INNER AMENABLE LOCALLY COMPACT GROUPS

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Abstract. In this paper we study the relationship between amenability, inner amenability and property P of a von Neumann algebra. We give necessary conditions on a locally compact group G to have an inner invariant mean m such that m(V) = 0 for some compact neighborhood V of G invariant under the inner automorphisms. We also give a sufficient condition on G (satisfied by the free group on two generators or an I.C.C. discrete group with Kazhdan's property T, e.g., SL(n, Z), n ≥ 3) such that each linear form on L^2(G) which is invariant under the inner automorphisms is continuous. A characterization of inner amenability in terms of a fixed point property for left Banach G-modules is also obtained.

Introduction

Let G be a locally compact group. Then G is called inner amenable if there exists a state m on L^∞(G), such that m(π(a)f) = m(f) for all a ∈ G and f ∈ L^∞(G), where

π(a)f(x) = f(a^{-1}xa), x ∈ G.

Amenable locally compact groups and [IN]-groups are inner amenable. The group G is [IN] if there exists a compact neighborhood V of the identity e in G such that a^{-1}Va = V for all a ∈ G. Furthermore when G is connected, then G is amenable if and only if G is inner amenable (see [17]). A recent account of amenability is given in [21].

Let H be a von Neumann algebra on a Hilbert space H and let H' be the commutant of H. For T ∈ B(H) (the space of bounded linear operators on H), let C_T be the weak*-closed convex subset of B(H) generated by \{U^*TU; U ∈ H^u\}, where H^u is the group of unitary elements in H. (Note that B(H) has a unique predual [28, p. 47].) H is said to have property P if C_T ∩ H' ≠ ∅ for each T ∈ B(H).

Let VN(G) denote the von Neumann algebra on L^2(G) generated by \{l_x; x ∈ G\} where l_xh(t) = h(xt), t ∈ G. A well-known result of Schwartz [29] asserts

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that if $G$ is discrete, then $G$ is amenable if and only if $\text{VN}(G)$ has property $P$. In §3 we study the relation between amenability, inner amenability, and property $P$ of a von Neumann algebra determined by $G$ and its action on a locally compact Hausdorff space $X$. In particular, we provide the missing link in the following well-known implications for a locally compact group $G$:

\[ \text{Amenability} \quad \rightarrow \quad \text{Inner Amenability} \quad \rightarrow \quad \text{Property } P \quad \text{(for } \text{VN}(G)) \]

In [20] Paschke proved that if $G$ is an infinite discrete group, then there exists an inner invariant mean on $l^\infty(G)$ different from the point evaluation at the identity if and only if the $C^*$-algebra generated by the unitaries on $l^2(G)$ corresponding to conjugation by elements in $G$ does not contain the projection on the space $C_0(e)$, where $e$ is the identity of $G$. In §4, we find necessary conditions for there to exist an inner invariant mean $m$ on $L^\infty(G)$ such that $m(1_V) = 0$ (when $V$ is a compact neighborhood of $G$ invariant under inner automorphisms). We also give a sufficient condition on $G$ (Theorem 4.4) such that each linear form $I$ on $L^2(G)$ which is invariant under inner automorphisms is continuous and has the form $I(f) = \frac{1}{\alpha} \int_V f dx$, where $\alpha = I(1_V)$. In particular (Corollary 4.5 and 4.6) if $G$ is the free group on two generators or a discrete group with Kazhdan’s property $T$ and every nontrivial conjugacy class in $G$ is infinite (e.g., $\text{SL}(n, \mathbb{Z})$, $n \geq 3$), then every inner invariant linear form on $l^2(G)$ is continuous. (See [18] for a discussion of similar problems.)

It is well known (see [6 or 26]) that amenability of a locally compact group $G$ may be characterized in terms of fixed points for affine maps on compact convex sets. In §5, we characterize inner amenability of $G$ in terms of a fixed point property for left Banach $G$-modules. Finally in §6, a few miscellaneous results on inner amenability are stated and proved.

The literature on inner amenability has grown substantially in recent years: see [1, 2, 7, 14, 16, 17, 20, 31].

2. Preliminaries and some notations

Throughout this paper $G$ denotes a locally compact group with a fixed left Haar measure $\lambda$. The spaces $L^p(G)$, $1 \leq p \leq \infty$, of measurable functions will be as defined in [13]. For each $a \in G$, $1 \leq p < \infty$, let $\pi(a)$ be the operator on $L^p(G)$ defined by

$$\pi(a)f(t) = f(a^{-1}ta)\Delta^{1/p}(a), \quad a, t \in G, \ f \in L^p(G),$$

where $\Delta$ is the modular function on $G$. The group $G$ is called amenable if there exists a mean $m$ on $L^\infty(G)$ (i.e., $m \in L^\infty(G)$, $m \geq 0$, and $||m|| = 1$) such that $m(\pi(a)f) = m(f)$ for all $a \in G$ and $f \in L^\infty(G)$. As is well known [8, Theorem 2.2.1], this is equivalent to the existence of a left invariant mean on $U_c(G)$, the space of bounded right uniformly complex-valued continuous
functions on \( G \) (as defined in \([13, \text{p. 21}]\)). All abelian groups and all compact
groups are amenable. However, if \( G \) contains the free group on two generators
as a closed subgroup (e.g., if \( G = \text{SL}(2, \mathbb{R}) \)), then \( G \) is not amenable (see \([8, 21, 23]\) for details).

Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space \( H \). If \( \mathcal{M} \) has property \( P \),
then there exists a projection of norm one \( E \) from \( \mathfrak{B}(H) \) onto \( \mathcal{M}' \) with
\( E(1) = 1 \) (see \([28, \text{p. 207, or 24, p. 136}]\)). Von Neumann algebras with this
latter property are called injective. As is well known \([4]\), injectivity and property \( P \) are equivalent. For a discussion of the various forms of amenability for von
Neumann algebras, see \([21, 2.35]\).

If \( X \) is a subset of a locally convex space \( E \) with topology \( \tau \), then \( \overline{co}^\tau X \)
will denote the closed convex hull of \( X \) in \( E \).

3. Inner amenability, amenability, and injectivity

A reference for the definitions below in the discrete cases is Zimmer \([32]\).
Let \( X \) be a locally compact Hausdorff space. Let \( G \) act invertibly on \( X \) on
the right such that the mapping \( X \times G \rightarrow X \) defined by \( (x, g) \rightarrow x \cdot g \), \( x \in X \),
\( g \in G \), is jointly continuous. Let \( \mu \) be a nonnegative quasi-invariant Radon
measure on \( X \). We define \( L^p(X \times G, \mu \times \lambda) \) or simply \( L^p(X \times G) \), \( 1 \leq p \leq \infty \),
as the usual \( L^p \)-spaces of Borel functions identified when they coincide off a
locally \((\mu \times \lambda)\)-null set in \( X \times G \). For each \( a \in G \), define \( \mu_a(E) = \mu(Ea) \).
Then, by quasi-invariance of \( \mu \), we have \( \mu_a \ll \mu \) for each \( a \in G \) and there is,
by the Radon-Nikodým theorem, a locally \( \mu \)-integrable function \( r(\cdot, a) \) such
that

\[
\int f(xa^{-1}) d \mu(x) = \int f(x) r(x, a) d \mu(x)
\]

for all \( f \in L^1(X) \) \((= L^1(X, \mu))\). It follows that \( r(x, ab) = r(x, a)r(xa, b) \)
for \( a, b \in G \), and \( r(x, e) = 1 \). For \( u \in G \) and \( \phi \in L^\infty(X) \), define the
operators \( U_a, V_a, M_\phi, \) and \( N_\phi \) on \( L^2(X \times G) \) by

\[
\begin{align*}
U_a f(x, b) &= f(xa, ba)r(x, a)^{1/2} \Delta(a)^{1/2}, \\
V_a f(x, b) &= f(x, a^{-1}b), \\
M_\phi f(x, b) &= \phi(x)f(x, b), \\
N_\phi f(x, b) &= \phi(xb^{-1})f(x, b),
\end{align*}
\]

where \( f \in L^2(X \times G) \).

Then each \( U_a, V_a \) is a unitary operator on \( L^2(X \times G) \). Let \( \mathcal{S} \) be the von
Neumann algebra generated by the operators \( V_a, N_\phi \) \((a \in G, \phi \in L^\infty(X))\), and
\( \mathcal{R} \) be the von Neumann algebra generated by the operators \( U_a, M_\phi \) \((a \in G, \\
\phi \in L^\infty(X))\). If \( J \in \mathcal{R}(L^2(X \times G)) \) is given by

\[
(Jf)(x, b) = f(xb^{-1}, b^{-1})r(x, b^{-1})^{1/2} \Delta(b^{-1})^{1/2},
\]
then $J^2f = f$, $JV_aJ = U_a$, and $JN_aJ = M_a$. So $J$ implements a spatial isomorphism between $\mathcal{L}$ and $\mathcal{R}$. Therefore $\mathcal{L}$ has property $P$ if and only if $\mathcal{R}$ has property $P$.

If $a \in G$, let $\delta_a$ denote the Dirac measure on $G$ concentrated at $a$. For any $f \in L^\infty(X)$, the function $(\delta_a \Box f)(x) = f(xa)$ is defined $\mu$-locally almost everywhere on $X$ (see [10, Lemma 2.1]). Furthermore, $\delta_a \Box f \in L^\infty(X)$. A linear functional $m$ on $L^\infty(X)$ is called a mean if $m(1) = 1$ and $m(f) \geq 0$ whenever $f \geq 0$. A mean $m$ is $G$-invariant if $m(\delta_g \Box f) = m(f)$ for all $g \in G$.

Also for any $\phi \in L^1(X)$ and $x \in G$, let $\delta_x \ast \phi \in L^1(X)$ be defined by

$$\delta_x \ast \phi(\xi) = \left(\frac{d\mu_x}{d\mu}\right)(\xi)\phi(x^{-1}\xi) \quad \mu\text{-a.e. on } X$$

(see [10, Lemma 2.2]), where $\mu_x(E) = \mu(x^{-1}E)$.

Theorem 3.1 below, in the special case where $X$ is a singleton, is proved in [21, p. 85].

**Theorem 3.1.** Let $G$, $X$, and $\mathcal{L}$ be as above. Then the following are equivalent:

(a) $G$ is amenable.

(b) $\mathcal{L}$ is injective, $L^\infty(X)$ has a $G$-invariant mean, and $G$ is inner amenable.

**Proof.** (a) $\Rightarrow$ (b) If $G$ is amenable, then $G$ is inner amenable since every invariant mean on $G$ is inner invariant. It follows from [10, Theorem 3.1] that $L^\infty(X)$ has a $G$-invariant mean.

To see that $\mathcal{L}$ is injective, we first note that the von Neumann algebra generated by $\{N_\phi; \phi \in L^\infty(X)\}$ has property $P$ (by the Markov-Katutani fixed point theorem). Hence $\mathcal{D} = \{N_\phi; \phi \in L^\infty(X)\}'$ is injective [28, Proposition 4.4.15]. (In fact, $\mathcal{D}$ is the von Neumann algebra generated by the $N_\phi$.) It suffices to show that there is a norm one projection from $\mathcal{D}$ onto $\mathcal{L}' = \{V_a; a \in G\}' \cap \mathcal{D}$. For then $\mathcal{L}'$ is injective and so $\mathcal{L}$ is also injective.

Let $T \in \mathcal{D}$ and $a \in G$. Then $V_a^{-1}TV_a \in \mathcal{D}$. Indeed $\mathcal{D}$ is generated by the $N_\phi$'s; hence we may assume $T = N_\phi$. If $x \in X$, $b \in G$, and $f \in L^2(X \times G)$, we have

$$(V_a^{-1}N_\phi V_a f)(x, b) = (N_\phi V_a f)(x, ab)$$

$$= \phi(xb^{-1}a^{-1})(V_a f)(x, ab) = (N_a^{-1} \phi f)(x, b),$$

i.e., $V_a^{-1}N_\phi V_a = N_a^{-1} \phi \in D$. The result follows.

Let $K_T$ denote the $w^*$-compact convex subset of $\mathcal{D}$. Consider the action of $G$ on $K_T$ defined by

$$(a, S) \mapsto V_a^{-1}SV_a.$$ 

Then the action is separately continuous in the weak operator topology WOT, which agrees with the $w^*$-topology on $K_T$. Indeed, if $a_\alpha \to a_0$ and $S \in K_T$, then ...
then \( V_{a^{-1}} \to V_a \) and \( V_{a^{-1}} \to V_{a^{-1}} \) in the strong operator topology (SOT). In particular, \( SV_{a^{-1}} \to SV_a \) in the SOT, and so \( V_{a^{-1}}SV_{a^{-1}} \to V_{a^{-1}}SV_a \) in the SOT (since multiplication is jointly continuous on bounded sets in the SOT). Hence \( V_{a^{-1}}SV_{a^{-1}} \to V_{a^{-1}}SV_a \) in the WOT. Now if \( a \in G \), and \( S_a \to S \) in the WOT, then for any \( \eta, \xi \in L_2(G \times X) \),
\[
\langle V_{a^{-1}}SV_a \xi, \eta \rangle = \langle S_a V_a \xi, V_a \eta \rangle = \langle SV_a \xi, V_a \eta \rangle = \langle V_{a^{-1}}SV_a \xi, \eta \rangle,
\]
i.e., \( V_{a^{-1}}SV_a \to V_{a^{-1}}SV_a \) in the WOT. Apply now Rickert's generalization of Day's fixed point theorem to obtain \( S \in K_T \) such that \( V_{a^{-1}}SV_a = S \) for all \( a \in S \), i.e., \( SV_a = V_a \) for all \( a \in S \). So \( S \in \{ V_a : a \in S \} \cap \mathcal{D} \). Consequently, there exists a projection \( Q : \mathcal{D} \to \{ V_a : a \in G \} \cap \mathcal{D} \) such that \( Q(T) \in K_T \) for all \( T \in \mathcal{D} \), \( Q(I) = I \), and \( ||Q|| = 1 \) by Yeadon's Theorem [30].

(b) \Rightarrow (a) Define a left and a right action of \( G \) on \( L^\infty(G \times X) \) by

\[
(1) \quad (F_a)(x, b) = F(x, ab), \quad (aF)(x, b) = F(xa, ba).
\]

Using (1) and the equalities \( r(x, ab) = r(x, a)r(xa, b) \) a.e. \( x \), and \( r(x, e) = 1 \) for all \( x \in G \), one shows that

\[
(2) \quad \langle F, V_a f \rangle = \langle Fa, f \rangle, \quad \langle F, U_a f \rangle = \langle a^{-1}F, f \rangle
\]

\((F \in L^\infty(G \times X), f \in L^1(G \times X))\). Here (with a slight abuse of notation),

\[
V_a f(x, b) = f(x, a^{-1}b), \quad U_a f(x, b) = f(xa, ba)r(x, a)\Delta(a)
\]

\((f \in L^1(G \times X), x \in X, a, b \in G)\).

We now show that there exists a positive linear functional \( m' \) with \( ||m'|| = 1 \) such that

\[
(3) \quad m'(aFa^{-1}) = m'(F)
\]

for all \( a \in G \) and \( F \in L^\infty(G \times X) \).

Since \( L^\infty(X) \) has a \( G \)-invariant mean, an argument similar to that of Namioka [19] shows that there exists a net \( \{ \phi_\alpha \} \) in \( P_1(X) = \{ \phi \in L^1(X) : \phi \geq 0 \text{ and } ||\phi||_1 = 1 \} \) such that \( ||\delta_a * \phi_\alpha - \phi_\alpha|| \to 0 \) for each \( a \in G \). Also since \( G \) is inner amenable, there exists a net \( \{ \mu_\beta \} \) in \( P_1(G) \) such that \( ||\delta_a * \mu_\beta * \delta_a^{-1} - \mu_\beta||_1 \to 0 \) (see [17, Proposition 1]). Let

\[
m_{\alpha, \beta}(F) = \int F d(\phi_\alpha \times \mu_\beta),
\]

where \( F \in L^\infty(G \times X) \). Then \( \{ m_{\alpha, \beta} \} \) is bounded in \( L^\infty(G \times C)^* \). Further-
more, if \( a \in G \) and \( F \in L^\infty(X \times G) \), then

\[
\|\langle m_{\alpha, \beta}, aF a^{-1} \rangle - \langle m_{\alpha, \beta}, F \rangle\|
\]

\[
\leq \left| \iint F(xa, a^{-1}ba) d\phi_\alpha(x) d\mu_\beta(b) - \iint F(x, b) d\phi_\alpha(x) d\mu_\beta(b) \right|
\]

\[
+ \left| \iint F(x, b) d\phi_\alpha(x) d\mu_\beta(b) - \iint F(xa, a^{-1}ba) d\phi_\alpha(x) d\mu_\beta(b) \right|
\]

\[
= \|\delta_\alpha * \phi_\beta - \phi_\alpha\| F_\infty + \|\delta_\alpha * \mu_\beta * \delta_\alpha^{-1} - \mu_\beta\| F_\infty
\]

which converges to zero. Hence if \( m' \) is any weak*-cluster point of the \( \{m_{\alpha, \beta}\} \), then \( m' \) satisfies (3).

By (3) and an idea of Namioka [19] there exists a net \( \{f_\delta\} \) in \( L_1(X \times G) \), \( f_\delta \geq 0 \), \( \|f_\delta\|_1 = 1 \) such that \( \|V_{a^{-1}} - U_a\| b_\delta \rightarrow 0 \). Let \( g_\delta = f_\delta^{1/2} \). Note that \( g_\delta \in L_2(X \times G), g_\delta \geq 0, \|g_\delta\|_2 = 1 \). Then \( (V_{a^{-1}} g_\delta)^{1/2} = V_a g_\delta \), \( (U_a g_\delta)^{1/2} = U_a g_\delta \), and hence

\[
\|V_{a^{-1}} - U_a\| g_\delta \rightarrow 0 \quad \text{for all } a \in G.
\]

For each \( F \in L^\infty(X \times G) \), let \( L_F \in B(L_2(X \times G)) \) be defined by

\[
L_F f(x, b) = F(x, b)f(x, b).
\]

Then, as readily checked,

\[
V_a L_F V_a^{-1} = L_{F a^{-1}}
\]

for each \( a \in G \). Let \( H \) denote the group of unitary elements in the von Neumann algebra \( \mathscr{R} \) with the strong operator topology. Let \( \psi_\delta \) be a function on \( H \) defined by \( \psi_\delta(F)(U) = \langle UL_F U^* g_\delta, g_\delta \rangle \) \( (U \in H) \). Then \( \psi_\delta \in \mathcal{U}_c(H) \). Also

\[
\psi_\delta(F a^{-1})(U) = \langle UL_{F a^{-1}} U^* g_\delta, g_\delta \rangle
\]

\[
= \langle UV_a L_F V_a^{-1} U^* g_\delta, g_\delta \rangle
\]

\[
= \langle UL_F U^*(V_{a^{-1}} g_\delta), V_a g_\delta \rangle
\]

using (5) and the fact that each \( V_a \) is in the commutant of \( \mathscr{R} \). Also

\[
\psi_\delta(F)V_{a^{-1}}(U) = \langle UL_F U^*(V_a g_\delta), V_a g_\delta \rangle.
\]

So

\[
\|\psi_\delta(F a^{-1}) - \psi_\delta(F)U_{a^{-1}}\|(U)\|
\]

\[
= \|\langle UL_F U^* V_{a^{-1}} g_\delta, V_{a^{-1}} g_\delta \rangle - \langle UL_F U^* V_a g_\delta, V_a g_\delta \rangle\|
\]

\[
= \|\langle UL_F U^*(V_{a^{-1}} - V_a) g_\delta, V_a g_\delta \rangle
\]

\[
+ \langle UL_F U^* V_{a^{-1}} g_\delta, (V_{a^{-1}} - V_a) g_\delta \rangle\|
\]

\[
\leq 2\|F\| \|V_{a^{-1}} - V_a\| g_\delta \|2.
\]
Since \( \mathcal{L} \) is injective, \( \mathcal{L}^* \) must have property \( P \). So \( \mathcal{R} \) also has property \( P \). By a result of de la Harpe [12], there exists a left invariant mean \( m \) on \( U_r(H) \), the space of bounded right uniformly continuous functions on \( H \). Hence using (4) and (7), we have

\[
|m(\psi_\delta(Fa^{-1}))-m(\psi_\delta(F))| \to 0.
\]

Let \( n_\delta = m \circ \psi_\delta \). Then \( n_\delta \) is a mean on \( L^\infty(X \times G) \). Let \( n \) be a weak*-cluster point of \( \{n_\delta\} \). Then \( n(Fa^{-1}) = n(F) \) for all \( F \in L^\infty(X \times G) \) and \( a \in G \). Define

\[
\tilde{n}(\phi) = n(1 \otimes \phi), \quad \phi \in L^\infty(G).
\]

Then \( \tilde{n} \) is a left invariant mean on \( L^\infty(G) \). Hence \( G \) is amenable. \( \Box \)

A well-known result of Schwartz [29] asserts that if \( G \) is discrete then \( G \) is amenable if and only if \( VN(G) \) has property \( P \). Letting \( G \) act trivially on a set consisting of one point, we obtain from Theorem 3.1 the following [21, 2.35]:

**Corollary 3.2.** Let \( G \) be a locally compact group. The following are equivalent:

(a) \( G \) is amenable.
(b) \( VN(G) \) is injective and \( G \) is inner amenable.

**Corollary 3.3.** Let \( G \) be an \([IN]\)-group. Then \( VN(G) \) is injective if and only if \( G \) is amenable.

**Corollary 3.4** (Losert and Rindler [17]). Let \( G \) be a connected locally compact group. Then \( G \) is amenable if and only if \( G \) is inner amenable.

**Proof.** If \( G \) is inner amenable, let \( U \) be a compact neighborhood of \( e \). Then \( G_0 = \bigcup_{n=1}^\infty U^n \) is an open (and hence closed), compactly generated subgroup of \( G \). Since \( G \) is connected, \( G = G_0 \). Let \( K \) be a compact normal subgroup such that \( G/K \) is separable metrizable (see [13, p. 71]). Clearly \( G/K \) is connected and inner amenable (Proposition 6.2). However \( VN(G/K) \) is injective [5, p. 112]. So \( G/K \) is amenable by Theorem 3.1. Hence \( G \) is also amenable. \( \Box \)

4. \([IN]\)-GROUPS AND INNER AMENABILITY

Let \( G \) be an \([IN]\)-group. Then there exists a compact neighborhood \( V \) of \( e \) such that \( x^{-1}Vx = V \) for each \( x \in G \). In this section we find necessary conditions such that there exists an inner invariant mean \( m \) on \( L^\infty(G) \) with \( m(1_V) = 0 \). We first establish the following general lemma.

**Lemma 4.1.** Let \( G \) be a locally compact group. Let \( \{\pi, H\} \) be a continuous unitary representation of \( G \). Let \( \eta_0 \in H \), \( \eta_0 \neq 0 \), and \( \pi(x)\eta_0 = \eta_0 \) for all \( x \in G \). Let \( H_0 = \{\eta \in H; \langle \eta, \eta_0 \rangle = 0\} \) and \( Q \in \mathcal{B}(H) \) be defined by \( Q(\eta) = \langle \eta, \eta_0 \rangle \eta_0/||\eta_0||^2 \). The following are equivalent:

(a) \( Q \notin C^*_\pi(G) \) (the \( C^* \)-algebra generated by \( \{\pi(x); x \in G\} \)).
(b) **There exists a net** \( \theta_\alpha \in H_0 \) **such that** \( \| \theta_\alpha \| = 1 \), **and** \( \| \pi(x) \theta_\alpha - \theta_\alpha \| \to 0 \) **for each** \( x \in G \).

(c) **There exists a state** \( \omega \) **on** \( B(H) \) **such that** \( \omega(\pi(x)) = 1 \) **for each** \( x \in G \) **and** \( \omega(Q) = 0 \).

**Proof.** (a) \( \Rightarrow \) (b) We follow an idea contained in the proof of [3, Theorem 1.1]. Suppose (b) fails; then we can find \( y_1, \ldots, y_M \in G \) and \( \varepsilon > 0 \), such that for all \( \theta \in H_0 \), \( \| \theta \| = 1 \), there exists some \( i \), \( 1 \leq i \leq M \), such that \( \| \pi(y_i) \theta - \theta \| > \varepsilon \).

Let \( x_1 = e \), the identity of \( G \), and \( x_2 = y_1, \ldots, x_{M+1} = y_M \). Let \( N = M + 1 \) and \( A = N^{-1} \sum_{k=1}^{N} \pi(x_k) \). We claim that \( \| A \|_{H_0} < 1 \). If not, we can find a sequence \( \theta_n \in H_0 \), \( \| \theta_n \| = 1 \), such that

\[
\| A(\theta_n) \|_2^2 = \langle A(\theta_n), A(\theta_n) \rangle = \frac{1}{N^2} \sum_{i,j} \langle \pi(x_j^{-1} x_i) \theta_n, \theta_n \rangle \to 1.
\]

Since \( \| \langle \pi(x_j^{-1} x_i) \theta_n, \theta_n \rangle \| \leq 1 \) for each \( i, j \), we conclude that

\[
\text{Re}(\pi(x_j^{-1} x_i) \theta_n, \theta_n) \to 1.
\]

But then

\[
\| \pi(x_j) \theta_n - \pi(x_i) \theta_n \|_2^2 = 2 - \text{Re}(\pi(x_j^{-1} x_i) \theta_n, \theta_n) \to 0
\]

as \( n \to \infty \). In particular, since \( x_1 = e \) and \( x_{k+1} = y_k, \ k = 1, \ldots, M \), we conclude that

\[
\lim_n \| \pi(y_k) \theta_n - \theta_n \|_2 = 0 \quad \text{for each } k, \ 1 \leq k \leq m.
\]

This contradicts the choice of \( y, \ldots, y_M \). Thus \( \| A \|_{H_0} < 1 \) as claimed.

Observe now that if \( \eta \in H \), then

(1) \( Q(\eta) = A^m(Q(\eta)) \). Indeed, if \( x \in G \), then

\[
\pi(x) Q(\eta) = \frac{1}{\| \eta_0 \|^2} \langle \eta, \eta_0 \rangle \pi(x)(\eta_0) = \frac{1}{\| \eta_0 \|^2} \langle \eta, \eta_0 \rangle \eta_0 = Q(\eta)
\]

by the invariance of \( \eta_0 \).

(2) \( \eta - Q(\eta) \in H_0 \). Indeed,

\[
\langle \eta - Q(\eta), \eta_0 \rangle = \langle \eta, \eta_0 \rangle - \frac{1}{\| \eta_0 \|^2} \langle \eta, \eta_0 \rangle \langle \eta_0, \eta_0 \rangle = 0.
\]

Hence we have for \( m \) fixed and \( \eta \in H \),

\[
\| (A^m - Q)\eta \|_2 = \| A^m(\eta - Q\eta) \| \quad \text{(by (1))}
\]

\[
\leq \| A^m \|_{H_0} \| \eta - Q\eta \| \quad \text{(by (2))}
\]

\[
\leq 2 \| A \|_{H_0}^m \| \eta \|.
\]

\[
\therefore \| A^m - Q \| \leq 2 \| A \|_{H_0}^m \to 0, \quad \text{i.e., } Q \in C_\pi^*(G).
\]
(b) ⇒ (c) Let \( \omega_\alpha = (T\theta_\alpha, \theta_\alpha) \) and \( \omega \) be a weak*-cluster point of \( \{\omega_\alpha\} \) in \( \mathcal{B}(H) \). Then clearly \( \omega(\pi(x)) = 1 \) for each \( x \in G \), and \( \omega(Q) = 0 \).

(c) ⇒ (a) If \( X = \sum_{i=1}^n \lambda_i(x_i) \), then
\[
\|X - Q\| \geq |\omega(X) - \omega(Q)| = \left| \sum_{i=1}^n \lambda_i \right|
\]
and
\[
\|X - Q\| \geq |\langle (X - Q)\theta, \theta \rangle| \quad \text{(where } \theta = \frac{\eta_0}{\|\eta_0\|})
\]
\[
= \left| \frac{1}{\|\eta_0\|^2} \left( \sum \lambda_i \eta_0, \eta_0 \right) - \left( \frac{1}{\|\eta_0\|^2} \left( \frac{\eta_0}{\|\eta_0\|}, \eta_0 \right) \eta_0, \frac{\eta_0}{\|\eta_0\|} \right) \right|
\]
\[
= \left| \sum \lambda_i - \frac{1}{\|\eta_0\|^2} \eta_0, \eta_0 \eta_0, \eta_0 \right|
\]
\[
= \left| \sum \lambda_i - 1 \right|
\]

Hence \( \|X - Q\| \geq \max\{\|\sum x_i\|, |1 - \sum \lambda_i|\} \geq \frac{1}{2} \). \( Q \notin C_\pi^*(G) \).  □

For each \( x \in G \), let \( \pi(x)f(t) = f(x^{-1}tx)\Delta(x)^{1/2}, t \in G, f \in L^2(G) \). Then \( \{\pi, L^2(G)\} \) is a continuous unitary representation of \( G \). Let \( C_\pi^*(G) \) denote the \( C^* \)-algebra generated by \( \{\pi(x); x \in G\} \) in \( \mathcal{B}(L^2(G)) \). A discrete version of the following result is proved in [20].

**Theorem 4.2.** Let \( G \) be a locally compact group and \( V \) be a compact neighborhood of \( e \) such that \( x^{-1}Vx = V \) for all \( x \in G \). Let \( L^2_0(V) = \{g \in L^2(G); \int_V g(x) \, dx = 0\} \). Consider the following conditions on \( G \):

(a) The operator \( Q_V(f) = \frac{1}{|V|^2} \int_V f(x) \, dx \cdot 1_V \) is not in \( C_\pi^*(G) \).

(b) There exists a net \( \{h_\alpha\} \) in \( L^2_0(V) \) such that \( \|h_\alpha\|_2 = 1 \) and 
\[
\|\pi(x)h_\alpha - h_\alpha\|_2 \to 0 \quad \text{for each } x \in G.
\]

(c) There exists a state \( \omega \) on \( \mathcal{B}(H) \) such that \( \omega(\pi(x)) = 1 \) for each \( x \in G \), and \( \omega(Q) = 0 \).

(d) There exists an inner invariant mean \( m \) on \( L^\infty(G) \) such that \( m(l_V) = 0 \).

Then \( (a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \).

**Proof.** That \( (a) \Leftrightarrow (b) \Leftrightarrow (c) \) follows from Lemma 4.1.

(d) ⇒ (b) Indeed, as in Losert and Rindler, there exists a net \( \nu_\alpha \in L^1(G) \), \( \nu_\alpha \geq 0, \|\nu_\alpha\|_1 = 1, \nu_\alpha(V) = 0 \), and \( \|\pi(x)\nu_\alpha\|_1 \to 0 \). Let \( h_\alpha = \nu_\alpha^{1/2} \); then 
\[
\|\pi(x)h_\alpha - h_\alpha\|_2 \to 0 \quad \text{for all } x \in G, \|h_\alpha\|_2 = 1.
\]
Furthermore,
\[
\left| \int_V h_\alpha \, dx \right| = \langle h_\alpha 1_V, 1_V \rangle \leq \left( \int_V h_\alpha^2 \, dx \right)^{1/2} = 0,
\]
i.e., \( h_\alpha \in L^2_0(V) \). □
Open Problem. Is (d) equivalent to the other conditions in Theorem 4.2? (This is the case when $G$ is discrete and $V = \{e\}$ as shown in [20].)

Lemma 4.3. Let $G$, $\{\pi, H\}$, $\eta_0$, $H_0$, and $Q$ be as in Lemma 4.1. If $Q \in C^*_\pi(G)$, then each linear form $I$ on $H$ which is invariant under $\{\pi(x) : x \in G\}$ is continuous, and has the form

$$I(\eta) = \frac{\alpha}{\|\eta_0\|^2} \langle \eta, \eta_0 \rangle,$$

where $\alpha = I(\eta_0)$.

Proof. As in the proof of Lemma 4.1, (a) $\Rightarrow$ (b), there exists $x_1, \ldots, x_{N+1} \in G$, such that $x_1 = e$, and the operator $A = (N + 1)^{-1} \sum_{k=1}^{N+1} \pi(x_k)\alpha$ satisfies $\|A\|_{H_0} < 1$. In particular, for each $\theta_0 \in H_0$, the series $\theta = \sum_{n=0}^{\infty} A^n(\theta_0)$ converges in $H_0$. Also,

$$\theta_0 = \theta - A\theta = \frac{N\theta}{N+1} - \frac{1}{N+1} \sum_{i=2}^{N+1} \pi(x_i)\alpha \theta$$

$$= \sum_{i=2}^{N+1} (\gamma - \pi(x_i)\gamma) \quad \text{with} \quad \gamma = \frac{\theta}{N+1}.$$

Let $\eta \in H$; then $\theta_0 = \eta - Q(\eta) \in H_0$. So

$$\eta = \eta - Q(\eta) + Q(\eta) = \sum_{i=2}^{N+1} (\gamma - \pi(x_i)\gamma) + Q(\eta).$$

So if $I$ is invariant on $H$, then

$$I(\eta) = I(Q(\eta)) = \frac{1}{\|\eta_0\|^2} \langle \eta, \eta_0 \rangle I(\eta_0).$$

The following is an analogue of the main result in [27].

Theorem 4.4. Let $G$ be a locally compact group and $V$ be a neighborhood of $e$ such that $x^{-1}Vx = V$ for all $x \in G$, $0 < \lambda(V) < \infty$. If $Q_V \in C^*_\pi(G)$, then each linear form $I$ on $L^2(G)$ which is invariant under inner automorphism is continuous and has the form

$$I(f) = \frac{\alpha}{\lambda(V)} \int_V f \, dx,$$

where $\alpha = I(1_V)$.

Proof. This follows from Lemma 4.3. □

Corollary 4.5. Let $G$ be an I.C.C. discrete group with Kazhdan's property T. Then every inner invariant linear form on $L^2(G)$ is continuous.

Proof. In this case $\delta_e$ is the only inner invariant mean on $L^\infty(G)$. By Paschke's Theorem [20], $Q_V \in C^*_\pi(G)$ when $V = \{e\}$. Apply Theorem 4.4. □

Corollary 4.6. Let $G$ be the free group on two generators. Then every inner invariant form on $L^2(G)$ is continuous.

Proof. By the result of Effros [7], $\delta_e$ is the only inner invariant mean on $L^\infty(G)$. Apply now Paschke's Theorem [20] and Theorem 4.4. □
Let \( V \) be a measurable subset of a locally compact group \( G \). Let \( L^2(V) = \{ f \in L^2(G) : f|_V = 0 \} \). Then \( L^2(V) \) is a closed subspace of \( L^2(G) \) and \( L^2(G) = L^2(V) \oplus L^2(G \sim V) \). Let \( P_V \) be the orthogonal projection of \( L^2(V) \).

**Proposition 4.7.** Let \( G \) be a locally compact group. Let \( V \) be a measurable subset of \( G \) such that \( xVx^{-1} = V \) for all \( x \in G \). Suppose there exist inner invariant means \( m, n \) such that \( m(V) = 0 \) and \( n(G \sim V) = 0 \). Then \( \| T - P_A \| \geq \frac{1}{2} \) for each \( T \in C^*_\pi(G) \).

**Proof.** Using an idea of Namioka [19], we may find nets \( \{f_\delta\} \) and \( \{g_\alpha\} \) of positive norm one functions in \( L^1(G) \) such that \( f_\delta(A) = 0 \), \( g_\alpha(G \sim A) = 0 \), \( \| \pi(x)f_\delta - f_\delta \|_1 \to 0 \), and \( \| \pi(x)g_\alpha - g_\alpha \|_1 \to 0 \) (here \( \pi(x)f(t) = f(x^{-1}tx)\Delta(x) \), \( f \in L^1(G) \) \( (x, t \in G) \)). Let \( f_\delta' = f_\delta^{1/2} \) and \( g_\alpha' = g_\alpha^{1/2} \). Then \( f_\delta' \) and \( g_\alpha' \) are positive norm one functions in \( L^2(G) \), \( f_\delta'(A) = 0 \), \( g_\alpha'(G \sim A) = 0 \), \( \| \pi(x)f_\delta' - f_\delta' \|_2 \to 0 \), and \( \| \pi(x)g_\alpha' - g_\alpha' \|_2 \to 0 \). Let \( x_1, \ldots, x_n \in G, \alpha_1, \ldots, \alpha_n \in C, \) and \( T = \sum_{i=1}^n \alpha_i \pi(x_i) \). Then

\[
\| T - P_A \| \geq \lim \sup_{\delta} \| Tf_\delta - P_A f_\delta \|_2
= \lim \sup_{\delta} \left\| \sum_{i=1}^n \alpha_i \pi(x_i)f_\delta \right\|_2 = \left\| \sum_{i=1}^n \alpha_i \right\|.
\]

Also

\[
\| T - P_A \| \geq \lim \sup_{\alpha} \| Tg_\alpha - P_A g_\alpha \|_2
= \lim \sup_{\alpha} \left\| \sum_{i=1}^n \alpha_i \pi(x_i)g_\alpha - g_\alpha \right\| + \left( \sum_{i=1}^n \alpha_i - 1 \right) g_\alpha
= \left\| \sum_{i=1}^n \alpha_i \right\| - \left( \sum_{i=1}^n \alpha_i - 1 \right).
\]

Hence

\[
\| T - P_A \| \geq \max \left\{ \left\| \sum_{i=1}^n \alpha_i \right\|, \left\| \sum_{i=1}^n \alpha_i - 1 \right\| \right\} \geq \frac{1}{2}. \quad \square
\]

## 5. A FIXED POINT PROPERTY

Let \( G \) be a locally compact group. A left Banach \( G \)-module \( X \) is a Banach space \( X \) which is a left \( G \)-module such that

(i) \( \| a \cdot x \| \leq \| x \| \) for all \( x \in X \) and \( a \in G \).

(ii) For all \( x \in X \), the map \( a \to a \cdot x \) is continuous from \( G \) into \( X \).

In this case, we define \( \langle f \cdot a, x \rangle = \langle f, a \cdot x \rangle \) for each \( f \in X^*, a \in G \), and \( x \in X \).

If \( \mu \in M(G) \) and \( f \in X^* \), we define

\[
\langle f \cdot \mu, x \rangle = \int \langle f, a \cdot x \rangle \, d\mu(a), \quad x \in X.
\]
Then \( f \cdot \mu \in X^* \), \( f \cdot \mu = f \cdot a \) if \( \mu = \delta_a \), and \( (f \cdot \mu_1) \cdot \mu_2 = f \cdot (\mu_1 \ast \mu_2) \) for \( \mu_1, \mu_2 \in M(G) \). Finally if \( a \in G \), \( \mu \in M(G) \), and \( m \in X^{**} \), we also define

\[
\langle a \cdot m, f \rangle = \langle m, f \cdot a \rangle \quad \text{and} \quad \langle \mu \cdot m, f \rangle = \langle m, f \cdot \mu \rangle
\]

for all \( f \in X^* \).

By the weak* operator topology (\( W^* \)OT) on \( \mathcal{B}(X^{**}) \), we shall mean the weak* topology of \( \mathcal{B}(X^{**}) \) when it is identified with the dual space \( (X^{**} \otimes X^*)^* \) in the obvious way. This topology is determined by the seminorms \( \{P_{f,m}; f \in X^*, m \in X^{**}\} \) where \( P_{f,m}(T) = |\langle Tm, f \rangle| \). Of course, the unit ball in \( \mathcal{B}(X^{**}) \) is compact in the \( W^* \)OT.

For each \( \phi \in L^1(G) \), let \( T_\phi \in \mathcal{B}(X^{**}) \) be defined by \( T_\phi(m) = \phi \cdot m \), \( m \in X^{**} \). Let \( \mathcal{P}_x^{**} \) denote the closure of \( \{T_\phi; \phi \geq 0, \|\phi\|_1 = 1\} \) in the \( W^* \)OT. Then \( \mathcal{P}_x^{**} \) with the \( W^* \)OT is compact and convex. Also if \( a \in G \), let \( T_a \in \mathcal{B}(X^{**}) \) be defined by \( T_a(m) = a \cdot m \), \( m \in X^{**} \). Inner amenability can be characterized by the following “fixed point property”.

**Theorem 5.1.** Let \( G \) be a locally compact group. The following are equivalent:

(a) \( G \) is inner amenable.

(b) Whenever \( X \) is a left Banach \( G \)-module there exists \( T \in \mathcal{P}_x^{**} \) such that \( T_aT = TT_a \) for all \( a \in G \).

**Proof.** (a) \( \Rightarrow \) (b) Let \( \{\phi_a\} \) be a net in \( L^1(G) \), \( \phi_a \geq 0 \), \( \|\phi_a\|_1 = 0 \), such that \( \|\delta_a \ast \phi_a - \phi_a \ast \delta_a\|_1 \to 0 \) for each \( a \in G \) [17, Proposition 1]. Since \( \{T_{\phi_a}\} \) is contained in the unit ball of \( \mathcal{B}(X^{**}) \) and the unit ball is compact in the \( W^* \)OT, we may assume by passing to a subnet if necessary that \( T_{\phi_a} \to T \) in the \( W^* \)OT, \( T \in \mathcal{B}(X^{**}) \) and \( \|T\| \leq 1 \). Now if \( a \in G \) and \( m \in X^{**} \), then

\[
\|T_aT_{\phi_a}m - T_{\phi_a}T_a m\| = T_{\phi_a \ast \delta_a}(m) - T_{\phi_a \ast \delta_a}(m) \leq \|\delta_a \ast \phi_a - \phi_a \ast \delta_a\|_1 \|m\| \to 0.
\]

On the other hand, \( T_aT_{\phi_a} \to T_aT \) and \( T_{\phi_a}T_a \to TT_a \) in the \( W^* \)OT. In particular \( T_aT = TT_a \).

(b) \( \Rightarrow \) (a) Let \( X = L^1(G) \) and consider \( L^1(G) \) as a left \( G \)-module where \( a \cdot h = l_{a^{-1}}h \), \( a \in G \), \( h \in L^1(G) \). Given \( m \in L^\infty(G)^* \), \( f \in L^\infty(G) \), define \( m_L(f) \in L^\infty(G) \) by

\[
\langle m_L(f), \phi \rangle = \langle m, \frac{1}{\Delta} \hat{f} \ast \phi \rangle, \quad \phi \in L_1(G).
\]

Define \( \langle \tilde{T}_n(m), f \rangle = \langle n, m_L(f) \rangle \), \( n \in L^\infty(G)^* \), \( f \in L^\infty(G) \). Then, as readily checked, \( \tilde{T}_\phi = T_\phi \) for each \( \phi \in L^1(G) \). Furthermore, the map \( n \to \tilde{T}_n \) from \( L^\infty(G)^* \) into \( \mathcal{B}(L^\infty(G)^*) \) is continuous when \( L^\infty(G)^* \) has the weak* topology and \( \mathcal{B}(L^\infty(G)^*) \) has the \( W^* \)OT. Hence

\[
\mathcal{P}_{L^\infty(G)^*} = \{\tilde{T}_n; n \in L^\infty(G)^*, n \geq 0, \text{ and } \|n\| = 1\}.
\]
By assumption, there exists \( n \in L^\infty(G)^* \), \( n \geq 0 \), \( \|n\| = 1 \), such that
\[
(1) \quad T_a \tilde{T}_n = \tilde{T}_n T_a \quad \text{for all } a \in G.
\]

Next we observe that
\[
(2) \quad \langle \langle T_a m \rangle_L(f), \phi \rangle = \langle m_L(f), \phi \ast \delta_{a^{-1}} \rangle
\]
for each \( a \in G \), \( m \in L^\infty(G)^* \), and \( f \in L^\infty(G) \).

Hence if \( \{\psi_a\} \) is a bounded approximate identity of \( L^1(G) \) and \( m \) is a weak* cluster point of \( \psi_a \), then (by (2))
\[
\langle T_a(m)_L(f), \phi \rangle = \langle m_L(f), \phi \ast \delta_a \rangle = \lim_{a} \langle \psi_a, \frac{1}{\Delta} (\phi \ast \delta_a)^* \ast f \rangle
= \langle \phi \ast \delta_a, f \rangle = \langle r_a f, \phi \rangle
\]
for any \( f \in L^\infty(G) \) and \( \phi \in L^1(G) \), i.e.,
\[
(3) \quad T_a(m)_L(f) = r_a f
\]
Also
\[
\langle T_a \tilde{T}_n(m), f \rangle = \langle \tilde{T}_n(m), l_a f \rangle = \langle n \circ m, l_a f \rangle
= \langle n, m_L(l_a f) \rangle = \langle n, l_a m_L(f) \rangle
= \langle l_a^n, f \rangle = \langle n, l_a f \rangle.
\]
Combining this with (1) and (3), we obtain that \( \langle n, l_a f \rangle = \langle n, r_a f \rangle \) for any \( f \in L^\infty(G) \) and \( a \in G \), i.e., \( n \) is an inner invariant mean. \( \square \)

6. Miscellaneous results

Proposition 6.1. Let \( G \) be a separable connected group. Then the following are equivalent:

(a) \( G \) admits a countably additive inner invariant mean.

(b) \( G \) is an \([IN]\)-group.

(c) \( G \) is an extension of a compact group by a vector group.

Proof. (a) \( \Rightarrow \) (b) Let \( B(G) = \{x \in G: \text{the conjugacy class of } x \text{ has relatively compact closure}\} \). By [9, Theorem 1.4], there exists a layering of \( G \) that terminates with the closed subgroup \( B(G) \), i.e., a sequence
\[
B(G) = X_0 \subset X_1 \subset \cdots \subset X_m = G
\]
such that each \( X_k \) is a closed subset of \( G \) invariant under the inner automorphisms and every point \( x \in X_k \) has a relative neighborhood in \( X_k \) with infinitely many disjoint conjugates. Suppose that \( m \) is a countably additive inner invariant mean and suppose that \( m(B(G)) = 0 \). Then \( m(X_k \sim X_{k-1}) > 0 \) for some \( k \). By separability, there exists a relatively open set \( U \) in \( X_k \sim X_{k-1} \).
with \( m(U) > 0 \) and a sequence \( \{x_n\} \) with \( \{x_n Ux_n^{-1}\} \) pairwise disjoint. This contradicts \( m(G) = 1 \). So \( m(B(G)) > 0 \), and hence \( \lambda(B(G)) > 0 \), where \( \lambda \) is the left Haar measure on \( G \). Consequently \( B(G) \) is an open \([FC]\)-subgroup of \( G \). In particular \( G \) is an \([IN]\)-group \([15, \text{Corollary 2.2}]\).

That \((b) \Leftrightarrow (c)\) for connected groups is well known \([11, \text{Corollary 2.8}]\). Also \((b) \Rightarrow (a)\) is clear. \(\Box\)

**Proposition 6.2.** Let \( G \) be a locally compact group and \( H \) be a closed normal subgroup of \( G \). If \( G \) is inner amenable, then \( G/H \) is also inner amenable.

**Proof.** Define a map \( \phi: L^\infty(G/H) \to L^\infty(G) \) by \( \phi(f) = f \circ \theta \), where \( \theta \) is the quotient map of \( G \) onto \( G/H \). Then, as is well known (see \([25, \text{pp. 66 and 82}]\)), \( \rho \) is a linear isometry from \( L^\infty(G/H) \) into the subspace \( A \) of \( L^\infty(G) \), where

\[
A = \{ f \in L^\infty(G) ; r_x f = f \text{ for all } x \in G \}.
\]

Furthermore \( \rho(\pi(x)f) = (\pi(x)f) \circ \theta \) for each \( x \in G \), where \( \pi = \pi G \). Let \( m \) be an inner invariant mean on \( L^\infty(G) \). Define \( m'(f) = m(\rho(f)) \), \( f \in L^\infty(G/H) \). Then, as is readily checked, \( m' \) is an inner invariant mean on \( L^\infty(G/H) \). \(\Box\)

**References**