properties of the Smirnov class $N_*(\mathbb{U}^n)$ of the unit polydisk $\mathbb{U}^n$ in $\mathbb{C}^n$ are studied. All multipliers of $N_*(\mathbb{U}^n)$ into the Hardy spaces $H_p(\mathbb{U}^n)$, $0 < p \leq \infty$, are described. A representation of the continuous linear functionals on $N_*(\mathbb{U}^n)$ is obtained. The Fréchet envelope of $N_*(\mathbb{U}^n)$ is constructed. It is proved that if $n > 1$, then $N_*(\mathbb{U}^n)$ is not isomorphic to $N_*(\mathbb{U}^1)$.

1. INTRODUCTION

The Smirnov classes are well-known spaces of analytic functions that arise naturally in many contexts of the geometric theory of H_p-spaces. The Smirnov class N_* of the unit disk in the complex plane was extensively studied by Yanagihara [16, 17], who described all continuous linear functionals on N_* and found all multipliers of N_* into Hardy spaces. It can be observed that the crucial step in the proofs of Yanagihara's results are the best possible estimates of the Taylor coefficients of functions in N_*. See [18], Stoll [15] obtained analogues of Yanagihara's estimates for functions in the Smirnov class $N_*(D)$ of an arbitrary irreducible bounded symmetric domain D in $\mathbb{C}^n$, which is best possible if D is the unit ball in $\mathbb{C}^n$ (see [8]). The present paper is a study of the topological vector space structure of the Smirnov class $N_*(\mathbb{U}^n)$ of the unit polydisk $\mathbb{U}^n$ in $\mathbb{C}^n$.

The paper is organized as follows. §2 contains preliminary definitions and notation. In §3, we study the mean growth of the Taylor coefficients of functions from the class $N_*(\mathbb{U}^n)$. This leads us to the construction of a nuclear Fréchet sequence space $M[\beta]$, which is basic for the rest of the paper.

In §4 we prove our main result (Theorem 4.5), which describes all multipliers of $N_*(\mathbb{U}^n)$ into the Hardy spaces of $\mathbb{U}^n$. This result is applied in §5 to obtain a representation of the continuous linear functionals on $N_*(\mathbb{U}^n)$. It turns out that the dual space of $N_*(\mathbb{U}^n)$ can be identified with the dual of $M[\beta]$. This implies

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that $M[\beta]$ is the so-called Fréchet envelope of $N_*(\mathbb{U}^n)$, i.e., the completion of $N_*(\mathbb{U}^n)$ equipped with the strongest locally convex topology on $N_*(\mathbb{U}^n)$ which is weaker than the original topology of $N_*(\mathbb{U}^n)$. We show that the Fréchet envelope of $N_*(\mathbb{U}^1)$ is not isomorphic to the Fréchet envelope of any class $N_*(\mathbb{U}^n)$, $n > 1$. In consequence, $N_*(\mathbb{U}^1)$ is not isomorphic to $N_*(\mathbb{U}^n)$ if $n > 1$.

2. Preliminaries

Throughout this paper, $\mathbb{U}$ will denote the open unit disk in the complex plane $\mathbb{C}$, $\mathbb{T}$ the unit circle, $\mathbb{I} = (0, 1)$ the unit interval, dm the normalized Lebesge measure on $\mathbb{T}$, and $\mathbb{Z}_+$ the set of all nonnegative integers. Moreover, for a natural number n, $\mathbb{U}^n$, $\mathbb{T}^n$, $\mathbb{I}^n$, dm_n, and $\mathbb{Z}_+^n$ will denote the n-fold products of $\mathbb{U}$, $\mathbb{T}$, $\mathbb{I}$, dm, and $\mathbb{Z}_+$, respectively.

We will denote by $H(\mathbb{U}^n)$ the space of all analytic functions on $\mathbb{U}^n$ endowed with the compact-open topology κ.

If $f \in H(\mathbb{U}^n)$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, then the αth Taylor coefficient of f will be denoted by $\hat{f}(\alpha)$.

If $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $r = (r_1, \dots, r_n) \in \mathbb{I}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, $s \in \mathbb{I}$, the following abbreviations will be used: $rz := (r_1 z_1, \dots, r_n z_n)$, $sz := (sz_1, \dots, sz_n)$, $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

A function $f \in H(\mathbb{U}^n)$ is in the *Nevanlinna class* $N(\mathbb{U}^n)$ provided that

$$\sup_{0<r<1} \int_{\mathbb{T}^n} \log^+ |f(r\omega)|\, dm_n(\omega) < \infty.$$

The *Smirnov class* $N_*(\mathbb{U}^n)$ is the subspace of $N(\mathbb{U}^n)$ consisting of those f for which the family $\{\log^+ |f_r| : r \in \mathbb{I}\}$ is uniformly integrable on $\mathbb{T}^n$, where $f_r(\omega) = f(r\omega)$.

It is well known that the functional

$$\|f\| = \sup_{0<r<1} \int_{\mathbb{T}^n} \log(1 + |f_r|)\, dm_n$$

is a complete F-norm on $N_*(\mathbb{U}^n)$, and so the balls $B(\varepsilon) = \{f \in N_*(\mathbb{U}^n) : \|f\| \le \varepsilon\}$, $\varepsilon > 0$, are a base of neighborhoods of zero for a complete metrizable vector topology, ν, on $N_*(\mathbb{U}^n)$. Thus, $(N_*(\mathbb{U}^n), \nu)$ is an F-space. Moreover, for each $f \in N_*(\mathbb{U}^n)$ the radial limits $\lim_{r\to 1^-} f(r\omega) = f^*(\omega)$ exist for almost all $\omega \in \mathbb{T}^n$ and $\|f\| = \int_{\mathbb{T}^n} \log(1 + |f^*|)\, dm_n$. The reader is referred to [11] for information on $N_*(\mathbb{U}^n)$.

The same arguments as in the case $n = 1$ (we use the n-subharmonity of $\log(1 + |f|)$) show that

$$\log(1 + |f(z)|) \le \frac{2^n \|f\|}{(1 - |z_1|) \cdots (1 - |z_n|)} \tag{2.1}$$

for all $f \in N_*(\mathbb{U}^n)$ and $z = (z_1, \dots, z_n) \in \mathbb{U}^n$. This estimate suggests a study

of the space $F_*(\mathbb{U}^n)$ of all analytic functions f on $\mathbb{U}^n$ for which

$$\|f\|^c = \sup_{z\in\mathbb{U}^n} |f(z)| \exp\left(-c\prod_{i=1}^{n}(1-|z_i|)^{-1}\right) < \infty$$

for all $c > 0$. It is easy to see that $F_*(\mathbb{U}^n)$ endowed with the topology determined by the family of norms $\{\|\cdot\|^c : c > 0\}$ is a Fréchet space (locally convex F-space). Note that $F^+ = F_*(\mathbb{U}^1)$ is the Fréchet envelope of $N^+ = N_*(\mathbb{U}^1)$ invented by Yanagihara [16] (see also [15, §6]).

As a simple consequence of inequality (2.1), we obtain

Proposition 2.1. *$N_*(\mathbb{U}^n)$ is contained in $F_*(\mathbb{U}^n)$, and the inclusion mapping is continuous.*

3. Taylor coefficients

If $\beta = [\beta_m(\alpha) : m \in \mathbb{N}, \alpha \in \mathbb{Z}_+^n]$ is a matrix whose entries are positive numbers, then a family $x = \{x(\alpha)\}_{\alpha\in\mathbb{Z}_+^n}$ of complex numbers is in the Köthe space $M[\beta]$ provided that

$$\|x\|_m = \sup_{\alpha\in\mathbb{Z}_+^n} |x(\alpha)|\beta_m(\alpha) < \infty \tag{3.0}$$

for all $m \in \mathbb{N}$. $M[\beta]$ equipped with the topology determined by the sequence of norms $\{\|\cdot\|_m : m \in \mathbb{N}\}$ is a Fréchet space (see [10, p. 17]).

Proposition 3.1 [10, Proposition 7.4.8]. *If the matrix $\beta = [\beta_m(\alpha)]$ satisfies*

$$\sum_\alpha \beta_m(\alpha)/\beta_{m+1}(\alpha) < \infty \quad \textit{for each } m \in \mathbb{N}, \tag{3.1}$$

then:

(a) *Each continuous linear functional T defined on the space $M[\beta]$ is of the form*

$$Tx = \sum_\alpha \lambda(\alpha)x(\alpha), \qquad x = \{x(\alpha)\} \in M[\beta],$$

where $\lambda = \{\lambda(\alpha)\}_{\alpha\in\mathbb{Z}_+^n}$ is a family of complex numbers such that

$$\sup |\lambda(\alpha)|/\beta_m(\alpha) < \infty \quad \textit{for some } m \in \mathbb{N}.$$

(b) *The space $M[\beta]$ is nuclear.*

In the sequel, we define and then we fix for the rest of the paper a very special matrix $\beta = [\beta_m(\alpha)]$. In §5, we prove that the Köthe space $M[\beta]$ defined by β is isomorphic to the Fréchet envelope of $N_*(\mathbb{U}^n)$.

If $\alpha \in \mathbb{Z}_+^n$ we let $\alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$ be the nonincreasing rearrangement of α and define

$$a_{m,j}(\alpha) := (\alpha_1^* \cdot \ldots \cdot \alpha_j^*/2^m)^{1/(j+1)}$$

where $m \in \mathbb{N}$, $1 \le j \le n$. Moreover, we define

$$L_m(\alpha) := \max\{k \in \{1, \dots, n\} : a_{m,j}(\alpha)/\alpha_j^* \le 1/2 \text{ for all } j \le k\}, \tag{3.2}$$

where $\alpha \in \mathbb{Z}_+^n \backslash \{0\}$, $m \in \mathbb{N}$, $0/0 = 1$. The integer $L_m(\alpha)$ will be called the m th *essential length of* α. If we take

$$a_m(\alpha) := a_{m, L_m(\alpha)}(\alpha), \tag{3.3}$$

then it is clear that

$$a_m(\alpha)/\alpha_j^* \leq 1/2 \quad \text{for all } j \leq L_m(\alpha), \tag{3.4}$$

$$\alpha_j^* \leq 2^{3/2} a_m(\alpha) \quad \text{for all } j > L_m(\alpha). \tag{3.5}$$

It is obvious that $L_1(\alpha) \leq L_2(\alpha) \leq \cdots \leq n$ for each $\alpha \in \mathbb{Z}_+^n \backslash \{0\}$.

Finally, we define

$$b_m(\alpha) = \begin{cases} 2^{-m} a_m(\alpha) & \text{if } \alpha \in \mathbb{Z}_+^n \backslash \{0\}, \\ 2^{-m} & \text{if } \alpha = 0 \end{cases} \tag{3.6}$$

and

$$\beta_m(\alpha) := \exp(-b_m(\alpha)) \tag{3.7}$$

for all $\alpha \in \mathbb{Z}_+^n$ and $m \in \mathbb{N}$.

Lemma 3.2. (a)

$$a_{m+1}(\alpha)/a_m(\alpha) \leq \begin{cases} 2^{1/2} & \textit{if } L_m(\alpha) < L_{m+1}(\alpha), \\ 2^{-1/(n+1)} & \textit{if } L_m(\alpha) = L_{m+1}(\alpha) \end{cases}$$

for all $m \in \mathbb{N}$, $\alpha \in \mathbb{Z}_+^n \backslash \{0\}$;

(b) $a_{n+m+l}(\alpha) \leq 2^{n/2 - l/(n+1)} a_m(\alpha)$ *for all* l, $m \in \mathbb{N}$, $\alpha \in \mathbb{Z}_+^n \backslash \{0\}$;

(c) $b_{m+1}(\alpha)/b_m(\alpha) \leq 2^{-1/2}$ *for all* $m \in \mathbb{N}$, $\alpha \in \mathbb{Z}_+^n \backslash \{0\}$;

(d) $\sum_\alpha \beta_m(\alpha)/\beta_{m+1}(\alpha) < \infty$ *for each* $m \in \mathbb{N}$.

Proof. (a) If $L_m(\alpha) = L_{m+1}(\alpha) =: k$, then $a_{m+1}(\alpha)/a_m(\alpha) = 2^{-1/(k+1)} \leq 2^{-1/(n+1)}$.

Suppose now $L_m(\alpha) < L_{m+1}(\alpha) =: i$. Then, using (3.4) and (3.5), we obtain $a_{m+1}(\alpha)/a_m(\alpha) \leq \alpha_i^*/2a_m(\alpha) \leq 2^{1/2}$.

(c) is an immediate consequence of (a), while (b) follows from (a) and the fact that the set $\{m \in \mathbb{N}\colon L_m(\alpha) < L_{m+1}(\alpha)\}$ has no more than n elements.

(d) we have

$$\begin{aligned} \beta_m(\alpha)/\beta_{m+1}(\alpha) &= \exp(-b_m(\alpha) + b_{m+1}(\alpha)) \\ &\leq \exp(-(1 - 2^{-1/2}) b_m(\alpha)). \end{aligned}$$

Therefore, for the proof of (d), it is enough to show that

$$\sum_{k=1}^{n} \sum_{L_m(\alpha)=k} \exp(-c(\alpha_1^* \cdots \alpha_k^*)^{1/(k+1)}) < \infty \quad \text{for all } c > 0.$$

However,

$$\begin{aligned}\sum_{L_m(\alpha)=k} &\exp(-c(\alpha_1^* \cdots \alpha_k^*)^{1/(k+1)}) \\ &\le k! \sum_{\alpha_1 \ge \cdots \ge \alpha_k \ge 1} \exp(-c(\alpha_1 \cdots \alpha_k)^{1/(k+1)}) \\ &\le k! \sum_{\alpha_1=1}^{\infty} \alpha_1^k \exp(-c\alpha_1^{1/(k+1)}) < \infty.\end{aligned}$$

The proof is complete.

Proposition 3.3. *The mapping* $f \xrightarrow{\tau} \{\hat{f}(\alpha)\}$ *is a continuous embedding of* $F_*(\mathbb{U}^n)$ *into* $M[\beta]$.

In §5 we show that in fact τ is a topological linear isomorphism of $F_*(\mathbb{U}^n)$ onto $M[\beta]$.

Proof. For a moment, fix $\alpha \in \mathbb{Z}_+^n \backslash \{0\}$, $m \in \mathbb{N}$, and an arbitrary function $f \in H(\mathbb{U}^n)$ satisfying

$$|f(z)| \le \exp\left(2^{-m} \prod_{j=1}^{n} (1 - |z_j|)^{-1}\right), \qquad z \in \mathbb{U}^n. \tag{3.8}$$

For the sake of convenience, let us assume that $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n$. It is well known that for all $r = (r_1, \dots, r_n) \in \mathbb{I}^n$

$$|\hat{f}(\alpha)| \le r_1^{-\alpha_1} \cdots r_n^{-\alpha_n} \max\{|f(z_1, \dots, z_n)| : |z_j| = r_j,\ j = 1, \dots, n\}.$$

Therefore, if $r \in \mathbb{I}^n$ then

$$|\hat{f}(\alpha)| \le r_1^{-\alpha_1} \cdots r_n^{-\alpha_n} \exp\left(2^{-m} \prod_{j=1}^{n} (1 - r_j)^{-1}\right).$$

Take $r_j := 1 - x_j$, where $x_j := a_m(\alpha)/\alpha_j$ for $j = 1, 2, \dots, L_m(\alpha) =: k$ and $x_j = 1/2$ for $j > L_m(\alpha)$ (see (3.4)). Using the inequality $1 - x \ge e^{-2x}$, $0 \le x \le 1/2$, we obtain:

$$|\hat{f}(\alpha)| \le \exp\left(2 \sum_{j=1}^{n} \alpha_j x_j + \frac{2^{-m}}{x_1 \cdots x_n}\right).$$

However, by (3.4) and (3.5),

$$\begin{aligned}2 \sum_{j=1}^{n} \alpha_j x_j + \frac{2^{-m}}{x_1 \cdots x_n} &= 2k a_m(\alpha) + \sum_{j=k+1}^{n} \alpha_j + 2^{n-k} \frac{(\alpha_1 \cdots \alpha_k / 2^m)}{a_m(\alpha)^k} \\ &\le (2k + (n-k)2^{3/2} + 2^{n-k}) a_m(\alpha) \le A a_m(\alpha),\end{aligned}$$

where A is an absolute constant. Therefore, we have proved that

$$|(\hat{f}\alpha)| \le \exp(A a_m(\alpha)) \tag{3.9}$$

for all $\alpha \in \mathbb{Z}_+^n \backslash \{0\}$, $m \in \mathbb{N}$ and f satisfying (3.8).

Observe that in order to prove that τ is continuous, it is enough to show that each norm $\|\cdot\|_\mu$, $\mu \in \mathbb{N}$, (see (3.0)) is bounded on the τ-image of some neighborhood of zero in $F_*(\mathbb{U}^n)$.

Fix $\mu \in \mathbb{N}$, choose $\ell \in \mathbb{N}$ so large that $A2^{n/2-l/(n+1)} \leq 2^{-\mu}$, and take $m := \mu + n + \ell$. Then, by (3.9) and Lemma 3.2(b), we obtain

$$|\hat{f}(\alpha)| \leq \exp(Aa_m(\alpha)) \leq \exp(2^{-\mu}a_\mu(\alpha)) = \exp(b_\mu(\alpha))$$

for all $\alpha \in \mathbb{Z}_+^n \backslash \{0\}$ and f belonging to the neighborhood of zero $V = \{f \in F_*(\mathbb{U}^n) \colon |f(z)| \leq \exp(2^{-m}\prod_{j=1}^n (1-|z_j|)^{-1})$ for all $z \in \mathbb{U}^n\}$ in $F_*(\mathbb{U}^n)$. Obviously, $\|\cdot\|_\mu$ is bounded on $\tau(V)$.

4. Multipliers of $N_*(\mathbb{U}^n)$

Recall that the Hardy space $H_p(\mathbb{U}^n)$, $0 < p < \infty$, is the subspace of $H(\mathbb{U}^n)$ consisting of all functions f for which

$$\|f\|_{H_p} = \sup_{0<r<1} \int_{\mathbb{T}^n} |f(r\omega)|^p \, dm_n(\omega) < \infty,$$

while $H_\infty(\mathbb{U}^n)$ is the space of all bounded analytic functions on $\mathbb{U}^n$.

A family $\lambda = \{\lambda(\alpha)\}_{\alpha \in \mathbb{Z}_+^n}$ of complex numbers is a *multiplier* of $N_*(\mathbb{U}^n)$ into $H_p(\mathbb{U}^n)$ if for each $f(z) = \sum_\alpha \hat{f}(\alpha) z^\alpha \in N_*(\mathbb{U}^n)$ the function $\tilde{\lambda} f(z) = \sum_\alpha \lambda(\alpha)\hat{f}(\alpha) z^\alpha$ belongs to $H_p(\mathbb{U}^n)$. It is well known that for each multiplier λ the induced linear operator $f \to \tilde{\lambda} f$ is continuous.

In this section, we describe all multipliers of $N_*(\mathbb{U}^n)$ into the Hardy spaces $H_p(\mathbb{U}^n)$, $0 < p \leq \infty$. For the proof of our main result (Theorem 4.5), we need a few technical lemmas. We use notation concerning the space $M[\beta]$ introduced in §2.

Lemma 4.1 [5, Chapter 4, §6].

$$\int_{\mathbb{T}} |1 - r\omega|^{-2} \, dm(\omega) = O((1-r)^{-1}) \quad \textit{as } r \to 1_-.$$

Definition 4.2. For each $k \in \{1, \ldots, n\}$, $r = (r_j) \in \mathbb{I}^k$, $c > 0$, we define

$$f_{k,r,c}(z) = c \exp\left(c \prod_{j=1}^k \frac{1 - r_j}{(1 - r_j z_j)^2} \right), \qquad z \in \mathbb{U}^n.$$

Lemma 4.3.

$$\lim_{c \to 0+} \sup_{k,r} \|f_{k,r,c}\| = 0.$$

Proof. In the proof we follow [3]. Fix an $\varepsilon > 0$. Using Lemma 4.1 we can find a $K > 0$ such that

$$\int_{\mathbb{T}^n} \prod_{j=1}^k |1 - r_j||1 - r_j\omega_j|^{-2} \, dm_n(\omega) \leq K < \infty, \tag{4.1}$$

for all $r \in \mathbb{I}^k$ and $k \in \{1, \dots, n\}$. Choose a subarc J of $\mathbb{T}$ with the midpoint 1 so small that $m_n(J_n)\log 4 < \varepsilon/3$, where $J_n := \{\omega = (\omega_j) : \omega_j \in J$ for some $j = 1, \dots, n\}$. Using (4.1) and the inequality $\log(1+cx) \le \log(1+c) + \log 2 + \log x$ $(c > 0, x \ge 1)$ with $x = \exp(c\prod_{j=1}^k |1-r_j||1-r_j\omega_j|^{-2})$, we obtain

$$\int_{J_n} \log(1 + |f^*_{k,r,c}|)\, dm_n \le m_n(J_n)\log(1+c) + m_n(J_n)\log 2 + cK.$$

Choose $c_0 > 0$ so small that the right side of the above inequality is not greater than $2\varepsilon/3$ for all $0 < c < c_0$. It is easy to see that

$$\lim_{c\to 0+} \sup\{|f^*_{k,r,c}(\omega)| : \omega \in \mathbb{T}^n \backslash J_n,\ r \in \mathbb{I}^k,\ k = 1, \dots, n\} = 0.$$

Therefore, there is $0 < c_1 < c_0$ such that $\int_{\mathbb{T}^n\backslash J_n} \log(1 + |f^*_{k,r,c}|)\, dm_n < \varepsilon/3$ for all k, r and $0 < c < c_1$. Finally, $\|f_{k,r,c}\| < \varepsilon$ for all k, r, and $0 < c < c_1$. The proof is complete.

Lemma 4.4. *For every $\varepsilon > 0$, there is a $\mu \in \mathbb{N}$ such that*

$$\inf_{\alpha\in\mathbb{Z}^n_+} \sup\{|\hat{f}(\alpha)| : f \in N_*(\mathbb{U}^n),\ \|f\| \le \varepsilon\}\beta_\mu(\alpha) > 0. \tag{4.2}$$

Proof. Fix $\varepsilon > 0$. There is $0 < d_0 < 1$ such that $\sup_{k,r}\|f_{k,r,d}\| < \varepsilon$ for all $0 < d < d_0$, where $f_{k,r,d}$ is the function described in Definition 4.2 (see Lemma 4.3). Take $m \in \mathbb{N}$ so large that $d := 4^n/2^m < d_0$. For each permutation ρ of $\{1, \dots, n\}$, the operator $f(z_1, z_2, \dots, z_n) \to f(z_{\rho(1)}, z_{\rho(2)}, \dots, z_{\rho(n)})$ is an isometric automorphism of $N_*(\mathbb{U}^n)$. Moreover, $b_\mu(\alpha) = b_\mu(\alpha\circ\rho)$ for each $\alpha \in \mathbb{Z}^n_+$. Therefore, to prove (4.2), it suffices to consider only those $\alpha \in \mathbb{Z}^n_+\backslash\{0\}$ that are nonincreasing. For each α of this type, we define

$$g_\alpha(z) := \exp\left(d\prod_{i=1}^k \frac{1-r_i}{(1-r_iz_i)^2}\right), \qquad z \in \mathbb{U}^n,$$

where $k := L_m(\alpha)$, $r_i := 1 - x_i$, $x_i := a_m(\alpha)/\alpha_i$ for $i = 1, 2, \dots, k$ (observe that by equation (3.4), $0 \le x_i \le 1/2$). Then we have

$$\begin{aligned}
g_\alpha(z) &= \sum_{j=0}^\infty \frac{d^j}{j!}\prod_{i=1}^k [(1-r_i)^j(1-r_iz_i)^{-2j}] \\
&= 1 + \sum_{j=1}^\infty \frac{d^j}{j!}\prod_{i=1}^k \left[x_i^j \sum_{\beta_i=0}^\infty \binom{2j+\beta_i-1}{\beta_i}(r_iz_i)^{\beta_i}\right] \\
&= 1 + \sum_{j=1}^\infty \frac{d^j}{j!} \sum_{\beta=(\beta_1,\dots,\beta_k)\in\mathbb{Z}^k_+} \prod_{i=1}^k \left[x_i^j \binom{2j+\beta_i-1}{\beta_i}(r_iz_i)^{\beta_i}\right] \\
&= 1 + \sum_{\beta\in\mathbb{Z}^k_+} \left\{\sum_{j=1}^\infty \frac{d^j}{j!}\prod_{i=1}^k \left[x_i^j \binom{2j+\beta_i-1}{\beta_i} r_i^{\beta_i}\right]\right\} z^\beta
\end{aligned}$$

for all $z \in \mathbb{U}^n$ (here we identify $\mathbb{Z}_+^k$ with $\mathbb{Z}_+^k \times \{0\} \times \cdots \times \{0\} \subset \mathbb{Z}_+^n$). Therefore,

$$\begin{aligned} u_\alpha &:= |\hat{g}_\alpha(\alpha_1, \ldots, \alpha_k, 0, \ldots, 0)| \\ &\geq \frac{d^j}{j!} \prod_{i=1}^k \left[x_i^j \binom{2j+\alpha_i-1}{\alpha_i} r_i^{\alpha_i} \right], \end{aligned}$$

for all $j \in \mathbb{N}$. It is obvious that

$$\binom{2j+\alpha_i-1}{\alpha_i} \geq \frac{\alpha_i^{2j-1}}{(2j)!}.$$

Thus,

$$u_\alpha \geq \frac{d^j}{(j!)[(2j)!]^k} \prod_{i=1}^k [x_i^j r_i^{\alpha_i} \alpha_i^{2j-1}]$$

for all $j \in \mathbb{N}$. Let us set $j = j_\alpha =$ the integer part of $a_m(\alpha)$. Then, using Stirling's formula we obtain

$$\begin{aligned} \log u_\alpha &\geq j \log d - j \log j + j - O(\log j) - k(2j \log 2j - 2j + O(\log 2j)\} \\ &\quad + \sum_{i=1}^k [j \log x_i + \alpha_i \log r_i + j \log \alpha_i^2 - \log \alpha_i] \\ &= j \left[2k + 1 + \log \left(\frac{d}{4^k j^{2k+1}} \prod_{i=1}^k x_i \alpha_i^2 \right) \right] \\ &\quad + \sum_{i=1}^k \alpha_i \log r_i - O(\log(\alpha_1 \cdots \alpha_k)) \end{aligned}$$

as $(\alpha_1 \cdot \ldots \cdot \alpha_k) \to \infty$. However,

$$\frac{d}{4^k j^{2k+1}} \prod_{i=1}^k x_i \alpha_i^2 \geq \frac{d}{4^k [a_m(\alpha)]^{2k+1}} [a_m(\alpha)]^k (\alpha_1 \cdots \alpha_k / 2^m) 2^m = \frac{d 2^m}{4^k} = 1.$$

Moreover,

$$\sum_{i=1}^k \alpha_i \log r_i = \sum_{i=1}^k \alpha_i \log(1 - x_i) \geq -2 \sum_{i=1}^k \alpha_i x_i = -2k a_m(\alpha).$$

Consequently,

$$\begin{aligned} \log u_\alpha &\geq (2k+1)(a_m(\alpha) - 1) - 2k a_m(\alpha) - O(\log a_m(\alpha)) \\ &\geq a_m(\alpha) - O(\log a_m(\alpha)). \end{aligned}$$

This implies that there are γ, $\delta > 0$ such that

$$|\hat{g}_\alpha(\alpha_1, \ldots, \alpha_{L_m(\alpha)}, 0, \ldots, 0)| \geq \delta \exp(\gamma 2^{-m} a_m(\alpha)) \geq \delta \exp(\gamma b_m(\alpha))$$

for all $\alpha \in \mathbb{Z}_+^n \backslash \{0\}$ with $\alpha_1 \geq \cdots \geq \alpha_n$. Now, for the function $h_\alpha(z) := d g_\alpha(z) z_{k+1}^{\alpha_{k+1}} \cdots z_n^{\alpha_n}$ (where $k := L_m(\alpha)$), we have $\|h_\alpha\| = \|d g_\alpha\| = \|f_{k,r,d}\| \leq \varepsilon$, and

$$\hat{h}_\alpha(\alpha) \geq d\delta \exp(\gamma b_m(\alpha)) \quad \text{for all } \alpha.$$

Lemma 3.2(c) implies that there exists $\mu \in \mathbb{N}$ such that $b_\mu(\alpha) \le \gamma b_m(\alpha)$ for all α. Finally, $\hat{h}_\alpha(\alpha) \ge d\delta \exp(b_\mu(\alpha))$ for all α. The proof is finished.

Theorem 4.5. *A complex family $\lambda = \{\lambda(\alpha)\}_{\alpha \in \mathbb{Z}_+^n}$ is a multiplier of $N_*(\mathbb{U}^n)$ into $H_p(\mathbb{U}^n)$, $0 < p \le \infty$, if and only if*

$$\sup_\alpha \frac{|\lambda(\alpha)|}{\beta_\mu(\alpha)} < \infty \quad \textit{for some } \mu \in \mathbb{N}.$$

Proof. Throughout the proof the letter C will denote many various positive constants, which are always independent of any individual analytic function or multi-index.

Suppose that λ is a multiplier of $N_*(\mathbb{U}^n)$ into $H_p(\mathbb{U}^n)$. We may assume $0 < p \le 1$. The operator $\tilde{\lambda}$ associated with λ is continuous, so there is an $\varepsilon > 0$ such that

$$\|\tilde{\lambda} f\|_{H_p} \le 1 \quad \text{for all } f \in N_*(\mathbb{U}^n), \ \|f\| \le \varepsilon. \tag{4.2}$$

However, for all $g \in H_p(\mathbb{U}^n)$ we have

$$|\hat{g}(\alpha)| \le C \left(\prod_{\substack{i=1 \\ \alpha_i \ne 0}}^n \alpha_i^{1/p-1} \right) \|g\|_{H_p} \quad \text{for all } \alpha \in \mathbb{Z}_+^n \tag{4.3}$$

(see [6, Theorem 5]). Thus,

$$|\lambda(\alpha)\hat{f}(\alpha)| \le C \left(\prod_{\substack{i=1 \\ \alpha_i \ne 0}}^n \alpha_i^{1/p-1} \right) \tag{4.4}$$

for all $\alpha \in \mathbb{Z}_+^n$ and $f \in N_*(\mathbb{U}^n)$, $\|f\| \le \varepsilon$. By Lemma 4.4, there are $\delta > 0$, $m \in \mathbb{N}$, and a family $\{f_\alpha\}_{\alpha \in \mathbb{Z}_+^n} \subset N_*(\mathbb{U}^n)$ such that $\|f_\alpha\| \le \varepsilon$ and $|\hat{f}_\alpha(\alpha)| \ge \delta \exp(b_m(\alpha))$ for all α. Finally, using Lemma 3.2, we obtain

$$\begin{aligned}
|\lambda(\alpha)| &\le C \left(\prod_{\substack{i=1 \\ \alpha_i \ne 0}}^n \alpha_i^{1/p-1} \right) \exp(-b_m(\alpha)) \le C(\alpha_1^*)^{n(1/p-1)} \exp(-b_m(\alpha)) \\
&\le C \exp(2^{-(m+1)}(\alpha_1^*/2^{m+1})^{1/(n+1)}) \exp(-b_m(\alpha)) \\
&\le C \exp(b_{m+1}(\alpha) - b_m(\alpha)) \le C \exp((2^{-1/2} - 1) b_m(\alpha)) \\
&\le C \exp(-b_{m+8}(\alpha)) = C\beta_{m+8}(\alpha)
\end{aligned}$$

for all $\alpha \in \mathbb{Z}_+^n$.

Suppose now that $\lambda = \{\lambda(\alpha)\}$ satisfies $|\lambda(\alpha)| \leq C \cdot \beta_\mu(\alpha)$ for all $\alpha \in \mathbb{Z}_+^n$. Proposition 3.3 shows that $|\hat{f}(\alpha)| \leq C/\beta_{\mu+1}(\alpha)$ for all α and all f belonging to some neighborhood of zero V in $N_*(\mathbb{U}^n)$. Consequently, $\sum_\alpha |\lambda(\alpha)\hat{f}(\alpha)| \leq C\sum_\alpha \beta_\mu(\alpha)/\beta_{\mu+1}(\alpha) < \infty$ (see Lemma 3.2(d)), and so the function $\tilde{\lambda}f(z) = \sum_\alpha \lambda(\alpha)\hat{f}(\alpha)z^\alpha$ is analytic on $\mathbb{U}^n$ and continuous on $\overline{\mathbb{U}}^n$ for every $f \in V$, hence $\tilde{\lambda}(V) \subset H_\infty(\mathbb{U}^n)$. This completes the proof since neighborhoods of zero are absorbing.

Remark 4.6. In the case $n = 1$, it is easily seen that for each $m \in \mathbb{N}$ the sequence $\{\beta_m(k)\}_{k\in\mathbb{Z}_+}$ is equivalent to the sequence $\{\exp(-ck^{1/2})\}_{k\in\mathbb{Z}_+}$ for some $c > 0$. Thus, Yanagihara's description of multipliers of N_* into H_p, $0 < p \leq \infty$, (cf. [17, Theorem 2]) is a particular case of Theorem 4.5.

5. Linear functionals and the Fréchet envelope

Theorem 5.1. *To each continuous linear functional T on $N_*(\mathbb{U}^n)$, there corresponds a unique complex family λ such that*

$$Tf = \sum_\alpha \hat{f}(\alpha)\lambda(\alpha) \quad \textit{for every } f \in N_*(\mathbb{U}^n) \tag{5.1}$$

and

$$\sup_\alpha \frac{|\lambda(\alpha)|}{\beta_\mu(\alpha)} < \infty \quad \textit{for some } \mu \in \mathbb{N}. \tag{5.2}$$

Conversely, for each sequence $\{\lambda(\alpha)\}$ satisfying (5.2), formula (5.1) defines a continuous linear functional on $N_*(\mathbb{U}^n)$.

Proof. The second part of the proof of Theorem 4.5 shows that if λ satisfies (5.2), then the series (5.1) defining Tf is absolutely convergent for all f and T is bounded on some neighborhood of zero in $N_*(\mathbb{U}^n)$. Hence, T is well defined and continuous.

Now suppose that T is a continuous linear functional on $N_*(\mathbb{U}^n)$ and let $\lambda(\alpha) = Tz^\alpha$, $\alpha \in \mathbb{Z}_+^n$. Then,

$$Tf = \lim_{r\to 1-} Tf_r = \lim_{r\to 1-} \sum_\alpha \hat{f}(\alpha)T(z^\alpha)r^\alpha = \lim_{r\to 1-} \sum_\alpha \hat{f}(\alpha)\lambda(\alpha)r^\alpha \tag{5.3}$$

for all $f \in N_*(\mathbb{U}^n)$, where $r \in \mathbb{I}$ and $r^\alpha = r^{\alpha_1+\cdots+\alpha_n}$. As has already been noted, if we show that λ satisfies (5.2), it will follow that the series $\sum \hat{f}(\alpha)\lambda(\alpha)$ is absolutely convergent. Consequently, the limits in (5.3) will be equal to the sum of this series (i.e., (5.1) will hold).

By Theorem 4.5, for the proof of (5.2) it is enough to show that λ is a multiplier of $N_*(\mathbb{U}^n)$ into $H_\infty(\mathbb{U}^n)$.

For each ζ, $z \in \mathbb{U}^n$ and $f \in N_*(\mathbb{U}^n)$ we define $f_\zeta(z) = f(\zeta z)$. It is easily seen that $\|f_\zeta\| \leq \|f\|$ for each $f \in N_*(\mathbb{U}^n)$, so the set $\{f_\zeta : \zeta \in \mathbb{U}^n\}$ is bounded

in $N_*(\mathbb{U}^n)$. Moreover,

$$\tilde{\lambda} f(\zeta) = \sum_\alpha \lambda(\alpha)\hat{f}(\alpha)\zeta^\alpha = T\left(\sum_\alpha \hat{f}(\alpha)\zeta^\alpha z^\alpha\right) = T(f_\zeta),$$

so $\tilde{\lambda} f \in H_\infty(\mathbb{U}^n)$ for each $f \in N_*(\mathbb{U}^n)$. In other words, λ is a multiplier of $N_*(\mathbb{U}^n)$ into $H_\infty(\mathbb{U}^n)$.

Let us recall that if $X = (X, \tau)$ is an F-space whose topological dual X' separates the points of X, then its *Fréchet envelope* $\widehat{X}$ is defined to be the completion of the space (X, τ^c), where τ^c is the strongest locally convex topology on X that is weaker than τ. In fact, it is known that τ^c is equal to the Mackey topology of the dual pair (X, X'). See [13]. For each metrizable locally convex topology ξ on X, (X, ξ) is a Mackey space, i.e., ξ coincides with the Mackey topology of the dual pair (X, X'_ξ) (cf. [12, Chapter IV, 3.4]), so the Fréchet envelope $\widehat{X}$ of X is up to an isomorphism uniquely defined by the following conditions:

(FE1) $\widehat{X}$ is a Fréchet space,

(FE2) there exists a continuous embedding j of X onto a dense subspace of $\widehat{X}$,

(FE3) the mapping $\gamma \mapsto \gamma \circ j$ is a linear isomorphism of $\widehat{X}'$ onto X'.

Theorem 5.2. (a) *$M[\beta]$ is the Fréchet envelope of $N_*(\mathbb{U}^n)$.*

(b) *The operator $f \xrightarrow{\tau} \{\hat{f}(\alpha)\}$ is an isomorphism of $F_*(\mathbb{U}^n)$ onto $M[\beta]$.*

Proof. Let i be the inclusion mapping of $N_*(\mathbb{U}^n)$ into $F_*(\mathbb{U}^n)$ (see Proposition 2.1) and let e_α be the αth unit vector in $M[\beta]$, i.e., $e_\alpha(\alpha) = 1$ and $e_\alpha(\beta) = 0$ if $\alpha \neq \beta$. It is well known that $\{e_\alpha\}_{\alpha \in \mathbb{Z}_+^n}$ is a basis of the nuclear space $M[\beta]$. Let $j = \tau \circ i$. Then $j(z^\alpha) = e_\alpha$ for each $\alpha \in \mathbb{Z}_+^n$, so j is a continuous embedding of $N_*(\mathbb{U}^n)$ into $M[\beta]$ and $j(N_*(\mathbb{U}^n))$ is dense in $M[\beta]$ (see Proposition 3.3). Condition (FE3) is satisfied because of Proposition 3.1 and Theorem 5.1. Thus, $M[\beta]$ is the Fréchet envelope of $N_*(\mathbb{U}^n)$. Obviously, the projective topology induced on $N_*(\mathbb{U}^n)$ by j coincides with the projective topology induced by i. Therefore, τ is an isomorphism of $F_*(\mathbb{U}^n)$ onto $M[\beta]$ because of the density of $N_*(\mathbb{U}^n)$ in $F_*(\mathbb{U}^n)$.

The preceding description of the Fréchet envelope of $N_*(\mathbb{U}^n)$ has a nice application in the proof of the following:

Theorem 5.3. *If $n > 1$ then $N_*(\mathbb{U}^n)$ is not isomorphic to any complemented subspace of $N_*(\mathbb{U})$.*

Before proving the theorem, let us recall that a nondecreasing sequence $\gamma = \{\gamma_j\}$ is said to be a *nuclear exponential sequence of finite type* if $\lim(\log j)/\gamma_j = 0$. For each such sequence γ, the *finite type power series space* $\Lambda_1(\gamma)$ is defined to be the space of all complex sequences $x = \{x_j\}$ such that

$$q_k(x) = \sup |x_j| \exp(-\gamma_j/k) < \infty \quad \text{for each } k \in \mathbb{N}.$$

See [4]. We say that γ is *stable* if $\sup(\gamma_{2j}/\gamma_j) < \infty$.

Observe that if $n = 1$, then the Köthe matrix $\beta = \beta^{(1)} = [\beta_m(j) \colon j \in \mathbb{Z}_+]$, which defines the Fréchet envelope $M[\beta^{(1)}]$ of $N_*(\mathbb{U})$, is very simple. Indeed, $\beta_m(j) = \exp(-2^{-3m/2} j^{1/2})$ (see equation (3.7)) and $\{j^{1/2}\}$ is a stable nuclear exponential sequence, so we have

Lemma 5.4 [16, Theorem 1]. *The Fréchet envelope $M[\beta^{(1)}]$ of $N_*(\mathbb{U})$ is isomorphic to $\Lambda_1(\{j^{1/2}\})$.*

Lemma 5.5 [4, Proposition 3]. *Let $\delta = \{\delta_j\}$ be a stable nuclear exponential sequence of finite type. Then $\Lambda_1(\gamma)$ is isomorphic to a subspace of $\Lambda_1(\delta)$ if and only if $\sup_j \delta_j/\gamma_j < \infty$.*

We apply these lemmas to prove

Proposition 5.6. *If $n > 1$, then the Fréchet envelope $M[\beta^{(n)}]$ of $N_*(\mathbb{U}^n)$ is not isomorphic to any subspace of $\Lambda_1(\{j^{1/2}\})$.*

Proof. It is easily seen that the space $M[\beta^{(2)}]$ is isomorphic to a subspace of $M[\beta^{(n)}]$ when $n \geq 2$. Therefore, for the proof of the proposition we can assume that $n = 2$.

Let $A = \{(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2 \colon \alpha_2 \leq \alpha_1 \leq \alpha_2^2\}$ and $A_k = \{\alpha \in A \colon \alpha_1\alpha_2 = k\}$ for $k \in \mathbb{Z}$ (note that the set A_k may be empty). Moreover, let j be a unique bijection of A onto $\mathbb{Z}_+$ such that

(a) $\max j(A_k) < \min j(A_l)$ if $A_k, A_l \neq \varnothing$ and $k < 1$,

(b) if $\alpha, \alpha' \in A_k$, $\alpha = (i, k/i)$, $\alpha' = (i', k/i')$ and $i < i'$, then $j(\alpha) < j(\alpha')$.

Define a sequence $\gamma = \{\gamma_j\}$ by

$$\gamma_j = (\alpha_1\alpha_2)^{1/3} \quad \text{if } j = j(\alpha_1, \alpha_2), \quad j = 1, 2, \ldots.$$

We will show that γ is a nuclear exponential sequence of finite type. Indeed, it is obvious that γ is nondecreasing. Moreover, $j(\alpha_1, \alpha_2) \leq |\{(s, t) \in \mathbb{Z}_+^2 \colon t \leq s \leq t^2 \& st \leq \alpha_1\alpha_2\}| \leq (\alpha_1\alpha_2)^2$ for all $(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$. Thus,

$$\frac{\log j(\alpha_1, \alpha_2)}{\gamma_j(\alpha_1, \alpha_2)} \leq \frac{\log(\alpha_1\alpha_2)^2}{(\alpha_1\alpha_2)^{1/3}} \to 0 \quad \text{as } j(\alpha_1, \alpha_2) \to \infty.$$

Now, let E be the closed subspace of $M[\beta^{(2)}]$ spanned by the set of the unit vectors $\{e_\alpha \colon \alpha \in A\}$. It is easily seen that for each $\alpha \in A$ and $m \in \mathbb{N}$, $m \geq 3$, $b_m(\alpha) = 2^{-4m/3}\gamma_{j(\alpha)}$ (see (3.6)). Therefore, E is isomorphic to $\Lambda_1(\gamma)$. Now, by Lemma 5.5 and Lemma 5.4, for the proof of the proposition it suffices to show that $\sup[j^{1/2}/\gamma_j] = \infty$. This is simple. Taking $j_k = j(k, k^2)$, $k = 1, 2, \ldots,$

we obtain $\gamma_{j_k} = k$ and

$$\begin{aligned} j_k &\geq |\{(s, t) \in A: j(s, t) \leq j(k, k^2)\}| \\ &\geq |\{(s, t) \in \mathbb{Z}_+^2 \backslash \{0\}: t \leq s \leq t^2 \leq k^2\}| \\ &\geq \sum_{t=1}^{k} (t^2 - t) \geq \frac{1}{12} k^3. \end{aligned}$$

Finally, $j_k^{1/2}/\gamma_{j_k} \to \infty$. The proof is complete.

Proof of Theorem 5.3. Suppose that there exists a continuous projection P of $N_*(\mathbb{U})$ onto its subspace X isomorphic to $N_*(\mathbb{U}^n)$, where $n > 1$. Then P remains continuous if we equip the spaces $N_*(\mathbb{U})$ and X with their own Mackey topologies. This implies that the Mackey topology of X is induced by the Mackey topology of the whole space $N_*(\mathbb{U})$. Let $\overline{X}$ be the closure of X in the Fréchet envelope $\Lambda_1(\{j\}^{1/2})$ of $N_*(\mathbb{U})$. Then $\overline{X}$ is isomorphic to its Fréchet envelope, which in turn is isomorphic to $M[\beta^{(n)}]$. However, this is impossible because of Proposition 5.6.

Problem 5.7. Are there any distinct n, $m \in \mathbb{N}$ such that $N_*(\mathbb{U}^n)$ is isomorphic to $N_*(\mathbb{U}^m)$?

Added in proof. After this paper had been completed the author showed that if $n/ \neq m$ then $N_*(\mathbb{U}^n)$ is not isomorphic to $N_*(\mathbb{U}^m)$ (see *The nonisomorphism of the Smirnov classes of different balls and polydiscs*, Bull. Soc. Math. Belg. Sér B **41** (1989), 307–315).

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