LOCALLY FLAT 2-KNOTS IN $S^2 \times S^2$
WITH THE SAME FUNDAMENTAL GROUP

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ABSTRACT. We consider a locally flat 2-sphere in $S^2 \times S^2$ representing a primitive homology class $\xi$, which is referred to as a 2-knot in $S^2 \times S^2$ representing $\xi$. Then for any given primitive class $\xi$, there exists a 2-knot in $S^2 \times S^2$ representing $\xi$ with simply-connected complement. In this paper, we consider the classification of 2-knots in $S^2 \times S^2$ whose complements have a fixed fundamental group. We show that if the complement of a 2-knot $S$ in $S^2 \times S^2$ is simply connected, then the ambient isotopy type of $S$ is determined. In the case of nontrivial $\pi_1$, however, we show that the ambient isotopy type of a 2-knot in $S^2 \times S^2$ with nontrivial $\pi_1$ is not always determined by $\pi_1$.

1. Introduction

Let $\zeta$ and $\eta$ be natural generators of $H_2(S^2 \times S^2; \mathbb{Z})$ represented by the cross-section and fiber of the projection $S^2 \times S^2 \to S^2$ onto the first factor with $\zeta \cdot \zeta = \eta \cdot \eta = 0$ and $\zeta \cdot \eta = \eta \cdot \zeta = 1$. A 2-knot $S$ in $S^2 \times S^2$ is a locally flat submanifold of $S^2 \times S^2$ homeomorphic to $S^2$. The fundamental group of the complement of $S$ is referred to as the fundamental group of $S$. The exterior of $S$ is the closure of the complement of a tubular neighborhood of $S$ in $S^2 \times S^2$. Two 2-knots in $S^2 \times S^2$ are equivalent if they are ambient isotopic, that is, there exists an isotopic deformation $F: (S^2 \times S^2) \times I \to (S^2 \times S^2) \times I$ such that the homeomorphism $F_1$ takes one to the other. Kuga and Freedman have characterized those homology classes in $S^2 \times S^2$ that can be represented by 2-knots in $S^2 \times S^2$ as follows. Kuga has shown in [10] that the homology class $\xi = p\zeta + q\eta$, $p, q \in \mathbb{Z}$, can be represented by a smooth 2-knot in $S^2 \times S^2$ if and only if $|p| \leq 1$ or $|q| \leq 1$. Meanwhile, Freedman has shown in [6] that if $p$ and $q$ are relatively prime integers, then $\xi$ can be represented by a 2-knot in $S^2 \times S^2$.

Since the problem of classifying 2-knots in $S^2 \times S^2$ is interesting, we consider in this paper the problem of whether the equivalence class of a 2-knot in $S^2 \times S^2$ is determined by its fundamental group. For any integer $p$, let $\rho_p: S^2 \to S^2$ be the canonical smooth map of degree $p$, and let $\phi_p: S^2 \to S^2 \times S^2$ be...
the embedding defined by $\phi_p(x) = (x, \rho_p(x))$. Then if we write $\Sigma_p$ for the image $\phi_p(S^2)$, $\Sigma_p$ is the standard smooth 2-knot in $S^2 \times S^2$ representing $\zeta + p\eta$. We obtained in [13] the following result: If the complement of a 2-knot $S$ in $S^2 \times S^2$ representing $\zeta + p\eta$ is simply connected, then $S$ and $\Sigma_p$ are equivalent. In this paper we prove the unknotting theorem in more general cases: If the complement of a 2-knot $S$ in $S^2 \times S^2$ representing $p\zeta + q\eta$ is simply connected, then the equivalence class of $S$ is determined. Moreover, we prove that the equivalence class of a 2-knot in $S^2 \times S^2$ is not always determined by the fundamental group itself.

This paper is organized as follows. In §2, we consider the case that the fundamental group of a 2-knot is trivial. We show that for any relatively prime integers $p$ and $q$, there is a 2-knot representing $p\zeta + q\eta$ with simply-connected complement, and prove the unknotting theorem. We consider in §3 the case that the fundamental group of a 2-knot is nontrivial. We prove that there exist distinct 2-knots with the same fundamental group. In §4, we consider the problem of whether a homology 3-sphere bounds a smooth acyclic 4-manifold or not, and by using Kuga's theorem and our technique in §2, we present a family of homology 3-spheres that cannot bound smooth acyclic 4-manifolds.

The author would like to express his gratitude to Professors M. Kato and T. Kanenobu. He would also like to thank Professor O. Saeki for helpful conversations.

2. 2-KNOTS IN $S^2 \times S^2$ WITH TRIVIAL $\pi_1$

It is easy to see that if the homology class represented by a 2-knot $S$ is not primitive, then $H_1(S^2 \times S^2 - S; \mathbb{Z})$ is nonzero. We begin with the following proposition.

**Proposition 2.1.** Let $p$ and $q$ be relatively prime integers. Then there exists a 2-knot in $S^2 \times S^2$ representing $p\zeta + q\eta$ with simply-connected complement.

**Proof.** Since $\text{g.c.d}(p, q) = 1$, there are two integers $a, b$ such that $bp - aq = 1$. We consider the 3-manifold $M$ obtained by surgery on the framed link $L$ illustrated in Figure 1. The link $L$ consists of two trivial knots $K_1$ and $K_2$. Since $|(2pq)(2ab) - (bp + aq)| = 1$, $M$ is a homology 3-sphere, so that $M$ bounds a topological contractible 4-manifold $V$. See [6]. Let $W$ be the 4-manifold obtained by attaching two 2-handles $h_1$ and $h_2$ to the 4-disk $D^4$ along the framed link $L$ (i.e., $W = D^4 \cup h_1 \cup h_2$). Set $X = W \cup_M V$, and $X$ is a topological closed 4-manifold. Let $B_i$ be a smooth 2-disk in $D^4$ which is the trivial knot $K_i$ bounds, and let $D_i$ be the core of $h_i$ ($i = 1, 2$). Then $S_i = B_i \cup D_i \subset W$ is diffeomorphic to $S^2$. Since the framing of $K_i$ is $2pq$, a closed tubular neighborhood of $S_i$ is the $D^2$-bundle $D(2pq)$ over $S^2$ with Euler number $2pq$. Since $K_i$ is trivial, $W$ is the 4-manifold obtained by attaching the 2-handle $h_2$ to $D(2pq)$. Hence, by the duality of handle-
where \( n \) denotes \( n \)-times full-twist.

**Figure 1**

decompositions, we can view \( W \) as \((M \times I \cup h^*_2) \cup \partial D(2pq)\), where \( h^*_2 \) is the dual handle of \( h_2 \) and \( \partial \) is the lens space \( L(2pq, 2pq - 1) \). Therefore, \( X = Y \cup \partial D(2pq) \), where \( Y = V \cup_{M \times I} h^*_2 \). Then by van Kampen's theorem, \( \pi_1(Y) = 1 \) and \( \pi_1(X) = 1 \). Moreover, \( \pi_1(X - S_1) \cong \pi_1(Y) = 1 \). \( H_2(X; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \) is generated by \([S_1]\) and \([S_2]\), and \( X \) has the intersection form

\[
A = \begin{pmatrix}
2pq & bp + aq \\
bp + aq & 2ab
\end{pmatrix}
\]

with respect to these generators. Since the form \( A \) is even and indefinite, \( A \) is equivalent over \( \mathbb{Z} \) to

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

In fact, if we let \((\mathbb{Z} \oplus \mathbb{Z}, A)\) and \((\mathbb{Z} \oplus \mathbb{Z}, (\begin{array}{c}0 \\ 1 \end{array}))\) be bilinear form spaces, then the matrix

\[
B = \begin{pmatrix}
p & a \\
q & b
\end{pmatrix}
\]

gives an isomorphism between them. Let \{\( u, v \)\} and \{\( \zeta, \eta \)\} be bases for the bilinear form spaces \((\mathbb{Z} \oplus \mathbb{Z}, A)\) and \((\mathbb{Z} \oplus \mathbb{Z}, (\begin{array}{c}0 \\ 1 \end{array}))\), respectively. The matrix \( B \) takes \( u \) to \( p\zeta + q\eta \). Thus \( X \) has the intersection form \((\mathbb{Z} \oplus \mathbb{Z}, (\begin{array}{c}0 \\ 1 \end{array}))\), and the homology class of \( S_1 \), \([S_1]\), is \( p\zeta + q\eta \). By Freedman's theorem, there is a homeomorphism \( h: X \to S^2 \times S^2 \). Then the induced isomorphism \( h_*: H_2(X; \mathbb{Z}) \to H_2(S^2 \times S^2; \mathbb{Z}) \) gives an automorphism of \((\mathbb{Z} \oplus \mathbb{Z}, (\begin{array}{c}0 \\ 1 \end{array}))\). Since the automorphism group of this form space is \( \{C \in GL(2, \mathbb{Z}); \ C(\begin{array}{c}0 \\ 1 \end{array}) = (\begin{array}{c}0 \\ 1 \end{array})\} = \{\pm(\begin{array}{c}1 \\ 0 \end{array}), \pm(\begin{array}{c}0 \\ 1 \end{array})\} \), \( h_* = \pm(\begin{array}{c}1 \\ 0 \end{array}) \) or \( \pm(\begin{array}{c}0 \\ 1 \end{array}) \). Thus the image \( h(S_1) \) is a locally flat 2-sphere in \( S^2 \times S^2 \) representing \( \pm(p\zeta + q\eta) \) or \( \pm(q\zeta + p\eta) \).
and $\pi_1(S^2 \times S^2 - h(S_1)) \cong \pi_1(X - S_1) = 1$. After changing the orientation of $S^2 \times S^2$ and/or the orientation of $\zeta$ and $\eta$ (if necessary), $h(S_1)$ may represent $p\zeta + q\eta$. Therefore, $h(S_1)$ is a required 2-knot in $S^2 \times S^2$.

Our key lemma in this section is the following.

**Lemma 2.2.** Let $p$ and $q$ be relatively prime integers, and let $S_1$ and $S_2$ be 2-knots in $S^2 \times S^2$ representing $p\zeta + q\eta$. If the complements of $S_1$ and $S_2$ are simply connected, then there exists a homeomorphism of $S^2 \times S^2$ taking $S_1$ to $S_2$.

Since we can prove this lemma in the same manner as [13], we only sketch the proof.

**Proof (sketch).** Let $N_i$ be a closed tubular neighborhood of $S_i$ and $E_i$ the exterior of $S_i$ $(i = 1, 2)$. Then $N_i$ is homeomorphic to $D(2pq)$, and so the boundary $\partial E_i$ of $E_i$ is the lens space $L(2pq, 2pq - 1)$, where $L(0, -1) = S^2 \times S^2$. Hence $(S^2 \times S^2, S_i)$ is pairwise homeomorphic to $(D(2pq) \cup_{\gamma_i} E_i, \nu(S^2))$, where $\gamma_i : L(2pq, 2pq - 1) \to L(2pq, 2pq - 1)$ is some gluing homeomorphism and $\nu : S^2 \to D(2pq)$ is the zero section. By the isotopy extension theorem, it is easily seen that the homeomorphism type of 2-knots with exterior $E_i$ depends only on the isotopy class of the homeomorphism $\gamma_i$. To prove Lemma 2.2, we need the following lemma.

**Lemma 2.3.** Suppose $E_1$ and $E_2$ are simply connected. Then $E_1$ is homeomorphic to $E_2$. In particular if $(p, q) = (\pm 1, 0)$ or $(0, \pm 1)$, then $E_1$ and $E_2$ are homeomorphic to $S^1 \times D^2$.

**Proof.** We give $E_i$ the orientation opposite to the one inherited from $S^2 \times S^2$. It follows that the intersection form $(H_2(E_i; \mathbb{Z}), \cdot)$ is isomorphic to $(\mathbb{Z}, (2pq))$, where $(2pq) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is the bilinear form defined by $(2pq)(1, 1) = 2pq$. Hence, $E_i$ is a simply-connected compact 4-manifold with boundary $L(2pq, 2pq - 1)$ and the intersection form $(\mathbb{Z}, (2pq))$. In [2], Boyer calculated the set of all oriented homeomorphism types of simply-connected compact 4-manifolds with given boundary and given intersection form. In the case of $(p, q) = (\pm 1, 0)$ or $(0, \pm 1)$, Remarks (5.3) of [2] say that $E_1$ and $E_2$ are homeomorphic to $S^2 \times D^2$. Next we consider the case of $pq \neq 0$. Since $\text{g.c.d}(p, q) = 1$, there are two integers $a, b$ such that $bp - aq = 1$. If we set $u_i = [S_i] = p\zeta + q\eta$ and $v = a\zeta + b\eta$, then $u_i$ and $v$ generate $H_2(S^2 \times S^2; \mathbb{Z})$. Let $w_i$ be a generator of $H_2(E_i, \partial E_i; \mathbb{Z}) \cong \mathbb{Z}$. Since $u_i \cdot v = bp + aq$, $\partial w_i \in H_1(\partial E_i; \mathbb{Z}) = H_1(L(2pq, 2pq - 1); \mathbb{Z})$ is represented by $(bp + aq)$-times the $\partial D^2$-fiber of the $D^3$-bundle $N_i$ over $S_i$. Since $bp - aq = 1$, $(bp + aq)^2 \equiv 1$ (mod $2pq$). Hence, it follows from Example 5.4 and Remarks 5.6 of [2] that $E_1$ is homeomorphic to $E_2$. □

Return to the proof of Lemma 2.2. Since the complements of $S_1$ and $S_2$ are simply connected, there is a homeomorphism $h : E_1 \to E_2$. Let $h$ be the
restriction of $h$ to $\partial E_1$. If the homeomorphism $\gamma_2^{-1} h \gamma_1 : \partial D(2pq) \rightarrow \partial D(2pq)$ extends to a homeomorphism $g$ of $(D(2pq), \nu(S^2))$, we have the following required homeomorphism:

$$\varphi : (D(2pq) \cup_{\gamma_1} E_1, \nu(S^2)) \rightarrow (D(2pq) \cup_{\gamma_2} E_2, \nu(S^2))$$

by setting

$$\varphi = \begin{cases} g & \text{on } D(2pq), \\
 h & \text{on } E_1. \end{cases}$$

Now we remark that in the case of $pq = 0$, $\gamma_2^{-1} h \gamma_1 : S^2 \times S^1 \rightarrow S^2 \times S^1$ is not isotopic to the twist $\tau : S^2 \times S^1 \rightarrow S^2 \times S^1$ defined by $\tau((\theta, \phi), \psi) = ((\theta + \psi, \phi), \psi)$, since $E_1$ and $E_2$ are homeomorphic to $S^2 \times D^2$ and the second Stiefel-Whitney class of $S^2 \times S^2$ is trivial. Hence, by investigating the homeotopy group of $L(2pq, 2pq - 1)$, it follows that there is an extension $g$ as the above. See [1], [8] and [9]. This completes the proof. \qed

**Theorem 2.4.** Let $S_1$ and $S_2$ be 2-knots in $S^2 \times S^2$ as in Lemma 2.2. If the complements of $S_1$ and $S_2$ are simply connected, then $S_1$ and $S_2$ are equivalent, i.e., ambient isotopic.

**Proof.** In the case when $|p| = 1$ or $|q| = 1$, we proved in [13]. We may assume that $|p| \geq 2$ and $|q| \geq 2$. It follows from [14] that the homeotopy group of $S^2 \times S^2$ corresponds to the subgroup of $GL(2; \mathbb{Z})$ consisting of

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with respect to generators $\zeta$ and $\eta$.

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are orientation preserving, while

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are orientation reversing. By Lemma 2.2, there is a homeomorphism $\phi$ of $S^2 \times S^2$ taking $S_1$ to $S_2$. Since $\phi_*(p\zeta + q\eta) = \phi_*([S_1]) = \pm[S_2] = \pm(p\zeta + q\eta)$, $\phi_* = \pm\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We consider the case of $\phi_* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $\phi_{|S_1}$ is orientation reversing. Let $-S_2$ be $S_2$ with opposite orientation. We decompose $(S^2 \times S^2, \pm S_2)$ as in Lemma 2.2: $(S^2 \times S^2, \pm S_2) = (D(2pq) \cup_{\gamma} E^\pm, \nu(S^2))$. Here we may assume that $\gamma^\pm$ is the identity map. Let $g : D(2pq) \rightarrow D(2pq)$ be the orientation-preserving homeomorphism such that its restriction to $\nu(S^2)$ is the antipodal map and its restriction to each fiber is the map induced on the unit disk in the complex plane by complex conjugation. Then $g' = g|_{\partial D(2pq)}$ is a homeomorphism of $\partial D(2pq)$ such that $g_*(\partial w_+) = \pm \partial w_-$, where $w_\pm$ is a generator of $H_2(E^\pm, \partial E^\pm; \mathbb{Z}) \cong \mathbb{Z}$. Since Boyer's results are based on
a theorem that gives necessary and sufficient conditions for the existence of a homeomorphism between simply-connected 4-manifolds extending a given homeomorphism of their boundaries, the fact that \( g'_i(\partial w_+) = \pm \partial w_- \) implies that there is an orientation-preserving homeomorphism \( h: E^+ \to E^- \) such that \( h|_{\partial E} = g' \). See [2]. Let \( \psi: S^2 \times S^2 \to S^2 \times S^2 \) be the orientation-preserving homeomorphism defined from \( g \) and \( h \). From the definition of \( g \), it is easily seen that \( \psi(S_2) = -S_2 \). Hence \( \psi \cdot \phi \) is a homeomorphism of \( S^2 \times S^2 \) taking \( S_1 \) to \( S_2 \) such that \( (\psi \cdot \phi)_* \) is the identity map.

Thus, we have a homeomorphism \( \phi' \) of \( S^2 \times S^2 \) taking \( S_1 \) to \( S_2 \) such that \( \phi'_* \) is the identity map, so \( \phi' \) is isotopic to the identity map. Therefore, \( S_1 \) and \( S_2 \) are equivalent. This completes the proof. \( \square \)

**Remark 2.5.** Let \( K \) be a 2-knot in \( S^4 \) and \( S \) a 2-knot in \( S^2 \times S^2 \). Then we obtain another 2-knot in \( S^2 \times S^2 \) by forming the connected sum of pairs \((S^2 \times S^2, S)\) and \((S^4, K)\). However, we do not always get a new 2-knot in \( S^2 \times S^2 \) in this manner. In fact, Theorem 2.2 says that if \( \pi_1(S^2 \times S^2) = 1 \), then the connected sum of \( S \) with any 2-knot in \( S^4 \) is always equivalent to the original 2-knot \( S \). See [13].

**Remark 2.6.** Let \( S_1 \) and \( S_2 \) be 2-knots in \( S^2 \times S^2 \) representing \( p\xi + q\eta \), where \( p \neq q \) and \( pq \neq 0 \). If there is a homeomorphism \( g \) of \( S^2 \times S^2 \) taking \( S_1 \) to \( S_2 \) such that \( g|_{S^2} \) is orientation preserving, then \( S_1 \) and \( S_2 \) are equivalent.

3. 2-KNOTS IN \( S^2 \times S^2 \) WITH NONTRIVIAL \( \pi_1 \)

We describe a construction of 2-knots in \( S^2 \times S^2 \) from [11] and [13]. Let \( K \) be a 2-knot in \( S^4 \) and \( C \) a smoothly embedded circle in \( S^4 - K \). Since we may assume that \( C \) is standardly embedded in \( S^4 \) up to ambient isotopy, the closure of the complement of a tubular neighborhood of \( C \) in \( S^4 \) is \( S^2 \times D^2 \). Then \( K \) is contained in \( S^2 \times D^2 \), so that this gives us a 2-knot \( S \) in \( S^2 \times S^2 = S^2 \times D^2 \cup S^2 \times D^2 \). If \( C \) is homologous in \( S^4 - K \) to a meridian of \( K \), then the 2-knot \( S \) represents \( \zeta \) [13]. Moreover, by van Kampen's theorem \( \pi_1(S^2 \times S^2 - S) \) is isomorphic to \( \pi_1(S^4 - K)/H \), where \( H \) is the normal closure of the element represented by \( C \) in \( \pi_1(S^4 - K) \).

We are concerned with the following two 2-knots in \( S^2 \times S^2 \) representing \( \zeta \). Let \( K \subset S^4 \) be the 5-twist spun 2-knot of the trefoil [15]. Then \( \pi_1(S^4 - K) \cong \mathcal{D} \times \mathbb{Z} \), where \( \mathcal{D} \) is the binary dodecahedral group

\[ \langle a, b; a^3 = b^5 = (ab)^2 \rangle \]

and \( \mathbb{Z} \) is generated by \( \mu \) which is homologous to a meridian of \( K \). The group \( \mathcal{D} \) is perfect and of order 120. The center of \( \mathcal{D} \) is generated by \( c = a^3 \) in \( \mathcal{D} \), and it is of order 2. Let \( C_1 \) and \( C_2 \) be embedded circles representing \( \mu \) and \( \mu c^{-1} \) in \( \pi_1(S^4 - K) \), respectively. Let \( S_1 \) be the 2-knot in \( S^2 \times S^2 \) constructed
from $K$ and $C_1$, and let $S_2$ be the 2-knot in $S^2 \times S^2$ constructed from $K$ and $C_2$. Let $E_1$ and $E_2$ be exteriors of $S_1$ and $S_2$, respectively. Then both $S_1$ and $S_2$ represent $\zeta$, and $\pi_1(S^2 \times S^2 - S_1) \cong \pi_1(S^2 \times S^2 - S_2) \cong D$. Thus $S_1$ and $S_2$ are 2-knots in $S^2 \times S^2$ that represent $\zeta$ and whose fundamental groups are isomorphic to $D$.

Now we investigate meridian elements in $D$ of the preceding 2-knots in $S^2 \times S^2$. We note that the group of the 5-twist spun 2-knot of the trefoil, $\pi_1(S^4 - K)$, has the following presentation:

$$\left\langle u, v; uvu = uvu, v = u^{-5}vu^5 \right\rangle,$$

where $u$ is a meridian and the second relation comes from the 5-twisting. Zeeman showed in [15] that $\pi_1(S^4 - K)$ is isomorphic to

$$\left\langle x, y, z; x^5 = (xy)^3 = (xyx)^2, z^{-1}xz = y, z^{-1}yz = yx^{-1} \right\rangle,$$

by making the substitution $u \rightarrow z$, $v \rightarrow xz$. Then $z$ is a meridian. By making the substitution $x \rightarrow b$, $xy \rightarrow a$, this group is isomorphic to

$$\left\langle a, b, z; a^3 = b^5 = (ab)^2, z^{-1}bz = b^{-1}a, z^{-1}b^{-1}az = b^{-1}ab^{-1} \right\rangle \cong \left\langle a, b, z, \mu; a^3 = b^5 = (ab)^2, \mu = ab^{-1}z, [\mu, a] = [\mu, b] = 1 \right\rangle \cong D \times \mathbb{Z}.$$

Therefore, $ba^{-1}$ and $ba^2$ in $D$ are meridian elements of 2-knots $S_1$ and $S_2$, respectively. Since $a^3$ in $D$ is of order 2, $ba^{-1}$ is of order 10. Also, since $ba^2 = a^3(ba^{-1})$ and $a^3$ is an element in the center of $D$, $ba^2$ is of order 5. Thus the order of a meridian element of $S_1$ is different from that of $S_2$, so that there is not a $\partial$-preserving homotopy equivalence $f : (E_1, \partial E_1) \rightarrow (E_2, \partial E_2)$, that is, two 2-knots $S_1$ and $S_2$ are inequivalent. Thus we have

**Theorem 3.1.** There exists 2-knots in $S^2 \times S^2$ representing $\zeta$ with fundamental group isomorphic to the binary dodecahedral group, but whose exteriors are not $\partial$-preserving homotopy equivalent.

**Remark 3.2.** The complements of 2-knots $S_1$ and $S_2$ in $S^2 \times S^2$ as given earlier are not $K(\pi, 1)$. In fact, $\pi_2(S^2 \times S^2 - S_i) \neq 0$ $(i = 1, 2)$. Let $S$ be either $S_1$ or $S_2$, and let $X$ be the complement of $S$. Then, since $S$ represents $\zeta \in H_2(S^2 \times S^2; \mathbb{Z}), \ H_2(X; \mathbb{Z}) \cong \mathbb{Z}$. If we let $p : \tilde{X} \rightarrow X$ be the universal covering, then we have a homomorphism $\tau : H_2(x; \mathbb{Z}) \rightarrow H_2(\tilde{X}; \mathbb{Z})$ such that $p_* \tau(\alpha) = 120\alpha$. Here $p_*$ is the homomorphism $H_2(\tilde{X}; \mathbb{Z}) \rightarrow H_2(X; \mathbb{Z})$ induced by the projection $p$, and $\alpha$ is a generator $H_2(X; \mathbb{Z}) \cong \mathbb{Z}$. Hence, $\pi_2(X) \cong \pi_2(\tilde{X}) \cong H_2(\tilde{X}; \mathbb{Z})$ is not trivial.

4. **Concluding remarks**

We consider in this section the problem of whether or not a given homology 3-sphere bounds a smooth acyclic 4-manifold. We have the Rohlin invariant
$\mu : H^3 \to \mathbb{Z}/2\mathbb{Z}$, where $H^3$ is the homology cobordism group of homology 3-spheres. If a homology 3-sphere $M$ bounds a smooth acyclic 4-manifold, then $\mu(M) = 0$. Some families of homology 3-spheres that bound smooth acyclic (or contractible) 4-manifolds are known. Meanwhile, the celebrated work of Donaldson [4] implies that if a homology 3-sphere $M$ bounds a smooth 4-manifold with nonstandard definite intersection form, then $M$ cannot bound a smooth acyclic 4-manifold. Also, Fintushel and Stern showed that if the invariant $R(a_1, \ldots, a_n)$ defined in [5] is positive, then the Seifert fibered homology 3-sphere $\Sigma(a_1, \ldots, a_n)$ cannot bound a smooth $\mathbb{Z}/2\mathbb{Z}$-acyclic 4-manifold. However, we note that every homology 3-sphere bounds a topological contractible 4-manifold. See [6].

**Definition 4.1.** Let $L$ be the following framed link in $S^3$ consisting of two knots $J$ and $K$ with linking number $t$ and with framing $m$ and $n$.

\[
M(L; t: m, n) =
\]

![Figure 2](image)

Then $M(L; t: m, n)$ is defined as a 3-manifold obtained by Dehn surgery on the framed link $L$.

The order of $H_1(M(L; t: m, n); \mathbb{Z})$ is $|mn - t^2|$. Hence, if $|mn - t^2| = 1$, then $M(L; t: m, n)$ is a homology 3-sphere.

Before stating the main result in this section, we notice the following. Since Donaldson's result in [3] extends without change to 4-manifolds with arbitrary fundamental groups [4], Kuga's result in [10] also extends to such 4-manifolds, that is,

**Theorem 4.2.** Let $X$ be a closed smooth 4-manifold with the intersection form

\[
\left(\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right)
\]

with respect to $\zeta$ and $\eta$ of $H_2(X ; \mathbb{Z})/\text{torsion} \cong \mathbb{Z} \oplus \mathbb{Z}$. Then the homology class $p\zeta + q\eta$ cannot be represented by a smoothly embedded 2-sphere in $X$ provided $|p| \geq 2$ and $|q| \geq 2$. 

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Proof. This follows in the same manner as in [10].

Our main result in this section is the following.

**Theorem 4.3.** Let \( t \) be a positive odd integer. Let \( J \) and \( K \) be slice knots. Suppose that \( m \) and \( n \) are positive even integers such that \( mn - t^2 = -1 \). If \( |m - t| > 1 \) or \( |n - t| > 1 \), then \( M = M(L; t; m, n) \) cannot bound a smooth compact 4-manifold \( V \) with \( \tilde{H}_4(V; \mathbb{Q}) = 0 \).

Hence, such an \( M \) does not bound a smooth acyclic 4-manifold.

**Proof.** Suppose that there is such a smooth 4-manifold \( V \). Let \( W \) be the smooth 4-manifold obtained by attaching two 2-handles to \( D^4 \) along the framed link \( L = J \cup K \). Then \( X = W \cup_M V \) is a closed smooth 4-manifold with the intersection form

\[
\begin{pmatrix}
m & t \\
t & n
\end{pmatrix}
\]

with respect to some generators of \( H_2(X; \mathbb{Z})/\text{torsion} \cong \mathbb{Z} \oplus \mathbb{Z} \). Then there are \( x \) and \( y \) in \( H_2(X; \mathbb{Z}) \) such that \( x^2 = m \), \( y^2 = n \) and \( x \cdot y = t \), and both \( x \) and \( y \) are represented by smoothly embedded 2-spheres in \( X \). Since \( m \) and \( n \) are even integers with \( mn - t^2 = -1 \), \( A \) is equivalent over \( \mathbb{Z} \) to

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

Hence, \( X \) has the intersection form

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

with respect to generators \( \zeta \) and \( \eta \) of \( H_2(X; \mathbb{Z})/\text{torsion} \). For some integers \( p \), \( q \), \( r \) and \( s \), \( x = p\zeta + q\eta \) and \( y = r\zeta + s\eta \). Since \( |m - t| > 1 \) or \( |n - t| > 1 \), it is seen that either \( \min(|p|, |q|) \) or \( \min(|r|, |s|) \) is greater than 1. Hence, there is a smoothly embedded 2-sphere in \( X \) representing \( a\zeta + b\eta \) with \( |a| \geq 2 \), and \( |b| \geq 2 \), contradicting Theorem 4.2. This completes the proof. \( \square \)

**Remark 4.4.** (1) Let \( J \) and \( K \) be any knots, and let \( m \) and \( n \) be even integers with \( mn - t^2 = -1 \). Then \( \mu(M(L; t; m, n)) = 0 \). (2) When \( J \) and \( K \) are trivial knots, \( M(L; t; m, n) \) is the Brieskorn homology 3-sphere \( \Sigma(t, |m - t|, |n - t|) \) if \( |m - t| > 1 \) and \( |n - t| > 1 \). Moreover,

\[
R(t, |m - t|, |n - t|) = 1.
\]

(3) If \( J \) and \( K \) are slices, then \( M = M(L; \pm 1; 0, 0) \) is embedded smoothly in \( S^4 \). See [7]. Hence, \( M \) bounds a smooth acyclic 4-manifold.

We can find the following lemma in [12].

**Lemma 4.5.** If a homology 3-sphere \( M \) is embedded smoothly in \( S^2 \times S^2 \), then \( M \) bounds a smooth acyclic 4-manifold.

Since every homology 3-sphere admits a locally flat embedding into \( S^4 \), it also admits such an embedding into \( S^2 \times S^2 \). However, Theorem 4.3 and Lemma 4.5 imply the following proposition.
**Proposition 4.6.** There exists a $\mu$-invariant 0 homology 3-sphere that cannot be embedded smoothly in $S^2 \times S^2$.

**References**