ON HERMITE-FEJÉR INTERPOLATION IN A JORDAN DOMAIN

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ABSTRACT. The Hermite-Fejer interpolation problem on a Jordan domain is studied. Under certain mild conditions on the smoothness of the boundary curve, we give both uniform and $L^p$, $0 < p < \infty$, estimates on the rate of convergence. Our estimates are sharp even for the unit disk setting.

1. INTRODUCTION

Let $D$ be a Jordan domain in the complex plane $\mathbb{C}$ with boundary $\Gamma$ and $z_k = z_{nk}$, $k = 1, \ldots, n$, be sample points chosen on $\Gamma$. Also, let $q$ be a non-negative integer and $N = N_n := (q + 1)n - 1$. In this paper we will consider the interpolation problem:

$$
\tilde{H}^N_N(f; z_k) = f(z_k), \quad \tilde{H}^{(j)}_N(f; z_k) = a^{(j)}_k,
$$

$k = 1, \ldots, n$ and $j = 1, \ldots, q$, where $f$ belongs to the class $A(D)$ of functions analytic in $D$ and continuous on $\overline{D} = D \cup \Gamma$, and $\tilde{H}^N_N(f; \cdot) \in \pi_N$, the space of all polynomials with degree at most $N$. Note that since $f$ is not necessarily differentiable at $z_k$ relative to $\overline{D}$ and the family of data values $\{a^{(j)}_k\}$ is arbitrarily given, the problem under consideration is different from the Hermite interpolation problem. In particular, by choosing $a^{(j)}_k = 0$ for all $k = 1, \ldots, q$, the problem

$$
H^N_N(f; z_k) = f(z_k), \quad H^{(j)}_N(f; z_k) = 0,
$$

$k = 1, \ldots, n$ and $j = 1, \ldots, q$, where $f \in A(D)$ and $H^N_N(f; \cdot) \in \pi_N$, is usually called the $(0, 1, \ldots, q)$ Hermite-Fejér Interpolation Problem.

It is well known that even for the unit disk $U = \{z : |z| < 1\}$, any $q$, and $z_k = e^{i2\pi k/n}$, there exists an $f \in A(U)$ such that $H^N_N(f; \cdot)$ does not converge uniformly on $\overline{U}$ to $f$ (see [13]). In this paper, under certain smoothness conditions on the Jordan curve $\Gamma$, we will first give a necessary and sufficient

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condition on \( f \in A(D) \) that guarantees uniform convergence of \( \tilde{H}_N(f; \cdot) \) to \( f \) on \( D \) for \( a^{(j)}_n = o(n^j/\ln n) \), and derive the order of uniform convergence on \( \overline{D} \) of the Hermite-Fejér interpolatory polynomials \( H_N(f; \cdot) \) to \( f \) in terms of the modulus continuity of \( f \). We will next show that for \( a^{(j)}_n = o(n^j) \), \( \tilde{H}_N(f; \cdot) \) always converges in \( L^p(\Gamma) \) to \( f \in A(\overline{D}) \), \( 0 < p < \infty \), and, in fact, a sharp order of convergence of \( H_N(f; \cdot) \) in \( L^p(\Gamma) \), \( 0 < p < \infty \), will be given.

Of course, for \( q = 0 \), problems (1.1) and (1.2) become the Lagrange interpolation problem:

\[
L_N(f; z_k) = f(z_k),
\]

for any \( p \), \( 0 < p < \infty \). Recently, this result was further improved by the second author and Zhong [10] to a Jordan curve \( \Gamma \) of class \( C^{1+\delta} \) where the order of approximation \( O(\omega(f; \frac{1}{N})) \) is also given. Here and throughout, \( \omega(f; \delta) \) denotes the modulus of continuity of \( f \) on \( \Gamma \) using the uniform norm. We remark that the \( L^p \), \( 0 < p < \infty \), modulus of continuity cannot be used even for the \( L^p \) estimate of \( \|L_N(f; \cdot) - f\|_p \).

The only result in the literature for Hermite-Fejér interpolation on a Jordan curve different from the circle was obtained by Gaier [6], where an analytic curve \( \Gamma \) and \( q = 1 \) are considered and the convergence is only uniform on compact subsets of \( D \). Various recent results concerning convergence on the unit disk of Hermite-Fejér interpolatory polynomials at the \( n \)th roots of unity can be found in Szabados and Varma [11], Varma [12], and the second author [8, 9].

### 2. Main results

Throughout this paper, \( w = \Phi(z) \) denotes the exterior conformal map from \( \mathbb{C} \setminus \overline{D} \) onto \( |w| > 1 \) such that \( \Phi(\infty) = \infty \) and \( \Phi'(\infty) > 0 \). Let \( \Psi = \Phi^{-1} \) and write

\[
z = \Psi(w) = dw + a_0 + a_1 w^{-1} + \cdots,
\]

where \( d = \Psi'(\infty) > 0 \). It will be clear that by a standard transformation, we may assume, without loss of generality, that \( d = 1 \). Extend \( \Psi \) to a continuous function on \( |w| \geq 1 \) and set \( z_k = z_{n_k} = \Psi(w_{n_k}) \) where \( w_{n_k} = w_k = e^{i2\pi k/n} \). Recall that the \( z_{n_k} \)'s are usually called the Fejér points on \( \Gamma = \partial D \). We need some assumptions on the smoothness of \( \Gamma \).
Definition. (i) \( \Gamma \) is said to be of class \( j_1 \) if \( \Psi'(w) \) exists and is continuous on \( |w| \geq 1 \), and its (uniform) modulus of continuity \( \sigma_1(t) \) on \( |w| = 1 \) satisfies the condition

\[
\int_0^a \frac{\sigma_1(t)}{t} |\ln t|^2 \, dt < \infty, \quad a > 0.
\]

(ii) \( \Gamma \) is said to be of class \( j_2 \) if \( \Psi''(w) \) exists and is continuous on \( |w| \geq 1 \), and its (uniform) modulus of continuity \( \sigma_2(t) \) on \( |w| = 1 \) satisfies the condition

\[
\int_0^a \frac{\sigma_2(t)}{t} |\ln t| \, dt < \infty, \quad a > 0.
\]

It is well known [1] that if \( \Gamma \) belongs to class \( j_1 \), then \( \Psi \) satisfies:

\[
0 < C_1 \leq \left| \frac{\Psi(w) - \Psi(u)}{w - u} \right| \leq C_2
\]

for all \( w \neq u \) and \( |w|, |u| \geq 1 \). We remark that in [1] it is shown that (2.4) already holds for those \( \Gamma \) with

\[
\int_0^a \frac{\sigma_1(t)}{t} \, dt < \infty.
\]

In addition, it is shown in the same paper that

\[
0 < C_1 \leq |\Psi'(w)| \leq C_2
\]

for all \( w, |w| \geq 1 \).

Let

\[
\omega_n(z) = \prod_{j=1}^n (z - z_j).
\]

Then for each \( k \), \( (z - z_k)/\omega_n(z) \) is analytic at \( z_k \), so that we can write

\[
\left( \frac{z - z_k}{\omega_n(z)} \right)^{q+1} = \sum_{\nu=0}^{\infty} \alpha_{k\nu}(z - z_k)^\nu,
\]

where \( \alpha_{k\nu} = \alpha_{k\nu}(q, n), q = 0, 1, \ldots \). In the following, we will give an asymptotic estimate of \( \alpha_{k\nu} = \alpha_{k\nu}(q, n) \) as \( n \to \infty \). We need the notation

\[
\Omega_n(w) = \prod_{k=1}^n \frac{z - z_k}{w - w_k}, \quad z = \Psi(w).
\]

Theorem 1. Let \( \Gamma \) belong to class \( j_2 \). Then for each \( \nu \) and \( q = 0, 1, \ldots \),

\[
\alpha_{k\nu} = \alpha_{k\nu}(q, n) = O \left( \frac{1}{n^{q+1-\nu}} \right)
\]

and the estimate is uniform in \( k, 1 \leq k \leq n \), as \( n \to \infty \).

Here and throughout, \( \sum_{l \neq k} \) denotes the summation over all \( l = 1, \ldots, n \) with \( l \neq k \). To construct the interpolatory polynomials \( \tilde{H}_n(f; \cdot) \) and \( H_n(f; \cdot) \)
we introduce the fundamental functions:

\( A_{kj}(z) = \left( \frac{\omega_n(z)}{z - z_k} \right)^q \sum_{\nu=0}^{q-j} \alpha_{k\nu} (z - z_k)^\nu \),

where \( j = 0, \ldots, q \) and \( l = 1, \ldots, n \). It is obvious that \( A_{kj} \in \pi_N \) and we will verify that they satisfy

\( A_{kj}(z_t) = \delta_{kl}\delta_{
u j}, \quad k, l = 1, \ldots, n; \quad \nu, j = 0, \ldots, q, \)

where, as usual, \( \delta_{ij} \) denotes the Kronecker delta.

**Theorem 2.** For any \( f \in A(\overline{D}) \), any nonnegative integer \( q \), and arbitrary complex numbers \( a_k^{(j)}, k = 1, \ldots, n; j = 1, \ldots, q \), there exists a unique \( \tilde{H}_N(f; \cdot) \in \pi_N \) satisfying the interpolation conditions (1.1). Furthermore, \( \tilde{H}_N(f; \cdot) \) is given by

\[
\tilde{H}_N(f; \cdot) = \sum_{k=1}^{n} f(z_k)A_{k0}(\cdot) + \sum_{k=1}^{n} \sum_{j=1}^{q} a_k^{(j)} A_{kj}(\cdot).
\]

In addition, under the assumption that \( \Gamma \) belongs to the class \( j_2 \), the fundamental functions \( A_{kj} \) satisfy the following estimates:

\[
\max_{z \in D} \sum_{k=1}^{n} |A_{kj}(z)| = O \left( \frac{\ln n}{n^j} \right), \quad j = 0, \ldots, q,
\]

and for \( 1 < p < \infty \),

\[
\left\| \sum_{k=1}^{n} b_k A_{kj}(\cdot) \right\|_p = O \left( \frac{1}{n^j} \right) \max_{1 \leq k \leq n} |b_k|, \quad j = 0, \ldots, q,
\]

for any sequence \( \{b_k\}, k = 1, \ldots, n \).

Of course, if we choose \( a_k^{(j)} = 0 \), then the polynomials \( \tilde{H}_N(f; \cdot) \) become \( H_N(f; \cdot) \) that satisfy the Hermite-Fejér interpolation condition (1.2). It is well known that even for the case \( D = U \), the unit disk, there exists an \( f \in A(U) \) such that \( H_N(f; \cdot) \) does not converge uniformly to \( f \) on \( U \). We have the following result on the order of uniform approximation.

**Theorem 3.** Let \( \Gamma \) belong to class \( j_2 \) and \( f \in A(\overline{D}) \). Then for any nonnegative integer \( q \),

\[
\max_{z \in D} |f(z) - H_N(f; z)| = O \left( \omega \left( f; \frac{1}{n} \right) \ln n \right).
\]

We remark that this result is sharp as shown by the second author in [9] for \( D = U \). For nonzero \( a_k^{(j)} \), we have the following result.
Theorem 4. Let $\Gamma$ belong to class $j_2$, $f \in A(\overline{D})$, and $q$ be any nonnegative integer. Suppose that
\begin{equation}
\max_{1 \leq k \leq n} |a_k^{(j)}| = o \left( \frac{n^j}{\ln n} \right), \quad j = 1, \ldots, q,
\end{equation}
and
\begin{equation}
\lim_{\delta \to 0} \omega(f; \delta) \ln \delta = 0.
\end{equation}
Then
\begin{equation}
\lim_{N \to \infty} \max_{z \in \overline{D}} |f(z) - \tilde{H}_N(f; z)| = 0.
\end{equation}

For $L^p$ convergence, $0 < p < \infty$, we no longer need $\ln n$ in (2.15) as in the following

Theorem 5. Let $\Gamma$ belong to class $j_2$, $f \in A(\overline{D})$, $q$ be any nonnegative integer, and $0 < p < \infty$. Then
\begin{equation}
\|f - H_N(f; \cdot)\|_p = O \left( \omega \left( f; \frac{1}{n} \right) \right).
\end{equation}

Again, this result is sharp even for $D = U$ as shown in [10]. For nonzero $a_k^{(j)}$, we have the following result.

Theorem 6. Let $\Gamma$ belong to class $j_2$, $f \in A(\overline{D})$, $q$ be any nonnegative integer, and $\{a_n^{(j)}\}$ satisfy
\begin{equation}
\max_{1 \leq k \leq n} |a_k^{(j)}| = o(n^j), \quad j = 1, \ldots, q.
\end{equation}
Then
\begin{equation}
\lim_{N \to \infty} \|f - \tilde{H}_N(f; \cdot)\|_p = 0, \quad 0 < p < \infty.
\end{equation}

3. Proof of Theorem 1

To establish Theorem 1, we need three lemmas.

Lemma 1. Let $\Psi''$ be continuous on $|w| \geq 1$. Then for each $k = 1, \ldots, n$,
\begin{equation}
\omega_n^{(j)}(z_k) = n \frac{\Omega_n'(w_k)}{\Psi'(w_k)w_k}
\end{equation}
and
\begin{equation}
\sum_{l \neq k} \frac{1}{z_k - z_l} = \frac{1}{2} \frac{\omega_n''(z_k)}{\omega_n'(z_k)} = \frac{1}{2\Psi'(w_k)} \left[ \frac{n - 1}{w_k} + \frac{2}{\Omega_n(w_k)} - \frac{\Psi''(w_k)}{\Psi'(w_k)} \right].
\end{equation}

Proof. From (2.6) and (2.8), we have
\begin{equation}
\omega_n(z) = (w'' - 1) \Omega_n(w),
\end{equation}
so that

\[(3.3)\quad \omega_n'(z) = [n w^{n-1} \Omega_n'(w) + (w^n - 1) \Omega_n'(w)] \cdot \frac{1}{\Psi'(w)},\]

from which (3.1) follows. To establish the two identities in (3.2), we first use logarithmic derivatives to obtain

\[
\frac{\beta_k'(z)}{\beta_k(z)} = \sum_{l \neq k} \frac{1}{z - z_l}
\]

with \( \beta_k(z) := \omega_n(z)/(z - z_k) \). Since \( \beta_k(z_k) = w_n'(z_k) \) and

\[
\beta_k'(z_k) = \lim_{z \to z_k} \frac{\omega_n'(z)(z - z_k) - \omega_n(z)}{(z - z_k)^2}
\]

\[
= \lim_{z \to z_k} \frac{\omega_n'(z)(z - z_k) - \left[ \omega_n'(z_k)(z - z_k) + \frac{\omega_n''(z_k)}{2}(z - z_k)^2 + o(z - z_k)^2 \right]}{(z - z_k)^2}
\]

\[
= \lim_{z \to z_k} \frac{w_n''(z_k) - \frac{w_n''(z_k)}{2}(z - z_k)^2 + o(z - z_k)^2}{(z - z_k)^2} = \frac{1}{2} \omega_n''(z_k),
\]

we have established the first identity in (3.2). To derive the second identity in (3.2), we first observe that

\[
\omega_n''(z_k) = n(n-1)w_k^{-2} \Omega_n'(w_k) + 2n w_k^{-1} \Omega_n'(w_k) \cdot \frac{1}{[\Psi'(w_k)]^2}
\]

\[
- n w_k^{-1} \Omega_n'(w_k) \left[ \Psi''(w_k) \right] \frac{1}{[\Psi'(w_k)]^3}
\]

by using (3.3) and the fact that \( w_n'' = 1 \). By substituting this quantity and the quantity in (3.1) into \( \omega_n''(z_k)/\omega_n'(z_k) \), we arrive at the second identity in (3.2). \( \square \)

In the following, we give certain estimates on \( \Omega_N \) and its relation with \( \Omega'_N \).

**Lemma 2.** If \( \Gamma \) belongs to class \( j_1 \), then

\[(3.4)\quad \max_{|w| \geq 1} |\Omega_n(w) - 1| = o \left( \frac{1}{\ln n} \right).
\]

Furthermore, if \( \Gamma \) belongs to class \( j_2 \), then

\[(3.5)\quad \max_{|w| \geq 1} \left| \frac{\Omega_n'(w)}{\Omega_n(w)} \right| = o(1).
\]

**Proof.** To prove (3.4), let

\[(3.6)\quad g(w, u) = \begin{cases} (\Psi(w) - \Psi(u))/(w - u) & \text{for } u \neq w, \\ \Psi'(w) & \text{for } u = w, \end{cases} \]

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where \(|u|, |w| \geq 1\). Hence, from the definition of \(\Omega_N(w)\) and \(g(w, u)\), we have

\[
\ln \Omega_N(w) = \sum_{k=1}^{n} \ln g(w, w_k),
\]

where the branch of the logarithm is taken so that \(\ln 1 = 0\). On the other hand, it is clear that

\[
\frac{\partial \ln g(w, u)}{\partial u} = -\frac{\Psi'(u)(w - u) + (\Psi(w) - \Psi(u))}{(w - u)(\Psi(w) - \Psi(u))}
\]

and

\[
\|\Psi(w) - \Psi(u) - \Psi'(u)(w - u)\| = \left| \int_{\gamma} [\Psi'(\xi) - \Psi'(u)] d\xi \right|
\leq C_1 \sigma_1(|w - u|) \int_{\gamma} |d\xi| \leq C_2 |w - u| \sigma_1(|w - u|),
\]

where \(\gamma\) is a contour joining \(u\) to \(w\) on \(|\xi| \geq 1\) with length bounded by \(\frac{1}{2} |u - w|\) and \(\sigma_1\) denotes the modulus of continuity of \(\Psi'\). By using (2.4), (3.8), and (3.9), we have

\[
\left| \frac{\partial \ln g(w, u)}{\partial u} \right| \leq C \sigma_1(|w - u|) \frac{|u|}{|w - u|},
\]

for \(|u|, |w| \geq 1\). Hence, from the hypothesis that \(\Gamma\) belongs to class \(j_1\), as a function of \(u\) on \(|u| = 1\), the function \(\ln g(w, u)\) satisfies the Dini condition uniformly on \(|w| = 1\). It follows that

\[
\ln g(w, u) = \sum_{j=1}^{\infty} \frac{a_j(w)}{u^j},
\]

uniformly on \(|u|, |w| \geq 1\). From the property

\[
\sum_{k=1}^{n} w_k^{-j} = \begin{cases} 0 & \text{if } n \nmid j, \\ n & \text{if } n \mid j \end{cases}
\]

of the \(n\)th roots of unity, we have, from (3.7),

\[
\ln \Omega_n(w) = n \sum_{l=1}^{\infty} a_l(w)
\]

uniformly on \(|w| \geq 1\). To estimate \(a_j(w)\), since \(\Gamma\) belongs to class \(j_1\) we may use the Hardy-Littlewood inequality (cf. [4, p. 100])

\[
|\Psi''(u)| \leq C \frac{\sigma_1(|u| - 1)}{|u| - 1}, \quad |u| > 1.
\]
Indeed, letting $1 < \rho \leq \frac{3}{2}$, we have from (3.8), for $|w| = 1$,

$$\int_{|u|=\rho} \left| \frac{\partial^2 \ln g(w, u)}{\partial u^2} \left| du \right| \right.$$

$$= \int_{|u|=\rho} \left| \frac{\Psi''(u)}{\Psi(u) - \Psi(w)} + \frac{(\Psi'(u))^2}{(\Psi(u) - \Psi(w))^2} - \frac{1}{(w - u)^2} \right| |du|$$

$$\leq \int_{|u|=\rho} \left| \frac{\Psi''(u)}{\Psi(n) - \Psi(w)} \right| |du|$$

$$+ \int_{|u|=\rho} \left| \frac{\Psi'(u)}{\Psi(u) - \Psi(w)} - \frac{1}{w - u} \right| \left| \frac{\Psi'(u)}{\Psi(u) - \Psi(w)} + \frac{1}{w - u} \right| |du|$$

$$:= I_1 + I_2,$$

where by applying (2.4) and (3.12), we have

$$I_1 \leq C_1 \frac{\sigma_1(\rho - 1)}{\rho - 1} \int_{|u|=\rho} \frac{du}{w - u} \leq C_2 \frac{\sigma_1(\rho - 1)}{\rho - 1} \ln \frac{1}{\rho - 1},$$

and by using (3.9) and (2.4), we also have

$$I_2 \leq C_1 \int_{|u|=\rho} \frac{\sigma_1(|w - u|)}{|w - u|^2} |du| \leq C_2 \frac{\sigma_1(\rho - 1)}{\rho - 1} \ln \frac{1}{\rho - 1}.$$

That is,

$$(3.13) \quad \int_{|u|=\rho} \left| \frac{\partial^2 \ln g(w, u)}{\partial u^2} \left| du \right| \right. \leq C \frac{\sigma_1(\rho - 1)}{\rho - 1} \ln \frac{1}{\rho - 1}, \quad \rho > 1.$$

By taking the second derivative of the power series (3.10) and applying the estimate in (3.13), we have, for $j = 2, 3, \ldots$,

$$|a_j(w)| = \left| \frac{1}{2\pi i} \int_{|u|=\rho} \frac{1}{j(j + 1)} \frac{\partial^2 \ln g(w, u)}{\partial u^2} u^{j+1} du \right|$$

$$\leq \rho^{j+1} \frac{1}{2\pi j^2} \int_{|u|=\rho} \left| \frac{\partial^2 \ln g(w, u)}{\partial u^2} \right| |du| \leq C \rho^{j+1} \frac{\sigma_1(\rho - 1)}{j^2(\rho - 1)} \ln \frac{1}{\rho - 1}.$$

By taking $\rho = 1 + \frac{1}{j}$, it follows that

$$\max_{|w|=1} \left| a_j(w) \right| \leq C \frac{\sigma_1(j^{-1})}{j} \ln j.$$

We now apply this estimate to (3.11), yielding

$$\max_{|w|=1} |\ln \Omega_n(w)| \leq C n \sum_{i=1}^{\infty} \frac{\sigma_1(1/\ln t)}{\ln(\ln t)} \ln(\ln t)$$

$$\leq C \int_n^{\infty} \frac{\sigma_1(1/\ln t)}{t} \ln t dt = C \int_0^{1/n} \frac{\sigma_1(s)}{s} |\ln s| ds$$

$$\leq C \frac{1}{\ln n} \int_0^{1/n} \frac{\sigma_1(s)}{s} |\ln s|^2 ds = o \left( \frac{1}{\ln n} \right),$$
where (2.2) has been used. This estimate is equivalent to (3.4).

To prove (3.5) for $\Gamma$ belonging to class $j_2$, we will apply the inequality of Hardy-Littlewood

$$|\Psi''(u)| \leq C \frac{\sigma_2(|u| - 1)}{|u| - 1}, \quad |u| > 1,$$

(3.14)

where $\sigma_2$ denotes the modulus of continuity of $\Psi''(u)$. Set

$$g_1(w, u) = \frac{\Psi'(u)}{\Psi(w) - \Psi(u)} - \frac{1}{w - u}.$$

Then by taking the logarithmic derivative of $\Omega_n$ and $g(w, w_k)$ in (2.8) and (3.6), respectively, we have

$$\frac{\Omega_n'(w)}{\Omega_n(w)} = \sum_{k=1}^{n} \frac{g'(w, w_k)}{g(w, w_k)} = \sum_{k=1}^{n} g_1(w, w_k),$$

(3.15)

where $g_1(w, u) = \Psi'(u)/\Psi(u) - 1/(w - u)$, hence, in view of $\Gamma$ belonging to class $j_2$, we have

$$g_1(w, u) = \sum_{j=1}^{\infty} \frac{b_j(w)}{u^j}$$

(3.16)

uniformly on $|u|, |w| \geq 1$. By applying (3.14) we may obtain an estimate similar to that of (3.13), namely,

$$\int_{|u|=\rho} \left| \frac{\partial^2 g_1(w, u)}{\partial u^2} \right| |du| \leq C \frac{\sigma_2(\rho - 1)}{\rho - 1} \ln \frac{1}{\rho - 1},$$

where $\rho > 1$. Hence, as before, we have

$$\max_{|w|=1} |b_j(w)| \leq C \frac{\sigma_2(j^{-1})}{j} \ln j$$

and

$$\frac{\Omega_n'(w)}{\Omega_n(w)} = n \sum_{l=1}^{\infty} b_{ln}(w), \quad |w| = 1,$$

so that

$$\max_{|w|\geq 1} \left| \frac{\Omega_n'(w)}{\Omega_n(w)} \right| \leq C n \sum_{l=1}^{\infty} \frac{\sigma_2(1/ln)}{ln} \ln(ln) \leq \int_{0}^{1/n} \frac{\sigma_2(t)}{t} |\ln t| dt$$

which is $o(1)$ by (2.3). This completes the proof of the lemma.

Remark. In [10], where Lagrange interpolation (or $q = 0$) was considered, the Jordan curve $\Gamma$ was assumed to belong to $C^{1+\delta}$, $\delta > 0$. However, from our estimate (3.4) and the procedure in [10], it can be shown that the result there also holds for $\Gamma$ belonging to class $j_1$.

As a consequence of estimates (3.5) in Lemma 2, the identity (3.2) in Lemma 1 yields the following result.
Corollary 1. Let \( \Gamma \) belong to class \( j_2 \). Then

\[
(3.17) \quad \sum_{l \neq k} \frac{1}{z_k - z_l} = \frac{(n-1)}{2w_k \Psi'(w_k)} - \frac{\Psi''(w_k)}{2 \left[ \Psi'(w_k) \right]^2} + o(1)
\]

uniformly in \( k \), \( 1 \leq k \leq n \).

In the proof of Theorem 1, the following estimates will also be used.

Lemma 3. Let \( \Gamma \) belong to \( j_2 \). Then for \( k = 1, 2, \ldots, n \) and \( r = 0, 1, \ldots \)

\[
(3.18) \quad \sum_{l \neq k} \frac{1}{(z_k - z_l)^{r+1}} = O(n^{r+1})
\]

and

\[
(3.19) \quad \frac{d^r}{dz^r} \left( \frac{z - z_k}{\omega_n(z)} \right) \bigg|_{z = z_k} = O(n^{r-1})
\]

uniformly in \( k \), \( 1 \leq k \leq n \).

Proof. The estimate (3.18) for \( r = 0 \) can easily be deduced by (3.17). For \( r \geq 1 \), by using (2.4) we have

\[
\left| \sum_{l \neq k} \frac{1}{(z_k - z_l)^{r+1}} \right| \leq \sum_{l \neq k} \left| \frac{1}{z_k - z_l} \right|^{r+1} \leq C_2 \sum_{l \neq k} \left| \frac{1}{w_k - w_l} \right|^{r+1}
\]

\[
\leq 2C_2 \sum_{l=1}^{[(n-1)/2]} \frac{1}{(2 \sin \theta / n)^{r+1}}
\]

\[
\leq 2C_2 \sum_{l=1}^{[(n-1)/2]} \frac{1}{(4l/n)^{r+1}} = O(n^{r+1}).
\]

We are going to verify (3.19) by induction. For \( r = 0 \), by using (3.1), (3.4), and (2.5), we have (3.19). For \( r \geq 1 \), by the induction hypothesis and using (3.18), we obtain

\[
\left( \frac{z - z_k}{\omega_n(z)} \right)^{(s)} \bigg|_{z = z_k} = - \left( \frac{z - z_k}{\omega_n(z)} \sum_{l \neq k}^{n} \frac{1}{z - z_l} \right)^{(s)} \bigg|_{z = z_k}
\]

\[
= - \sum_{\nu=0}^{s} \binom{s}{\nu} \left( \frac{z - z_k}{\omega_n(z)} \right)^{(\nu)} \left( \sum_{l \neq k}^{n} \frac{1}{z - z_l} \right)^{(s-\nu)} \bigg|_{z = z_k}
\]

\[
= - \sum_{\nu=0}^{s} \binom{s}{\nu} \left( \frac{z - z_k}{\omega_n(z)} \right)^{(\nu)} \left( \sum_{l \neq k}^{n} \frac{(-1)^{s-\nu}(s-\nu)!}{(z - z_l)^{s-\nu+1}} \right) \bigg|_{z = z_k}
\]

\[
= \sum_{\nu=0}^{n} \binom{s}{\nu} O(n^{n-1}) \cdot O \left( n^{s-\nu+1} \right) = O(n^{s}).
\]

This completes the proof of the lemma. \( \square \)
We are now ready to prove Theorem 1.

Proof of Theorem 1. For $q = 0$, from (2.7), it is clear that

$$\sigma_{k,\nu}(0) = \frac{1}{\nu!} \frac{d^\nu}{dz^\nu} \left( \frac{z - z_k}{\omega_n(z)} \right)_{z=z_k}^{q+2},$$

which yields (2.9) by using (3.19) in Lemma 3. We will now use induction in $q$. Indeed, by the induction hypothesis and using (3.19) in Lemma 3 it follows that

$$\sigma_{k,\nu}(q + 1) = \frac{1}{\nu!} \frac{d^\nu}{dz^\nu} \left( \frac{z - z_k}{\omega_n(z)} \right)_{z=z_k}^{q+2} = \frac{1}{\nu!} \sum_{j=0}^{\nu} \binom{\nu}{j} j! O \left( \frac{1}{n^{q-j+1}} \right) O \left( n^{\nu-j-1} \right) = O \left( \frac{1}{n^{q+2-\nu}} \right).$$

This completes the proof of Theorem 1. □

4. PROOF OF THEOREM 2

We first establish the existence and uniqueness of $\tilde{H}_N(f; \cdot)$ for any given $f \in A(\overline{D})$. Since $N = (q + 1)n - 1$, it follows from the definition (2.11) that $A_{k_i} \in \pi_N$. Hence, $\tilde{H}_N(f; \cdot) \in \pi_N$ also. Next, we will establish (2.11). For $l \neq k$, it is clear from the first factor that $A_{k,l}^{(\nu)}(z_i) = 0$ for all $\nu, j = 0, \ldots, q$. We now consider the case $l = k$. From (2.10) and (2.7), it follows that

$$A_{kj}(z) = \left( \frac{\omega_n(z)}{z - z_k} \right)^{q+1} \left( \frac{z - z_k}{\omega_n(t)} \right)^{q+1} - \sum_{\mu=q-j+1}^{\infty} \alpha_{k,\mu}(z - z_k)^{\mu}.$$

Hence, for $\nu < j$, we have $A_{kj}^{(\nu)}(z_k) = 0$. For $\nu = j$, then $A_{kj}^{(j)}(z_k) = 1$. Finally, for $\nu > j$, we also have $A_{kj}^{(\nu)}(z_k) = 0$. This establishes the interpolatory property of $A_{kj}$ in (2.11). Thus, by defining $\tilde{H}_N(f; \cdot)$ as in (2.12), $\tilde{H}_N(f; \cdot)$ solves the interpolation problem (1.1). The uniqueness of $\tilde{H}_N(f; \cdot)$ is trivial. □

In order to establish the estimates (2.13) and (2.14), we need the following lemma.
Lemma 4. Let $\Gamma$ belong to class $j_1$. Then

\begin{align}
\max_{z \in \overline{D}} \sum_{k=1}^{n} \left| \frac{\omega_n(z)}{(z - z_k)\omega_n'(z_k)} \right| &= O(\ln n), \\
\max_{z \in \overline{D}} \sum_{k=1}^{n} \left| \frac{\omega_n(z)}{(z - z_k)\omega_n'(z_k)} \right|^{1+\delta} &= O(1), \quad \delta > 0,
\end{align}

and for $1 < p < \infty$,

\begin{equation}
\left\| \sum_{k=1}^{n} b_k \frac{\omega_n(z)}{(z - z_k)\omega_n'(z_k)} \right\|_p = O \left( \max_{1 \leq k \leq n} |b_k| \right).
\end{equation}

Proof. From (2.8) and (3.1), we have

\begin{equation}
\frac{\omega_n(z)}{(z - z_k)\omega_n'(z_k)} = \frac{w_k \Psi'(w_k) (w^n - 1)}{n(\Psi(w) - \Psi'(w_k))} \frac{\Omega_n(w)}{\Omega_n'(w_k)},
\end{equation}

where $z = \Psi(w)$ and $z_k = \Psi(w_k)$. For the unit disk, it is well known (cf. Gaier [6, pp. 80–81]) that

\begin{equation}
\max_{|w| \leq 1} \sum_{k=1}^{n} \left| \frac{w_k}{n} \frac{w^n - 1}{w - w_k} \right| = O(\ln n).
\end{equation}

In addition, for $\delta > 0$ it is also well known (cf. [9]) that

\begin{equation}
\max_{|w| \leq 1} \sum_{k=1}^{n} \left| \frac{w_k}{n} \frac{w^n - 1}{w - w_k} \right|^{1+\delta} = O(1).
\end{equation}

Hence, by applying (2.4), (2.5), and (3.4) in Lemma 2 to (4.4), we have both (4.1) and (4.2). Next, by (4.4), it follows that

\begin{align}
\sum_{k=1}^{n} b_k \frac{\omega_n(z)}{(z - z_k)\omega_n'(z_k)} &= \sum_{k=1}^{n} b_k \frac{w_k \Psi'(w_k) (w^n - 1)}{n(\Psi(w) - \Psi'(w_k))} \left( \frac{\Psi_n(w)}{\Psi_n'(w_k)} - 1 \right) + \sum_{k=1}^{n} b_k \frac{w_k}{n} \frac{w^n - 1}{w - w_k} \\
&\quad + \sum_{k=1}^{n} b_k \frac{w_k}{n} \frac{w^n - 1}{w - w_k} \left[ \frac{\Psi'(w_k)}{\Psi(w) - \Psi'(w_k)} - \frac{1}{w - w_k} \right] \\
&=: I_5 + I_6 + I_7.
\end{align}

By applying (4.1) and (3.4) in Lemma 2, we obtain

\begin{equation}
\max_{z \in \overline{D}} |I_5| = O \left( \max_{1 \leq k \leq n} |b_k| \right).
\end{equation}

In order to estimate $I_7$, we may assume, without loss of generality, that $|w - 1|$ is not greater than $|w - w_k|$, $k = 1, 2, \ldots, n - 1$, so that

\begin{equation}
|\arg w| \leq \frac{\pi}{n}, \quad \left| \frac{w^n - 1}{w - 1} \right| \leq n,
\end{equation}

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and
\[ \left| \frac{1}{w-w_k} \right| \leq \begin{cases} \pi/n, & 1 \leq k \leq n/2, \\ (n-k)/k, & n/2 < k \leq n-1. \end{cases} \]

Hence from (3.9), we have
\[ |I_{b_k} w_n^j (w^n - 1) \left[ \frac{\Psi'(w_k)}{\Psi(w) - \Psi(w_k)} - \frac{1}{w - w_k} \right] | \leq C |b_k| \]
and
\[ \left| \sum_{k=1}^{n-1} b_k \frac{w_k}{n} (w^n - 1) \left[ \frac{\Psi'(w_k)}{\Psi(w) - \Psi(w_k)} - \frac{1}{w - w_k} \right] \right| \leq C \max_{1 \leq k \leq n-1} |b_k| \frac{1}{n} \sum_{k=1}^{n-1} \frac{\sigma_1(|w - w_k|)}{|w - w_k|} \]
\[ \leq C \max_{1 \leq k \leq n-1} |b_k| \frac{1}{n} \sum_{k=1}^{n-1} \frac{\sigma_1(k/n)}{k/n} \]
\[ \leq C \max_{1 \leq k \leq n-1} |b_k| \int_0^1 \frac{\sigma_1(t)}{t} \, dt \leq C \max_{1 \leq k \leq n-1} |b_k|, \]
where the condition in (2.2) is used. Thus combining (4.5) and (4.6), we obtain
\[ \max_{|z| \in D} |I_{b_k}| = O \left( \max_{1 \leq k \leq n} |b_k| \right). \]

Finally, by the Marcinkiewicz-Zygmund inequality for \( 1 < p < \infty \) (cf. [15]), we have
\[ \|I_{b_k}\|_p = O \left( \frac{1}{n} \sum_{k=1}^{n} |b_k|^p \right)^{1/p} = O \left( \max_{1 \leq k \leq n} |b_k| \right). \]

This completes the proof of the lemma. \( \square \)

We now return to the estimates of (2.13) and (2.14) and obtain
\[ A_{k,j}(z) = \left( \frac{\omega_n(z)}{(z-z_k)} \right)^{q+1} \frac{1}{j!} \alpha_k^{q-j} (z-z_k)^q \]
\[ + \left( \frac{\omega_n(z)}{z-z_k} \right)^{q+1} \frac{(z-z_k)^{q-j-1}}{j!} \sum_{\nu=0}^{q-j-1} \alpha_{k,\nu}(z-z_k)^\nu \]
\[ := I_8(z) + I_9(z). \]

Hence, by (3.1) in Lemma 1, (3.4) in Lemma 2, (2.5), (2.9) in Theorem 1, and (4.1), (4.3) in Lemma 4, we have
\[ \sum_{k=1}^{n} |I_8(z)| = O \left( \frac{1}{n^j} \right) \max_{z \in D} \sum_{k=1}^{n} \left| \frac{\omega_n(z)}{(z-z_k)\omega_n(z_k)} \right| = O \left( \frac{\ln n}{n^j} \right) \]
and for \( 1 < p < +\infty \)
\[ \left\| \sum_{k=1}^{n} b_k I_q(z) \right\|_p = O(n) \max_{1 \leq k \leq n} |b_k\alpha_k^{q-j}| = O \left( \frac{1}{n^j} \right) \max_{1 \leq k \leq n} |b_k|. \]
Similarly, by (3.1) in Lemma 1, (3.4) in Lemma 2, (2.5), (2.9) in Theorem 1, and (4.2) in Lemma 4, we have

$$\sum_{k=1}^{n} |I_k(z)| = O \left( \frac{1}{n^l} \right).$$

By combining these estimates, we have established both (2.13) and (2.14). □

5. PROOF OF THEOREMS 3–6

Let \( \Gamma \) belong to class \( j_2 \) and \( f \in A(D) \). It is known (cf. Theorems 1 and 6 in [5, Chapter 9]) that there exists \( P_N \in \pi_N \) such that

$$\max_{z \in D} |f(z) - P_N(z)| = O \left( \omega \left( f; \frac{1}{N} \right) \right)$$

and

$$\max_{z \in D} |P_N^{(m)}(z)| = O \left( N^m \omega \left( f; \frac{1}{N} \right) \right), \quad m = 1, 2, \ldots.$$

By using the first part of Theorem 2, we have

$$f(z) - H_N(f; z) = f(z) - P_N(z) + \sum_{k=1}^{N} (P_N(z_k) - f(z_k)) A_{k0}(z) + \sum_{k=1}^{q} \sum_{j=1}^{n} A_{kj}(z).$$

Hence, by applying (2.13) and (2.14) of Theorem 2 and (5.1), (5.2) above, we have completed the proof of Theorem 3. Next, we write

$$f(z) - \tilde{H}_N(f; z) = f(z) - H_N(f; z) + \sum_{k=1}^{N} \sum_{j=1}^{q} a_{kj}(z).$$

Here, by using the hypothesis (2.17) and Theorem 3, we have

$$\max_{z \in D} |f(z) - H_N(f; z)| \to 0,$$

and by using the hypothesis (2.16) and applying (2.13) in Theorem 2, we also have

$$\max_{z \in D} \left( \sum_{k=1}^{N} \sum_{j=1}^{q} a_{kj}(z) \right) \to 0.$$

This completes the proof of Theorem 4. The proofs of Theorems 5 and 6 are similar simply by applying (2.14) in Theorem 2, noting that Hölder’s inequality can be applied for \( 0 < p \leq 1 \) and using the result for \( p = 2 \). □

6. FINAL REMARKS

In this section, we give examples of the domain \( D \) whose boundary curve \( \Gamma \) belongs to classes \( j_1 \) and \( j_2 \). Let \( \Gamma \) be of class \( C^1 \) and denote its angle of inclination as a function of arc length \( s \) by \( \theta(s), \quad 0 \leq s \leq |\Gamma| \), the length of \( \Gamma \).
Proposition 1. If $\Gamma$ satisfies
\begin{equation}
\int_0^a \frac{\omega(\theta; t)}{t} |\ln t|^3 \, dt < \infty, \quad a > 0,
\end{equation}
then $\Gamma$ belongs to class $J_1$.

Of course, every $\Gamma$ of class $C^{1+\delta}$ for some $\delta > 0$ satisfies (6.1). We also have the following

Proposition 2. If $\Gamma$ satisfies
\begin{equation}
\int_0^a \frac{\omega'(\theta; t)}{t} |\ln t|^2 \, dt < \infty, \quad a > 0,
\end{equation}
then $\Gamma$ belongs to class $J_2$.

Of course, every $\Gamma$ of class $C^{2+\delta}$ for some $\delta > 0$ satisfies (6.2).

To prove these results, we need the following result in [14]: If
\begin{equation}
\int_0^a \frac{\omega^{(n)}(\theta; t)}{t} \, dt < \infty, \quad a > 0,
\end{equation}
then $\Psi^{(n+1)}$ is continuous on $|w| \geq 1$ and

\begin{equation}
\omega(\Psi^{(n+1)}; t) = O\left( \int_0^t \frac{\omega^{(n)}(\theta; \tau)}{\tau} \, d\tau \right. \\
+ t \int_t^a \frac{\omega^{(n)}(\theta; \tau)}{\tau^2} \, d\tau \left. + t \ln \frac{1}{t} \right), \quad a > 0.
\end{equation}

Let $n = 0$. If (6.1) is satisfied, so is (6.3), and hence $\Psi'$ is continuous on $|w| \geq 1$. Using (6.4) for $n = 0$, we have

\begin{align*}
\int_0^a \frac{\omega(\Psi; t)}{t} |\ln t|^2 \, dt &= O\left( \int_0^a \left( \int_0^t \frac{\omega(\theta; \tau)}{\tau} \, d\tau \right) \frac{|\ln t|^2}{t} \, dt \\
&\quad + \int_0^t \left( t \int_t^a \frac{\omega(\theta; \tau)}{\tau^2} \, d\tau \right) \frac{|\ln t|^2}{t} \, dt + \int_0^a t \ln(1/t) \ln |t|^2 \, dt \right) \\
&= O\left( \int_0^a \left( \int_0^a \frac{|\ln t|^2}{t} \, dt \right) \frac{\omega(\theta; \tau)}{\tau} \, d\tau \\
&\quad + \int_0^a \left( \int_0^\tau |\ln t|^2 \, dt \right) \frac{\omega(\theta; \tau)}{\tau^2} \, d\tau \right) + O(1) \\
&= O\left( \int_0^a \frac{\omega(\theta; t)}{t} |\ln t|^2 \, dt \right) + O(1) < \infty.
\end{align*}
That is, $\Gamma$ is of class $j_1$. This completes the proof of Proposition 1. The proof of Proposition 2 is similar by applying $n = 1$ in (6.3) and (6.4), using the condition (6.2).

We conclude this paper by posing three open problems.

1. In this paper, we consider the $(0, 1, \ldots, q)$ Hermite-Fejér interpolation problem where the interpolatory polynomials $H_N(f; \cdot)$ satisfy $H_N^{(j)}(f; z_k) = 0$, $j = 1, \ldots, q$ and $k = 1, \ldots, n$. It is interesting to study if the convergence and estimates in this paper are still valid if we impose a more general interpolatory condition:

$$H_N^{(j)}(f; z_k) = 0 \quad \text{for} \quad j = 1, \ldots, q, \quad k = 1, \ldots, n,$$

where $q_k = q_k(n)$ satisfies $\max_{1 \leq k \leq n} q_k(n) \leq M < \infty$ for all $n$.

2. How much can the Fejér points $z_k = z_{nk}$ be perturbed on $\Gamma$ so that the convergence and estimates in this paper are still valid?

3. If $D$ is different from the unit disk, do there exist $(0, m_1, \ldots, m_q)$ Birkhoff-Fejér interpolants $B_N(f; \cdot)$; that is,

$$B_N(f; z_k) = f(z_k) \quad \text{and} \quad B_N^{(m_j)}(f; z_k) = 0,$$

for $j = 1, \ldots, q$ and $k = 1, \ldots, n$? If $B_N(f; \cdot)$ exist, do they converge to $f$ in $L^p$, $0 < p < \infty$? For the unit disk, results on convergence and estimates have been obtained in [10].

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