MICROLOCAL HOLMGREN'S THEOREM FOR A CLASS OF HYPO-ANALYTIC STRUCTURES

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ABSTRACT. A microlocal version of Holmgren's Theorem is proved for a certain class of the hypo-analytic structures of Baouendi, Chang, and Treves.

1. INTRODUCTION

In [4] Sjöstrand gave a simpler proof of a result of Schapira [3] concerning a microlocal version of Holmgren's theorem for real analytic data. Inspired by [4], in this paper we will extend Schapira's result to a certain class of hypo-analytic structures. The paper is organized as follows: In §2 we discuss the Cauchy-Kovalevskva theorem for maximal hypo-analytic structures. In §3 we introduce a class of hypo-analytic structures which we call real hypo-analytic, give a statement of the main theorem of this article, and derive two corollaries. A lemma is included in the same section and is used in the proof of the main theorem which appears in §4.

2. CAUCHY-KOVALEVSKA FOR HYPO-ANALYTIC STRUCTURES

We are interested in the hypo-analytic structures introduced by Baouendi, Chang, and Treves in [1]. We briefly recall the relevant concepts here.

Let $\Omega$ be a smooth manifold of dimension $m$. A hypo-analytic structure of maximal dimension on $\Omega$ is the data of an open covering $\{U_\alpha\}$ of $\Omega$ and for each index $\alpha$, of $m$ $C^\infty$ functions $Z_\alpha^1, \ldots, Z_\alpha^m$ satisfying the following two conditions:

1. $dZ_\alpha^1, \ldots, dZ_\alpha^m$ are linearly independent at each point of $U_\alpha$;
2. if $U_\alpha \cap U_\beta \neq \emptyset$, there are open neighborhoods $O_\alpha$ of $Z_\alpha(U_\alpha \cap U_\beta)$ and $O_\beta$ of $Z_\beta(U_\alpha \cap U_\beta)$ and a holomorphic map $F_\beta^\alpha$ of $O_\alpha$ onto $O_\beta$ such that $Z_\beta = F_\beta^\alpha \circ Z_\alpha$ on $U_\alpha \cap U_\beta$.

We will use the notation $Z_\alpha = (Z_\alpha^1, \ldots, Z_\alpha^m): U_\alpha \mapsto C^m$. A distribution $h$ defined in an open neighborhood of a point $p_0$ of $\Omega$ is hypo-analytic at $p_0$ if there is a chart $(U_\alpha, Z_\alpha)$ of the above type whose domain contains $p_0$ and a
holomorphic function \( \tilde{h} \) defined on an open neighborhood of \( Z_\alpha(p_0) \) in \( C^m \) such that \( h = \tilde{h} \circ Z_\alpha \) in a neighborhood of \( p_0 \). By a hypo-analytic local chart we mean an \( m+1 \)-tuple \((U, Z^1, \ldots, Z^m)\) [abbreviated \((U, Z)\)] consisting of an open subset \( U \) of \( \Omega \) and of \( m \) hypo-analytic functions whose differentials are linearly independent at every point of \( U \).

In [2] we introduced hypo-analytic differential operators which by definition map hypo-analytic functions to hypo-analytic functions. A linear differential operator \( P \) on \( \Omega \) is hypo-analytic if and only if for every hypo-analytic local chart \((U, Z^1, \ldots, Z^m)\), \( U \) sufficiently small, and vector fields \( M_1, \ldots, M_m \) satisfying \( M_j Z^k = \delta^k_j \) we have: \( P = \sum_{|a| \leq n} a_n(x)M^n \), where each \( a_n \) is a hypo-analytic function on \( U \). Let \( p \) be an arbitrary point of \( \Omega \). The differentials of the germs of hypo-analytic functions at \( p \) make up a complex vector subspace of the complex cotangent space \( CT^*_p \Omega \). This subspace, which we denote by \( T'_p \), has dimension \( m \). Condition (2) in the definition of hypo-analytic structures implies that the subspace \( T'_p \) makes up a smooth vector subbundle \( T' \) of the complex cotangent bundle \( CT^* \Omega \). \( T' \) will be referred to as the structure bundle.

We now introduce the concept of hypo-analytic submanifolds. By a submanifold of \( \Omega \) we mean a subset of \( \Omega \) equipped with a \( C^\infty \) structure such that the natural injection into \( \Omega \) is a \( C^\infty \) map with injective differential. Let \( M \) be a submanifold of \( \Omega \). We shall denote by \( \pi_M \) the natural map \( T^* \Omega |_M \rightarrow T^* M \) and by \( \pi^C_M \) the analogous map of the complex cotangent bundles. In general, \( T'_M = \pi^C_M(T') \) is not a vector bundle.

**Definition 2.1.** A submanifold \( M \) of \( \Omega \) is called a hypo-analytic submanifold if it is equipped with a hypo-analytic structure whose structure bundle is identical to \( T'_M \) and which has the following property: Given any hypo-analytic function \( f \) on an open set \( \Omega' \subset \Omega \) which intersects \( M \), the restriction of \( f \) to \( M \cap \Omega' \) is hypo-analytic.

Simple examples show that the second property in the above definition is not redundant.

**Proposition 2.1.** Suppose \( \Sigma \) is a hypo-analytic submanifold of \( \Omega \) whose structure bundle has dimension \( m-k \). Then each point \( q \in \Sigma \) is contained in a hypo-analytic chart \((U; Z^1, \ldots, Z^m)\) of \( \Omega \) with \( Z^{m-k+1}, \ldots, Z^m \) all vanishing on \( U \cap \Sigma \).

**Proof.** Let \( q \in \Sigma \) and \((U; W^1, \ldots, W^m)\) be a hypo-analytic chart for \( \Omega \) around \( q \). Since the differentials \( dW^1, \ldots, dW^m \) span \( CT^*U \), without loss of generality we may assume that \( \pi^C_{\Sigma}(dW^1), \ldots, \pi^C_{\Sigma}(dW^{m-k}) \) span \( CT^*(U \cap \Sigma) \).

Moreover, \((U \cap \Sigma, W^1_{|\Sigma}, \ldots, W^{m-k}_{|\Sigma})\) is a hypo-analytic chart in \( \Sigma \) since \( \Sigma \) is a hypo-analytic submanifold of \( \Omega \).
Now $W^{m-k+1}, \ldots, W^m$ all restrict to hypo-analytic functions in $\Sigma$. Therefore, there are holomorphic functions $H_1, \ldots, H_k$ such that $W^{m-k+j}(x) = H_j(W^1(x), \ldots, W^{m-k}(x))$ for each $x \in \Sigma \cap U$ and $1 \leq j \leq k$. Here the set $U$ may have to be contracted. For $x \in U$, let

$$Z^j(x) = W^j(x), \quad 1 \leq j \leq m - k,$$

and

$$Z^l(x) = W^{m-k+l}(x) - H_l(W^1(x), \ldots, W^{m-k}(x))$$

when $m - k < l \leq m$.

Then $(U; Z^1, \ldots, Z^m)$ is a hypo-analytic chart on $\Omega$ satisfying the properties in the proposition.

**Remark 2.1.** If $\Sigma$ is a hypo-analytic submanifold of $\Omega$, then the dimension of $\Sigma$ is the same as the dimension of its structure bundle.

Suppose now $P$ is a hypo-analytic differential operator on $\Omega$. We would like to introduce the concept of noncharacteristic hypersurfaces. Let $\Sigma$ be a hypo-analytic hypersurface of $\Omega$. By Proposition 2.1, $\Sigma$ is locally given by $H(x) = 0$, where $H$ is hypo-analytic and $dH \neq 0$. If $(U; Z^1, \ldots, Z^m)$ is a hypo-analytic chart for $\Omega$ near a central point $q \in \Sigma$, then $P$ can be written as $P = \sum_{|\alpha| \leq k} a_{\alpha}(Z(x)) M^\alpha$ and $H(x) = \hat{H}(Z(x))$ for some holomorphic functions $a_{\alpha}$ and $\hat{H}$ in a neighborhood of $Z(q)$ in $C^m$. We push everything by the map $Z$ into $C^m$ near $Z(q)$ and write $P^Z(z, \frac{\partial}{\partial z}) = \sum_{|\alpha| \leq k} a_{\alpha}(z)(\frac{\partial}{\partial z})^\alpha$ and $\Sigma^Z = \{z \in C^m : \hat{H}(z) = 0\}$.

Since $dH \neq 0$, $\Sigma^Z$ is a complex submanifold of $C^m$ of complex codimension 1 passing through $Z(q)$.

If $(V; W^1, \ldots, W^m)$ is another hypo-analytic chart about $q$, let $G$ be a biholomorphism near $Z(q)$ in $C^m$ such that $(W^1, \ldots, W^m) = G(Z^1, \ldots, Z^m)$. Then $P^W_k(w, \frac{\partial}{\partial w})$ and $\Sigma^W$ are the expressions of $P^Z_k(z, \frac{\partial}{\partial z})$ and $\Sigma^Z$ in the coordinates $w^1, \ldots, w^m$ of $C^m$. Hence, in particular, $\Sigma^Z$ is noncharacteristic with respect to $P^Z$ if and only if $\Sigma^W$ is noncharacteristic with respect to $P^W$.

This observation justifies the following definition in which we use the same notations as above.

**Definition 2.2.** We say $\Sigma$ is noncharacteristic with respect to $P$ at a point $q \in \Sigma$ if $\Sigma^Z$ is noncharacteristic with respect to $P^Z(z, \frac{\partial}{\partial z})$ at $Z(q)$ for some hypo-analytic chart $(U; Z^1, \ldots, Z^m)$ about $q$.

We can now formulate a Cauchy-Kovalevska theorem for a hypo-analytic differential operator and hypo-analytic Cauchy data on a noncharacteristic hypo-analytic hypersurface.

Suppose now $P$ is a hypo-analytic differential operator and $\Sigma$ is a noncharacteristic hypo-analytic hypersurface with respect to $P$ at the point $q \in \Sigma$. Let
the order of $P$ near $q = k$. Suppose $L$ is a hypo-analytic vector field not belonging to $CT\Sigma$ at the point $q$ (and hence near $q$). Then we have:

**Theorem 2.1.** There is an open neighborhood $\Omega'$ of $q$ in $\Omega$ such that to every hypo-analytic function $f$ in $\Omega'$ and to every set of $k$ hypo-analytic functions $u_0, \ldots, u_{k-1}$ on $\Sigma \cap \Omega'$, there is a unique hypo-analytic function $u$ in $\Omega'$ such that

$$Pu = f \quad \text{in } \Omega',$$

and for every $j = 0, \ldots, k-1$, $L^j u = u_j$ in $\Sigma \cap \Omega'$.

**Proof.** By Proposition 2.1, $q \in \Sigma$ is contained in a hypo-analytic chart $(U; Z^1, \ldots, Z^m)$ of $\Omega$ with $Z^m$ vanishing on $U \cap \Sigma$. Let $M_1, \ldots, M_m$ be the vector fields in $U$ satisfying $M_j Z^k = \delta^k_j$. Then in the chart $(U, Z)$, we may write $P = \sum_{|\alpha| \leq k} a_{\alpha}(x) M^\alpha$ and $L = \sum_j c_j(x) M_j$, where the coefficients are all hypo-analytic. The condition $L \notin CT\Sigma$ near $q$ is equivalent to $c_m(x) \neq 0$ for $x$ near $q$.

Let $\hat{u}_j$, $\hat{f}$, $\hat{a}_\alpha$, and $\hat{c}_j$ be the holomorphic functions defined near $Z(q) \in C^m$ such that $u_j(x) = \hat{u}_j(Z(x))$ etc.

Set

$$P^Z \left(z, \frac{\partial}{\partial z} \right) = \sum_{|\alpha| \leq k} a_{\alpha}(z) \left( \frac{\partial}{\partial z} \right)^\alpha,$$

$$L^Z = \sum_{j=1}^m \hat{c}_j(z) \frac{\partial}{\partial z_j} \quad \text{and} \quad \Sigma^Z = \{ z \in C^m : z_m = 0 \}.$$

The assumptions on $\Sigma$ and $L$ imply that $\Sigma^Z$ is noncharacteristic for $P^Z$ and that $\hat{c}_m(z) \neq 0$ for $z$ near $Z(q)$. Therefore the existence part of Theorem 2.1 follows from the existence part of the holomorphic version of the Cauchy-Kovalevska theorem applied to the problem

$$P^Z \hat{u} = \hat{f} \quad \text{near } Z(q) \text{ in } C^m$$

and for $0 \leq j \leq k-1$,

$$(L^Z)^j \hat{u} = \hat{u}_j \quad \text{near } Z(q) \text{ in } \Sigma^Z \text{ (see [7]).}$$

We just set $u(x) = \hat{u}(Z(x))$ and observe that $M_j u(x) = \frac{\partial u}{\partial z_j}(Z(x))$ for each $j = 1, \ldots, m$. To see the uniqueness, suppose $u'$ is another solution and set $v = u - u'$. Then

$$Pv = 0 \quad \text{in } \Omega' \quad \text{and} \quad L^j v = 0 \quad \text{in } \Sigma \cap \Omega'$$

and $v$ is hypo-analytic. Since $M_1, \ldots, M_{m-1}$ all belong to $CT\Sigma$ and $v = 0$ on $\Sigma$, it follows that $M_j v = \cdots = M_{m-1} v = 0$ on $\Sigma$ (near $q$). Now $L = \sum_{j=1}^m c_j(x) M_j$ with $c_m(x) \neq 0$ and $L v = 0$ on $\Sigma$. Therefore $M_m v = 0$ on $\Sigma$. Moreover, from $L^j v = 0$ for $0 \leq j \leq k-1$, we deduce that $M^\alpha v = 0$ for $|\alpha| \leq k-1$ on $\Sigma$. Next, since the coefficient of $M^k_m$ in $P = \sum_{|\alpha| \leq k} a_{\alpha}(x) M^\alpha$
is nonzero, it follows that on \( \Sigma \), \( M^\alpha v = 0 \) for \( |\alpha| \leq k \). Finally, applying the vector fields \( M_j \) to the equation \( P v = 0 \), we see that \( M^\alpha v = 0 \) on \( \Sigma \) for all indices \( \alpha \). Now let \( \hat{v} \) be the holomorphic function near \( Z(q) \) in \( C^m \) satisfying \( v(x) = \hat{v}(Z(x)) \).

We write the power series of \( v \) around \( Z(q) \) as

\[
\hat{v}(z) = \sum a_\alpha (z - Z(q))^\alpha, \quad \text{where} \quad a_\alpha = \frac{1}{\alpha!} \left( \frac{\partial}{\partial z} \right)^\alpha \hat{v}(Z(q)).
\]

But then

\[
\left( \frac{\partial}{\partial z} \right)^\alpha \hat{v}(Z(q)) = (M^\alpha v)(q) = 0 \quad \forall \alpha.
\]

Therefore, \( \hat{v} \equiv 0 \) near \( Z(q) \). Hence \( v \equiv 0 \) in \( \Omega' \).

3. Real Hypo-Analytic Structures and Statement of the Main Result

We will continue to look at a maximal hypo-analytic structure on \( \Omega \). We noted that a hypersurface \( \Sigma \) is hypo-analytic if and only if \( \Sigma \) is the zero set of a hypo-analytic function \( f \) with nonzero differential. We now strengthen this condition and introduce the following:

**Definition 3.1.** \( \Sigma \) is said to be a real hypo-analytic hypersurface if every point \( p \in \Sigma \) has a neighborhood \( U_p \) in \( \Omega \), a hypo-analytic function \( h \) of a nonzero differential defined on \( U_p \), and \( \epsilon > 0 \) such that:

1. \( \Sigma \cap U_p = \{ x \in U_p : h(x) = 0 \} \).
2. For \( c \in C \), \( |c| < \epsilon \), the set \( \Sigma_c = \{ x \in U_p : h(x) = c \} \) is either \( \emptyset \) or a hypersurface.
3. \( \bigcup \Sigma_c \) is a neighborhood in \( U_p \) of \( p ; |c| < \epsilon \).

We note that near each point of \( \Sigma \), the above definition gives a local foliation of \( \Omega \) by means of hypo-analytic hypersurfaces.

**Example 1.** Suppose \( \Omega \) is a real analytic structure. The real analytic structure can be viewed as a hypo-analytic structure and in this case, any real analytic hypersurface is real hypo-analytic.

**Example 2.** Consider a hypo-analytic local chart \((U, Z)\) around 0 in a maximal hypo-analytic structure on \( R^m \). Suppose \( Z_j = x_j + \sqrt{-1} \phi_j(x), \) \( j = 1, \ldots, m - 1 \), and \( Z_m = x_m + \sqrt{-1} \phi_m(x_m) \), where \( \phi = (\phi_1, \ldots, \phi_m) \) is real-valued, with zero differential at 0, and \( \phi(0) = 0 \).

Assume that \( U \) is small enough so that the mapping \( Z = (Z_1, \ldots, Z_m) : U \rightarrow C^m \) is a diffeomorphism of \( U \) onto \( Z(U) \). Then \( \Sigma = \{ x \in U : x_m = 0 \} \) is a real hypo-analytic hypersurface. In this case, the defining function can be taken to be \( Z_m \).

Lemma 3.2 will show that Example 2 is a typical example.
The proof of the main theorem will use two equivalent formulations of microlocal hypo-analyticity that were developed in [1]. We briefly recall them here.

**Sato's Microlocalization.** We consider a hypo-analytic local chart \((U, Z)\) of the maximal structure \(\Omega\).

In the sequel \(\Gamma\) is a nonempty, acute, and open cone in \(R^m\setminus\{0\}\). For \(A\) an open subset of \(U\) and \(\delta > 0\), let

\[ N_\delta(A, \Gamma) = \{Z(x) + \sqrt{-1} Z_x(x)v : x \in A, v \in \Gamma, |v| < \delta\}. \]

Let \(B_\delta(A, \Gamma)\) denote the space of holomorphic functions on \(N_\delta(A, \Gamma)\) of tempered growth. More precisely, a holomorphic function \(f\) with domain \(N_\delta(A, \Gamma)\) is in \(B_\delta(A, \Gamma)\) if it satisfies the condition: to every compact subset \(K\) of \(N_\delta(A, \Gamma)\) there are an integer \(k \geq 0\) and a constant \(c > 0\) such that

\[ |f(z)| \leq c(\text{dist}[z, Z(A)])^{-k} \]

for all \(z\) in \(K\).

In [1] it was shown that if \(A\) is sufficiently small and \(f \in B_\delta(A, \Gamma)\), then for every \(\psi \in C^\infty_c(A)\),

\[ \lim_{t \to +0} \int_A f(Z(x) + \sqrt{-1} Z_x(x)tv) \psi(x) \, dZ(x) \]

exists and is independent of \(v \in \Gamma\). Let \(bf\) denote the limit distribution.

**Definition 3.2.** Let \(u \in D'(U)\) and \((x, \xi) \in U \times R_m\setminus\{0\}\). We say that \(u\) is microlocally hypo-analytic at \((x, \xi)\) if there are an open neighborhood \(A \subseteq U\) of \(x\), \(\delta > 0\) and a finite collection of nonempty acute open cones \(\Gamma_k\) in \(R_m\setminus\{0\}\) \((k = 1, \ldots, r)\) satisfying \(\langle v, \xi \rangle < 0\) for every \(v\) in each \(\Gamma_k\) and such that the following hold:

- for each \(k\) there is \(f_k \in B_\delta(A, \Gamma_k)\) such that in \(A\),
  \[ u = bf_1 + \cdots + bf_r. \]

The above definition of microlocal hypo-analyticity in the cotangent space does not depend on the choice of the chart \((U, Z)\) (see [1]).

**Definition 3.3.** Let \(u \in D'(\Omega)\). The hypo-analytic wavefront set of the distribution \(u\) is denoted by \(WF_{ha}u\) and is defined as

\[ WF_{ha}u = \{(x, \xi) \in T^*\Omega : u \text{ is not hypo-analytic at } (x, \xi)\}. \]

**The FBI Transform.** We continue to work in a chart \((U, Z)\) of the maximal structure \(\Omega\). Assume that \(Z = (Z_1, \ldots, Z_m) : U \to C^m\) is a diffeomorphism of \(U\) onto \(Z(U)\) and that \(U\) is the domain of local coordinates \(x_j\) \((1 \leq j \leq m)\) all vanishing at a "central point" which will be denoted by \(0\). We will suppose \(Z(0) = 0\) and by substituting \(Z_x(0)^{-1}Z(x)\) for \(Z(x)\) if necessary, we may assume that

\[ Z_x(0) = \text{the identity matrix}. \]
Let $u$ be a compactly supported distribution in $U$. We shall refer to
\[ F(u, z, \zeta) = \int_y \exp(\sqrt{-1} \cdot z - Z(y)) - \langle \zeta \rangle (z - Z(y))^2 u(y) dZ(y) \]
as the Fourier-Bros-Iagolnitzer (in short, FBI) transform of $u$. Here $z \in C^m, \zeta \in C_m$ with $|\text{Im}\ \zeta| < |\text{Re}\ \zeta|$, and
\[ \langle \zeta \rangle^2 = \zeta_1^2 + \cdots + \zeta_m^2. \]
In [1], the authors established the following FBI transform criterion for hypoanalyticity. We will state it here in a form that will be of convenience to us.

**Theorem 3.1.** The following two properties of a compactly supported distribution are equivalent:

(i) $u$ is microlocally hypo-analytic at $(0, \zeta_0) \in T^* U \setminus \{0\}$.

(ii) There is an open neighborhood $V$ of $0$ in $C^m$, a conic open neighborhood $\mathbb{C}_0$ of $\zeta_0$ in $C_m$, and constants $c, r > 0$ such that $|F(u, z, \zeta)| \leq c \exp(-r |\zeta|)$ for all $z$ in $V$ and for all $\zeta$ in $\mathbb{C}_0$.

We are now ready to state the main theorem of this paper.

**Theorem 3.2.** Let $P$ be a hypo-analytic differential operator and $\Sigma$ a real hypoanalytic hypersurface which is noncharacteristic for $P$. Assume $u \in D'(\Omega)$ such that $Pu$ is hypo-analytic. Suppose $\sigma \in T^* \Omega \mid \Sigma$ for which the hypo-analytic Cauchy data of $u$ are microlocally hypo-analytic at $\pi_\Sigma(\sigma)$. Then $\sigma \notin \text{WF}_h u$.

**Remark 3.2.** The proof will actually show that it is sufficient to have $Pu$ microlocally hypo-analytic at $\sigma$.

From Theorem 3.2 we deduce the following consequences. $\Sigma$ and $P$ will be as in Theorem 3.2.

**Corollary 3.1.** Suppose $Pu$ is hypo-analytic at $q \in \Sigma$ and the hypo-analytic Cauchy data of $u$ is also hypo-analytic at $q$. Then $u$ is hypo-analytic at $q$.

**Proof.** Since the hypo-analytic Cauchy data is hypo-analytic at $q$, it is microlocally hypo-analytic in every direction in $T_q^* \Sigma \setminus \{0\}$. (See [1] for a proof.) Therefore, by Theorem 3.2, $u$ is microlocally hypo-analytic in every direction in $T_q^* \Omega$. Hence by [1], $u$ is hypo-analytic at $q$.

**Corollary 3.2.** Suppose $Pu = 0$ and the hypo-analytic Cauchy data of $u$ on $\Sigma$ is $0$. Then $u \equiv 0$.

**Proof.** By Corollary 3.1, $u$ is hypo-analytic. But then by the uniqueness part of Theorem 2.1, $u \equiv 0$.

The following lemmas will be used in the proof of Theorem 3.2.

**Lemma 3.1.** Let $P$ be a hypo-analytic differential operator and $\sigma \notin \text{Char } P$. If $u \in D'(\Omega)$ for which $\sigma \notin \text{WF}_h Pu$, then $\sigma \notin \text{WF}_h u$.

**Proof.** We reason in a chart $(U, Z)$ around $0$ where we assume that $Z(0) = 0, dZ(0) = \text{Id}$, $\sigma = (0, \zeta^0) \in T^* U$, and $U$ is the domain of local coordinates
We can then take $\Re Z_j$ as new coordinates in which $Z(x) = x + \sqrt{-1}\phi(x)$, $\phi(0) = 0$, $d\phi(0) = 0$ and $\phi = (\phi_1, \ldots, \phi_m)$ is real-valued. Moreover, the functions $Z_j$ may be selected so that all the derivatives of order 2 of the $\phi_j$ vanish at 0. Indeed, if this is not already so it suffices to replace each $Z_j$ by

$$Z_j - \frac{\sqrt{-1}}{2} \sum_{k=1}^{m} \sum_{l=1}^{m} \frac{\partial^2 \phi_j}{\partial x_k \partial x_l}(0) Z_k Z_l.$$ 

Let $M_j$ $(1 \leq j \leq m)$ be the vector fields satisfying $M_j Z_k = \delta_j^k$. To prove the lemma, we will use the FBI transform. First we note that for any $f \in C^1(U)$,

$$\langle df, M_k \rangle = M_k f = \sum_j \langle (M_j f) dZ_j, M_k \rangle \quad \forall k.$$ 

It follows that

$$df = \sum_{j=1}^{m} (M_j f) dZ_j.$$ 

Therefore, if $g$ or $h$ has compact support in $U$, by Stokes' theorem we have

$$0 = \int_{\partial U} h g dZ_1 \wedge \cdots \wedge dZ_m \wedge \cdots \wedge dZ_m = (-1)^{j-1} \left[ \int_U [(M_j h) g + h(M_j g)] dZ_1 \wedge \cdots \wedge dZ_m \right].$$ 

Hence

$$\int_U (M_j h) g dZ_1 \wedge \cdots \wedge dZ_m = \int_U h(M_j g) dZ_1 \wedge \cdots \wedge dZ_m.$$ 

If $U$ is sufficiently small, in the chart $(U, Z)$ we may write

$$P = \sum_{|\alpha| \leq k} a_\alpha(x) M^\alpha,$$

where each $a_\alpha$ is hypo-analytic on $U$.

Since $\sigma = (0, \xi^0) \notin WF_{ha} Pu$, Theorem 3.1 tells us that

$$F(Pu, z, \zeta) = \int_U \exp(\sqrt{-1} \zeta \cdot (z - Z(y)) - \langle \zeta \rangle (z - Z(y))^2) \sum_{|\alpha| \leq k} a_\alpha(y) M^\alpha u(y) dZ(y)$$

has an exponential decay for $z$ near 0 and $\zeta$ in a complex conic neighborhood of $\xi^0$.

Since $y \mapsto \exp(\sqrt{-1} \zeta \cdot (z - Z(y)) - \langle \zeta \rangle (z - Z(y))^2)$ is hypo-analytic, for each $j = 1, \ldots, m$,

$$M_j(\exp h(z, \zeta, y)) = \left[-\sqrt{-1} \zeta_j + 2 \langle \zeta \rangle (z_j - Z_j(y)) \right] \exp(h(z, \zeta, y)).$$
where 
\[ h(z, \zeta, y) = \sqrt{-1} \zeta \cdot (z - Z(y)) - \langle \zeta \rangle (z - Z(y))^2. \]

This observation together with the integration by parts formula (3.1) imply the existence of a hypo-analytic amplitude \( Q(z, \zeta, y) \) elliptic at \( \sigma \) such that
\[
F(Pu, z, \zeta) = \int_U \exp(\sqrt{-1} \zeta \cdot (z - Z(y)) - \langle \zeta \rangle (z - Z(y))^2)Q(z, \zeta, y)u(y)\,dZ.
\]

By the results of [5], we conclude that \( \sigma \notin WF_{h\alpha}u. \)

**Lemma 3.2.** Suppose \( \Sigma \) is a real hypo-analytic hypersurface of \( \Omega \). Then each point \( p \in \Sigma \) is contained in a hypo-analytic chart \( (U, Z_1, \ldots, Z_m) \), where \( U \) is the domain of local coordinates \( (U, x_1, \ldots, x_m) \) in which
\[
Z_j = x_j + \sqrt{-1} \phi_j(x) \quad \text{for} \quad 1 \leq j < m
\]
and \( Z_m = x_m + \sqrt{-1} \Psi(x_m) \), where \( (\phi_1, \ldots, \phi_{m-1}, \Psi) \) is real-valued and \( \Sigma \cap U = \{ x \in U : x_m = 0 \} \).

**Proof.** By Proposition 2.1, there is a chart \( (U, Z) \) of \( \Omega \) near \( p \) such that 
\[
L \cap U = \{ x \in U : x_m = 0 \}.
\]
Since \( d(Z_1|_\Sigma), \ldots, d(Z_{m-1}|_\Sigma) \) are linearly independent, by making linear substitutions if necessary, we may assume that \( d( \mathbb{R}Z_1|_\Sigma), \ldots, d( \mathbb{R}Z_{m-1}|_\Sigma) \) are independent.

We may then take \( \mathbb{R}Z_1, \ldots, \mathbb{R}Z_{m-1} \) as coordinates on \( \Sigma \). By multiplying \( Z_m \) by \( \sqrt{-1} \) if necessary, we may also assume that \( \mathbb{R}Z_1, \ldots, \mathbb{R}Z_m \) are coordinates in \( U \) (all this locally near \( p \)).

Then
\[
Z_j = x_j + \sqrt{-1} \phi_j, \quad Z_m = x_m + \sqrt{-1} \Psi(x), \quad 1 \leq j < m,
\]
and since \( Z_m|_{\Sigma \cap U} = 0 \), we have
\[
\Sigma \cap U = \{ x \in U : x_m = 0 \}.
\]

Next let \( h \) be the defining function of \( \Sigma \) near \( p \) satisfying the conditions of Definition 3.1. Write \( h(x) = \hat{h}(Z(x)) \), where \( \hat{h} \) is holomorphic.

Since \( h|_\Sigma = \hat{h}(Z_1, \ldots, Z_{m-1}, 0)|_\Sigma = 0 \) and the image of \( \Sigma \) under \( (Z_1, \ldots, Z_{m-1}) \) is a totally real manifold of maximal dimension in \( C^{m-1} \), it follows that
\[
h(x) = \hat{h}(Z_1(x), \ldots, Z_{m-1}(x)) = \hat{h}(Z_m(x)).
\]

Now since \( dh \neq 0 \), \( \hat{h} \) is invertible. Hence, for any constant \( c \in C \),
\[
h(x) = c \quad \text{iff} \quad Z_m(x) = \hat{h}^{-1}(c).
\]

It now follows from Definition 3.1 that \( Z_m = x_m + \sqrt{-1} \Psi(x_m) \).
4. PROOF OF THEOREM 3.2

Lemma 3.2 permits us to reason in a local hypo-analytic chart \((U, Z)\), where \(U\) is also the domain of local coordinates \((U, x_1, \ldots, x_m)\) centered at 0 with \(Z_j = x_j + \sqrt{-1}\phi_j(x), 1 \leq j < m, Z_m = x_m + \sqrt{-1}\Psi(x_m), \Sigma\) is given by \(x_m = 0\) and \(\sigma = (0, \xi_0)\).

We may also assume that \(Z(0) = 0, dZ(0) = \text{Id}, \phi''(0) = 0, \text{and } \Psi''(0) = 0\).

Let \(M_j (1 \leq j \leq m)\) be the vector fields satisfying \(M_j Z_k = \delta_{jk}\). If \(p \in \Sigma\) and \(1 \leq j < m\), then \((M_j)_p \in CT_p \Sigma\). Moreover, after multiplication by a nonvanishing hypo-analytic function, \(P\) will have the form

\[ P = M_m^n + \sum_{|\alpha| \leq n, \alpha_m < n} a_\alpha(x) M^\alpha, \]

where the \(a_\alpha\) are all hypo-analytic functions. Since \(Pu\) is hypo-analytic, it follows that \(u\) is a \(C^\infty\) function of \(x_m\) valued in the space of distributions in the variable \(x' = (x_1, \ldots, x_{m-1})\) (see [8]). In particular, the trace of \(u\) on \(\Sigma\) is well defined.

We may therefore restate the theorem as:

Suppose \(Pu\) is hypo-analytic and \((0', \xi') \in T_{0'}^* \Sigma\) such that \((0', \xi_0') \notin WF_{ha}(M_m^j u(x', 0))\) for \(0 \leq j < n\). Then \((0, (\xi', \xi_n')) \notin WF_{ha} u\).

Since the statement is purely local, we may assume that the support of \(u\) is contained in a set of the form

\[ \{x' : |x'| \leq T/2\} \times (-T, T) \quad \text{and} \quad \{(x', 0) : |x'| \leq T\} \subseteq \Sigma. \]

For \(t \in (-T, T)\), let \(\Sigma_t = \Sigma \times \{t\}\) and \(\Omega_t = \{(x', x_m) : |x'| < T/2, 0 \leq x_m < t \text{ or } t < x_m \leq 0\}\).

We observe that for any \(j, k\), and \(l\),

\[ M_j (M_k Z_l) = 0 = M_k (M_j Z_l). \]

Since the differentials \(dZ_1, \ldots, dZ_m\) span \(CT^* U\), it follows that the vector fields \(M_j\) commute pairwise. This observation together with the integration by parts formula of §3 and the fact that for each \(t\) and \(j < m, M_j \in CT \Sigma_t\), yield:

\[
\int_{\Omega_t} (Pu) w \, dZ_1 \wedge \cdots \wedge dZ_m - \int_{\Omega_t} u(tPw) \, dZ_1 \wedge \cdots \wedge dZ_m
= \sum_{j+k \leq n-1} \int_{\Sigma_t} (M_m^j u)(B_{jk}(x, M^l)M_m^k w) \, dZ_1 \wedge \cdots \wedge dZ_{m-1}
- \sum_{j+k \leq n-1} \int_{\Sigma_0} (M_m^j u)(B_{jk}(x, M^l)M_m^k w) \, dZ_1 \wedge \cdots \wedge dZ_{m-1},
\]

where the \(B_{jk}\) are hypo-analytic differential operators in \(M_1, \ldots, M_{m-1}\) of order \(n-1-j-k\).
For $\alpha = (z_0', \xi') \in C^{m-1} \times (R_{m-1} \setminus \{0\})$ and $\tau \in C.,$ satisfying $1 < |\tau| < C_0$, $|3\tau| < \varepsilon |\tau|$ ($\varepsilon$ and $C_0$ to be determined later), set

$$V_{\alpha, \tau}(z') = \exp(\sqrt{-1}(z_0' - z') \cdot \xi' - \tau |\xi'|^2(z_0' - z')^2).$$

Since $\gamma P$ is a hypo-analytic differential operator, let

$$\gamma P = \sum_{|\alpha| \leq n} c_{\alpha}(x) M^\alpha,$$

where each $c_{\alpha}(x) = \tilde{c}_{\alpha}(Z(x))$ for holomorphic $\tilde{c}_{\alpha}$. Set

$$\gamma P \left( z, \frac{\partial}{\partial z} \right) = \sum_{|\alpha| \leq n} \tilde{c}_{\alpha}(z) \left( \frac{\partial}{\partial z} \right)^\alpha.$$

Let $\Sigma_t = \{(z', t) \in C^{m-1} \times \{t\} : |z'| \leq T \}$.

The Cauchy-Kovalevskaya theorem tells us that there is $t_0 > 0$ such that if $t \in [-t_0, t_0]$ we can find a solution $\tilde{w}(z) = \tilde{w}_{\alpha, \tau, t}(z)$ in a neighborhood of $\{(z', x_m) \in C^{m-1} \times R : |z'| \leq T, |x_m| < t_0 \}$ of the problem

$$\gamma P \left( z, \frac{\partial}{\partial z} \right) \tilde{w} = 0, \quad \tilde{w}|_{\Sigma_t} = \cdots = \left( \frac{\partial}{\partial z_m} \right)^{n-2} \tilde{w}|_{\Sigma_t} = 0$$

(4.2)

$$\left( \frac{\partial}{\partial z_m} \right)^{n-1} \tilde{w}|_{\Sigma_t} = V_{\alpha, \tau}. $$

The solution $\tilde{w} = \tilde{w}_{\alpha, \tau, t}$ can be estimated in terms of the Cauchy data on $\Sigma_t$. Indeed, the Ovchinnikov method (see [6]) implies

$$\exists c > 0 \text{ independent of } t, \tau, \alpha \text{ such that}$$

$$|\tilde{w}_{\alpha, \tau, t}(w', z_m)| \leq c \sum_{|\beta'| \leq n} \sup_{|z'| - w' | \leq c |z_m - t|} |\partial_{z'} \gamma P_{\alpha, \tau}(z')|. $$

For $|\beta'| \leq n$ we have

(4.4)

$$|\partial_{z'} \gamma P_{\alpha, \tau}(z')| \leq c_1 (1 + |\xi'|)^n \exp(\langle 3(z' - z'_0), \xi' \rangle - |\xi'|^2 |\Re \{ (\Re z' - \Re z'_0)^2 \}
$$

$$- 2 \Re \tau \Re (z' - z'_0) \cdot 3(z' - z'_0)) \rangle.$$

We are going to be interested in $z', z'_0$, where $3z'$ is small compared to $\Re z'$ and $z'_0$ is close enough to $0'$. This consideration together with a sufficiently small choice of $\varepsilon$ in the definition of $\tau$ imply for $|\beta'| \leq n$

(4.5)

$$|\partial_{z'} \gamma P_{\alpha, \tau}(z')| \leq c_1 (1 + |\xi'|)^n \exp \left( \langle 3(z' - z'_0), \xi' \rangle - \frac{\Re \tau}{2} |\xi'|^2 |\Re (z' - z'_0)^2 - (3z' - 3z'_0)^2| \right).$$
Application of (4.5) to (4.3) yields

\[ |\hat{w}_{\alpha, \tau, t}(z', x_m + i\psi(x_m))| \leq c_1 (|z'|)^n \exp \left( \frac{(\Re(z' - z_0'), \xi') - \frac{\Re \tau}{2} |\xi'|}{\zeta_1} \times [((\Re z' - \Re z_0')^2 - (\Im z' - \Im z_0')^2) + c|\xi'||x_m - t] \right). \]

Let \( w_{\alpha, \tau, t}(x) = \hat{w}_{\alpha, \tau, t}(Z(x)) \). For \( \alpha = (z_0', \xi') \) in a sufficiently small conic neighborhood of \( (0', \xi_0') \) and with \( w = w_{\alpha, \tau, t} \) we wi. estimate the term

\[ \int_{\Omega} (Pu) w dZ_1 \wedge \cdots \wedge dZ_m \text{ in (4.1)}. \]

(4.2) tells us that \( w = w_{\alpha, \tau, t} \) solves

\[ (4.2') \quad i P(x, M) w = 0, \quad w|_{\Sigma_1} = \cdots = M^{n-2} w|_{\Sigma_1} = 0, \]

and

\[ M_m^{n-1} w(x', t) = V_{\alpha, \tau}(Z(x', t)). \]

Since \( Pu \) and \( w = w_{\alpha, \tau, t} \) are hypo-analytic, we can deform the integration contour from \( \Omega_1 \) to the image of \( \Omega_1 \) under the map

\[ (x', x_m) \mapsto \theta(x', x_m) = Z(x', x_m) - \sqrt{-1} \left( d\chi(x') \frac{\xi'}{|\xi'|}, 0 \right), \]

where \( \chi(x') \) is a cutoff function \( \equiv 1 \) near \( \Re z_0' \) and \( d \) is chosen so that we stay inside the domain of hypo-analyticity.

Along this contour, (4.6) gives the following estimate on \( w = w_{\alpha, \tau, t} \):

\[ \|w\| \leq c_1 (|z'|)^n \exp \left( e^{-d\chi(x')|\xi'| + (\phi'(x), \xi') - \frac{\Re \tau}{2} |\xi'| |(x' - \Re z_0')^2 (\phi'(x) - d\chi(x') \frac{\xi'}{|\xi'|} - \Re z_0')^2 + c|\xi'||x_m - t|} \right). \]

(Here \( \phi' = (\phi_1, \ldots, \phi_{m-1}) \).)

By using the term \( (x' - \Re z_0')^2 \) when \( x' \) is away from \( \Re z_0' \) and the term \( d\chi(x')|\xi'| \) when \( x' \) is near \( \Re z_0' \), we see that \( w \) is exponentially decaying along this contour. The latter may require shrinking of the interval \([0', 0] \) to a smaller interval which we will still call \([-t_0, t_0] \).

It follows that we can find a sufficiently small \( t > 0 \) and a sufficiently large \( c_2 > 0 \) such that

\[ \left| \int_{\Omega_1} (Pu) w_{\alpha, \tau, t} dZ_1 \wedge \cdots \wedge dZ_m \right| \leq c_2 \exp \left( -\frac{|\xi'|}{c_2} \right) \]

for \( |t| \leq t_0 \) and \( \alpha = (z_0', \xi') \) in a small conic neighborhood of \( (0', \xi_0') \).
Since \( w = w_{\alpha, t, t} \) solves (4.2'), formula (4.1) reduces to (4.9)
\[
i(-1)^{n+1} \int_{\Omega} (Pu)w \, dZ_1 \wedge \cdots \wedge dZ_m
= \int_{|x'| \leq T} e^{(\sqrt{-1}(z_0' - Z'(x', t), \xi') - \tau |\xi'| |z_0' - Z'(x', t)|^2)} \\
\times u(x', t) \, dZ_1 \wedge \cdots \wedge dZ_{m-1}(x', t)
+ i(-1)^n \sum_{j+k \leq n-1} \int_{\Sigma_0} (M^j_M u)(B_{jk}(x, M')M^K_M w) \, dZ_1 \wedge \cdots \wedge dZ_{m-1}(x', 0).
\]

We consider now the integrals over \( \Sigma_0 = \Sigma \). Fix \( j \) and \( k \leq j + k \leq n - 1 \). Since by assumption \((0', \xi') \notin WF_{ha}(M^j_M u|\Sigma_0)\), without loss of generality we may assume
\[
M^j_M u|\Sigma_0 = \lim_{x \to 0} f_j(Z'(x', 0) + \sqrt{-1}sZ'(x', 0)v)
\]
\((Z' = (Z_1, \ldots, Z_{m-1}))\) for some tempered holomorphic function \( f_j \), and \( v \) is in a cone \( \Gamma_j \subseteq R^{m-1} \) satisfying
\[
\langle v, \xi_0' \rangle < 0.
\]

Hence, in the integral over \( \Sigma_0 \), we may deform a contour to \( Z(x', 0) + \sqrt{-1}s\chi(x')Z(x', 0)v \), where \( s \) is chosen sufficiently small and \( \chi(x') \) is selected as before.

Estimates analogous to (4.6) are also valid for the derivatives \( \{M^k_M w\}_k \). Such estimates and the new contour for each \( j \) yields, after enlarging \( c_2 \) if necessary,
\[
\int_{\Sigma_0} (M^j_M u)(B_{jk}(x, M')M^K_M w) \, dZ_1 \wedge \cdots \wedge dZ_{m-1} \leq c_2 \exp \left( -\frac{|\xi'|}{c_2} \right)
\]
for \( t \in [-t_0, 0] \) and \( \alpha = (z^0_0, \xi_0') \) in a small conic neighborhood of \((0', \xi_0')\).

It follows that (after modifying \( t_0 \) and \( c_2 \))
\[
\int_{|x'| \leq T} u(x', t) \exp(\sqrt{-1}(z_0' - Z'(x', t), \xi') - \tau |\xi'| (Z'(x', t) - z_0')^2) \, dZ'
\]
\[
\leq c_2 \exp \left( -\frac{|\xi'|}{c_2} \right)
\]
for \( t \in [-t_0, 0] \) and \( \alpha = (z^0_0, \xi') \) in a small conic neighborhood of \((0', \xi_0')\).

Let \( I(t, \tau, z_0', \xi') \) = the integral (without the absolute value) in (4.11). Suppose (4.11) holds in a cone \( \Gamma' \subseteq R^{m-1} \) containing \( \xi_0' \).
Let $z_0 = (z_0^', z_0^m) \in C^{m-1} \times C$ and $\xi = (\xi', \xi_m) \in R_{m-1} \times R$. In order to examine $WF_{ha} u$ at $(0, (\xi_0', \xi_m))$, we have to estimate the FBI:

$$F(z_0, \xi) = \int_{|t| \leq t_0} \int_{|x'| \leq T} \exp(\sqrt{-1}(z_0 - Z(x', t), \xi)) - |\xi|(z_0 - Z(x', t))^2) u(x', t) \, dZ.$$

But since $Z_m$ depends only on $t$, we get

$$F(z_0, \xi) = \int_{|t| \leq t_0} \exp((z_0^m - Z_m(t))\xi_m - |\xi|(z_0^m - Z^m)^2) \times I(t, |\xi|/|\xi'|, z_0^', \xi') \, dZ_m(t).$$

We now select $C_0$ as follows. Since $(0, \xi_0') \notin \text{Char} P$, by Lemma 3.1 there exists a constant $C_0 > 1$ such that if $|\xi| \geq C_0|\xi'|$, then $F(z_0, \xi)$ decays exponentially for $z_0$ near 0 in $C^m$.

Let $\Gamma = \Gamma' \times R$. Pick $\xi = (\xi', \xi_m) \in \Gamma$. To finish the proof, we consider two cases:

**Case (i).** $|\xi| \geq C_0|\xi'|$. This was just taken care of.

**Case (ii).** $|\xi| \leq C_0|\xi'|$. Then $|\xi| = (|\xi|/|\xi'|)|\xi'| = \tau|\xi'|$ with $1 < \tau < C_0$.

Hence (4.11) and (4.12) guarantee the exponential decay of $F(z_0, \xi)$ for $z_0$ near 0 in $C^m$.

**REFERENCES**