ON COMPLETING UNIMODULAR POLYNOMIAL VECTORS
OF LENGTH THREE

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Abstract. It is shown that if $R$ is a local ring of dimension three, with $\frac{1}{2} \in R$, then a polynomial three vector $(v_0(X), v_1(X), v_2(X))$ over $R[X]$ can be completed to an invertible matrix if and only if it is unimodular. In particular, if $1/3! \in R$, then every stably free projective $R[X_1, \ldots, X_n]$-module is free.

1. INTRODUCTION

In [6] A. Suslin queries

A. Suslin's question ($S_r(R)$). Let $R$ be a local ring. If $1/r! \in R$, can every unimodular $(r + 1)$-vector over $R[X]$ be completed to an invertible matrix?

In this note we settle $S_r(R)$ when $R$ is a noetherian local ring of Krull dimension three.

Let us briefly recapitulate known results on $S_r(R)$. Let $R$ be a two dimensional noetherian local ring. A beautiful theorem of L. N. Vaserstein in [8] identifies the set $Um_3(R[X])/E_3(R[X])$ with the Elementary Symplectic Witt group $W_E(R[X])$. If $1/2 \in R$, a well-known theorem of M. Karoubi asserts that any invertible alternating matrix over a polynomial ring $R[X]$ is stably congruent to its constant form. In particular, the Symplectic Witt group $W(R[X]) \equiv 0$. M. P. Murthy had remarked that these two facts could be used to prove that every $v \in Um_3(R[X])$ can be completed to an invertible matrix. We expanded on this theme of M. P. Murthy, in [3], to show that $S_d(R)$ holds. Here we extend the methods in [3] to prove

Theorem. Let $R$ be a noetherian, local ring of Krull dimension three with $1/2 \in R$. Then every unimodular 3-vector over $R[X]$ can be completed to an invertible matrix.

The reader can also find some very interesting results on A. Suslin’s question, due to M. Roitman, in positive prime characteristics in [5]. The present approach had its genesis in [2], (of course, with roots in Vaserstein theory developed in [8], and guided by M. P. Murthy’s remark), where I could extend some of M. Roitman’s results in dimensions $\leq 4$.

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2. Preliminary remarks and calculations

All rings $A$ considered in this article will be commutative with an identity element and noetherian. A vector $v = (v_0, v_1, \ldots, v_r) \in A^{r+1}$ is said to be unimodular if there is a vector $w = (w_0, w_1, \ldots, w_r) \in A^{r+1}$ such that $v_0w_0 + \cdots + v_rw_r = 1$. $\text{Um}_{r+1}(A)$ will denote the set of all unimodular vectors $v \in A^{r+1}$. The group $Gl_{r+1}(A)$ of invertible matrices acts on $A^{r+1}$ in a natural way: if $v \in A^{r+1}$, $\sigma \in Gl_{r+1}(A)$ then $\sigma$ will map $v$ to $v\sigma$. Under this action $\text{Um}_{r+1}(A)$ is mapped onto itself; and so $Gl_{r+1}(A)$ acts on $\text{Um}_{r+1}(A)$. We let $\sim$ denote equivalence of two vectors under this action. Let $Er_{r+1}(A)$ denote the subgroup of $Gl_{r+1}(A)$ consisting of all the elementary matrices, i.e. those matrices which are a finite product of matrices of the form $E_{ij}(\lambda)$, $i \neq j$, $\lambda \in A$, which has all its diagonal entries one, has one off-diagonal entry in the $(i, j)$th position equal $\lambda$, and has all other entries zero. $v \sim w$ will denote that $v$ can be elementarily transformed to $w$. Let $\text{Um}_{r+1}(A)/Er_{r+1}(A)$ be the set of equivalence classes of vectors $v$ under the equivalence $\sim$ induced by the action of $Er_{r+1}(A)$ on $\text{Um}_{r+1}(A)$; and let $[v]$ denote the equivalence class of $v \in \text{Um}_{r+1}(A)$ in $\text{Um}_{r+1}(A)/Er_{r+1}(A)$.

(2.1) W. Van der Kallen’s group structure on $\text{Um}_{d+1}(A)/E_{d+1}(A)$. If $A$ is a ring whose maximal spectrum $\text{Max}(A)$ is a finite union of subsets $V_i$ where each $V_i$, when endowed with the (topology induced from the) Zariski topology is a space of Krull dimension $\leq d$ we shall say that $A$ is essentially of dimension $d$. For instance, a ring of Krull dimension $d$ is obviously essentially of dimension $\leq d$; a local ring of dimension $d$ is essentially of dimension $0$; whereas a polynomial extension $R[X]$ of a local ring $R$ of dimension $d \geq 1$ has dimension $d + 1$ but is essentially of dimension $d$ as $\text{Max}(R[X]) = \text{Max}(R/(a)[X]) \cup \text{Max}(R_a[X])$ for any non-zero-divisor $a \in R$.

In [9, Theorem 3.6], W. Van der Kallen has described how one could have an abelian group structure on $\text{Um}_{d+1}(A)/E_{d+1}(A)$. In the sequel we shall always refer to this group structure on $\text{Um}_{d+1}(A)/E_{d+1}(A)$; and let $*$ denote the group multiplication henceforth. One has

(2.1.1) Remark. Let $A$ be essentially of dimension $d \geq 2$, and let $C_{d+1}(A)$ denote the set of all completable $(d + 1)$-vectors in $\text{Um}_{d+1}(A)$. Then,

(i) The map $\sigma \rightarrow [e_i, \sigma]$, where $e_i = (1, 0, \ldots, 0) \in \text{Um}_{d+1}(A)$, is a group homomorphism $S_{d+1}(A) \rightarrow \text{Um}_{d+1}(A)/E_{d+1}(A)$.

(ii) $C_{d+1}(A)/E_{d+1}(A)$ is a subgroup of $\text{Um}_{d+1}(A)/E_{d+1}(A)$.

Proof. (i) follows from [9, Theorem 3.16(iv)]. Since any $v \in C_{d+1}(A)$ can be completed to a matrix of determinant one, $C_{d+1}(A)/E_{d+1}(A)$ is the image of $S_{d+1}(A)$ under the homomorphism mentioned in (i); whence it is a subgroup of $\text{Um}_{d+1}(A)/E_{d+1}(A)$.
(2.2) On A. Suslin’s procedure for completing \((a_0, a_1, a_2, \ldots, a_r)\). In [6, Proposition 1.6] A. Suslin shows that if \((a_0, a_1, \ldots, a_r) \in \text{Um}_{r+1}(A)\) then \((a_0, a_1, a_2^2, \ldots, a_r^2)\) can be completed. His proof, as observed by M. P. Murthy in [1, Chapter V, Proposition 1.2], actually demonstrates,

(2.2.1) Proposition. Let \((a_0, a_1, \ldots, a_r) \in \text{Um}_{r-1}(A)\). Suppose that \((\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{r+1})\) is completable in \(\bar{A} = A/(a_r)\). Then \((\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_r^2)\) is completable.

As an application of this proposition we have

(2.2.2) Proposition. Let \(R\) be a local ring of dimension 3 with \(1/2 \in R\). Let \(v = (v_0, v_1, v_2, v_3) \in \text{Um}_4(R[X])\). Then \(v\) is completable if and only if \(v^{(2)} = (v_0^2, v_1, v_2, v_3)\) is completable.

Proof. By [3, Example 1.5.3 and Lemma 1.3.1],

\[ [v^{(2)}] = [v] * [v] \]

in \(\text{Um}_4(R[X])/E_4(R[X])\). By Remark 2.1.1, \(v\) is completable implies that \(v^{(2)}\) is also completable.

Conversely, let \(v^{(2)}\) be completable. By [3, Proposition 1.4.4],

\[ v \sim (w_0, w_1, w_2, c) \]

with \(c \in R\) a non-zero-divisor. As mentioned in the introduction (or cf. [3, Theorem 2.5]), since \(\dim R/(c) = 2\) and \(1/2 \in R\),

\[ (\bar{w}_0, \bar{w}_1, \bar{w}_2) \in e_1S I_3(R/(c)[X]). \]

By Proposition 2.2.1, \((w_0, w_1, w_2, c^3)\) is completable. Thus,

(i) \((v_0, v_1, v_2, v_3^3) \sim (w_0, w_1, w_2, c^3)\) by [10, Theorem],

(ii) \([v]^n = [(v_0, v_1, v_2, v_3^n)]\) for all \(n\) by [3, Example 1.5.3 and Lemma 1.3.1].

Hence \([v]^2 = [v^{(2)}] \in C_4(R[X])/E_4(R[X])\), and \([v]^3 = [(w_0, w_1, w_2, c^3)] \in C_4(R[X])/E_4(R[X])\). By Remark 2.1.1, \([v] \in C_4(R[X])/E_4(R[X])\), i.e. \(v\) is completable.

(2.3) The elementary symplectic Witt group \(W_E(A)\). If \(\alpha \in M_r(A)\), \(\beta \in M_s(A)\) are matrices then \(\alpha \perp \beta\) denotes the matrix \(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in M_{r+s}(A)\). \(\psi_1\) will denote \(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in E_2(Z)\), and \(\psi_r\) is inductively defined by \(\psi_r = \psi_{r-1} \perp \psi_1 \in E_{2r}(Z)\), for \(r \geq 2\).

A skew-symmetric matrix whose diagonal elements are zero is called an alternating matrix. If \(\varphi \in M_{2r}(A)\) is alternating then \(\det(\varphi) = (\text{pf}(\varphi))^2\) where \(\text{pf}\) is a polynomial (called the Pfaffian) in the matrix elements with coefficients \(\pm 1\). Note that we need to fix a sign in the choice of \(\text{pf}\); so insist \(\text{pf}(\psi_r) = 1\) for all \(r\). For any \(\alpha \in M_{2r}(A)\) and any alternating matrix \(\varphi \in M_{2r}(A)\) we have \(\text{pf}(\alpha^t \phi \alpha) = \text{pf}(\varphi) \det(\alpha)\). For alternating matrices \(\varphi, \psi\) it is easy to check that \(\text{pf}(\varphi \perp \psi) = (\text{pf}(\varphi))(\text{pf}(\psi))\).
Two matrices $\alpha \in M_r(A)$, $\beta \in M_r(A)$ are said to be equivalent (w.r.t. $EA$) if there is an $\varepsilon \in E_{2(r+4+l)}(A)$, for some $l$, such that $\alpha \perp \psi_{r+l} = \varepsilon^t(\beta \perp \psi_{r+l})\varepsilon$, (the $t$ stands for ‘transpose’). Denote this by $\alpha \sim \beta$. $\sim$ is an equivalence relation; denote by $[\alpha]$ the orbit of $\alpha$ under this relation. Moreover, a matrix equivalent to an alternating matrix is itself alternating and has the same Pfaffian.

It is easy to see (cf. [8, p. 945]) that $\perp$ induces the structure of an abelian group on the set of all equivalence classes of alternating matrices with Pfaffian 1; this group is called the Elementary Symplectic Witt group and is denoted by $W_E A$.

(2.4) M. Karoubi’s theorem and square roots in $W_E(R[X])$. A famous theorem of M. Karoubi asserts that any invertible alternating matrix $V(X)$ over a polynomial ring $R[X]$ is stably congruent to its constant form if $1/2 \in R$, i.e. there is an $l$, and a $\sigma \in SL_2(R[X])$, for suitable $s$, such that $\sigma^t(V(X) \perp \psi_l)\sigma = V(0) \perp \psi_l$. The machination of M. Karoubi’s proof (cf. [8, §3]) gives

(2.4.1) Proposition. Let $R$ be a local ring with $1/2k \in R$, and let $[V] \in W_E(R[X])$. Then $[V]$ has a $k$th root, i.e. there is a $[W] \in W_E(R[X])$ such that $[V] = [W]^k$ in $W_E(R[X])$.

**Proof.** Since $R$ is local $W_E(R) \equiv 0$, so we may assume that $V(0) = \psi_r$ for some $r$. Let me describe M. Karoubi’s process showing $V$ is stably congruent to $V(0)$; for details consult [8, §3]. The first step is to “stably make $V(X)$ linear” (known as the “Higman trick”)—i.e. find an $\varepsilon \in E_{2(r+l)}(R[X])$ such that

$$\varepsilon^t(V \perp \psi_l) = \psi_{r+l} + nX,$$

for some $t \geq 0$, some $n \in M_{2(r+l)}(R)$.

Since $\gamma = I_{r+l} - \psi_{r+l}nX \in SL_{2(r+l)}(R[X])$, $\psi_{r+l}n$ is nilpotent, i.e. $(\psi_{r+l}n)^l \equiv 0$ for some $l$. Hence, if $1/2k \in R$, we can extract a $k$th root of $\gamma$ (say $\beta^{2k}$) for some $\beta \in SL_{2(r+l)}(R[X])$. Now M. Karoubi pointed out that

$$(*) \quad \varepsilon^t(V \perp \psi_l)\varepsilon = \psi_{r+l}\gamma = \psi_{r+l}\beta^{2k} = (\beta^{2k})^t\psi_{r+l}\beta^{2k}.$$

Let $W = \beta^t\psi_{r+l}\beta$. Then applying Whitehead’s lemma one can check that

$W \perp W \perp \cdots \perp W$ (k times) $\sim V$, i.e. $[V] = [W]^k$ in $W_E(R[X])$.

(2.5) The antipodal vectors equality in $Um_3(R[X])$ in small dimensions. In [3, Lemma 1.3.1] we showed that if a $v = (v_0, v_1, \ldots, v_d) \in Um_{d+1}(A)$, where $A$ is essentially of dimension $d$, can be elementarily transformed to (its antipodal vector) $-v = (-v_0, v_1, \ldots, -v_d)$ then for all $n$, $[v^n] = [v]$ in $Um_{d+1}(A)/E_{d+1}(A)$. There are many examples of vectors which cannot be elementarily transformed to their antipodal vector; but in [3, §1.5] we showed that if $A = R[X]$, $R$ a local ring of dimension 2 with $1/2 \in R$, then for any $v \in Um_3(R[X])$, $v \sim -v$. Here, by a a different argument, we show that
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(2.5.1) **Proposition.** Let $R$ be a local ring of dimension $\leq 4$ with $1/2 \in R$ and let $v = (v_0, v_1, v_2) \in \text{Um}_3(R[X])$. Then $v = (v_0, v_1, v_2) \sim (-v_0, -v_1, -v_2) = -v$.

**Proof.** Choose a $w = (w_0, w_1, w_2)$ such that $v_0w_0 + v_1w_2 + v_2w_2 = 1$, and consider the alternating matrix $V$ with Pfaffian 1 given by

$$
V(v, w) = \begin{pmatrix}
0 & v_0 & v_1 & v_2 \\
-v_0 & 0 & w_2 & -w_1 \\
v_1 & -w_2 & 0 & w_0 \\
-v_2 & w_1 & -w_0 & 0
\end{pmatrix} \in SL_4(R[X]).
$$

Since $1/2 \in R$, by M. Karoubi's theorem (cf. §2.4) there is a $\beta \in SL_4(R[X])$, for some $l$, such that $\beta^l(V \downarrow \psi_i)\beta = \psi_{i+2}$. Since $\dim R \leq 4$, by [7, Theorem 2.6], $\text{Um}_r(R[X]) = e_1 E_r(R[X])$ for all $r \geq 6$. Hence on applying [8, Lemma 5.5 and Lemma 5.6] we can find a $\beta^* \in SL_4(R[X])$ such that $(\beta^*)^l V \beta^* = \psi_2$.

Let $\delta = \text{diagonal} (-1, 1, -1, 1) \in E_4(R)$. Then $\delta^l \psi_2 \delta = -\psi_2$. Thus

$$
\delta^l (\beta^*)^l V \beta^* \delta = \delta^l \psi_2 \delta = -\psi_2 = \psi_2^l = [(\beta^*)^l V \beta^*]^l = (\beta^*)^l V^l \beta^*,
$$

and so if $\sigma = (\beta^*)^l$ then $(\sigma^{-1} \delta^l \sigma) V (\sigma^{-1} \delta^l \sigma)^l = -V$.

By [7, Corollary 1.4] $\sigma^{-1} \delta^l \sigma \in E_4(R[X])$. Now the equation (*) will prove the proposition on applying [11, Theorem 10].

(2.5.2) **Remark.** The above argument can be suitably modified to show that if $[V] \in \text{W}_E(R[X])$, where $R$ is a local ring with $1/2 \in R$, then $[V] = [-V]$ in $\text{W}_E(R[X])$.

(2.6) “Coordinate squares” in $\text{W}_E(R[X])$. Let us say that an invertible alternating matrix $V$ is a “coordinate kth power” if the first row of $V$ has the form $(0, v_1^k, v_2, \ldots, v_{2r-1})$. It would be of interest to know if, under congenial conditions, the above fact, proven in Proposition 2.4.1, that every $[V] \in \text{W}_E(R[X])$ is a kth power in $\text{W}_E(R[X])$ (under suitable hypothesis on $R$) can be translated to read that $[V]$ has a representative $V^*$ which is a coordinate kth power and which, moreover, has the same size as that of $V$. We give some evidence for this here.

Firstly recall some multiplicative relations in $\text{W}_E(A)$ observed by L. N. Vasertstein in [8, Theorem 5.2(a$_2$)].

(2.6.1) **The Vaserstein Rule.** Let $v_1 = (a_0, a_1, a_2), v_2 = (a_0, b_1, b_2)$ be unimodular vectors. Suppose that $a_0a'_0 + a_1a'_1 + a_2a'_2 = 1$, and that

$$
v_3 = (a_0, (b_1, b_2)(a_1, a_2)) \in \text{Um}_3(A).
$$

Then for any $w_1, w_2, w_3$ such that $v_i \cdot w_i = 1$, $i = 1, 2, 3$, we have

$$
[V(v_1, w_1)] \perp [V(v_2, w_2)] = [V(v_3, w_3)] \text{ in } \text{W}_E(A).
$$

(Note. $V(v, w)$ is defined in Proposition 2.5.1, and $[V(v, w)]$ is well defined in $\text{W}_E(A)$ via [8, Lemma 5.1].)
(2.6.2) **Corollary.** (i) Let \( v_1 = (a_0, a_1, a_2), \ v_2 = (b_0, a_1, a_2) \) be unimodular vectors. Suppose that \( a_0a'_0 + a_1a'_1 + a_2a'_2 = 1 \) and that \( v_3 = (a_0b_0 + a'_0, a_1, a_2) \in \text{Um}_3(A) \). Then for any \( w_1, w_2, w_3 \) such that \( v_iw_i^t = 1, \ i = 1, 2, 3 \), we have
\[
[V(v_1, w_1)] \perp [V(v_2, w_2)] = [V(v_3, w_3)] \quad \text{in } \text{WE}(A).
\]
(ii) Let \( v_1 = (a_0, a_1, a_2), \ v_2 = (b_0^2, a_1, a_2) \) be unimodular vectors. Suppose that \( v_3 = (a_0b_0^2, a_1, a_2) \) and that \( w_1, w_2, w_3 \) are such that \( v_iw_i^t = 1, \ i = 1, 2, 3 \), then
\[
[V(v_1, w_1)] \perp [V(v_2, w_2)] = [V(v_3, w_3)] \quad \text{in } \text{WE}(A).
\]
**Proof.** (i) is immediate from the Vaserstein Rule. We refer the reader to [9, Theorem 3.16(iii)] for deriving (ii) from (i). Note: You may need the Roitman lemma in [5, Lemma 1].

(2.6.3) **The “antipodal vectors equality” lemma in \( \text{WE}(A) \).** Let \( v = (v_0, v_1, v_2) \) be a unimodular vector and assume that \( v \sim E v \sim E (-v_0, -v_1, -v_2) \). Let \( v_1^{(2)} = (v_0^2, v_1, v_2) \) and let \( w, w_1 \) be such that \( v \cdot w^t = 1 \). Then
\[
[V(v, w)]^2 = [V(v^{(2)}, w_1)] \quad \text{in } \text{WE}(A).
\]
**Proof.** Imitate the argument in [3, Lemma 1.3.1] in \( \text{WE}(A) \). (Note. You will need Corollary 2.6.2(ii) above.)

Finally, we give some conditions under which we can extract “coordinate squares” in \( \text{WE}(R[X]) \):

(2.6.4) **Corollary.** Let \( R \) be a local ring of dimension \( \leq 4 \) with \( 1/2 \in R \) and let \( v = (v_0, v_1, v_2), \ v_1^{(2)} = (v_0^2, v_1, v_2) \) be unimodular \( R[X] \)-vectors. Let \( w, w_1 \) such that \( v \cdot w^t = v^{(2)} \cdot w_1^t = 1 \). Then,
\[
[V(v, w)]^2 = [V(v^{(2)}, w_1)] \quad \text{in } \text{WE}(R[X]).
\]
**Proof.** This will follow from Proposition 2.5.1 and Lemma 2.6.3.

(2.6.5) **Proposition.** Let \( R \) be a local ring of dimension \( \leq 3 \) with \( 1/2 \in R \) and let \( V \in S_{14}(R[X]) \) be an alternating matrix with Pfaffian 1. Then \([V] = [V^*] \) in \( \text{WE}(R[X]) \) with \( V^* \in S_{14}(R[X]) \) a coordinate square. Consequently, there is a stably elementary \( \gamma \in S_{14}(R[X]) \) such that \( V = \gamma^t V^* \gamma \).

**Proof.** By Proposition 2.4.1, \([V] = [W]^2 \) for some \([W] \in \text{WE}(R[X]) \). By [7, Theorem 2.6] \( \text{Um}_r(R[X]) = e_1 E_r(R[X]) \) for all \( r \geq 5 \), and so on applying [8, Lemma 5.3 and Lemma 5.5] a few times, if necessary, we can find an alternating matrix \( W^* \in S_{14}(R[X]) \) (with Pfaffian 1) such that \([W] = [W^*] \). Now apply Corollary 2.6.4 to find \( V^* \) as required. The last statement follows as above (only applying [8, Lemma 5.5 and Lemma 5.6] instead).
3. The main theorem

(3.1) Theorem. Let $R$ be a local ring of Krull dimension three with $1/2 \in R$ and let $v = (v_0, v_1, v_2)$ be a unimodular 3-vector over $R[X]$. Then $v$ can be completed to an invertible matrix.

Proof. Choose a $w = (w_0, w_1, w_2)$ such that $v_0w_0 + v_1w_1 + v_2w_2 = 1$, and consider the alternating matrix $V$ with Pfaffian 1 given by

$$V = \begin{pmatrix} 0 & v_0 & v_1 & v_2 \\ -v_0 & 0 & w_2 & -w_1 \\ -v_1 & -w_2 & 0 & w_0 \\ -v_2 & w_1 & -w_0 & 0 \end{pmatrix} \in SL_4(R[X]).$$

Since $1/2 \in R$, by M. Karoubi's theorem (see (*) in Proposition 2.4.1) there is a $a \in SL_4+(R[X])$, for some $l$, such that $a^t (V \perp \psi) a = \psi_{l+2}$.

Since $\dim R = 3$, by [7, Theorem 2.6] $Um_4(R[X]) = e_4E_4(R[X])$ for all $r \geq 6$. Hence on applying [8, Lemma 5.5 and Lemma 5.6] we can find an $a \in SL_4(R[X])$ such that $a^t V a = \psi_2$. Consider $e_4a^t$, where $e_4 = (0, 0, 0, 1)$.

By [3, Proposition 1.4.4] $e_4a^t \sim (a_0(X), a_1(X), a_2(X), c)$, where $c \in R$ is a non-zero-divisor in $R$. Let the ‘overbar’ denote ‘modulo (c)’.

By Proposition 2.2.2, $(a_0(X), a_1(X), a_2(X), c) \sim (b_0(X)^2, b_1(X), b_2(X))$, for some $b_0(X), b_1(X), b_2(X) \in R[X]$. On “lifting” this elementary map, and after an appropriate elementary transformation further, we can arrange that $e_4a^t \sim (b_0(X)^2, b_1(X), b_2(X), c)$.

By Proposition 2.2.2, $(b_0(X), b_1(X), b_2(X), c)$ can be completed to an invertible matrix, say $\beta \in SL_4(R[X])$ with $e_4\beta = (b_0(X), b_1(X), b_2(X), c)$.

Finally, Remark 1.1.1 follows that $e_4\beta^{-2}a^t = [e_4\beta^{-2}] * [e_4a^t] = [e_4\beta^{-2}] * [e_4a^t]$

$$= (([b_0(X), b_1(X), b_2(X), c])^2)^{-1} * [e_4a^t] = [e_4a^t]^{-1} * [e_4a^t] \equiv 1,$$

the last equality being deduced via [3, Example 1.5.3 and Lemma 1.5.1]. Thus, $\beta^{-2}a^t = \epsilon \delta'$ for some $\epsilon \in E_4(R[X])$ and $\delta' = \left(\begin{array}{cc} 1 & 0 \\ 0 & \delta \end{array}\right)$ with $\delta \in SL_3(R[X])$.

Now $\psi_2 = a^t V a = (\beta^2 \epsilon \delta') V (\beta^2 \epsilon \delta') = \beta^2 V^* (\beta^2)^t$, where $e_1V^* = (0, v \delta^t \epsilon)$ for some $\epsilon \in E_4(R[X])$—this will follow as $\delta' = \left(\begin{array}{cc} 1 & 0 \\ 0 & \delta \end{array}\right)$ and via [11, Theorem 10].

By Proposition 2.6.5 there is a stably elementary $\gamma \in SL_4(R[X])$ such that $\beta V^* \beta^t = \gamma V^* \gamma$, with $V^* \in SL_4(R[X])$ a coordinate square. Let $e_1V^* = (0, a^2, b, c)$, and let $\alpha_0$ (cf. §2.2) be a completion of $(a^2, b, c)$.

Since $c_1V^* = e_1 \left(\begin{array}{cc} 1 & 0 \\ 0 & \alpha_0 \end{array}\right) \psi_2 \left(\begin{array}{cc} 1 & 0 \\ 0 & \alpha_0 \end{array}\right)$

it follows via [8, Lemma 5.1] that

$$V^* = e_1 \left(\begin{array}{cc} 1 & 0 \\ 0 & \alpha_0 \end{array}\right) \psi_2 \left(\begin{array}{cc} 1 & 0 \\ 0 & \alpha_0 \end{array}\right) e_1.$$
for some \( e_1 \in E_4(R[X]) \). Thus,

\[
\beta V^* \beta^t = \gamma^t V^{**} \gamma = \gamma^t e_1^t \begin{pmatrix} 1 & 0 \\ 0 & \alpha_0 \end{pmatrix} \psi_2 \begin{pmatrix} 1 & 0 \\ 0 & \alpha_0 \end{pmatrix} e_1 \gamma.
\]

Hence,

\[
\beta^{-1} \left[ \left( \begin{pmatrix} 1 & 0 \\ 0 & \alpha_0 \end{pmatrix} \right)^t (\epsilon_1^{-1})^t (\gamma^{-1})^t \right] \beta V^* \beta^t \left[ \gamma^{-1} \epsilon_1^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_0 \end{pmatrix} \right] (\beta^{-1})^t
\]

\[
= \beta^{-1} \psi_2 (\beta^{-1})^t = \beta^{-1} (\beta^2 V^* (\beta^2)^t (\beta^{-1})^t = \beta V^* \beta^t = \gamma^t V^{**} \gamma;
\]

and so if

\[
\theta = \beta^{-1} \epsilon_1^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_0 \end{pmatrix} (\beta^{-1})^{-1} \gamma^{-1}, \text{ then } \theta^t V^* \theta = V^{**}.
\]

Compute \( e_4 \theta^t \) in the abelian group \( \text{Um}_4(R[X])/E_4(R[X]) \) via Remark 2.1.1 to get \( [e_4 \theta^t] = [e_4 (\gamma')^{-1}]^2 \). But \( \gamma \) is stably elementary and so via [3, Proposition 2.6] \( [e_4 (\gamma')^{-1}]^2 = 1 \); hence \( [e_4 \theta^t] = 1 \), i.e. \( e_4 \theta^t \sim e_4 \) Hence

\[
\theta^t \epsilon' = \begin{pmatrix} 1 & 0 \\ 0 & (\theta')^t \end{pmatrix}
\]

for some \( \theta' \in \text{SL}_3(R[X]) \), \( \epsilon' \in E_4(R[X]) \).

Now

\[
\theta^t V^* \theta = \begin{pmatrix} 1 & 0 \\ 0 & (\theta')^t \end{pmatrix} (\epsilon')^{-1} V^* ((\epsilon')^{-1})^t \begin{pmatrix} 1 & 0 \\ 0 & \theta' \end{pmatrix} = V^{**},
\]

and so via [11, Theorem 10] we can deduce that there is an \( e'' \in E_3(R[X]) \) such that \( \nu e'' \theta' = (a^2, b, c) \). Since \( (a^2, b, c) \) is completable, so is \( v \).

\textbf{Remark.} Let us, following M. Krusemeyer, say that a vector \( v \in \text{Um}_r(A) \) is skew-completable if there is an invertible alternating matrix \( V \in \text{SL}_{r+1}(A) \) with its first row \( e_i V = (0, v) \).

By making some appropriate modifications in the argument used to prove Theorem 3.1 one can show that,

\textbf{(3.2) Theorem.} Let \( R \) be a local ring of Krull dimension \( d \) with \( 1/2 \in R \), and let \( v = (v_0, v_1, \ldots, v_{d-1}) \) be a skew-completable vector over \( R[X] \). Then \( v \) can be completed to an invertible matrix.

Finally, using the well-known "Quillen-Suslin" Monic inversion and Local-Global principles, one can derive from \( S_d(R) \) and Theorem 3.1 that,

\textbf{(3.3) Corollary.} Let \( R \) be a noetherian ring of dimension 3 with \( 1/6 \in R \). Then any stably extended projective module over \( R[X_1, \ldots, X_n] \) is extended.

\textbf{Note added in proof.} The contents (especially the mode of proof of the main result) of this note seems of interest in connection with the following problem:

(i) Let \( V : \text{Um}_3(A)/E_3(A) \to W_E(A) \) be the Vaserstein symbol. Is this map injective if \( \dim A = 3 \)?
I also hope that, after incorporation of some additional theories, the techniques used here will provide some insight towards settling,

(a) Let \( R \) be a local ring with \( \frac{1}{2} \in R \). Is every \( v \in Um_3(R[X]) \) completable?

(b) Let \( A \) be a smooth affine algebra over the field \( C \) of complex numbers of dimension \( d \). Is a stably free \( A \)-module of rank \( (d - 1) \) a free module?

In an article entitled On some actions of stably elementary matrices on alternating matrices we prove that

"Let \( A \) have Krull dimension \( \leq 5 \), and let \( V \in SL_4(A) \cap E_5(A) \) be a stably elementary alternating matrix of Pfaffian one. Then \( V^8 \in E_4(A) \)."

Note. One needs to show that \( V \in E_4(A) \) to settle (i) above.

We also give some examples of 3 dimensional affine algebras for which the Vaserstein symbol \( V \) is bijective.

REFERENCES


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