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SIMILARITY ORBITS AND THE RANGE OF THE GENERALIZED DERIVATION $X \to MX - XN$

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ABSTRACT. If M and N are bounded operators on infinite dimensional complex Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, let $\tau(X) = MX - XN$ for X in $\mathscr{L}(\mathscr{K}, \mathscr{H})$. The closure of the range of τ is characterized when M and N are normal. There is a close connection between the range of τ and operators C for which $\left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$ is in the closure of the similarity orbit of $\left[\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}\right]$. This latter set is characterized and compared with the closure of the range of τ.

1. INTRODUCTION

Let M and N be bounded operators on infinite dimensional complex Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively. The generalized derivation (or Rosenblum operator) τ_{MN} is defined by $\tau_{MN}(X) = MX - XN$. It is an operator on the space of linear operators $\mathscr{L}(\mathscr{K}, \mathscr{H})$, and hereafter it will simply be denoted by τ. The range $\mathscr{R}(\tau)$ of such an operator was studied by Fialkow [2] and previously by M. Rosenblum [7], where it was proved that if M and N are Hermitian, then C belongs to $\mathscr{R}(\tau)$ if and only if the operator matrices $\left[\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$ are similar. Previously this was shown when M and N are finite matrices over a field [8], and subsequently when M and N are normal operators [9]. One direction in these results is immediate, for observe that $\left[\begin{smallmatrix} I & -X \\ 0 & I \end{smallmatrix}\right]\left[\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}\right]\left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right] = \left[\begin{smallmatrix} M & MX-XN \\ 0 & N \end{smallmatrix}\right]$.

Hereafter it will be assumed that M and N are normal. The present paper investigates what happens when we pass to closures. That is, what is the relationship between those C in $\mathscr{R}(\tau)^-$ and those C for which $\left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$ is in the closure of the similarity orbit of $\left[\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}\right]$. This latter set we denote by $\mathscr{S}_{co}$. Clearly we always have $\mathscr{R}(\tau)^- \subset \mathscr{S}_{co}$, even if M and N are not normal. Also note in passing that $\left[\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}\right]$ is always in the closure of the orbit of $\left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$, because $\left[\begin{smallmatrix} I & 0 \\ 0 & (1/\varepsilon)I \end{smallmatrix}\right]\left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]\left[\begin{smallmatrix} I & 0 \\ 0 & \varepsilon I \end{smallmatrix}\right] = \left[\begin{smallmatrix} M & \varepsilon C \\ 0 & N \end{smallmatrix}\right]$.

The strategy in this paper will be to characterize $\mathscr{R}(\tau)^-$ (see Theorem 2.2), characterize $\mathscr{S}_{co}$ (see Theorem 3.6), and then compare the results (see

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Theorem 4.1). The investigation of $\mathscr{S}_{co}$ in §3 relies heavily on the major effort of D. A. Herrero et al. to determine closures of similarity orbits [1, 3].

Notation to be used includes $\sigma(M)$, $\sigma_p(M)$, and $\sigma_e(M)$ for the spectrum, point spectrum, and essential spectrum of M, respectively. Also, let E_M denote the spectral measure associated with the normal operator M. Thus, if S is any Borel measurable set in the plane, then $E_M(S)$ is a projection that reduces M. The spectrum of the restriction of M to the range of $E_M(S)$ is contained in S^-. The notation $E_M(\lambda)$ will be used in place of $E_M(\{\lambda\})$ when S consists of a single point λ. Throughout, Δ_n will designate the open disk with center λ and radius $1/n$.

In what follows, frequent use will be made of what is sometimes called Rosenblum's corollary.

Theorem 1.1 (Rosenblum's theorem, apparently first proved by D. C. Kleinecke —see [4, 6]). $\sigma(\tau) = \sigma(M) - \sigma(N)$.

Corollary 1.2 (Rosenblum's corollary—see [4]). *If $\sigma(M) \cap \sigma(N) = \varnothing$, then τ is invertible. In particular, τ is onto.*

2. The closure of the range of τ

In this section we describe the closure of the range of the Rosenblum operator $\tau\colon X \to MX - XN$ when M and N are normal. An immediate consequence of this will be Fialkow's characterization [2, Proposition 4.2] of those τ that have dense range.

We begin with a lemma.

Lemma 2.1. *If the operators A_n converge to zero in the strong operator topology and C is compact, then the operators A_nC converge to zero in norm.*

Proof. First observe that the conclusion holds if C has finite rank, for in that case one can apply A_n to each of a finite number of vectors in an orthonormal basis for the range of C. For the general case, let K be a bound for $\{A_n\}$ and write $C = PC + (I-P)C$ where P is a finite-rank projection and $\|(I-P)C\| < \varepsilon/2K$. Select n so that $\|A_nP\| < \varepsilon/2\|C\|$. As a result, $\|A_nC\| < \varepsilon$.

Theorem 2.2. *If M and N are normal, then the following are equivalent*:

(a) $C \in \mathscr{R}(\tau)^-$;
(b) (i) $\lambda \in \sigma_p(M) \cap \sigma_p(N) \Rightarrow E_M(\lambda)CE_N(\lambda) = 0$, *and*
(ii) $\lambda \in \sigma_e(M) \cap \sigma_e(N) \Rightarrow \|E_M(\Delta_n)CE_N(\Delta_n)\| \to 0$;
(c) $\lambda \in \sigma(M) \cap \sigma(N) \Rightarrow \|E_M(\Delta_n)CE_N(\Delta_n)\| \to 0$.

Proof. (a) $\Rightarrow$ (b): If $\lambda \in \sigma_p(M) \cap \sigma_p(N)$, then for any X we have

$$\begin{aligned}\|MX - XN - C\| &\geq \|E_M(\lambda)[(M-\lambda)X - X(N-\lambda) - C]E_N(\lambda)\| \\ &= \|E_M(\lambda)CE_N(\lambda)\|.\end{aligned}$$

Consequently, if $C \in \mathscr{R}(\tau)^-$, we conclude that $E_M(\lambda)CE_N(\lambda) = 0$.

If $\lambda \in \sigma_e(M) \cap \sigma_e(N)$, then for any X we have

$$\begin{aligned}\|MX - XN - C\| &\geq \|E_M(\Delta_n)[(M-\lambda)X - X(N-\lambda) - C]E_N(\Delta_n)\| \\ &\geq \|E_M(\Delta_n)CE_N(\Delta_n)\| - \|E_M(\Delta_n)(M-\lambda)XE_N(\Delta_n)\| \\ &\quad - \|E_M(\Delta_n)X(N-\lambda)E_N(\Delta_n)\| \\ &\geq \|E_M(\Delta_n)CE_N(\Delta_n)\| - 2\|X\|/n\end{aligned}$$

since both $\|E_M(\Delta_n)(M-\lambda)\|$ and $\|(N-\lambda)E_N(\Delta_n)\|$ are less than or equal to $1/n$. Thus, if $C \in \mathscr{R}(\tau)^-$, $\|E_M(\Delta_n)CE_N(\Delta_n)\|$ can be made arbitrarily small by first selecting X to make $\|MX - XN - C\|$ small and then picking n to make $2\|X\|/n$ small.

(b) $\Rightarrow$ (c): Assume (b) and suppose that $\lambda \in \sigma(M) \cap \sigma(N)$. If $\lambda \in \sigma_e(M) \cap \sigma_e(N)$, there is nothing to prove. Otherwise λ is not in the essential spectrum of, say (without loss of generality), N. This means that λ is an isolated eigenvalue of finite multiplicity for N. In view of (b)(i), we might as well also assume that λ is not an eigenvalue of M. Thus $E_M(\lambda) = 0$, and when n is large we have $E_N(\Delta_n) = E_N(\lambda)$. Now, $E_M(\Delta_n)CE_N(\Delta_n) = [E_M(\Delta_n) - E_M(\lambda)]CE_N(\Delta_n)$, which is $E_M(\Delta_n - \{\lambda\})CE_N(\lambda)$ for large n. This goes to zero with n by Lemma 2.1, because $CE_N(\lambda)$ has finite rank, and the projections $E_M(\Delta_n - \{\lambda\})$ decrease strongly to zero.

(c) $\Rightarrow$ (a):

Case I. Assume $\sigma(M) \cap \sigma(N)$ is finite. Use (c) to cover $\sigma(M) \cap \sigma(N)$ with open disks $D_1, D_2, \ldots, D_k$ with nonintersecting closures such that

$$\|E_M(D_i)CE_N(D_i)\| < \varepsilon \quad \text{for } 1 \leq i \leq k .$$

Also set $D_0 = (\bigcup D_i)'$. The D's are disjoint measurable sets that fill the complex plane.

Decompose $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_k$ and $\mathscr{K} = \mathscr{K}_0 \oplus \mathscr{K}_1 \oplus \cdots \oplus \mathscr{K}_k$, where $\mathscr{H}_i$ and $\mathscr{K}_i$ are the ranges of the projections $E_M(D_i)$ and $E_N(D_i)$, respectively. These decompositions reduce M and N. Then for $0 \leq i, j \leq k$, let C_{ij} represent the compression $E_M(D_i)CE_N(D_j)$, and let M_i and N_j represent the restrictions of M and N to the ranges of $E_M(D_i)$ and $E_N(D_j)$, respectively. For $i \neq j$, consider the operator $M_iX_{ij} - X_{ij}N_j$. Since $\sigma(M_i) \cap \sigma(N_j) \subseteq \overline{D}_i \cap \overline{D}_j = \varnothing$, Rosenblum's corollary tells us there is a bounded X_{ij} for which $MX_{ij} - X_{ij}N = C_{ij}$. Likewise, there is a bounded X_{00} for which $M_0X_{00} - X_{00}N_0 = C_{00}$, since the spectra of M and N do not intersect on $\overline{D}_0$ $(= D_0)$. Finally, let $X_{ii} = 0$ for $i > 0$.

Letting $X = (X_{ij})$, we obtain

$$MX - XN - C = \begin{bmatrix} 0 & 0 & 0 & \cdots & \\ 0 & C_{11} & 0 & \cdots & \\ 0 & 0 & C_{22} & & \\ \vdots & \vdots & & \ddots & \\ & & & & C_{kk} \end{bmatrix}$$

an operator whose norm is $\max \|C_{ii}\| = \max \|E_M(D_i)CE_N(D_i)\|$, which does not exceed ε. Thus, $C \in \mathscr{R}(\tau)^-$.

Case II. Assume $\sigma(M) \cap \sigma(N)$ is contained in an arc. Cover this intersection with open disks D_i, each centered on the arc, so that $\|E_M(D_i)CE_N(D_i)\| < \varepsilon$ for $1 \le i \le k$. To obtain disjoint regions let $S_1 = D_1$, $S_2 = D_2 \backslash D_1$, $S_3 = D_3 \backslash (D_1 \cup D_2)$, etc. Also let $S_0 = (\bigcup S_i)'$. Decompose $\mathscr{H}$ and $\mathscr{K}$ as in Case I. Observe that $\sigma(M_i) \cap \sigma(N_j) \subset \sigma(M) \cap \overline{S}_i \cap \sigma(N) \cap \overline{S}_j = \sigma(M) \cap \sigma(N) \cap \overline{S}_i \cap \overline{S}_j$. This intersection is finite if $i \ne j$, so by Case I we can obtain X_{ij} so that $\|M_i X_{ij} - X_{ij} N_j - C_{ij}\| < \varepsilon/(k+1)^2$. Also as in Case I, there exists X_{00} for which $M_0 X_{00} - X_{00} N_0 - C_{00} = 0$. Finally, set $X_{ii} = 0$ for $i > 0$. Let $X = (X_{ij})$ and observe that $MX - XN - C$ is $dg(0, C_{11}, C_{22}, \dots, C_{kk})$ plus an operator with norm less than $(k+1)^2 \varepsilon/(k+1)^2$. Consequently $\|MX - XN - C\| < 2\varepsilon$.

Case III. Now let $\sigma(M) \cap \sigma(N)$ be arbitrary. Proceed as in Case II. Begin with a covering of $\sigma(M) \cap \sigma(N)$ with n open disks and from them construct disjoint sets, $S_0, \dots, S_n$, of which any two may have closures intersecting in an arc. For $i \ne j$, use Case II to obtain X_{ij} so that $\|M_i X_{ij} - X_{ij} N_j - C_{ij}\| < \varepsilon/(n+1)^2$; define X_{00} so that $M_0 X_{00} - X_{00} N_0 - C_{00} = 0$ (the spectra of M_0 and N_0 do not intersect); and let $X_{ii} = 0$ for $i > 0$. As a result $\|MX - XN - C\|$ is less than $\varepsilon + (n+1)^2 \varepsilon/(n+1)^2 = 2\varepsilon$. Thus $C \in \mathscr{R}(\tau)^-$.

Comment. For a finite dimensional version of this theorem see [6, Theorem 4.6].

Corollary 2.3 (Fialkow [2]). *$\mathscr{R}(\tau)$ is dense if and only if both $\sigma_p(M) \cap \sigma_p(N)$ and $\sigma_e(M) \cap \sigma_e(N)$ are empty.*

Proof. Clearly if both intersections are empty, then condition (b) implies the range of τ is dense.

Conversely, if there is a λ in both point spectra, select unit vectors f_λ and g_λ in the eigenspaces $E_M(\lambda)$ and $E_N(\lambda)$, respectively, and consider the rank-one partial isometry C for which $Cg_\lambda = f_\lambda$. A glance at condition (b) shows that C is not in $\mathscr{R}(\tau)^-$.

Likewise, if there is a λ in both essential spectra, select an orthonormal sequence $\{f_n\}$ such that f_n is in $E_M(\Delta_n)$ and an orthonormal sequence $\{g_n\}$ with g_n belonging to $E_N(\Delta_n)$. Define a partial isometry C by $Cg_n = f_n$ and $C \equiv 0$ on the orthogonal complement of the closed linear span $\bigvee\{g_n\}$. Any C so defined does not belong to $\mathscr{R}(\tau)^-$, since $\|E_M(\Delta_n)CE_N(\Delta_n)\| \ge \|f_n\| = 1$. Thus, if either intersection is not empty, the range of τ is not dense.

3. Similarity orbits

Again assume M and N are normal. In this section we wish to characterize $\mathscr{S}_{co}$—those operators C for which $\left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$ is in the closure of the similarity orbit of $\left[\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}\right]$. Intense study of the closures of similarity orbits was initiated by D. A. Herrero. Much of the resulting theory is contained in the two volumes

[1, 3]. Fortunately, some obstacles vanish in the case of normal operators, since normal operators do not possess "megambic" points. Nevertheless, even with normal operators, the theory is far from trivial. In order to state necessary and sufficient conditions, a fair amount of notation must now be introduced (see [1, 3] for a fuller presentation).

Use $\sigma(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$, and $\sigma_{re}(T)$ to denote the spectrum, the essential spectrum, the left essential spectrum, and the right essential spectrum of T. The Wolf spectrum, $\sigma_{lre}(T)$, is simply the intersection $\sigma_{le}(T) \cap \sigma_{re}(T)$. Its complement, the semi-Fredholm domain, will be denoted by $\rho_{\text{s-F}}(T)$. Recall that λ belongs to the semi-Fredholm domain of T if $\lambda - T$ is semi-Fredholm, that is, if $\lambda - T$ has closed range and either $\lambda - T$ or $(\lambda - T)^*$ has finite nullity. For such λ the semi-Fredholm index is defined by

$$\operatorname{ind}(\lambda - T) = \operatorname{nul}(\lambda - T) - \operatorname{nul}(\lambda - T)^* .$$

Also define

$$\min \operatorname{ind}(\lambda - T) = \min\{\operatorname{nul}(\lambda - T), \operatorname{nul}(\lambda - T)^*\} .$$

By $\sigma_0(T)$ we denote the normal eigenvalues of T, that is, spectral points which are isolated in $\sigma(T)/\sigma_e(T)$. Equivalently, λ is a normal eigenvalue of T if λ is isolated in the spectrum and the associated Riesz spectral invariant subspace $\mathscr{H}(\lambda; T)$ is finite dimensional.

Within $\mathscr{L}(\mathscr{H})$, the algebra of bounded operators on a Hilbert space $\mathscr{H}$, the compact operators $\mathscr{K}(\mathscr{H})$ form a closed, two-sided ideal. Consequently the quotient algebra (in this case the Calkin algebra) $\mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ is a C^*-algebra. For T in $\mathscr{L}(\mathscr{H})$, let $\widetilde{T}$ be the corresponding element in the Calkin algebra. If λ is an isolated point in $\sigma_e(T)$, and ρ is any faithful, unital $*$-representation of the Calkin algebra, then the Riesz decomposition theorem [5, Chapter XI] implies that $\rho(\widetilde{T})$ is similar to $(\lambda + Q_\lambda) \oplus R_\lambda$, where λ is not in the spectrum of R_λ, and Q_λ is quasinilpotent. Single out those λ for which Q_λ is nilpotent by defining

$$\sigma_{ne}(T) = \{\lambda | \lambda \text{ is isolated in } \sigma_e(T), \text{ and } Q_\lambda \text{ is nilpotent}\} .$$

Finally, define

$$k(\lambda; \widetilde{T}) = \begin{cases} 0, & \text{if } \lambda \text{ is not an isolated point in } \sigma_e(T); \\ n, & \text{if } \lambda \in \sigma_{ne}(T), \text{ and } Q_\lambda \text{ is nilpotent of order } n; \\ \infty, & \text{if } \lambda \text{ is isolated in } \sigma_e(T), \text{ but } Q_\lambda \text{ is not nilpotent.} \end{cases}$$

Observe that $k(\lambda; \widetilde{T})$ can only be 0 or 1 if T is normal.

Now we are in a position to state the theorem of Apostol et al. on closures of similarity orbits—here adapted for normal operators.

Theorem 3.1 (cf. [1, Theorem 9.1]). *If T is normal, then A is in the closure of the similarity orbit of T if and only if*

(S) $\sigma_0(A) \subset \sigma_0(T)$ *and each component of* $\sigma_{lre}(A) \backslash \sigma_{ne}(A)$ *intersects* $\sigma_e(T) \backslash \sigma_{ne}(T)$;

(F) $\rho_{\text{s-F}}(A) \subset \rho_{\text{s-F}}(T)$, $\operatorname{ind}(\lambda - A) = 0$ *for all* $\lambda \in \rho_{\text{s-F}}(A)$, *and*

$$\min \operatorname{ind}(\lambda - A) \geq \min \operatorname{ind}(\lambda - T) \quad \textit{for all } \lambda \in \rho_{\text{s-F}}(A) \ ;$$

and

(A) $\dim \mathscr{H}(\lambda; A) = \dim \mathscr{H}(\lambda; T)$ *for all* $\lambda \in \sigma_0(A)$, $k(\lambda; \widetilde{A}) \leq k(\lambda; \widetilde{T})$ *for all* $\lambda \in \mathbf{C}$, *and if* $\lambda \in \sigma_{ne}(A)$ *is an isolated point in* $\sigma(A)$ *then the Riesz spectral invariant subspace* $\mathscr{H}(\lambda; A)$ *is an eigenspace for* A.

In the present situation with $T = \left[\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}\right]$ and $A = \left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$, the above conditions will simplify considerably. First, observe that the corresponding parts of the spectra are the same.

Lemma 3.2. *If* M *and* N *are normal, then* $\left[\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$ *have the same spectrum, the same left essential spectrum, and the same right essential spectrum.*

Proof. The proofs of these statements are straightforward and similar. To illustrate one of these we show that the left essential spectra are identical. If $\left[\begin{smallmatrix} R & S \\ T & U \end{smallmatrix}\right]\left[\begin{smallmatrix} M-\lambda & 0 \\ 0 & N-\lambda \end{smallmatrix}\right]$ is a compact perturbation of the identity, then it is readily seen that $\left[\begin{smallmatrix} R & -RCU \\ 0 & U \end{smallmatrix}\right]\left[\begin{smallmatrix} M-\lambda & C \\ 0 & N-\lambda \end{smallmatrix}\right]$ is also. Conversely, suppose that $\left[\begin{smallmatrix} R & S \\ T & U \end{smallmatrix}\right]\left[\begin{smallmatrix} M-\lambda & C \\ 0 & N-\lambda \end{smallmatrix}\right] = I + K$ for some compact K. Then $R(M-\lambda) = I + K_1$. So $M - \lambda$, being left invertible in the Calkin algebra, has finite dimensional kernel M_λ and closed range $\mathscr{R}(M-\lambda)$. Since $T(M-\lambda) = K_2$, we know the restriction of T to the range of $M-\lambda$ is compact. By the normality of M, we have the decomposition $\mathscr{H} = M_\lambda \oplus \mathscr{R}(M-\lambda)$. Thus T itself is compact, because M_λ, i.e., $\ker(M-\lambda)$, is finite dimensional. Consequently, $U(N-\lambda) = I + K_3 - TC = I + K_4$, and it follows that $\left[\begin{smallmatrix} R & 0 \\ 0 & U \end{smallmatrix}\right]\left[\begin{smallmatrix} M-\lambda & 0 \\ 0 & N-\lambda \end{smallmatrix}\right]$ is identity plus compact.

Also observe that the normality of $T = \left[\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}\right]$ implies that $\min \operatorname{ind}(\lambda - T) = \operatorname{nul}(\lambda - M) + \operatorname{nul}(\lambda - N)$; $\dim \mathscr{H}(\lambda; T) = \operatorname{nul}(\lambda - M) + \operatorname{nul}(\lambda - N)$; and $k(\lambda; \widetilde{T})$ can only assume the values 0 and 1. It will be shown in the proof of Lemma 3.5 that $k(\lambda; \widetilde{A})$ can only assume the values 0, 1, and 2 when $A = \left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$, from which it follows that $\sigma_{ne}(A) = \sigma_{ne}(T)$.

Now we may restate Theorem 3.1 as it applies to $\left[\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$. Notice that condition (S) is satisfied for all C in view of the preceding comments and Lemma 3.2.

Theorem 3.3. *Assume* M *and* N *are normal. Let* $A = \left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$ *and let* $T = \left[\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}\right]$. *Then* A *is in the closure of the similarity orbit of* T *if and only if*

(F′) $\operatorname{ind}(\lambda - A) = 0$ *and* $\min \operatorname{ind}(\lambda - A) \geq \operatorname{nul}(\lambda - M) + \operatorname{nul}(\lambda - N)$ *for all* $\lambda \in \sigma(T)$ *which are isolated eigenvalues of finite multiplicity*;

(A′) $\dim \mathscr{H}(\lambda; A) = \operatorname{nul}(\lambda - M) + \operatorname{nul}(\lambda - N)$ *for all* $\lambda \in \sigma_0(A)$; $k(\lambda; \widetilde{A}) = 1$ *for each* λ *isolated in the essential spectrum of* A; *and* $\mathscr{H}(\lambda; A)$ *is an eigenspace for* A *whenever* $\lambda \in \sigma_{ne}(A)$ *is an isolated point in* $\sigma(A)$.

In the characterization of $\mathscr{S}_{co}$, the role of points isolated in the spectrum of A and points isolated in the essential spectrum of A is critical. We turn our attention now to these points with a pair of lemmas.

Lemma 3.4. *Suppose λ is an isolated point of the spectrum of $A = \left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$, where M and N are normal. Then $\mathscr{H}(\lambda; A)$ coincides with the eigenspace of A associated with λ if and only if $E_M(\lambda)CE_N(\lambda) = 0$.*

Proof. We know $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathscr{H}(\lambda; \left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right])$ if and only if

$$\left\| \begin{bmatrix} M-\lambda & C \\ 0 & N-\lambda \end{bmatrix}^n \begin{bmatrix} f \\ g \end{bmatrix} \right\|^{1/n} \to 0$$

(see [5, p. 424]). This condition is

$$\left\| \begin{bmatrix} (M-\lambda)^n & Q \\ 0 & (N-\lambda)^n \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \right\|^{1/n} \to 0,$$

where Q is of the form $(M-\lambda)^{n-1}C + R_n(N-\lambda)$. This, in turn, is equivalent to $\|(M-\lambda)^n f + Qg\|^{1/n} \to 0$ and $\|(N-\lambda)^n g\|^{1/n} \to 0$. The latter simply means g belongs to the eigenspace N_λ, since N is normal. When $g \in N_\lambda$ the former reduces to

$$\|(M-\lambda)^n f + (M-\lambda)^{n-1} Cg\|^{1/n} \to 0,$$

or

$$\|(M-\lambda)^{n-1}[(M-\lambda)f + Cg]\|^{1/n} \to 0.$$

By the normality of M, this is equivalent to the statement that $(M-\lambda)f + Cg$ belongs to the eigenspace M_λ. Thus, the Riesz spectral invariant subspace $\mathscr{H}(\lambda; A)$ is given by

$$\mathscr{H}(\lambda; A) = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} : g \in N_\lambda,\ (M-\lambda)f + Cg \in M_\lambda \right\},$$

while the eigenspace A_λ is given by

$$A_\lambda = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} : g \in N_\lambda,\ (M-\lambda)f + Cg = 0 \right\}.$$

These two sets are identical if and only if $Cg \in \mathscr{R}(M-\lambda)$ whenever $g \in N_\lambda$, that is, if and only if $E_M(\lambda)CE_N(\lambda) = 0$.

Lemma 3.5. *If λ is isolated in the essential spectrum of $A = \left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$, then $k(\lambda; \widetilde{A}) \le 2$. Furthermore, $k(\lambda; \widetilde{A}) = 1$ if and only if the compression $E_M(\Delta_n)CE_N(\Delta_n)$ is compact for some open disk Δ_n with radius $1/n$ and center λ.*

Proof. Assume λ is isolated in the essential spectra of normal operators M and N. Let Δ_n be the open disk with center λ and radius $1/n$. For n sufficiently large, Δ_n will contain only λ in the essential spectrum of M. It could also contain isolated eigenvalues of finite multiplicity which may converge to λ, but a compact perturbation of M will be normal and have λ as its only spectral point inside Δ_n. The same is true for N. Hence there is a compact

perturbation of $\left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$ which will have the following representation relative to $\mathscr{R}(E_M(\Delta_n))$, $\mathscr{R}(E_M(\Delta_n))^\perp$, $\mathscr{R}(E_N(\Delta_n))$, and $\mathscr{R}(E_N(\Delta_n))^\perp$:

$$\begin{bmatrix} \lambda & 0 & C_1 & C_2 \\ 0 & M_1 & C_3 & C_4 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & N_1 \end{bmatrix}.$$

This in turn is similar to

$$\left[\begin{array}{cc|cc} \lambda & 0 & C_1 & C_2 - \lambda X_2 + X_2 N_1 \\ 0 & M_1 & C_3 - M_1 X_3 + \lambda X_3 & * \\ \hline & & \lambda & 0 \\ & & 0 & N_1 \end{array}\right],$$

where the similarity is implemented by the invertible operator

$$\begin{bmatrix} I & 0 & X_1 & X_2 \\ 0 & I & X_3 & X_4 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

With the proper choice of X_2 and X_3, the expressions $C_2 - \lambda X_2 + X_2 N_1$ and $C_3 - M_1 X_3 + \lambda X_3$ vanish (the operators $X \to XN_1 - \lambda X$ and $X \to \lambda X - M_1 X$ are nonsingular since λ is not in the spectra of M_1 and N_1—Rosenblum's corollary again). Finally, a row and column interchange produce the similar operator

$$\begin{bmatrix} \lambda & C_1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & M_1 & * \\ 0 & 0 & 0 & N_1 \end{bmatrix},$$

where C_1 is the compression $E_M(\Delta_n) C E_N(\Delta_n)$.

Thus $k(\lambda; \widetilde{A})$ is at most 2, and $k(\lambda; \widetilde{A}) = 1$, if and only if C_1 is compact.

At last we are prepared to characterize $\mathscr{S}_{co}$. As usual, Δ_n designates the open disk with radius $1/n$ and center λ.

Theorem 3.6. *If M and N are normal, then $\left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$ is in the closure of the similarity orbit of $\left[\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}\right]$, if and only if*

(a) *$E_M(\lambda) C E_N(\lambda) = 0$ if λ is isolated in $\sigma(M) \cup \sigma(N)$, and*
(b) *$E_M(\Delta_n) C E_N(\Delta_n)$ is compact for all n sufficiently large if λ is isolated in $\sigma_e(M) \cup \sigma_e(N)$.*

Proof. First assume (a) and (b) hold. Suppose λ is isolated in $\sigma(M) \cup \sigma(N)$, and let M_λ and N_λ designate the respective eigenspaces. Condition (a) implies that if $(N - \lambda) g = 0$, then $Cg \in \mathscr{R}(M - \lambda)$, a closed range on which $M - \lambda$ is invertible. From this it follows that the kernel of $\left[\begin{smallmatrix} M-\lambda & C \\ 0 & N-\lambda \end{smallmatrix}\right]$ consists of all vectors $\left[\begin{smallmatrix} f - (M-\lambda)^{-1} Cg \\ g \end{smallmatrix}\right]$, where f and g are arbitrary vectors in M_λ and N_λ, respectively. Thus, the kernel has dimension equal to $\dim M_\lambda + \dim N_\lambda$. In the

same fashion, the adjoint of $\left[\begin{smallmatrix} M-\lambda & C \\ 0 & N-\lambda \end{smallmatrix}\right]$ has a kernel consisting of the vectors $\left[\begin{smallmatrix} f \\ g-(N^*-\bar{\lambda})^{-1}C^*f \end{smallmatrix}\right]$ with f in M_λ and g in N_λ, so the dimension of this kernel is also $\dim M_\lambda + \dim N_\lambda$.

This shows that $\left[\begin{smallmatrix} M-\lambda & C \\ 0 & N-\lambda \end{smallmatrix}\right]$ has index zero and minimum index matching that of $\left[\begin{smallmatrix} M-\lambda & 0 \\ 0 & N-\lambda \end{smallmatrix}\right]$ for λ in the semi-Fredholm resolvent. Thus, condition (F′) of Theorem 3.3 is satisfied.

For λ isolated in $\sigma(A)$ the Riesz space is an eigenspace (see Lemma 3.4). This implies

$$\dim \mathscr{H}(\lambda; A) = \dim \ker(A-\lambda) = \dim M_\lambda + \dim N_\lambda = \dim \mathscr{H}(\lambda; T).$$

For λ isolated in the essential spectrum, condition (b) and Lemma 3.5 show that $k(\lambda; \tilde{A}) = 1$. Thus, condition (A′) is satisfied.

On the other hand, suppose (F′) and (A′) are satisfied. Lemma 3.5 then gives us (b). If λ is isolated in $\sigma(T)$ and has finite multiplicity, then by (F′) we see that $\min \operatorname{ind}(\lambda - A) \geq \min \operatorname{ind}(\lambda - T) = \dim M_\lambda + \dim N_\lambda$, a finite value. The expressions obtained above for $\ker(\lambda - A)$ and $\ker(\bar{\lambda} - A^*)$ show that $\min \operatorname{ind}(\lambda - A)$ can reach this value only when CN_λ is in the range of $M-\lambda$, and C^*M_λ is in the range of $N^*-\bar{\lambda}$, that is, when $E_M(\lambda)CE_N(\lambda) = 0$. If λ is isolated in $\sigma(T)$ and has infinite multiplicity, then (A′) implies that $\mathscr{H}(\lambda; A)$ is an eigenspace for A. This in turn implies that $E_M(\lambda)CE_N(\lambda) = 0$ by Lemma 3.4. Thus (a) holds.

Corollary 3.7. *If M and N are normal, then $\mathscr{R}(\tau)^- \subseteq \mathscr{S}_{co}$.*

Proof. Of course, as observed in the introduction, this is a very simple result even if M and N are not normal, but it might be satisfying to derive it using the characterizations of $\mathscr{R}(\tau)^-$ and $\mathscr{S}_{co}$ given by Theorems 2.2 and 3.6. Observe that if $C \in \mathscr{R}(\tau)^-$ and λ is isolated in $\sigma(M) \cup \sigma(N)$, then $E_M(\lambda)CE_N(\lambda) = 0$ whether λ is in just one or both of the spectra. If λ is isolated in $\sigma_e(M) \cup \sigma_e(N)$, but only belongs to one of these essential spectra, then clearly $E_M(\Delta_n)CE_N(\Delta_n)$ is zero or finite rank for large n. If λ is (isolated) in both essential spectra, then Theorem 2.2 says $\|E_M(\Delta_n)CE_N(\Delta_n)\|$ goes to zero with n. Suppose k is sufficiently large that Δ_k contains only λ in $\sigma_e(M) \cup \sigma_e(N)$. Then, since Δ_k may also contain eigenvalues of finite multiplicity possibly accumulating at λ, $E_M(\Delta_k) - E_M(\Delta_n)$ and $E_N(\Delta_k) - E_N(\Delta_n)$ are finite rank projections for $n \geq k$. Now $\|E_M(\Delta_n)CE_N(\Delta_n)\| \to 0$ means

$$\|[E_M(\Delta_k) - E_M(\Delta_n)]CE_N(\Delta_n) + E_M(\Delta_k)C[E_N(\Delta_k) - E_N(\Delta_n)] \\ - E_M(\Delta_k)CE_N(\Delta_k)\| \to 0,$$

so $E_M(\Delta_k)CE_N(\Delta_k)$ is the limit of finite rank operators and is compact. Thus, C belongs to $\mathscr{S}_{co}$.

4. The relation between $\mathscr{R}(\tau)^-$ and $\mathscr{S}_{co}$

Let $\mathscr{S}_0$ be the set of C for which $\left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$ is in the similarity orbit of the Rosenblum operator $\tau\colon X \to MX - XN$. Let $\mathscr{S}_{co}$ be the set of C for which $\left[\begin{smallmatrix} M & C \\ 0 & N \end{smallmatrix}\right]$ is in the closure of that similarity orbit. In [9] it was shown that $\mathscr{R}(\tau)$ coincided with $\mathscr{S}_0$ when M and N were normal. This paper was motivated by the question: what is the relation between $\mathscr{R}(\tau)^-$ and $\mathscr{S}_{co}$? We are now in a position to answer that question. Observe that $\mathscr{S}_{co}$ is not, in general, the closure of $\mathscr{S}_0$.

Theorem 4.1. *Assume M and N are normal. Then $\mathscr{R}(\tau)^- = \mathscr{S}_{co}$, if and only if*

(a) $\lambda \in \sigma_p(M) \cap \sigma_p(N) \Rightarrow \lambda$ *is isolated in* $\sigma(M)$ *and* $\sigma(N)$, *and*
(b) $\lambda \in \sigma_e(M) \cap \sigma_e(N) \Rightarrow \lambda$ *is isolated in* $\sigma_e(M)$ *and* $\sigma_e(N)$.

Proof. Assume $C \in \mathscr{S}_{co}$ and (a) and (b) hold. If λ belongs to both $\sigma_p(M)$ and $\sigma_p(N)$, then λ is isolated in both spectra by (a). Then Theorem 3.6 implies $E_M(\lambda)CE_N(\lambda) = 0$. On the other hand, if λ belongs to both essential spectra, then (b) says it is isolated in both essential spectra. Consequently, by Theorem 3.6, there is a k such that $E_M(\Delta_n)CE_N(\Delta_n)$ is compact for $n \geq k$. Let $K = E_M(\Delta_k)CE_N(\Delta_k)$. Then for $n \geq k$,

$$\begin{aligned}
\|E_M(\Delta_n)CE_N(\Delta_n)\| &= \|E_M(\Delta_n)KE_N(\Delta_n)\| \\
&= \|[E_M(\Delta_n) - E_M(\lambda)]KE_N(\Delta_n) + E_M(\lambda)K[E_N(\Delta_n) - E_N(\lambda)] \\
&\qquad\qquad + E_M(\lambda)KE_N(\lambda)\| \\
&\leq \|[E_M(\Delta_n) - E_M(\lambda)]K\| + \|K[E_N(\Delta_n) - E_N(\lambda)]\|
\end{aligned}$$

which goes to zero with n by Lemma 2.1. By Theorem 2.2, $C \in \mathscr{R}(\tau)^-$. Since $\mathscr{R}(\tau)^-$ is always contained within $\mathscr{S}_{co}$, we have equality.

Conversely, if (a) is not satisfied, then there is a λ in both point spectra which is not isolated in both spectra. As in the proof of Corollary 2.3, select unit vectors f_λ and g_λ in the eigenspaces M_λ and N_λ, respectively, and consider the rank-one partial isometry C for which $Cg_\lambda = f_\lambda$. A glance at Theorem 3.6 and Theorem 2.2 shows that C is in $\mathscr{S}_{co}$ but is not in $\mathscr{R}(\tau)^-$.

Likewise, if (b) is not satisfied, then there is a λ in both essential spectra which is not isolated in both essential spectra. For this λ we select an orthonormal sequence $\{f_n\}$ such that f_n is in $E_M(\Delta_n)$ and an orthonormal sequence $\{g_n\}$ with g_n belonging to $E_N(\Delta_n)$. Define a partial isometry C by $Cg_n = f_n$ and $C \equiv 0$ on the orthogonal complement of the closed linear span $\bigvee\{g_n\}$. Any C so defined does not belong to $\mathscr{R}(\tau)^-$, since $\|E_M(\Delta_n)CE_N(\Delta_n)\| \geq \|f_n\| = 1$. To obtain such a C that belongs to $\mathscr{S}_{co}$ we must take care not to violate condition (a) of Theorem 3.6. This can be arranged by replacing f_n and g_n by subsequences if necessary. This completes the proof of Theorem 4.1.

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