ROOTS OF UNITY
AND THE ADAMS-NOVIKOV SPECTRAL SEQUENCE
FOR FORMAL $A$-MODULES

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Abstract. The cohomology of a Hopf algebroid related to the Adams-Novikov
spectral sequence for formal $A$-modules is studied in the special case in which $A$
is the ring of integers in the field obtained by adjoining $p$th roots of unity to $\mathbb{Q}_p$, the $p$-adic numbers. Information about these cohomology groups is used to give new proofs of results about the $E_2$ term of the Adams spectral sequence based on 2-local complex $K$-theory, and about the odd primary Kervaire invariant elements in the usual Adams-Novikov spectral sequence.

One of the most powerful tools used in the computation of stable homotopy groups is the Adams-Novikov spectral sequence. The $E_2$ term of this spectral sequence is a certain Ext group derived from a universal formal group law. In [R3] the corresponding Ext group for a universal formal $A$-module, for $A$ the ring of algebraic integers in an algebraic number field, $K$, or its $p$-adic completion, was introduced and certain conjectures about these groups were formulated. One of these conjectures (concerning the value of $\text{Ext}^{1,*}$) was confirmed in [J] using a Hopf algebroid (i.e., a generalized Hopf algebra in which the left and right units need not agree), $E_A T$, which generalizes the Hopf algebroid $K_* K$ of stable cooperations for complex $K$-theory. The present paper is concerned with the cohomology of $E_A T$ in the special case of $A = \mathbb{Z}_p[\zeta]$ where $\zeta$ is a $p$th root of unity and $\mathbb{Z}_p$ denotes the $p$-adic integers. We will show that in this case $E_A T$ is contained in an extension of Hopf algebroids

$E \overline{A} T \rightarrow E_A T \rightarrow E_{\overline{A}} T$

and that the cohomology of $E \overline{A} T$ can be completely described. This provides us with information about the cohomology of $E_A T$ via the Cartan-Eilenberg spectral sequence associated to this extension.

Two applications of this result are presented. In the case $p = 2$, $E_A T$ can be identified with the 2-adic completion of the Hopf algebroid $K_* K_{(2)}$ of stable cooperations for 2-primary complex $K$-theory. In this case the cohomology
of $\overline{E_A T}$ can also be described, so that we can completely describe the Cartan-Eilenberg spectral sequence (there are no nontrivial differentials for dimensional reasons). We therefore obtain a new proof of the results in [R1, K], computing $H^*(K_1 K_2)$.

A second application is the construction of nontrivial elements in the classical Adams-Novikov spectral sequence based on $BP$, the Brown-Peterson spectrum, which is a summand of $MU$, the spectrum representing complex cobordism. In [R3] a map of Hopf algebroids

$$\Psi: VT = BP_* BP \to V_A T$$

was described (here $V_A T$ is the Hopf algebroid generalizing $VT$ to the category of formal $A$-modules, and is constructed using isomorphisms of $A$-typical formal $A$-modules). Composing $\Psi$ with the Conner-Floyd map

$$\Phi: V_A T \to E_A T$$

constructed in [J] and the map $\rho$, we have a map

$$\chi: VT \to \overline{E_A T}$$

from $VT$ to a Hopf algebroid whose cohomology is known. We thus have a tool for identifying nonzero elements of $H^{**} VT$, the $E_2$ term of the classical Adams-Novikov spectral sequence. We apply this to give a new proof of Theorem 4 of [R1] concerning the odd primary Kervaire invariant elements.

1. AN EXTENSION CONTAINING $E_A T$

In this section, we recall the definition and some of the structure of $E_A T$. We describe the homogeneous components of $E_A T$ and construct two related Hopf algebroids, $\overline{C}_n$ and $\overline{C}_n$, with which we construct the extension described in the introduction. We conclude by computing the cohomology of $\overline{C}_n$.

The ring $A = \hat{\mathbb{Z}}_p[\zeta]$ is the ring of integers in the field $K = \hat{\mathbb{Q}}_p[\zeta]$, which is an extension of $\mathbb{Q}_p$ of degree $p - 1$. $A$ has a unique prime ideal $(\pi)$ whose generator may be taken to be $\pi = \zeta - 1$, and the residue field of $A$, i.e., $A/(\pi)$, is $\mathbb{Z}/p\mathbb{Z}$. $p$ is totally ramified in $A$, with $(\pi)^{p-1} = (p)$.

Recall from [J] that the Hopf algebroid $(E_A, E_A T)$ is defined by

$$E_A = A[t, t^{-1}]$$

$$E_A T = \{ f \in K[u, u^{-1}, v, v^{-1}] | f(at, bt) \in E_A, \text{ if } a, b \in A, a, b \equiv 1 (\pi) \}$$

and that $E_A, E_A T$ are graded with $\deg(t) = \deg(u) = \deg(v) = 2(p - 1)$. The structure maps for $(E_A, E_A T)$ are:

$$\eta_L(t) = u,$$

$$\eta_R(t) = v,$$

$$\psi(u) = u \otimes 1,$$

$$\psi(v) = 1 \otimes v,$$

$$c(u) = v,$$

$$c(v) = u,$$

$$\epsilon(u) = t,$$

$$\epsilon(v) = t.$$
If we denote the homogeneous component of $E_A T$ of degree $2 \cdot n \cdot (p - 1)$ by $(E_A T)_n$ then we obtain a Hopf algebroid $(A, (E_A T)_n)$. Let us also define

$$C_n = \{ f \in K[w, w^{-1}] | f(a) \in A \text{ if } a \in A, a \equiv 1 \text{ (} \pi \text{)} \}.$$ 

$(A, C_n)$ can be given the structure of a Hopf algebroid via the maps

$$\eta_L(1) = 1, \quad \eta_R(1) = w_n,$$

$$\psi(w) = w \otimes w, \quad c(w) = w^{-1}, \quad \epsilon(w) = 1.$$ 

We may define a map $C_n \rightarrow (E_A T)_n$ by $f \mapsto u^n \cdot f(v/u)$ and it is straightforward to check that this defines an isomorphism of Hopf algebroids. Thus, in particular we have

$$H^{s, 2n(p-1)}(E_A T) \simeq H^s(C_n).$$

We will do most of our computations using $C_n$ rather than $E_A T$, and write $C$ in place of $C_n$ if the choice of right unit is not relevant.

Let us also write $B = C \cap K[w]$. We may define a sequence of polynomials in $B$ inductively by

$$q_0 = (w - 1)/\pi, \quad q_{i+1} = (q_i^q - q_i)/\pi.$$ 

Also, let us denote

$$q^I(w) = q_0^{i_0} \cdots q_m^{i_m}$$

if $I = (i_0, \ldots, i_m)$ is a multi-index.

**Lemma 1.** The polynomials $\{q^I | 0 \leq i_j < p, m = 0, 1, 2, \ldots\}$ form a basis for $B$ as an $A$-module.

**Proof.** This is Proposition 7 of [J] (note that these polynomials are denoted there by $f_i$).

**Corollary 2.** The polynomials $\{q_i | i = 0, 1, \ldots\}$ generate $C$ over $A[w, w^{-1}]$.

It will be useful for us to have a slightly different generating set for $C$ available in addition to this one. Define inductively

$$\tilde{q}_0 = (w - 1)/\pi, \quad \tilde{q}_1 = (w^p - 1)/\pi^{p+1}, \quad \tilde{q}_{i+1} = (\tilde{q}_i^p - \tilde{q}_i)/\pi$$

and

$$q^I = \tilde{q}_0^{i_0} \cdots \tilde{q}_m^{i_m} \quad \text{if} \quad I = (i_0, \ldots, i_m).$$

**Lemma 3.** The polynomials $\{q^I | 0 \leq i_j < p, m = 0, 1, 2, \ldots\}$ form a basis for $B$ as an $A$-module.

**Proof.** Part of this lemma is, of course, that $\tilde{q}_i(w) \in B$. Since for any $a \in A$, $a^n - a \equiv O(\pi)$, it is sufficient for us to show that $\tilde{q}_i(w) \in B$. This, however, follows from [J, Lemma 17].

To see that this set forms a basis, note that the $(n + 1) \times (n + 1)$ matrix that expresses the polynomials $q^I$ with $\sum i_j p^j \leq n$ as a linear combination of the polynomials $q^I$ is triangular, with diagonal entries equal to $1$. Thus, it is
invertible over $A$, and so the polynomials $q^i$ span $B$. They are clearly linearly independent.

**Corollary 4.** The polynomials $\{q_i|i = 0, 1, \ldots\}$ generate $C$ over $A[w, w^{-1}]$.

Our interest in this second generating set is motivated by the fact, easily proved by induction, that $q_i(w)$ for $i \geq 1$ is a polynomial in $w^p$. If we denote $\tilde{C} = C \cap K[w^p, w^{-p}]$ and $\tilde{B} = \tilde{C} \cap K[w^p]$, then we have

**Corollary 5.** The polynomials $\{q^i|i_0 = 0, 0 \leq i_j < p, m = 1, 2, \ldots\}$ form a basis for $\tilde{B}$ as an $A$-module.

**Corollary 6.** The polynomials $\{q_i|i = 1, 2, \ldots\}$ generate $\tilde{C}$ over $A[w^p, w^{-p}]$. A third algebra related to $C$ and $\tilde{C}$ is

$$C = \{f \in K[x, x^{-1}]|f(a) \in A \text{ if } a \equiv 1 (\pi^{p+1})\}.$$ We make $(A, \hat{C}_n)$ into a Hopf algebroid by defining

$$\eta_L(1) = 1, \quad \eta_R(1) = x^n,$$
$$\psi(x) = x \otimes x, \quad \sigma(x) = x^{-1}, \quad e(x) = 1.$$ We also define $\hat{B} = \tilde{C} \cap K[x]$.

The analogs of the polynomials $q_i$ and $\hat{q}_i$ in this case are the polynomials defined by

$$\hat{q}_i = (x - 1)/\pi^{p+1}, \quad \hat{q}_{i+1} = (\hat{q}_i^p - \hat{q}_i)/\pi.$$ We also use the notation $q^I = q_1^{i_1} \cdots q_m^{i_m}$ if $I = (i_1, \ldots, i_m)$. The analog of Lemmas 1 and 3 is

**Lemma 7.** The polynomials $\{q^I|0 \leq i_j < p, m = 1, 2, \ldots\}$ form a basis for $\hat{B}$ as an $A$-module.

**Proof.** The map $K[x] \rightarrow K[x]$ defined by $g(x) \mapsto g((x - 1)/\pi^{p+1})$ maps the algebra of polynomials with the property that $g(a) \in A$ if $a \in A$ isomorphically to $\hat{B}$. Since it also maps the basis for this former algebra constructed in [J, Proposition 7], onto the set $\{q^I\}$, the latter must be a basis for $\hat{B}$.

**Corollary 8.** The polynomials $\{q_i|i = 1, 2, \ldots\}$ generate $\tilde{C}$ over $A[x, x^{-1}]$.

The connection between $\tilde{C}$ and the previous two Hopf algebras we have considered is given by

**Proposition 9.** The map from $\tilde{C}$ to $C$ that sends $x$ to $w^p$ is an injection of Hopf algebroids whose image is $\tilde{C}$.

**Proof.** Since this map sends $\hat{q}_i$ to $q_i$, the result is clear.

We next describe the Hopf algebroid $(\overline{E}_A, \overline{E}_A \overline{T})$, or rather we describe its homogeneous, degree $n \cdot 2 \cdot (p - 1)$ component, $\overline{C}_n$. Let $\overline{C}_n$ denote the dual of the group algebra of the cyclic group of order $p$:

The structure maps for $\overline{C}_n$ are, using $\delta$ to denote a generator for $\mathbb{Z}/p\mathbb{Z}$,
\[
\psi(f)(\delta^i \otimes \delta^j) = f(\delta^{i+j}), \quad \eta_L(1)(\delta^i) = 1, \quad \eta_R(1)(\delta^i) = \zeta^{ni},
\]
\[
c(f)(\delta^i) = f(\delta^{-i}), \quad \epsilon(f) = f(1).
\]

Let us also define a map of Hopf algebroids $\rho : (A, C_n) \to (A, \overline{C}_n)$ by $\rho(f)(\delta^i) = f(\zeta^i)$.

The critical fact about $\rho$ is

**Lemma 10.** $\rho$ is a normal map of Hopf algebroids.

**Proof.** It is straightforward that $\rho$ preserves the Hopf algebroid structure maps and so defines a map of Hopf algebroids; the question is whether it is normal. Referring to [R4, A1.1.10] we must verify that
\[
C_n \boxtimes \overline{C}_n A = A \boxtimes \overline{C}_n C_n
\]
where $\boxtimes$ denotes the cotensor product and, for $(A, \Gamma)$ a Hopf algebroid, $\Gamma'$ is the associated Hopf algebra, defined by
\[
\Gamma' = \Gamma/(\eta_R(a) - \eta_L(a)|a \in A).
\]

In the case $\Gamma = \overline{C}_n$, this becomes
\[
\overline{C}_n' = \begin{cases} 
\overline{C}_n & \text{if } p|n, \\
A & \text{if } (p, n) = 1.
\end{cases}
\]

To see this, note that if $p|n$, then
\[
(\eta_R(a) - \eta_L(a))(\delta^i) = a \cdot (\eta_R(1) - \eta_L(1))(\delta^i) = a \cdot (\zeta^{ni} - 1) = a \cdot (1 - 1) = 0
\]
while if $(n, p) = 1$, then the ideal generated by $\eta_R(a) - \eta_L(a)$ is
\[
I = \{\phi \in \overline{C}_n|\phi(1) = 0\}.
\]

Thus, the map $\overline{C}_n' = \overline{C}_n/I \to A$ that sends $\phi$ to $\phi(1)$ is an isomorphism.

Since $C_n \boxtimes_A A = A \boxtimes_A C_n = C_n$, we may assume that $p|n$. The cotensor product $C_n \boxtimes_{\overline{C}_n} A$ is defined to be the kernel of the map
\[
C_n \simeq C_n \otimes_A A \to C_n \otimes_A \overline{C}_n \otimes_A A
\]
which sends $f$ to $(1 \otimes \rho)(\psi(f) \otimes 1 - f \otimes \eta_L(1) \otimes 1)$. This kernel consists of those elements $f \in \overline{C}_n$ for which
\[
(1 \otimes \rho)(f(w \otimes w)) = f(w) \otimes 1
\]
in $C_n \otimes_A \overline{C}_n$. These are precisely those elements $f \in \overline{C}_n$ of the form $f(w) = g(w^p)$. Similarly, $A \boxtimes_{\overline{C}_n} C_n$ consists of those $f \in C_n$ for which
\[
(\rho \otimes 1)(f(w \otimes w)) = \eta_R(1) \otimes f(w)
\]
in $\overline{C}_n \otimes C_n$. Since $\eta_R(1) = 1$ in $\overline{C}_n$ when $p|n$, we see that this also consists of those $f \in \overline{C}_n$ of the form $f(w) = g(w^p)$.

If we define the sub-Hopf algebroid $(\hat{A}, \hat{C}_n)$ of $(A, C_n)$ by
\[
\hat{A} = A \Box_{\overline{C}_n} A, \quad \hat{C}_n = A \Box_{\overline{C}_n} C_n \Box_{\overline{C}_n} A,
\]
then, following [R4, A1.1.15], we have

**Corollary 11.** $(\hat{A}, \hat{C}_n) \xrightarrow{i} (A, C_n) \xrightarrow{\rho} (A, \overline{C}_n)$ is an extension of Hopf algebroids. (The fact that $i$ is an inclusion is [R4, A1.1.14].)

For this to be useful we must describe $(\hat{A}, \hat{C}_n)$. As noted in the proof of [R4, A1.1.14], we have
\[
\hat{A} = \{a \in A|L(a) = \eta_R(a) \text{ in } \overline{C}_n\},
\]
\[
\hat{C}_n = \{f \in C_n|(\rho \otimes 1 \otimes \rho)\psi^2 f = \eta_L(1) \otimes f \otimes \eta_R(1)\}.
\]
and so
\[
(\hat{A}, \hat{C}_n) = \begin{cases}
0 & \text{if } (n, p) = 1, \\
(A, \tilde{C}_n) \simeq (A, \tilde{C}_{n/p}) & \text{if } n|p.
\end{cases}
\]

The applications we have in mind for this extension involve the cohomology of $C_n$, which we approach via that of $\tilde{C}_n$ and $\overline{C}_n$. We conclude this section, therefore, by recalling the cohomology of $\overline{C}_n$. Let us define two homomorphisms $S, T: \overline{C}_n \to \overline{C}_n$ by
\[
S(f)(x) = f(\delta x) - f(x) \quad \text{and} \quad T(f)(x) = \sum_{i=0}^{p-1} f(\delta^i x).
\]

A straightforward computation yields

**Lemma 12.** $0 \to A \xrightarrow{\eta_L} \overline{C}_n \xrightarrow{S} \overline{C}_n \xrightarrow{T} \overline{C}_n \xrightarrow{S} \cdots$ is an injective resolution of $A$ considered as a left $C_n$ comodule.

**Corollary 13.** The cohomology of $\overline{C}_n$ is given by
\[
H^s(\overline{C}_n) = \begin{cases}
A/pA, & s \text{ odd,} \\
0, & s \text{ even,}
\end{cases}
\]
if $(n, p) = 1$, and by
\[
H^s(\overline{C}_n) = \begin{cases}
A, & s = 0, \\
A/pA, & s > 0, s \text{ even,} \\
0, & s \text{ odd,}
\end{cases}
\]
if $p|n$.

**Proof.** Applying the functor $A \Box_{C_n} (\ )$ to the resolution of $A$ and using the identification $A \Box_{C_n} C_n = A$ gives the complexes
\[
A \xrightarrow{\xi^{n-1}} A \xrightarrow{0} A \xrightarrow{\xi^{n-1}} A \xrightarrow{0} \cdots
\]
if $(n, p) = 1$, and
\[
A \xrightarrow{0} A \xrightarrow{p} A \xrightarrow{0} A \xrightarrow{p} A \rightarrow \ldots
\]
if $p|n$.

2. Applications

2.1. The cohomology of $K_{*}K_{(2)}$. If the prime $p$ is chosen to be 2, then $A = \hat{\mathbb{Z}}_2$ and $C_n$ can be described as
\[
C_n = \{ f \in \hat{\mathbb{Q}}_2[w, w^{-1}] | f(a) \in \hat{\mathbb{Z}}_2 \text{ if } a \equiv 1 \text{ (2)} \}.
\]
The description of $K_{*}K$ given in [AHS]
\[
K_{*}K = \{ f \in \mathbb{Q}[u, u^{-1}, v, v^{-1}] | f(at, bt) \in \mathbb{Z}[t, t^{-1}, 1/a, 1/b] \}
\]
if $a, b \in \mathbb{Z}, a, b \neq 0$
shows that $C_n$ can be identified with $(K_{*}K_{(2)})_n$ so that the $E_2$ term of the Adams spectral sequence based on 2-local complex $K$-theory has as its completion
\[
E_2^{*, n} = H^*(C_n) = \text{Ext}_{C_n}^*(\hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2).
\]
The Cartan-Eilenberg spectral sequence, [R4, A1.3.14], allows us to describe these groups in terms of the cohomology of $\tilde{C}_n$ and $\overline{C}_n$: Proposition 14. There is a spectral sequence converging to $\text{Ext}_{C_n}^*(\hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2)$ whose $E_2$ term is $E_2^{*, i} = \text{Ext}_{C_n}^{i*} (\hat{\mathbb{Z}}_2, \text{Ext}_{C_n}^{i} (\hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2))$.

Since $\text{Ext}_{C_n}^{*} (\hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2)$ is described at the end of §1, we turn to describing $\text{Ext}_{C_n}^{*} (\hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2)$. The key to this description is the following injective resolution, which is the analog at the prime 2 of a resolution constructed for odd primes in [B, §7].

Lemma 15. The sequence
\[
0 \rightarrow \hat{\mathbb{Z}}_2 \xrightarrow{p_0} \tilde{C}_n \xrightarrow{p_1} \tilde{C}_n \xrightarrow{p_2} \hat{\mathbb{Q}}_2 \rightarrow 0
\]
defined by $p_1(f) = f(9w) - f(w)$ and $p_2(\sum a_i w^i) = a_0$ is an injective resolution of $\hat{\mathbb{Z}}_2$.

(The left $\tilde{C}_n$ comodule structure of $\hat{\mathbb{Z}}_2$ and $\hat{\mathbb{Q}}_2$ is that defined by $\eta_L$.)

(The factor $9 = 2^3 + 1$ occurs here because it is a generator of
\[
(1 + 2^3 \hat{\mathbb{Z}}_2)/(1 + 2^2 \hat{\mathbb{Z}}_2)
\]
for $n \geq 4$.)

Proof. If $p_1(\sum a_i w^i) = \sum_i a_i (9^i - 1) \cdot w^i = 0$ then $a_i = 0$ for $i \neq 0$ and the integrality condition for $\tilde{C}_n$ shows that $a_0 \in \hat{\mathbb{Z}}_2$. Thus, $\ker(p_1) = \text{Im}(p_0)$.
The fact that the polynomials \((w^{2^k} - 1)/2^{k+3}\) are in \(\hat{C}_n\) shows that \(p_2\) is surjective.

It remains to verify that \(\ker(p_2) = \text{Im}(p_1)\). Suppose that
\[
f = \sum_{i \neq 0} a_i w^i \in \ker(p_2).
\]
For any \(a \in \hat{Q}_2\), the polynomial
\[
g(w) = a + \sum_{i \neq 0} \frac{a_i w^i}{9^i - 1}
\]
is mapped to \(f\) by \(p_1\). The question is whether \(a\) can be chosen so that \(g \in \hat{C}_n\). Choose \(a\) so that \(g(1) = 0\). Since \(p_1(g) = g(9w) - g(w) \in \hat{C}_n\), it follows by induction on \(k\) that \(g(9^k) \in A\) for any \(k\), and this is enough to imply \(g \in \hat{C}_n\). To see this, first note that there exists \(m\) such that \(2^m \cdot g \in \hat{Z}_2[w, w^{-1}]\), and such that if \(a, b \in \hat{Z}_2\), then \(g(b) \in \hat{Z}_2\). However, \((1 + 2^3\hat{Z}_2)/(1 + 2^m\hat{Z}_2)\) is cyclic, generated by 9. Thus, if \(a \in 1 + 2^3\hat{Z}_2\), then \(a \equiv 9^k \mod 2^m\) for some \(k\) and so \(g(a) \in \hat{Z}_2\).

Corollary 16.
(a) \[
\text{Ext}^s_{\hat{C}_n}(\hat{Z}_2, \hat{Z}_2) = \begin{cases} 
\hat{Z}_2, & s = 0, \\
\hat{Q}_2/\hat{Z}_2, & s = 2, \\
0, & \text{otherwise}; 
\end{cases}
\]
(b) for \(n \neq 0\)
\[
\text{Ext}^s_{\hat{C}_n}(\hat{Z}_2, \hat{Z}_2) = \begin{cases} 
\hat{Z}/2^{d(n)}\hat{Z}, & s = 1, \\
0, & \text{otherwise}; 
\end{cases}
\]
(c) \[
\text{Ext}^s_{\hat{C}_n}(\hat{Z}_2, \hat{Z}/2\hat{Z}) = \begin{cases} 
\hat{Z}/2\hat{Z}, & s = 0, 1, \\
0, & \text{otherwise}. 
\end{cases}
\]

Here \(d(n)\) is the largest integer such that \(2^{d(n)}\) divides \(2^3 \cdot n\).

Proof. \(\text{Ext}^s_{\hat{C}_n}(\hat{Z}_2, \hat{Z}_2)\) is the cohomology of the complex
\[
\hat{Z}_2 \square_{\hat{C}_n} \hat{C}_n \to \hat{Z}_2 \square_{\hat{C}_n} \hat{C}_n \to \hat{Z}_2 \square_{\hat{C}_n} \hat{Q}_2 \to 0
\]
If \(n = 0\) this complex is
\[
\hat{Z}_2 \xrightarrow{0} \hat{Z}_2 \xrightarrow{1} \hat{Q}_2 \to 0
\]
and, if \(n \neq 0\)
\[
\hat{Z}_2 \xrightarrow{9^{n-1}} \hat{Z}_2 \to 0 \to 0.
\]
These account for (a) and (b), since the highest power of 2 dividing $9^n - 1$ is $2^{d(n)}$. For (c), we are interested in the cohomology of

$$\mathbb{Z}_2 \mathcal{C}_n \cong A \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}_2 \mathcal{C}_n \cong A \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}_2 \mathcal{C}_n \cong A \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$  

This complex is, for any $n$, 

$$\mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$  

Combining these results with Proposition 14 and Corollary 13, we obtain

**Corollary 17.** The $E_2$ term of the spectral sequence of Proposition 14 is

$$E_2^{s,t} = \begin{cases} 
\mathbb{Z}_2, & \text{if } (s,t) = (0,0), \ n = 0, \\
\mathbb{Q}_2/\mathbb{Z}_2, & \text{if } (s,t) = (2,0), \ n = 0, \\
\mathbb{Z}/2^{d(m)}\mathbb{Z}, & \text{if } (s,t) = (1,0), \ n = 2m = 0, \\
\mathbb{Z}/2\mathbb{Z}, & \text{if } (s,t) = (0,2t'), \ n = 2m, \text{ or } (1,2t'), \\
0, & \text{otherwise}.
\end{cases}$$

**Corollary 18.**

$$\text{Ext}_K^{s,t}(\pi_*, K, \pi_* K) = \begin{cases} 
\mathbb{Z}(2), & \text{if } (s,t) = (0,0), \\
\mathbb{Z}/2^\infty\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } (s,t) = (2,0), \\
\mathbb{Z}/2^{d(m)}\mathbb{Z}, & \text{if } (s,t) = (1,2m) \neq (1,0), \\
\mathbb{Z}/2\mathbb{Z}, & \text{if } (s,t) = (s,2t') \neq (2,0), \ s \geq 2, \\
0, & \text{otherwise}.
\end{cases}$$

2.2. The odd primary Kervaire invariant elements. The Hopf algebroid $(V_A, V_A T)$ is constructed using isomorphisms of $A$-typical formal $A$-modules. If $A = \mathbb{Z}(p)$, then one obtains $(V, V T) = (BP_*, BP_* BP)$, the Hopf algebroid of Brown-Peterson homology. If $A$ is a $Z_{(p)}$ algebra as in the case $A = \mathbb{Z}_p[\zeta]$ with which we are concerned, then a formal $A$-module is also a formal $Z_{(p)}$ module. Thus, we obtain, as in [R3, 3.11], as map of Hopf algebroids

$$\Psi: (V, V T) \rightarrow (V_A, V_A T).$$

Composing this with the generalized Conner-Floyd map

$$\Phi: (V_A, V_A T) \rightarrow (E_A, E_A T)$$

of [J] and with $\rho: (E_A, E_A T) \rightarrow (E_A, E_A T)$ we obtain a map

$$\chi: (V, V T) \rightarrow (E_A, E_A T)$$

and so a map in cohomology

$$\chi^*: H^*VT \rightarrow H^*(E_A T).$$

We will show that a family of interesting elements in $H^*VT$, the odd primary Kervaire invariant elements, have nonzero image under this map.
Recall that \((V, VT)\) has the description
\[
V = \mathbb{Z}_{(p)}[v_1, v_2, \ldots], \quad VT = V[t_1, t_2, \ldots],
\]
and that \(V, VT\) are graded with \(\deg(v_i) = \deg(t_i) = 2(p^i - 1)\). The elements
\[h_0, b_i \in H^{1, 2(p-1)}(VT), \quad H^{2, 2(p-1)p^i+1}(VT),\]
respectively, are represented in the cobar complex of \(VT\) by \(h_0 = [t_1]\) and
\[
b_i = \frac{1}{p} \sum_{j=1}^{p^{i+1}-1} (p_j^{i+1})[t_1^i \otimes t_1^{p^{i+1}-j}].
\]

Our result is

**Proposition 19.** All monomials in \(h_0, b_i, \ i = 0, 1, 2, \ldots\), have nonzero image in \(H^*(E_A T)\) under \(\chi^*\).

**Proof.** It is straightforward to describe the map of cobar complexes induced by \(\chi\). We also need, however, a method of identifying cohomologically nontrivial elements in the cobar complex of \(E_A T\) or \(\overline{C}_n\). For this we define a chain map from the cobar complex of \(\overline{C}_n\) to the complex described in §1, Lemma 12.

Recall from [R4, A1.2.11] that the cobar resolution of \(A\) as a \(\overline{C}_n\) comodule has as its \(s\)th term \(\overline{C}_n \otimes (\ker(\epsilon))^{\otimes s}\) and that the differential is given by
\[
d(\gamma_0 \otimes \cdots \otimes \gamma_s) = \sum_{i=0}^{s} (-1)^i \gamma_0 \otimes \cdots \otimes \psi(\gamma_i) \otimes \cdots \otimes \gamma_s + (-1)^{s+1} \gamma_0 \otimes \cdots \otimes \gamma_s.
\]

If we identify elements of
\[
\overline{C}_n \otimes (\ker(\epsilon))^{\otimes s} \subseteq \overline{C}_n^{\otimes s+1} = \text{Hom}_A(A[\mathbb{Z}/p\mathbb{Z}], A)^{\otimes s+1}
\]
with multilinear maps from \(A[\mathbb{Z}/p\mathbb{Z}]^{s+1}\) to \(A\), then the differential becomes
\[
df(w_0, \ldots, w_{s+1}) = \sum_{i=0}^{s} (-1)^i f(w_0, \ldots, w_i \cdot w_{i+1}, \ldots, w_{s+1})
\]
\[+ (-1)^{s+1} f(w_0, \ldots, w_s).
\]

Using this identification we define a chain map, \(R\), from the cobar resolution of \(A\) over \(\overline{C}_n\) to the resolution described in Lemma 12.

\[
R(f)(w) = \left\{
\begin{array}{ll}
f(w, \zeta) & \text{if } s = 1, \\
\sum_{i_1, \ldots, i_{s-1} = 1}^{p-1} f(w, \zeta^{i_1}, \zeta, \zeta^{i_2}, \ldots, \zeta), & \text{s odd,} \\
\sum_{i_1, \ldots, i_{s-1} = 1}^{p-1} f(w, \zeta, \zeta^{i_1}, \zeta, \zeta^{i_2}, \ldots, \zeta^{i_{s-2}}), & \text{s even.}
\end{array}
\right.
\]

Applying \(A \square_{\overline{C}_n} (\ _\ )\) we obtain a map from the cobar complex of \(\overline{C}_n\) to the complex of Corollary 13. We denote this map by \(R\) as well. It is given by

\[
R(f) = \left\{
\begin{array}{ll}
f(\zeta), & s = 1, \\
\sum f(\zeta^{i_1}, \zeta, \zeta^{i_2}, \ldots, \zeta), & s \text{ odd,} \\
\sum f(\zeta^{i_1}, \zeta, \zeta^{i_2}, \ldots, \zeta^{i_s}), & s \text{ even.}
\end{array}
\right.
\]
Under the composition $\Phi \circ \Psi$, the elements $h_0$ and $b_i$ are mapped to $(w - 1)/\pi$ and

$$\frac{1}{p} \sum_{j=1}^{p^{i+1}-1} \left( \binom{p^{i+1}}{j} \left( \frac{w - 1}{\pi} \right)^j \otimes \left( \frac{w - 1}{p} \right)^{p^{i+1} - j} \right)$$

in the cobar complex of $C_n$. Under the composition $R \circ \rho$, these are mapped to $1$ and $\frac{m}{p - m}$ respectively. We denote the latter element of $A$ by $k_t$. This series of maps will send the monomial $h_0^{i_1} b_1^{i_2} \cdot \cdot \cdot b_m^{i_m}$ to $k_1^{i_1} \cdot \cdot \cdot k_m^{i_m}$. Showing that $k_t \equiv 0 \bmod \pi$ will, therefore, complete the proof of Proposition 19.

$$\frac{1}{p} \sum_{j=1}^{p^{i+1}-1} \left( \left( \frac{\zeta^j - 1}{\pi} + 1 \right)^{p^{i+1}} - \left( \frac{\zeta^j - 1}{\pi} \right)^{p^{i+1}} - 1 \right)$$

respectively. We denote the latter element of $A$ by $k_t$. This series of maps will send the monomial $h_0^{i_1} b_1^{i_2} \cdot \cdot \cdot b_m^{i_m}$ to $k_1^{i_1} \cdot \cdot \cdot k_m^{i_m}$. Showing that $k_t \equiv 0 \bmod \pi$ will, therefore, complete the proof of Proposition 19.

$$\frac{1}{p} \sum_{j=1}^{p^{i+1}-1} \left( \left( \frac{\zeta^j - 1}{\pi} + 1 \right)^{p^{i+1}} - \left( \frac{\zeta^j - 1}{\pi} \right)^{p^{i+1}} - 1 \right)$$

$$= \frac{1}{p} \sum_{j=1}^{p^{i+1}-1} \left( \binom{p^{i+1}}{j} \left( \frac{\zeta^j - 1}{\pi} \right)^{p^{i+1}} \right)$$

$$\equiv \frac{1}{p} \sum_{j=1}^{p^{i+1}-1} \left( \binom{p^{i+1}}{j} \left( \frac{\zeta^j - 1}{\pi} \right)^{p^{i+1}} \right) \bmod \pi$$

$$\equiv \frac{1}{p} \sum_{j=1}^{p^{i+1}-1} \left( \binom{p^{i+1}}{j} \left( \frac{\zeta^j - 1}{\pi} \right)^{p^{i+1}} \right) \bmod \pi$$

$$\equiv \frac{1}{p} \sum_{j=1}^{p^{i+1}-1} \left( \binom{p^{i+1}}{j} \left( \frac{\zeta^j - 1}{\pi} \right)^{p^{i+1}} \right) \bmod \pi$$

$$\equiv \frac{1}{p} \sum_{k=1}^{p^{i+1}-1} \left( \binom{p^{i+1}}{k} \left( \frac{\zeta^j - 1}{\pi} \right)^{p^{i+1}} \right) \bmod \pi$$

$$\equiv \frac{1}{p} \sum_{k=1}^{p^{i+1}-1} \left( \binom{p^{i+1}}{k} \left( \frac{\zeta^j - 1}{\pi} \right)^{p^{i+1}} \right) \bmod \pi$$

$$\equiv \frac{1}{p} \sum_{k=1}^{p^{i+1}-1} \left( \binom{p^{i+1}}{k} \left( \frac{\zeta^j - 1}{\pi} \right)^{p^{i+1}} \right) \bmod \pi$$

In these congruences, we have used the fact that $\binom{p^{i+1}}{k}$ is divisible by $p^2$ unless $k$ is divisible by $p^i$, that $\binom{p^{i+1}}{k p^i} \equiv \binom{p}{k} \bmod p$, that since $\zeta = \pi + 1$, $((\zeta^j - 1)/\pi) \equiv i \bmod \pi$, and, finally, that $\sum_{j=1}^{p-1} j^k \equiv 0 \bmod p$ if $k < p - 1$ and that $\sum_{j=1}^{p-1} j^{p-1} \equiv -1 \bmod p$. 

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