LOCALLY FLAT 2-KNOTS IN $S^2 \times S^2$
WITH THE SAME FUNDAMENTAL GROUP

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Abstract. We consider a locally flat 2-sphere in $S^2 \times S^2$ representing a primitive homology class $\xi$, which is referred to as a 2-knot in $S^2 \times S^2$ representing $\xi$. Then for any given primitive class $\xi$, there exists a 2-knot in $S^2 \times S^2$ representing $\xi$ with simply-connected complement. In this paper, we consider the classification of 2-knots in $S^2 \times S^2$ whose complements have a fixed fundamental group. We show that if the complement of a 2-knot $S$ in $S^2 \times S^2$ is simply connected, then the ambient isotopy type of $S$ is determined. In the case of nontrivial $\pi_1$, however, we show that the ambient isotopy type of a 2-knot in $S^2 \times S^2$ with nontrivial $\pi_1$ is not always determined by $\pi_1$.

1. Introduction

Let $\zeta$ and $\eta$ be natural generators of $H_2(S^2 \times S^2; \mathbb{Z})$ represented by the cross-section and fiber of the projection $S^2 \times S^2 \to S^2$ onto the first factor with $\zeta \cdot \zeta = \eta \cdot \eta = 0$ and $\zeta \cdot \eta = \eta \cdot \zeta = 1$. A 2-knot $S$ in $S^2 \times S^2$ is a locally flat submanifold of $S^2 \times S^2$ homeomorphic to $S^2$. The fundamental group of the complement of $S$ is referred to as the fundamental group of $S$. The exterior of $S$ is the closure of the complement of a tubular neighborhood of $S$ in $S^2 \times S^2$. Two 2-knots in $S^2 \times S^2$ are equivalent if they are ambient isotopic, that is, there exists an isotopic deformation $F: (S^2 \times S^2) \times I \to (S^2 \times S^2) \times I$ such that the homeomorphism $F_1$ takes one to the other. Kuga and Freedman have characterized those homology classes in $S^2 \times S^2$ that can be represented by 2-knots in $S^2 \times S^2$ as follows. Kuga has shown in [10] that the homology class $\xi = p\zeta + q\eta$, $p, q \in \mathbb{Z}$, can be represented by a smooth 2-knot in $S^2 \times S^2$ if and only if $|p| \leq 1$ or $|q| \leq 1$. Meanwhile, Freedman has shown in [6] that if $p$ and $q$ are relatively prime integers, then $\xi$ can be represented by a 2-knot in $S^2 \times S^2$.

Since the problem of classifying 2-knots in $S^2 \times S^2$ is interesting, we consider in this paper the problem of whether the equivalence class of a 2-knot in $S^2 \times S^2$ is determined by its fundamental group. For any integer $p$, let $\rho_p: S^2 \to S^2$ be the canonical smooth map of degree $p$, and let $\phi_p: S^2 \to S^2 \times S^2$ be
the embedding defined by \( \phi_p(x) = (x, \rho_p(x)) \). Then if we write \( \Sigma_p \) for the image \( \phi_p(S^2) \), \( \Sigma_p \) is the standard smooth 2-knot in \( S^2 \times S^2 \) representing \( \zeta + p\eta \). We obtained in [13] the following result: If the complement of a 2-knot \( S \) in \( S^2 \times S^2 \) representing \( \zeta + p\eta \) is simply connected, then \( S \) and \( \Sigma_p \) are equivalent. In this paper we prove the unknotting theorem in more general cases: If the complement of a 2-knot \( S \) in \( S^2 \times S^2 \) representing \( p\zeta + q\eta \) is simply connected, then the equivalence class of \( S \) is determined. Moreover, we prove that the equivalence class of a 2-knot in \( S^2 \times S^2 \) is not always determined by the fundamental group itself.

This paper is organized as follows. In \( \S 2 \), we consider the case that the fundamental group of a 2-knot is trivial. We show that for any relatively prime integers \( p \) and \( q \), there is a 2-knot representing \( p\zeta + q\eta \) with simply-connected complement, and prove the unknotting theorem. We consider in \( \S 3 \) the case that the fundamental group of a 2-knot is nontrivial. We prove that there exist distinct 2-knots with the same fundamental group. In \( \S 4 \), we consider the problem of whether a homology 3-sphere bounds a smooth acyclic 4-manifold or not, and by using Kuga’s theorem and our technique in \( \S 2 \), we present a family of homology 3-spheres that cannot bound smooth acyclic 4-manifolds.

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2. 2-KNOTS IN \( S^2 \times S^2 \) WITH TRIVIAL \( \pi_1 \)

It is easy to see that if the homology class represented by a 2-knot \( S \) is not primitive, then \( H_1(S^2 \times S^2 - S; \mathbb{Z}) \) is nonzero. We begin with the following proposition.

**Proposition 2.1.** Let \( p \) and \( q \) be relatively prime integers. Then there exists a 2-knot in \( S^2 \times S^2 \) representing \( p\zeta + q\eta \) with simply-connected complement.

**Proof.** Since \( \text{g.c.d}(p, q) = 1 \), there are two integers \( a \), \( b \) such that \( bp - aq = 1 \). We consider the 3-manifold \( M \) obtained by surgery on the framed link \( L \) illustrated in Figure 1. The link \( L \) consists of two trivial knots \( K_1 \) and \( K_2 \). Since \( |(2pq)(2ab) - (bp + aq)^2| = 1 \), \( M \) is a homology 3-sphere, so that \( M \) bounds a topological contractible 4-manifold \( V \). See [6]. Let \( W \) be the 4-manifold obtained by attaching two 2-handles \( h_1 \) and \( h_2 \) to the 4-disk \( D^4 \) along the framed link \( L \) (i.e., \( W = D^4 \cup h_1 \cup h_2 \)). Set \( X = W \cup_M V \), and \( X \) is a topological closed 4-manifold. Let \( B_i \) be a smooth 2-disk in \( D^4 \) which is the trivial knot \( K_i \) bounds, and let \( D_i \) be the core of \( h_i \) (\( i = 1, 2 \)). Then \( S_i = B_i \cup D_i \subset W \) is diffeomorphic to \( S^2 \). Since the framing of \( K_1 \) is \( 2pq \), a closed tubular neighborhood of \( S_1 \) is the \( D^2 \)-bundle \( D(2pq) \) over \( S^2 \) with Euler number \( 2pq \). Since \( K_i \) is trivial, \( W \) is the 4-manifold obtained by attaching the 2-handle \( h_2 \) to \( D(2pq) \). Hence, by the duality of handle-
decompositions, we can view $W$ as $(M \times I \cup h^*_2) \cup_d D(2pq)$, where $h^*_2$ is the dual handle of $h_2$ and $\partial$ is the lens space $L(2pq, 2pq - 1)$. Therefore, $X = Y \cup_d D(2pq)$, where $Y = V \cup_{M \times \{0\}} M \times I \cup h^*_2$. Then by van Kampen’s theorem, $\pi_1(Y) = 1$ and $\pi_1(X) = 1$. Moreover, $\pi_1(X - S_1) \cong \pi_1(Y) = 1$. $H_2(X; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $[S_1]$ and $[S_2]$, and $X$ has the intersection form

$$A = \begin{pmatrix} 2pq & bp + aq \\ bp + aq & 2ab \end{pmatrix}$$

with respect to these generators. Since the form $A$ is even and indefinite, $A$ is equivalent over $\mathbb{Z}$ to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

In fact, if we let $(\mathbb{Z} \oplus \mathbb{Z}, A)$ and $(\mathbb{Z} \oplus \mathbb{Z}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ be bilinear form spaces, then the matrix

$$B = \begin{pmatrix} p & a \\ q & b \end{pmatrix}$$

gives an isomorphism between them. Let $\{u, v\}$ and $\{\zeta, \eta\}$ be bases for the bilinear form spaces $(\mathbb{Z} \oplus \mathbb{Z}, A)$ and $(\mathbb{Z} \oplus \mathbb{Z}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$, respectively. The matrix $B$ takes $u$ to $p\zeta + q\eta$. Thus $X$ has the intersection form $(\mathbb{Z} \oplus \mathbb{Z}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$, and the homology class of $S_1$, $[S_1]$, is $p\zeta + q\eta$. By Freedman’s theorem, there is a homeomorphism $h: X \to S^2 \times S^2$. Then the induced isomorphism $h_*: H_2(X; \mathbb{Z}) \to H_2(S^2 \times S^2; \mathbb{Z})$ gives an automorphism of $(\mathbb{Z} \oplus \mathbb{Z}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$. Since the automorphism group of this form space is $\{C \in GL(2, \mathbb{Z}); C \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$, $h_* = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus the image $h(S_1)$ is a locally flat 2-sphere in $S^2 \times S^2$ representing $\pm (p\zeta + q\eta)$ or $\pm (q\zeta + p\eta)$,
and \( \pi_1(S^2 \times S^2 - h(S_1)) \cong \pi_1(X - S_1) = 1 \). After changing the orientation of \( S^2 \times S^2 \) and/or the orientation of \( \zeta \) and \( \eta \) (if necessary), \( h(S_1) \) may represent \( p\zeta + q\eta \). Therefore, \( h(S_1) \) is a required 2-knot in \( S^2 \times S^2 \).

Our key lemma in this section is the following.

**Lemma 2.2.** Let \( p \) and \( q \) be relatively prime integers, and let \( S_1 \) and \( S_2 \) be 2-knots in \( S^2 \times S^2 \) representing \( p\zeta + q\eta \). If the complements of \( S_1 \) and \( S_2 \) are simply connected, then there exists a homeomorphism of \( S^2 \times S^2 \) taking \( S_1 \) to \( S_2 \).

Since we can prove this lemma in the same manner as [13], we only sketch the proof.

**Proof (sketch).** Let \( N_i \) be a closed tubular neighborhood of \( S_i \) and \( E_i \) the exterior of \( S_i \) (\( i = 1, 2 \)). Then \( N_i \) is homeomorphic to \( D(2pq) \), and so the boundary \( \partial E_i \) of \( E_i \) is the lens space \( L(2pq, 2pq-1) \), where \( L(0, -1) = S^2 \times S^1 \). Hence \((S^2 \times S^2, S_i)\) is pairwise homeomorphic to \((D(2pq) \cup \gamma_i E_i, \nu(S^2))\), where \( \gamma_i : L(2pq, 2pq-1) \rightarrow L(2pq, 2pq-1) \) is some gluing homeomorphism and \( \nu : S^2 \rightarrow D(2pq) \) is the zero section. By the isotopy extension theorem, it is easily seen that the homeomorphism type of 2-knots with exterior \( E_i \) depends only on the isotopy class of the homeomorphism \( \gamma_i \). To prove Lemma 2.2, we need the following lemma.

**Lemma 2.3.** Suppose \( E_1 \) and \( E_2 \) are simply connected. Then \( E_1 \) is homeomorphic to \( E_2 \). In particular if \( (p, q) = (\pm 1, 0) \) or \( (0, \pm 1) \), then \( E_1 \) and \( E_2 \) are homeomorphic to \( S^1 \times D^2 \).

**Proof.** We give \( E_i \) the orientation opposite to the one inherited from \( S^2 \times S^2 \). It follows that the intersection form \((H_2(E_i; \mathbb{Z}), \cdot)\) is isomorphic to \((\mathbb{Z}, (2pq))\), where \((2pq) : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}\) is the bilinear form defined by \((2pq)(1, 1) = 2pq\). Hence, \( E_i \) is a simply-connected compact 4-manifold with boundary \( L(2pq, 2pq-1) \) and the intersection form \((\mathbb{Z}, (2pq))\). In [2], Boyer calculated the set of all oriented homeomorphism types of simply-connected compact 4-manifolds with given boundary and given intersection form. In the case of \( (p, q) = (\pm 1, 0) \) or \( (0, \pm 1) \), Remarks (5.3) of [2] say that \( E_1 \) and \( E_2 \) are homeomorphic to \( S^1 \times D^2 \). Next we consider the case of \( pq \neq 0 \). Since \( \gcd(p, q) = 1 \), there are two integers \( a, b \) such that \( bp - aq = 1 \). If we set \( u_i = [S_i] = p\zeta + q\eta \) and \( v = a\zeta + b\eta \), then \( u_i \) and \( v \) generate \( H_2(S^2 \times S^2; \mathbb{Z}) \). Let \( w_i \) be a generator of \( H_2(E_i, \partial E_i; \mathbb{Z}) \equiv \mathbb{Z} \). Since \( u_i \cdot v = bp + aq \), \( \partial w_i \in H_1(\partial E_i; \mathbb{Z}) = H_1(L(2pq, 2pq-1); \mathbb{Z}) \) is represented by \((bp + aq)\) times the \( \partial D^2 \)-fiber of the \( D^3 \)-bundle \( N_i \) over \( S_i \). Since \( bp - aq = 1 \), \((bp + aq)^2 \equiv 1 \) (mod \( 2pq \)). Hence, it follows from Example 5.4 and Remarks 5.6 of [2] that \( E_1 \) is homeomorphic to \( E_2 \). \( \square \)

Return to the proof of Lemma 2.2. Since the complements of \( S_1 \) and \( S_2 \) are simply connected, there is a homeomorphism \( h : E_1 \rightarrow E_2 \). Let \( h \) be the
restriction of $h$ to $\partial E_1$. If the homeomorphism $\gamma_2^{-1}h\gamma_1 : \partial D(2pq) \to \partial D(2pq)$ extends to a homeomorphism $g$ of $(D(2pq), \nu(S^2))$, we have the following required homeomorphism:

$$\varphi : (D(2pq) \cup \gamma_1 E_1, \nu(S^2)) \to (D(2pq) \cup \gamma_2 E_2, \nu(S^2))$$

by setting

$$\varphi = \begin{cases} g & \text{on } D(2pq), \\ h & \text{on } E_1. \end{cases}$$

Now we remark that in the case of $pq = 0$, $\gamma_2^{-1}h\gamma_1 : S^2 \times S^1 \to S^2 \times S^1$ is not isotopic to the twist $\tau : S^2 \times S^1 \to S^2 \times S^1$ defined by $\tau(\theta, \phi, \psi) = ((\theta + \psi, \phi, \psi)$, since $E_1$ and $E_2$ are homeomorphic to $S^2 \times D^2$ and the second Stiefel-Whitney class of $S^2 \times S^2$ is trivial. Hence, by investigating the homeotopy group of $L(2pq, 2pq - 1)$, it follows that there is an extension $g$ as the above. See [1], [8] and [9]. This completes the proof. □

**Theorem 2.4.** Let $S_1$ and $S_2$ be 2-knots in $S^2 \times S^2$ as in Lemma 2.2. If the complements of $S_1$ and $S_2$ are simply connected, then $S_1$ and $S_2$ are equivalent, i.e., ambient isotopic.

**Proof.** In the case when $|p| = 1$ or $|q| = 1$, we proved in [13]. We may assume that $|p| \geq 2$ and $|q| \geq 2$. It follows from [14] that the homeotopy group of $S^2 \times S^2$ corresponds to the subgroup of $GL(2; \mathbb{Z})$ consisting of

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with respect to generators $\zeta$ and $\eta$.

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are orientation preserving, while

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are orientation reversing. By Lemma 2.2, there is a homeomorphism $\phi$ of $S^2 \times S^2$ taking $S_1$ to $S_2$. Since $\phi_*([\zeta \cdot \eta \cdot \nu]) = \pm[S_2] = \pm[p\zeta + q\eta]$, $\phi_* = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We consider the case of $\phi_* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $\phi_*|S_1$ is orientation reversing. Let $-S_2$ be $S_2$ with opposite orientation. We decompose $(S^2 \times S^2, \pm S_2)$ as in Lemma 2.2: $(S^2 \times S^2, \pm S_2) = (D(2pq) \cup \gamma E^\pm, \nu(S^2))$. Here we may assume that $\gamma^\pm$ is the identity map. Let $g : D(2pq) \to D(2pq)$ be the orientation-preserving homeomorphism such that its restriction to $\nu(S^2)$ is the antipodal map and its restriction to each fiber is the map induced on the unit disk in the complex plane by complex conjugation. Then $g' = g|\partial D(2pq)$ is a homeomorphism of $\partial D(2pq)$ such that $g'_w(\partial w_+) = \pm w_-$, where $w_+$ is a generator of $H^2_2(E^\pm, \partial E^\pm; \mathbb{Z}) \cong \mathbb{Z}$. Since Boyer's results are based on
a theorem that gives necessary and sufficient conditions for the existence of a homeomorphism between simply-connected 4-manifolds extending a given homeomorphism of their boundaries, the fact that $g'(\partial w_+) = \pm \partial w_-$ implies that there is an orientation-preserving homeomorphism $h: E^+ \to E^-$ such that $h|_{\partial E} = g'$. See [2]. Let $\psi: S^2 \times S^2 \to S^2 \times S^2$ be the orientation-preserving homeomorphism defined from $g$ and $h$. From the definition of $g$, it is easily seen that $\psi(S_2) = -S_2$. Hence $\psi \cdot \phi$ is a homeomorphism of $S^2 \times S^2$ taking $S_1$ to $S_2$ such that $(\psi \cdot \phi)_* \text{is the identity map.}$

Thus, we have a homeomorphism $\phi'$ of $S^2 \times S^2$ taking $S_1$ to $S_2$ such that $\phi'_*$ is the identity map, so $\phi'$ is isotopic to the identity map. Therefore, $S_1$ and $S_2$ are equivalent. This completes the proof. □

**Remark 2.5.** Let $K$ be a 2-knot in $S^4$ and $S$ a 2-knot in $S^2 \times S^2$. Then we obtain another 2-knot in $S^2 \times S^2$ by forming the connected sum of pairs $(S^2 \times S^2, S)$ and $(S^4, K)$. However, we do not always get a new 2-knot in $S^2 \times S^2$ in this manner. In fact, Theorem 2.2 says that if $\pi_1(S^2 \times S^2 - S) = 1$, then the connected sum of $S$ with any 2-knot in $S^4$ is always equivalent to the original 2-knot $S$. See [13].

**Remark 2.6.** Let $S_1$ and $S_2$ be 2-knots in $S^2 \times S^2$ representing $p\xi + q\eta$, where $p \neq q$ and $pq \neq 0$. If there is a homeomorphism $g$ of $S^2 \times S^2$ taking $S_1$ to $S_2$ such that $g|_{S_1}$ is orientation preserving, then $S_1$ and $S_2$ are equivalent.

### 3. 2-KNOTS IN $S^2 \times S^2$ WITH NONTRIVIAL $\pi_1$

We describe a construction of 2-knots in $S^2 \times S^2$ from [11] and [13]. Let $K$ be a 2-knot in $S^4$ and $C$ a smoothly embedded circle in $S^4 - K$. Since we may assume that $C$ is standardly embedded in $S^4$ up to ambient isotopy, the closure of the complement of a tubular neighborhood of $C$ in $S^4$ is $S^2 \times D^2$. Then $K$ is contained in $S^2 \times D^2$, so that this gives us a 2-knot $S$ in $S^2 \times S^2 = S^2 \times D^2 \cup S^2 \times D^2$. If $C$ is homologous in $S^4 - K$ to a meridian of $K$, then the 2-knot $S$ represents $\xi$ [13]. Moreover, by van Kampen's theorem $\pi_1(S^2 \times S^2 - S)$ is isomorphic to $\pi_1(S^4 - K)/H$, where $H$ is the normal closure of the element represented by $C$ in $\pi_1(S^4 - K)$.

We are concerned with the following two 2-knots in $S^2 \times S^2$ representing $\xi$. Let $K \subset S^4$ be the 5-twist spun 2-knot of the trefoil [15]. Then $\pi_1(S^4 - K) \cong \mathcal{D} \times \mathbb{Z}$, where $\mathcal{D}$ is the binary dodecahedral group

$$\langle a, b; a^3 = b^5 = (ab)^2 \rangle$$

and $\mathbb{Z}$ is generated by $\mu$ which is homologous to a meridian of $K$. The group $\mathcal{D}$ is perfect and of order 120. The center of $\mathcal{D}$ is generated by $c = a^3$ in $\mathcal{D}$, and it is of order 2. Let $C_1$ and $C_2$ be embedded circles representing $\mu$ and $\mu c^{-1}$ in $\pi_1(S^4 - K)$, respectively. Let $S_1$ be the 2-knot in $S^2 \times S^2$ constructed
from $K$ and $C_1$, and let $S_2$ be the 2-knot in $S^2 \times S^2$ constructed from $K$ and $C_2$. Let $E_1$ and $E_2$ be exteriors of $S_1$ and $S_2$, respectively. Then both $S_1$ and $S_2$ represent $\zeta$, and $\pi_1(S^2 \times S^2 - S_1) \cong \pi_1(S^2 \times S^2 - S_2) \cong \mathcal{D}$. Thus $S_1$ and $S_2$ are 2-knots in $S^2 \times S^2$ that represent $\zeta$ and whose fundamental groups are isomorphic to $\mathcal{D}$.

Now we investigate meridian elements in $\mathcal{D}$ of the preceding 2-knots in $S^2 \times S^2$. We note that the group of the 5-twist spun 2-knot of the trefoil, $\pi_1(S^4 - K)$, has the following presentation:

$$\pi_1(S^4 - K) = \langle u, v ; uvu = vuv, v = u^{-5}vu^5 \rangle,$$

where $u$ is a meridian and the second relation comes from the 5-twisting. Zeeman showed in [15] that $\pi_1(S^4 - K)$ is isomorphic to

$$\langle x, y, z ; x^5 = (xy)^3 = (xyx)^2, z^{-1}xz = y, z^{-1}yz = xy^{-1} \rangle,$$

by making the substitution $u \rightarrow z$, $v \rightarrow xz$. Then $z$ is a meridian. By making the substitution $x \rightarrow b$, $xy \rightarrow a$, this group is isomorphic to

$$\langle a, b, z ; a^3 = b^5 = (ab)^2, z^{-1}bz = b^{-1}a, z^{-1}b^{-1}az = b^{-1}ab^{-1} \rangle$$

$$\cong \langle a, b, z, \mu ; a^3 = b^5 = (ab)^2, \mu = ab^{-1}z, [\mu, a] = [\mu, b] = 1 \rangle$$

$$\cong \mathcal{D} \times \mathbb{Z}.$$

Therefore, $ba^{-1}$ and $ba^2$ in $\mathcal{D}$ are meridian elements of 2-knots $S_1$ and $S_2$, respectively. Since $a^3$ in $\mathcal{D}$ is of order 2, $ba^{-1}$ is of order 10. Also, since $ba^2 = a^3(ba^{-1})$ and $a^3$ is an element in the center of $\mathcal{D}$, $ba^2$ is of order 5. Thus the order of a meridian element of $S_1$ is different from that of $S_2$, so that there is not a $\partial$-preserving homotopy equivalence $f : (E_1, \partial E_1) \rightarrow (E_2, \partial E_2)$, that is, two 2-knots $S_1$ and $S_2$ are inequivalent. Thus we have

**Theorem 3.1.** There exists 2-knots in $S^2 \times S^2$ representing $\zeta$ with fundamental group isomorphic to the binary dodecahedral group, but whose exteriors are not $\partial$-preserving homotopy equivalent.

**Remark 3.2.** The complements of 2-knots $S_1$ and $S_2$ in $S^2 \times S^2$ as given earlier are not $K(\pi, 1)$. In fact, $\pi_s(S^2 \times S^2 - S_i) \neq 0$ ($i = 1, 2$). Let $S$ be either $S_1$ or $S_2$, and let $X$ be the complement of $S$. Then, since $S$ represents $\zeta \in H_2(S^2 \times S^2; \mathbb{Z})$, $H_2(X; \mathbb{Z}) \cong \mathbb{Z}$. If we let $p : \tilde{X} \rightarrow X$ be the universal covering, then we have a homomorphism $\tau : H_2(x; \mathbb{Z}) \rightarrow H_2(\tilde{X}; \mathbb{Z})$ such that $p_* \tau(\alpha) = 120\alpha$. Here $p_*$ is the homomorphism $H_2(\tilde{X}; \mathbb{Z}) \rightarrow H_2(X; \mathbb{Z})$ induced by the projection $p$, and $\alpha$ is a generator $H_2(X; \mathbb{Z}) \cong \mathbb{Z}$. Hence, $\pi_2(X) \cong \pi_2(\tilde{X}) \cong H_2(\tilde{X}; \mathbb{Z})$ is not trivial.

4. **Concluding remarks**

We consider in this section the problem of whether or not a given homology 3-sphere bounds a smooth acyclic 4-manifold. We have the Rohlin invariant

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\( \mu : H^3 \rightarrow \mathbb{Z}/2\mathbb{Z} \), where \( H^3 \) is the homology cobordism group of homology 3-spheres. If a homology 3-sphere \( M \) bounds a smooth acyclic 4-manifold, then \( \mu(M) = 0 \). Some families of homology 3-spheres that bound smooth acyclic (or contractible) 4-manifolds are known. Meanwhile, the celebrated work of Donaldson [4] implies that if a homology 3-sphere \( M \) bounds a smooth 4-manifold with nonstandard definite intersection form, then \( M \) cannot bound a smooth acyclic 4-manifold. Also, Fintushel and Stern showed that if the invariant \( R(a_1, \ldots, a_n) \) defined in [5] is positive, then the Seifert fibered homology 3-sphere \( \Sigma(a_1, \ldots, a_n) \) cannot bound a smooth \( \mathbb{Z}/2\mathbb{Z} \)-acyclic 4-manifold. However, we note that every homology 3-sphere bounds a topological contractible 4-manifold. See [6].

**Definition 4.1.** Let \( L \) be the following framed link in \( S^3 \) consisting of two knots \( J \) and \( K \) with linking number \( t \) and with framing \( m \) and \( n \).

![Figure 2](image)

Then \( M(L; t: m, n) \) is defined as a 3-manifold obtained by Dehn surgery on the framed link \( L \).

The order of \( H_1(M(L; t: m, n); \mathbb{Z}) \) is \( |mn - t^2| \). Hence, if \( |mn - t^2| = 1 \), then \( M(L; t: m, n) \) is a homology 3-sphere.

Before stating the main result in this section, we notice the following. Since Donaldson's result in [3] extends without change to 4-manifolds with arbitrary fundamental groups [4], Kuga's result in [10] also extends to such 4-manifolds, that is,

**Theorem 4.2.** Let \( X \) be a closed smooth 4-manifold with the intersection form

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

with respect to \( \zeta \) and \( \eta \) of \( H_2(X; \mathbb{Z})/\text{torsion} \cong \mathbb{Z} \oplus \mathbb{Z} \). Then the homology class \( p\zeta + q\eta \) cannot be represented by a smoothly embedded 2-sphere in \( X \) provided \( |p| \geq 2 \) and \( |q| \geq 2 \).
Proof. This follows in the same manner as in [10].

Our main result in this section is the following.

**Theorem 4.3.** Let $t$ be a positive odd integer. Let $J$ and $K$ be slice knots. Suppose that $m$ and $n$ are positive even integers such that $mn - t^2 = -1$. If $|m - t| > 1$ or $|n - t| > 1$, then $M = M(L; t: m, n)$ cannot bound a smooth compact 4-manifold $V$ with $\tilde{H}^*(V; \mathbb{Q}) = 0$.

Hence, such an $M$ does not bound a smooth acyclic 4-manifold.

**Proof.** Suppose that there is such a smooth 4-manifold $V$. Let $W$ be the smooth 4-manifold obtained by attaching two 2-handles to $D^4$ along the framed link $L = J \cup K$. Then $X = W \cup_M V$ is a closed smooth 4-manifold with the intersection form

$$A = \begin{pmatrix} m & t \\ t & n \end{pmatrix}$$

with respect to some generators of $H_2(X; \mathbb{Z})/\text{torsion} \cong \mathbb{Z} \oplus \mathbb{Z}$. Then there are $x$ and $y$ in $H_2(X; \mathbb{Z})$ such that $x^2 = m$, $y^2 = n$ and $x \cdot y = t$, and both $x$ and $y$ are represented by smoothly embedded 2-spheres in $X$. Since $m$ and $n$ are even integers with $mn - t^2 = -1$, $A$ is equivalent over $\mathbb{Z}$ to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Hence, $X$ has the intersection form

$$(\xi, \eta)$$

with respect to generators $\xi$ and $\eta$ of $H_2(X; \mathbb{Z})/\text{torsion}$. For some integers $p$, $q$, $r$ and $s$, $x = p\xi + q\eta$ and $y = r\xi + s\eta$. Since $|m - t| > 1$ or $|n - t| > 1$, it is seen that either $\min(|p|, |q|)$ or $\min(|r|, |s|)$ is greater than 1. Hence, there is a smoothly embedded 2-sphere in $X$ representing $a\xi + b\eta$ with $|a| \geq 2$, and $|b| \geq 2$, contradicting Theorem 4.2. This completes the proof. □

**Remark 4.4.** (1) Let $J$ and $K$ be any knots, and let $m$ and $n$ be even integers with $mn - t^2 = -1$. Then $\mu(M(L; t: m, n)) = 0$. (2) When $J$ and $K$ are trivial knots, $M(L; t: m, n)$ is the Brieskorn homology 3-sphere $\Sigma(t, |m - t|, |n - t|)$ if $|m - t| > 1$ and $|n - t| > 1$. Moreover,

$$R(t, |m - t|, |n - t|) = 1.$$  

(3) If $J$ and $K$ are slices, then $M = M(L; \pm 1: 0, 0)$ is embedded smoothly in $S^4$. See [7]. Hence, $M$ bounds a smooth acyclic 4-manifold.

We can find the following lemma in [12].

**Lemma 4.5.** If a homology 3-sphere $M$ is embedded smoothly in $S^2 \times S^2$, then $M$ bounds a smooth acyclic 4-manifold.

Since every homology 3-sphere admits a locally flat embedding into $S^4$, it also admits such an embedding into $S^2 \times S^2$. However, Theorem 4.3 and Lemma 4.5 imply the following proposition.
Proposition 4.6. There exists a $\mu$-invariant 0 homology 3-sphere that cannot be embedded smoothly in $S^2 \times S^2$.

REFERENCES

10. K. Kuga, Representing homology classes of $S^2 \times S^2$, Topology 23 (1984), 133–137.

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