INNER AMENABLE LOCALLY COMPACT GROUPS

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Abstract. In this paper we study the relationship between amenability, inner amenability and property P of a von Neumann algebra. We give necessary conditions on a locally compact group G to have an inner invariant mean m such that \( m(V) = 0 \) for some compact neighborhood V of G invariant under the inner automorphisms. We also give a sufficient condition on G (satisfied by the free group on two generators or an I.C.C. discrete group with Kazhdan's property T, e.g., SL(n, \( \mathbb{Z} \)), \( n \geq 3 \)) such that each linear form on \( L^2(G) \) which is invariant under the inner automorphisms is continuous. A characterization of inner amenability in terms of a fixed point property for left Banach G-modules is also obtained.

Introduction

Let G be a locally compact group. Then G is called inner amenable if there exists a state \( m \) on \( L^\infty(G) \), such that \( m(\pi(a)f) = m(f) \) for all \( a \in G \) and \( f \in L^\infty(G) \), where

\[
\pi(a)f(x) = f(a^{-1}xa), \quad x \in G.
\]

Amenable locally compact groups and [IN]-groups are inner amenable. The group G is [IN] if there exists a compact neighborhood V of the identity e in G such that \( a^{-1}Va = V \) for all \( a \in G \). Furthermore when G is connected, then G is amenable if and only if G is inner amenable (see [17]). A recent account of amenability is given in [21].

Let \( \mathcal{M} \) be a von Neumann algebra on a Hilbert space H and let \( \mathcal{M}' \) be the commutant of \( \mathcal{M} \). For \( T \in \mathcal{B}(H) \) (the space of bounded linear operators on H), let \( C_T \) be the weak*-closed convex subset of \( \mathcal{B}(H) \) generated by \{ \( U^*TU ; U \in \mathcal{M}' \) \}, where \( \mathcal{M}' \) is the group of unitary elements in \( \mathcal{M} \). (Note that \( \mathcal{B}(H) \) has a unique predual [28, p. 47].) \( \mathcal{M} \) is said to have property P if \( C_T \cap \mathcal{M}' \neq \emptyset \) for each \( T \in \mathcal{B}(H) \).

Let VN(G) denote the von Neumann algebra on \( L^2(G) \) generated by \{ \( l_x ; x \in G \) \} where \( l_x h(t) = h(xt), \ T \in G \). A well-known result of Schwartz [29] asserts
that if \( G \) is discrete, then \( G \) is amenable if and only if \( \text{VN}(G) \) has property \( P \). In §3 we study the relation between amenability, inner amenability, and property \( P \) of a von Neumann algebra determined by \( G \) and its action on a locally compact Hausdorff space \( X \). In particular, we provide the missing link in the following well-known implications for a locally compact group \( G \):

\[
\text{Amenability} \quad \iff \quad \text{Inner Amenability} \quad \iff \quad \text{Property } P \text{ (for } \text{VN}(G)\text{)}
\]

In [20] Paschke proved that if \( G \) is an infinite discrete group, then there exists an inner invariant mean on \( l^\infty(G) \) different from the point evaluation at the identity if and only if the \( C^* \)-algebra generated by the unitaries on \( l^2(G) \) corresponding to conjugation by elements in \( G \) does not contain the projection on the space \( C_0(e) \), where \( e \) is the identity of \( G \). In §4, we find necessary conditions for there to exist an inner invariant mean \( m \) on \( L^\infty(G) \) such that \( m(1_v) = 0 \) (when \( V \) is a compact neighborhood of \( G \) invariant under inner automorphisms). We also give a sufficient condition on \( G \) (Theorem 4.4) such that each linear form \( I \) on \( L^2(G) \) which is invariant under inner automorphisms is continuous and has the form \( I(f) = \frac{1}{V} \int_V f \, d\rho \), where \( \rho = I(1_v) \).

In particular (Corollary 4.5 and 4.6) if \( G \) is the free group on two generators or a discrete group with Kazhdan’s property \( T \) and every nontrivial conjugacy class in \( G \) is infinite (e.g., \( \text{SL}(n, \mathbb{Z}), n \geq 3 \)), then every inner invariant linear form on \( l^2(G) \) is continuous. (See [18] for a discussion of similar problems.)

It is well known (see [6 or 26]) that amenability of a locally compact group \( G \) may be characterized in terms of fixed points for affine maps on compact convex sets. In §5, we characterize inner amenability of \( G \) in terms of a fixed point property for left Banach \( G \)-modules. Finally in §6, a few miscellaneous results on inner amenability are stated and proved.

The literature on inner amenability has grown substantially in recent years: see [1, 2, 7, 14, 16, 17, 20, 31].

2. Preliminaries and some notations

Throughout this paper \( G \) denotes a locally compact group with a fixed left Haar measure \( \lambda \). The spaces \( L^p(G) \), \( 1 \leq p < \infty \), of measurable functions will be as defined in [13]. For each \( a \in G \), \( 1 \leq p < \infty \), let \( \pi(a) \) be the operator on \( L^p(G) \) defined by

\[
\pi(a)f(t) = f(a^{-1}ta)\Delta^{1/p}(a), \quad a, t \in G, \quad f \in L^p(G),
\]

where \( \Delta \) is the modular function on \( G \). The group \( G \) is called amenable if there exists a mean \( m \) on \( L^\infty(G) \) (i.e., \( m \in L^\infty(G), m \geq 0 \), and \( \|m\| = 1 \)) such that \( m(l_a f) = m(f) \) for all \( a \in G \) and \( f \in L^\infty(G) \). As is well known [8, Theorem 2.2.1], this is equivalent to the existence of a left invariant mean on \( U_r(G) \), the space of bounded right uniformly complex-valued continuous
functions on $G$ (as defined in [13, p. 21]). All abelian groups and all compact groups are amenable. However, if $G$ contains the free group on two generators as a closed subgroup (e.g., if $G = \text{SL}(2, \mathbb{R})$), then $G$ is not amenable (see [8, 21, 23] for details).

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $H$. If $\mathcal{M}$ has property $P$, then there exists a projection of norm one $E$ from $\mathcal{B}(H)$ onto $\mathcal{M}'$ with $E(1) = 1$ (see [28, p. 207, or 24, p. 136]). Von Neumann algebras with this latter property are called injective. As is well known [4], injectivity and property $P$ are equivalent. For a discussion of the various forms of amenability for von Neumann algebras, see [21, 2, 35].

If $X$ is a subset of a locally convex space $E$ with topology $\tau$, then $\overline{co}^\tau X$ will denote the closed convex hull of $X$ in $E$.

3. Inner amenability, amenability, and injectivity

A reference for the definitions below in the discrete cases is Zimmer [32]. Let $X$ be a locally compact Hausdorff space. Let $G$ act invertibly on $X$ on the right such that the mapping $X \times G \to X$ defined by $(x, g) \to x \cdot g$, $x \in X$, $g \in G$, is jointly continuous. Let $\mu$ be a nonnegative quasi-invariant Radon measure on $X$. We define $L^p(X \times G, \mu \times \lambda)$ or simply $L^p(X \times G)$, $1 \leq p \leq \infty$, as the usual $L^p$-spaces of Borel functions identified when they coincide off a locally $(\mu \times \lambda)$-null set in $X \times G$. For each $a \in G$, define $\mu_a(E) = \mu(Ea)$. Then, by quasi-invariance of $\mu$, we have $\mu_a \ll \mu$ for each $a \in G$ and there is, by the Radon-Nikodym theorem, a locally $\mu$-integrable function $r(\cdot, a)$ such that

$$\int f(xa^{-1}) d\mu(x) = \int f(x) r(x, a) d\mu(x)$$

for all $f \in L^1(X)$ ($= L^1(X, \mu)$). It follows that $r(x, ab) = r(x, a)r(xa, b)$ for $a, b \in G$, and $r(x, e) = 1$. For $u \in G$ and $\phi \in L^\infty(X)$, define the operators $U_a$, $V_a$, $M_\phi$, and $N_\phi$ on $L^2(X \times G)$ by

$$U_a f(x, b) = f(xa, ba)r(x, a)^{1/2} \Delta(a)^{1/2},$$

$$V_a f(x, b) = f(x, a^{-1}b),$$

$$M_\phi f(x, b) = \phi(x)f(x, b),$$

$$N_\phi f(x, b) = \phi(xb^{-1})f(x, b),$$

where $f \in L^2(X \times G)$.

Then each $U_a$, $V_a$ is a unitary operator on $L^2(X \times G)$. Let $\mathcal{M}$ be the von Neumann algebra generated by the operators $V_a$, $M_\phi$ ($a \in G$, $\phi \in L^\infty(X)$), and $\mathcal{R}$ be the von Neumann algebra generated by the operators $U_a$, $M_\phi$ ($a \in G$, $\phi \in L^\infty(X)$). If $J \in \mathcal{R}(L^2(X \times G))$ is given by

$$(Jf)(x, b) = f(xb^{-1}, b^{-1})r(x, b^{-1})^{1/2} \Delta(b^{-1})^{1/2},$$

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then $J^2f = f$, $JV_aJ = U_a$, and $JN_aJ = M_a$. So $J$ implements a spatial isomorphism between $\mathcal{L}$ and $\mathcal{R}$. Therefore $\mathcal{L}$ has property $P$ if and only if $\mathcal{R}$ has property $P$.

If $a \in G$, let $\delta_a$ denote the Dirac measure on $G$ concentrated at $a$. For any $f \in L^\infty(X)$, the function $(\delta_a \square f)(x) = f(xa)$ is defined $\mu$-locally almost everywhere on $X$ (see [10, Lemma 2.1]). Furthermore, $\delta_a \square f \in L^\infty(X)$. A linear functional $m$ on $L^\infty(X)$ is called a mean if $m(1) = 1$ and $m(f) \geq 0$ whenever $f \geq 0$. A mean $m$ is $G$-invariant if $m(\delta_g \square f) = m(f)$ for all $g \in G$.

Also for any $\phi \in L^1(X)$ and $x \in G$, let $\delta_x \ast \phi \in L^1(X)$ be defined by

$$
\delta_x \ast \phi(\xi) = \left(\frac{d\mu_x}{d\mu}\right)(\xi)\phi(x^{-1}\xi) \quad \mu\text{-a.e. on } X
$$

(see [10, Lemma 2.2]), where $\mu_x(E) = \mu(x^{-1}E)$.

Theorem 3.1 below, in the special case where $X$ is a singleton, is proved in [21, p. 85].

**Theorem 3.1.** Let $G$, $X$, and $\mathcal{L}$ be as above. Then the following are equivalent:

(a) $G$ is amenable.

(b) $\mathcal{L}$ is injective, $L^\infty(X)$ has a $G$-invariant mean, and $G$ is inner amenable.

**Proof.** (a) $\Rightarrow$ (b) If $G$ is amenable, then $G$ is inner amenable since every invariant mean on $G$ is inner invariant. It follows from [10, Theorem 3.1] that $L^\infty(X)$ has a $G$-invariant mean.

To see that $\mathcal{L}$ is injective, we first note that the von Neumann algebra generated by $\{N_\phi; \phi \in L^\infty(X)\}$ has property $P$ (by the Markov-Katutani fixed point theorem). Hence $\mathcal{D} = \{N_\phi; \phi \in L^\infty(X)\}'$ is injective [28, Proposition 4.4.15]. (In fact, $\mathcal{D}$ is the von Neumann algebra generated by the $N_\phi$.) It suffices to show that there is a norm one projection from $\mathcal{D}$ onto $\mathcal{L}' = \{V_a; a \in G\}' \cap \mathcal{D}$.

For then $\mathcal{L}'$ is injective and so $\mathcal{L}$ is also injective.

Let $T \in \mathcal{D}$ and $a \in G$. Then $V_a^{-1}TV_a \in \mathcal{D}$. Indeed $\mathcal{D}$ is generated by the $N_\phi$'s; hence we may assume $T = N_\phi$. If $x \in X$, $b \in G$, and $f \in L^2(X \times G)$, we have

$$(V_{a^{-1}}N_\phi V_a f)(x, b) = (N_\phi V_a f)(x, ab) = \phi(xb^{-1}a^{-1})(V_a f)(x, ab) = (N_{a^{-1}\phi} f)(x, b),$$

i.e., $V_{a^{-1}}N_\phi V_a = N_{a^{-1}\phi} \in D$. The result follows.

Let $K_T$ denote the weak$^*$-compact convex subset of $\mathcal{D}$. Consider the action of $G$ on $K_T$ defined by

$$(a, S) \rightarrow V_a^{-1}SV_a.$$

Then the action is separately continuous in the weak operator topology WOT, which agrees with the $w^*$-topology on $K_T$. Indeed, if $a_\alpha \rightarrow a_0$ and $S \in K_T$,
then $V_{a_n} \to V_a$ and $V_{a_n}^{-1} \to V_{a_n}^{-1}$ in the strong operator topology (SOT). In particular, $SV_{a_n} \to SV_a$ in the SOT, and so $V_{a_n}^{-1}SV_{a_n} \to V_{a_n}^{-1}SV_a$ in the SOT (since multiplication is jointly continuous on bounded sets in the SOT). Hence $V_{a_n}^{-1}SV_{a_n} \to V_{a_n}^{-1}SV_a$ in the WOT. Now if $a \in G$, and $S_a \to S$ in the WOT, then for any $\eta, \xi \in L_2(G \times X),$

$\langle V_{a_n}^{-1}SV_a \xi, \eta \rangle = \langle S_a \xi, V_a \eta \rangle = \langle SV_a \xi, V_a \eta \rangle = \langle V_{a_n}^{-1}SV_a \xi, \eta \rangle,$

i.e., $V_{a_n}^{-1}SV_a \to V_{a_n}^{-1}SV_a$ in the WOT. Apply now Rickert's generalization of Day's fixed point theorem to obtain $S \in K_T$ such that $V_{a_n}^{-1}SV_a = S$ for all $a \in S$, i.e., $SV_a = V_a$ for all $a \in S$. So $S \in \{V_a: a \in S\} \cap \mathcal{D}$. Consequently, there exists a projection $Q: \mathcal{D} \to \{V_a: a \in G\} \cap \mathcal{D}$ such that $Q(T) \in K_T$ for all $T \in \mathcal{D}$, $Q(I) = I$, and $\|Q\| = 1$ by Yeadon's Theorem [30].

(b) $\Rightarrow$ (a) Define a left and a right action of $G$ on $L^\infty(X \times G)$ by

(1) $(Fa)(x, b) = F(x, ab), \quad (aF)(x, b) = F(xa, ba)$.

Using (1) and the equalities $r(x, ab) = r(x, a)r(xa, b)$ a.e. $x$, and $r(x, e) = 1$ for all $x \in G$, one shows that

(2) $(F, V_a f) = (Fa, f), \quad \langle F, U_a f \rangle = \langle a^{-1}F, f \rangle$

$(F \in L^\infty(X \times G), f \in L^1(X \times G))$. Here (with a slight abuse of notation),

$V_a f(x, b) = f(x, a^{-1}b), \quad U_a f(x, b) = f(xa, ba)r(x, a)\Delta(a)$

$(f \in L^1(X \times G), x \in X, a, b \in G)$.

We now show that there exists a positive linear functional $m'$ with $\|m'\| = 1$ such that

(3) $m'(aFa^{-1}) = m'(F)$

for all $a \in G$ and $F \in L^\infty(X \times F)$.

Since $L^\infty(X)$ has a $G$-invariant mean, an argument similar to that of Namioka [19] shows that there exists a net $\{\delta_a \phi \}_{a \in G}$ in $P_1(X) = \{\phi \in L^1(X): \phi \geq 0$ and $\|\phi\|_1 = 1\}$ such that $\|\delta_a \phi - \delta_a \phi\| \to 0$ for each $a \in G$. Also since $G$ is inner amenable, there exists a net $\{\mu_\beta\}$ in $P_1(G)$ such that $\|\delta_a \mu_\beta - \delta_a \mu_\beta\|_1 \to 0$ (see [17, Proposition 1]). Let

$m_{\alpha, \beta}(F) = \int F d(\phi_\alpha \times \mu_\beta),$

where $F \in L^\infty(X \times G)$. Then $\{m_{\alpha, \beta}\}$ is bounded in $L^\infty(X \times C)^*$. Further-
more, if \( a \in G \) and \( F \in L^\infty(X \times G) \), then
\[
\|\langle m_\alpha, \beta \rangle, a F a^{-1} \rangle - \langle m_\alpha, \beta \rangle, F \rangle\|
\leq \\left| \int \int F(xa, a^{-1}ba) d\phi_\alpha(x) d\mu_\beta(b) - \int \int F(x, b) d\phi_\alpha(x) d\mu_\beta(b) \right|
\leq \|\delta_\alpha * \phi_\alpha - \phi_\alpha\| \|F\|_\infty + \|\delta_\alpha * \mu_\beta * \delta_\alpha^{-1} - \mu_\beta\| \|F\|_\infty
\]
which converges to zero. Hence if \( m' \) is any weak*-cluster point of the \( \{m_\alpha, \beta\} \), then \( m' \) satisfies (3).

By (3) and an idea of Namioka [19] there exists a net \( \{f_\delta\} \) in \( L_1(X \times G) \), \( f_\delta \geq 0 \), \( \|f_\delta\|_1 = 1 \) such that \( \|(V a^{-1} - U a) b_\delta\|_1 \to 0 \). Let \( g_\delta = f_\delta^{1/2} \). Note that \( g_\delta \in L_2(X \times G) \), \( g_\delta \geq 0 \), and \( \|g_\delta\|_2 = 1 \). Then \( (V a^{-1} g_\delta)^{1/2} = V a g_\delta \), \( (U a g_\delta)^{1/2} = U a g_\delta \), and hence
\[
\|(V a^{-1} - U a) g_\delta\|_2 \to 0 \quad \text{for all} \ a \in G.
\]
For each \( F \in L^\infty(X \times G) \), let \( L_F \in B(L_2(X \times G)) \) be defined by
\[
L_F f(x, b) = F(x, b)f(x, b).
\]
Then, as readily checked,
\[
V a L_F V a^{-1} = L_{F a^{-1}}
\]
for each \( a \in G \). Let \( H \) denote the group of unitary elements in the von Neumann algebra \( B \) with the strong operator topology. Let \( \psi_\delta \) be a function on \( H \) defined by \( \psi_\delta(F)(U) = \langle U L_F U^* g_\delta, g_\delta \rangle \) \( (U \in H) \). Then \( \psi_\delta \in U_1(H) \).

Also
\[
\psi_\delta(F a^{-1})(U) = \langle U L_{F a^{-1}} U^* g_\delta, g_\delta \rangle
\]
\[
= \langle U V a L_F V a^{-1} U^* g_\delta, g_\delta \rangle
\]
\[
= \langle U L_F U^* (V a^{-1} g_\delta), V a^{-1} g_\delta \rangle
\]
using (5) and the fact that each \( V a^{-1} \) is in the commutant of \( B \). Also
\[
\psi_\delta(F)V a^{-1}(U) = \langle U L_F U^*(V a g_\delta), V a g_\delta \rangle.
\]
So
\[
\|[\psi_\delta(F a^{-1}) - \psi_\delta(F)U a^{-1}](U)\|
\leq \|U L_F U^* V a^{-1} g_\delta, V a^{-1} g_\delta\| - \|U L_F U^* V a g_\delta, V a g_\delta\|
\leq \|F\|_\infty \|V a^{-1} - V a\|_2 \|g_\delta\|_2.
\]
Since \( \mathcal{L} \) is injective, \( \mathcal{L} \) must have property \( P \). So \( \mathcal{R} \) also has property \( P \). By a result of de la Harpe [12], there exists a left invariant mean \( m \) on \( U_r(H) \), the space of bounded right uniformly continuous functions on \( H \). Hence using (4) and (7), we have

\[
|m(\psi_\delta(Fa^{-1})) - m(\psi_\delta(F))| \to 0.
\]

Let \( n_\delta = m \circ \psi_\delta \). Then \( n_\delta \) is a mean on \( L^\infty(X \times G) \). Let \( n \) be a weak*-cluster point of \( \{n_\delta\} \). Then \( n(Fa^{-1}) = n(F) \) for all \( F \in L^\infty(X \times G) \) and \( a \in G \). Define

\[
\tilde{n}(\phi) = n(1 \otimes \phi), \quad \phi \in L^\infty(G).
\]

Then \( \tilde{n} \) is a left invariant mean on \( L^\infty(G) \). Hence \( G \) is amenable. \( \square \)

A well-known result of Schwartz [29] asserts that if \( G \) is discrete then \( G \) is amenable if and only if \( VN(G) \) has property \( P \). Letting \( G \) act trivially on a set consisting of one point, we obtain from Theorem 3.1 the following [21, 2.35]:

**Corollary 3.2.** Let \( G \) be a locally compact group. The following are equivalent:

(a) \( G \) is amenable.

(b) \( VN(G) \) is injective and \( G \) is inner amenable.

**Corollary 3.3.** Let \( G \) be an \([IN]\)-group. Then \( VN(G) \) is injective if and only if \( G \) is amenable.

**Corollary 3.4** (Losert and Rindler [17]). Let \( G \) be a connected locally compact group. Then \( G \) is amenable if and only if \( G \) is inner amenable.

**Proof.** If \( G \) is inner amenable, let \( U \) be a compact neighborhood of \( e \). Then \( G_0 = \bigcup_{n=1}^{\infty} U^n \) is an open (and hence closed), compactly generated subgroup of \( G \). Since \( G \) is connected, \( G = G_0 \). Let \( K \) be a compact normal subgroup such that \( G/K \) is separable metrizable (see [13, p. 71]). Clearly \( G/K \) is connected and inner amenable (Proposition 6.2). However \( VN(G/K) \) is injective [5, p. 112]. So \( G/K \) is amenable by Theorem 3.1. Hence \( G \) is also amenable. \( \square \)

### 4. \([IN]\)-GROUPS AND INNER AMENABILITY

Let \( G \) be an \([IN]\)-group. Then there exists a compact neighborhood \( V \) of \( e \) such that \( x^{-1}Vx = V \) for each \( x \in G \). In this section we find necessary conditions such that there exists an inner invariant mean \( m \) on \( L^\infty(G) \) with \( m(1_V) = 0 \). We first establish the following general lemma.

**Lemma 4.1.** Let \( G \) be a locally compact group. Let \( \{\pi, H\} \) be a continuous unitary representation of \( G \). Let \( \eta_0 \in H, \eta_0 \neq 0, \) and \( \pi(x)\eta_0 = \eta_0 \) for all \( x \in G \). Let \( H_0 = \{\eta \in H; \langle \eta, \eta_0 \rangle = 0\} \) and \( Q \in \mathcal{B}(H) \) be defined by \( Q(\eta) = \langle \eta, \eta_0 \rangle \eta_0/||\eta_0||^2 \). The following are equivalent:

(a) \( Q \notin C^*_\pi(G) \) (the \( C^* \)-algebra generated by \( \{\pi(x); x \in G\} \)).
(b) There exists a net \( \theta_\alpha \in H_0 \) such that \( \| \theta_\alpha \| = 1 \), and \( \| \pi(x) \theta_\alpha - \theta_\alpha \| \to 0 \) for each \( x \in G \).

(c) There exists a state \( \omega \) on \( \mathcal{B}(H) \) such that \( \omega(\pi(x)) = 1 \) for each \( x \in G \) and \( \omega(Q) = 0 \).

Proof. (a) \( \Rightarrow \) (b) We follow an idea contained in the proof of [3, Theorem 1.1]. Suppose (b) fails; then we can find \( y_1, \ldots, y_M \in G \) and \( \epsilon > 0 \), such that for all \( \theta \in H_0 \), \( \| \theta \| = 1 \), there exists some \( i, 1 < i < M \), such that \( \| \pi(y_i)^\theta \| > \epsilon \).

Let \( x_1 = e \), the identity of \( G \), and \( x_2 = y_1, \ldots, x_{M+1} = y_M \). Let \( N = M + 1 \) and \( A = N^{-1} \sum_{k=1}^{N} \pi(x_k) \). We claim that \( \| A \|_{H_0} < 1 \).

Observe now that if \( \theta \in H \), then

\[
\| A(\theta) \|_2 = \langle A(\theta), A(\theta) \rangle = \frac{1}{N^2} \sum_{i,j} \langle \pi(x_j^{-1} x_i) \theta, \theta \rangle \to 1.
\]

Since \( \| \langle \pi(x_j^{-1} x_i) \theta, \theta \rangle \| \leq 1 \) for each \( i, j \), we conclude that

\[
\text{Re}(\langle \pi(x_j^{-1} x_i) \theta, \theta \rangle) \to 1.
\]

But then

\[
\| \pi(x_j) \theta - \pi(x_i) \theta \|_2^2 = 2 - \text{Re}(\langle \pi(x_j^{-1} x_i) \theta, \theta \rangle) \to 0
\]

as \( n \to \infty \). In particular, since \( x_1 = e \) and \( x_{k+1} = y_k, k = 1, \ldots, M \), we conclude that

\[
\lim_n \| \pi(y_k) \theta - \theta \|_2 = 0 \quad \text{for each } k, 1 \leq k \leq M.
\]

This contradicts the choice of \( y, \ldots, y_M \). Thus \( \| A \|_{H_0} < 1 \) as claimed.

Observe now that if \( \eta \in H \), then

(1) \( Q(\eta) = A^m(Q(\eta)) \). Indeed, if \( x \in G \), then

\[
\pi(x)Q(\eta) = \frac{1}{\| \eta_0 \|^2} \langle \eta, \eta_0 \rangle \pi(x)(\eta_0).
\]

by the invariance of \( \eta_0 \).

(2) \( \eta - Q(\eta) \in H_0 \). Indeed,

\[
\langle \eta - Q(\eta), \eta_0 \rangle = \langle \eta, \eta_0 \rangle - \frac{1}{\| \eta_0 \|^2} \langle \eta, \eta_0 \rangle \langle \eta_0, \eta_0 \rangle = 0.
\]

Hence we have for \( m \) fixed and \( \eta \in H \),

\[
\| (A^m - Q) \eta \|_2 = \| A^m (\eta - Q \eta) \| \quad \text{(by (1))}
\]

\[
\leq \| A^m \|_{H_0} \| \eta - Q \eta \| \quad \text{(by (2))}
\]

\[
\leq 2 \| A \|_{H_0}^m \| \eta \|.
\]

\[
\therefore \| A^m - Q \| \leq 2 \| A \|_{H_0}^m \to 0, \quad \text{i.e., } Q \in C^*_\pi(G).
\]
(b) $\Rightarrow$ (c) Let $\omega_\alpha = \langle \theta_\alpha, \theta_\alpha \rangle$ and $\omega$ be a weak*-cluster point of $\{\omega_\alpha\}$ in $\mathcal{B}(H)$. Then clearly $\omega(\pi(x)) = 1$ for each $x \in G$, and $\omega(Q) = 0$.

(c) $\Rightarrow$ (a) If $X = \sum_{i=1}^n \lambda_i(x_i)$, then
\[
\|X - Q\| \geq |\omega(X) - \omega(Q)| = \left| \sum_{i=1}^n \lambda_i \right|
\]
and
\[
\|X - Q\| \geq |\langle X - Q, \theta \rangle| \quad \text{(where $\theta = \frac{\eta_0}{\|\eta_0\|}$)}
\]
\[
= \left| \left( \frac{1}{\|\eta_0\|^2} \sum \lambda_i \eta_0, \eta_0 \right) - \left( \frac{1}{\|\eta_0\|^2} \sum \eta_0, \eta_0 \right) \eta_0 \right|
\]
\[
= \left| \sum \lambda_i - \frac{1}{\|\eta_0\|^2} \sum \frac{1}{\|\eta_0\|^2} \langle \eta_0, \eta_0 \rangle \eta_0 \right|
\]
\[
= \left| \sum \lambda_i - 1 \right|.
\]
Hence $\|X - Q\| \geq \max\{|\sum \lambda_i|, |1 - \sum \lambda_i|\} \geq \frac{1}{2}$. \(\Box\)

For each $x \in G$, let $\pi(x)f(t) = f(\pi^{-1}(tx))\Delta(x)^{1/2}$, $t \in G$, $f \in L^2(G)$. Then $\{\pi, L^2(G)\}$ is a continuous unitary representation of $G$. Let $\mathbb{C}_\pi^*(G)$ denote the $C^*$-algebra generated by $\{\pi(x); x \in G\}$ in $\mathcal{B}(L^2(G))$. A discrete version of the following result is proved in [20].

**Theorem 4.2.** Let $G$ be a locally compact group and $V$ be a compact neighborhood of $e$ such that $x^{-1}Vx = V$ for all $x \in G$. Let $L^2_0(V) = \{g \in L^2(G); \int_V g(x) \, dx = 0\}$. Consider the following conditions on $G$:

(a) The operator $Q_{\pi}(f) = \frac{1}{|V|} \int_V f(x) \, dx \cdot 1_V$ is not in $\mathbb{C}_\pi^*(G)$.

(b) There exists a net $\{h_\alpha\}$ in $L^2_0(V)$ such that $\|h_\alpha\|_2 = 1$ and $\|\pi(x)h_\alpha - h_\alpha\|_2 \to 0$ for each $x \in G$.

(c) There exists a state $\omega$ on $\mathcal{B}(H)$ such that $\omega(\pi(x)) = 1$ for each $x \in G$, and $\omega(Q) = 0$.

(d) There exists an inner invariant mean $m$ on $L^\infty(G)$ such that $m(1_V) = 0$.

Then (a) $\iff$ (b) $\iff$ (c) $\iff$ (d).

**Proof.** That (a) $\iff$ (b) $\iff$ (c) follows from Lemma 4.1.

(d) $\Rightarrow$ (b) Indeed, as in Losert and Rindler, there exists a net $\nu_\alpha \in L^1(G)$, $\nu_\alpha \geq 0$, $\|\nu_\alpha\|_1 = 1$, $\nu_\alpha(V) = 0$, and $\|\pi(x)\nu_\alpha\|_1 \to 0$. Let $h_\alpha = \nu_\alpha^{1/2}$, then $\|\pi(x)h_\alpha - h_\alpha\|_1 \to 0$ for all $x \in G$, $\|h_\alpha\|_2 = 1$. Furthermore,
\[
\left| \int_V h_\alpha \, dx \right| = \langle h_\alpha 1_V, 1_V \rangle \leq \left( \int_V h_\alpha^2 \, dx \right)^{1/2} \lambda(V)^{1/2} = 0,
\]
i.e., $h_\alpha \in L^2_0(V)$. \(\Box\)
Open Problem. Is (d) equivalent to the other conditions in Theorem 4.2? (This is the case when $G$ is discrete and $V = \{e\}$ as shown in [20].)

Lemma 4.3. Let $G$, $\{\pi, H\}$, $\eta_0$, $H_0$, and $Q$ be as in Lemma 4.1. If $Q \in C^*_\pi(G)$, then each linear form $I$ on $H$ which in invariant under $\{\pi(x): x \in G\}$ is continuous, and has the form

$$I(\eta) = \frac{\alpha}{\|\eta_0\|^2} \langle \eta, \eta_0 \rangle, \quad \text{where } \alpha = I(\eta_0).$$

Proof. As in the proof of Lemma 4.1, $(a) \Rightarrow (b)$, there exists $x_1, \ldots, x_{N+1} \in G$, such that $x_1 = e$, and the operator $A = (N + 1)^{-1} \sum_{k=1}^{N+1} \pi(x_k)$ satisfies $\|A\|_{H_0} < 1$. In particular, for each $\theta_0 \in H_0$, the series $\theta = \sum_{n=0}^{\infty} A^n(\theta_0)$ converges in $H_0$. Also,

$$\theta_0 = \theta - A\theta = \frac{N\theta}{N + 1} - \frac{1}{N + 1} \sum_{i=2}^{N+1} \pi(x_i)\theta$$

$$= \sum_{i=2}^{N+1} (\gamma - \pi(x_i)\gamma) \quad \text{with } \gamma = \frac{\theta}{N + 1}.$$

Let $\eta \in H$; then $\theta_0 = \eta - Q(\eta) \in H_0$. So

$$\eta = \eta - Q(\eta) + Q(\eta) = \sum_{i=1}^{n} (\gamma - \pi(x_i)\gamma) + Q(\eta).$$

So if $I$ is invariant on $H$, then

$$I(\eta) = I(Q(\eta)) = \frac{1}{\|\eta_0\|^2} \langle \eta, \eta_0 \rangle I(\eta_0). \quad \Box$$

The following is an analogue of the main result in [27].

Theorem 4.4. Let $G$ be a locally compact group and $V$ be a neighborhood of $e$ such that $x^{-1}Vx = V$ for all $x \in G$, $0 < \lambda(V) < \infty$. If $Q_V \in C^*_\pi(G)$, then each linear form $I$ on $L^2(G)$ which is invariant under inner automorphism is continuous and has the form

$$I(f) = \frac{\alpha}{\lambda(V)} \cdot \int_V f \, dx,$$

where $\alpha = I(1_V)$. \quad \Box

Proof. This follows from Lemma 4.3. \quad \Box

Corollary 4.5. Let $G$ be an I.C.C. discrete group with Kazhdan's property $T$. Then every inner invariant linear form on $L^2(G)$ is continuous.

Proof. In this case $\delta_e$ is the only inner invariant mean on $L^\infty(G)$. By Paschke's Theorem [20], $Q_V \in C^*_\pi(G)$ when $V = \{e\}$. Apply Theorem 4.4. \quad \Box

Corollary 4.6. Let $G$ be the free group on two generators. Then every inner invariant form on $L^2(G)$ is continuous.

Proof. By the result of Effros [7], $\delta_e$ is the only inner invariant mean on $L^\infty(G)$. Apply now Paschke's Theorem [20] and Theorem 4.4. \quad \Box
Let $V$ be a measurable subset of a locally compact group $G$. Let $L^2(V) = \{f \in L_2(G) : f|_V = 0\}$. Then $L^2(V)$ is a closed subspace of $L_2(G)$ and $L^2(G) = L^2(V) \oplus L^2(G \sim V)$. Let $P_V$ be the orthogonal projection of $L^2(V)$.

**Proposition 4.7.** Let $G$ be a locally compact group. Let $V$ be a measurable subset of $G$ such that $xVx^{-1} = V$ for all $x \in G$. Suppose there exist inner invariant means $m, n$ such that $m(V) = 0$ and $n(G \sim V) = 0$. Then $\|T - P_A\| \geq \frac{1}{2}$ for each $T \in C^*_\pi(G)$.

**Proof.** Using an idea of Namioka [19], we may find nets $\{f_\delta\}$ and $\{g_\alpha\}$ of positive norm one functions in $L^1(G)$ such that $f_\delta(A) = 0$, $g_\alpha(G \sim A) = 0$, $\|\pi(x)f_\delta - f_\delta\|_1 \to 0$, and $\|\pi(x)g_\alpha - g_\alpha\|_1 \to 0$ (here $\pi(x)f(t) = f(x^{-1}tx)\Delta(x)$, $f \in L_1(G)$, $x, t \in G$). Let $f_\delta' = f_\delta^{1/2}$ and $g_\alpha' = g_\alpha^{1/2}$. Then $f_\delta'$ and $g_\alpha'$ are positive norm one functions in $L^2(G)$, $f_\delta'(A) = 0$, $g_\alpha'(G \sim A) = 0$, $\|\pi(x)f_\delta' - f_\delta'\|_2 \to 0$, and $\|\pi(x)g_\alpha' - g_\alpha'\|_2 \to 0$. Let $x_1, \ldots, x_n \in G$, $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, and $T = \sum_{i=1}^n \alpha_i \pi(x_i)$. Then

$$\|T - P_A\| \geq \limsup_{\delta} \|Tf_\delta - P_Af_\delta\|_2 = \limsup_{\delta} \left\| \sum_{i=1}^n \alpha_i \pi(x_i)f_\delta \right\|_2 = \left\| \sum_{i=1}^n \alpha_i \right\|_2.$$

Also

$$\|T - P_A\| \geq \limsup_{\alpha} \|Tg_\alpha - P_Ag_\alpha\|_2 = \limsup_{\alpha} \left\| \sum_{i=1}^n \alpha_i \pi(x_i)g_\alpha - g_\alpha \right\|_2 = \left\| \sum_{i=1}^n \alpha_i - 1 \right\|_2.$$

Hence

$$\|T - P_A\| \geq \max \left\{ \left| \sum_{i=1}^n \alpha_i \right|, \left| \sum_{i=1}^n \alpha_i - 1 \right| \right\} \geq \frac{1}{2}. \square$$

**5. A FIXED POINT PROPERTY**

Let $G$ be a locally compact group. A **left Banach $G$-module** $X$ is a Banach space $X$ which is a left $G$-module such that

(i) $\|a \cdot x\| \leq \|x\|$ for all $x \in X$ and $a \in G$.

(ii) For all $x \in X$, the map $a \mapsto a \cdot x$ is continuous from $G$ into $X$.

In this case, we define $(f \cdot a, x) = (f, a \cdot x)$ for each $f \in X^*$, $a \in G$, and $x \in X$.

If $\mu \in M(G)$ and $f \in X^*$, we define

$$(f \cdot \mu, x) = \int (f \cdot a \cdot x) \, d\mu(a), \quad x \in X.$$
Then $f \cdot \mu \in X^*$, $f \cdot \mu = f \cdot a$ if $\mu = \delta_a$, and $(f \cdot \mu_1) \cdot \mu_2 = f \cdot (\mu_1 \ast \mu_2)$ for $\mu_1, \mu_2 \in M(G)$. Finally if $a \in G$, $\mu \in M(G)$, and $m \in X^{**}$, we also define

$$\langle a \cdot m, f \rangle = \langle m, f \cdot a \rangle \quad \text{and} \quad \langle \mu \cdot m, f \rangle = \langle m, f \cdot \mu \rangle$$

for all $f \in X^*$.

By the weak* operator topology ($W^*OT$) on $\mathcal{B}(X^{**})$, we shall mean the weak* topology of $\mathcal{B}(X^{**})$ when it is identified with the dual space $(X^{**} \otimes X^*)^*$ in the obvious way. This topology is determined by the seminorms $\{P_f, m; f \in X^*, m \in X^{**}\}$ where $p_{f,m}(T) = \|\langle Tm, f \rangle\|$. Of course, the unit ball in $\mathcal{B}(X^{**})$ is compact in the $W^*OT$.

For each $\phi \in L^1(G)$, let $T_\phi \in \mathcal{B}(X^{**})$ be defined by $T_\phi(m) = \phi \cdot m$, $m \in X^{**}$. Let $\mathcal{P}_{X^{**}}$ denote the closure of $\{T_\phi; \phi \geq 0, \|\phi\|_1 = 1\}$ in the $W^*OT$. Then $\mathcal{P}_{X^{**}}$ with the $W^*OT$ is compact and convex. Also if $a \in G$, let $T_a \in \mathcal{B}(X^{**})$ be defined by $T_a(m) = a \cdot m$, $m \in X^{**}$. Inner amenability can be characterized by the following “fixed point property”.

**Theorem 5.1.** Let $G$ be a locally compact group. The following are equivalent:

(a) $G$ is inner amenable.

(b) Whenever $X$ is a left Banach $G$-module there exists $T \in \mathcal{P}_{X^{**}}$ such that $T_a T = T T_a$ for all $a \in G$.

**Proof.** (a) $\Rightarrow$ (b) Let $\{\phi_\alpha\}$ be a net in $L^1(G)$, $\phi_\alpha \geq 0$, $\|\phi_\alpha\|_1 = 0$, such that $\|\delta_a \ast \phi_\alpha - \phi_\alpha \ast \delta_a\|_1 \to 0$ for each $a \in G$ [17, Proposition 1]. Since $\{T_\phi_\alpha\}$ is contained in the unit ball of $\mathcal{B}(X^{**})$ and the unit ball is compact in the $W^*OT$, we may assume by passing to a subnet if necessary that $T_\phi_\alpha \to T$ in the $W^*OT$, $T \in \mathcal{B}(X^{**})$ and $\|T\| \leq 1$. Now if $a \in G$ and $m \in X^{**}$, then

$$\|T_a T_\phi_\alpha m - T_\phi_\alpha T_a m\| = T_{\delta_a \ast \phi_\alpha} (m) - T_{\phi_\alpha \ast \delta_a} (m) \leq \|\delta_a \ast \phi_\alpha - \phi_\alpha \ast \delta_a\|_1 \|m\| \to 0.$$  

On the other hand, $T_a T_\phi_\alpha \to T_a T$ and $T_\phi_\alpha T_a \to T T_a$ in the $W^*OT$. In particular $T_a T = T T_a$.

(b) $\Rightarrow$ (a) Let $X = L^1(G)$ and consider $L^1(G)$ as a left $G$-module where $a \cdot h = l_a h$, $a \in G$, $h \in L^1(G)$. Given $m \in L^\infty(G)^*$, $f \in L^\infty(G)$, define $m_L(f) \in L^\infty(G)$ by

$$\langle m_L(f), \phi \rangle = \langle m, \frac{1}{\Delta} \hat{\phi} \ast f \rangle, \quad \phi \in L_1(G).$$

Define $\langle \widehat{T_n}(m), f \rangle = \langle n, m_L(f) \rangle$, $n \in L^\infty(G)^*$, $f \in L^\infty(G)$. Then, as readily checked, $\widehat{T_n} = T_\phi$ for each $\phi \in L^1(G)$. Furthermore, the map $n \to \widehat{T_n}$ from $L^\infty(G)^*$ into $\mathcal{B}(L^\infty(G)^*)$ is continuous when $L^\infty(G)^*$ has the weak*-topology and $\mathcal{B}(L^\infty(G)^*)$ has the $W^*OT$. Hence

$$\mathcal{P}_{L^\infty(G)^*} = \{\widehat{T_n}; n \in L^\infty(G)^*, n \geq 0, \text{ and } \|n\| = 1\}.$$
By assumption, there exists \( n \in L^\infty(G)^* \), \( n \geq 0 \), \( \|n\| = 1 \), such that

\[
T_a \tilde{T}_n = \tilde{T}_n T_a \quad \text{for all } a \in G.
\]

Next we observe that

\[
\langle (T_a m)_L(f), \phi \rangle = \langle m_L(f), \phi \ast \delta_a^{-1} \rangle
\]

for each \( a \in G \), \( m \in L^\infty(G)^* \), and \( f \in L^\infty(G) \).

Hence if \( \{\psi_a\} \) is a bounded approximate identity of \( L^1(G) \) and \( m \) is a

weak* cluster point of \( \psi_a \), then (by (2))

\[
\langle T_a(m)_L(f), \phi \rangle = \langle m_L(f), \phi \ast \delta_a \rangle = \left\langle m, \frac{1}{\Delta}(\phi \ast \delta_a)^{-1} \ast f \right\rangle
= \lim_{a} \left\langle \psi_a, \frac{1}{\Delta}(\phi \ast \delta_a)^{-1} \ast f \right\rangle = \lim_{a} \langle \phi \ast \delta_a \ast \psi_a, f \rangle
= \langle \phi \ast \delta_a, f \rangle = \langle r_a f, \phi \rangle
\]

for any \( f \in L^\infty(G) \) and \( \phi \in L^1(G) \), i.e.,

\[
T_a(m)_L(f) = r_a f
\]

Also

\[
\langle T_a \tilde{T}_n(m), f \rangle = \langle \tilde{T}_n(m), l_a f \rangle = \langle n \circ m, l_a f \rangle
= \langle n, m_l(l_a f) \rangle = \langle n, l_a m_L(f) \rangle
= \langle l_a^* n, f \rangle = \langle n, l_a f \rangle.
\]

Combining this with (1) and (3), we obtain that \( \langle n, l_a f \rangle = \langle n, r_a f \rangle \) for any

\( f \in L^\infty(G) \) and \( a \in G \), i.e., \( n \) is an inner invariant mean. \( \square \)

6. Miscellaneous results

**Proposition 6.1.** Let \( G \) be a separable connected group. Then the following are equivalent:

(a) \( G \) admits a countably additive inner invariant mean.

(b) \( G \) is an \([IN]\)-group.

(c) \( G \) is an extension of a compact group by a vector group.

**Proof.** (a) \( \Rightarrow \) (b) Let \( B(G) = \{x \in G: \text{the conjugacy class of } x \text{ has relatively compact closure}\} \). By [9, Theorem 1.4], there exists a layering of \( G \) that terminates with the closed subgroup \( B(G) \), i.e., a sequence

\[
B(G) = X_0 \subset X_1 \subset \cdots \subset X_m = G
\]

such that each \( X_k \) is a closed subset of \( G \) invariant under the inner automorphisms and every point \( x \in X_k \sim X_{k-1} \) has a relative neighborhood in \( X_k \) with infinitely many disjoint conjugates. Suppose that \( m \) is a countably additive inner invariant mean and suppose that \( m(B(G)) = 0 \). Then \( m(X_k \sim X_{k-1}) > 0 \) for some \( k \). By separability, there exists a relatively open set \( U \) in \( X_k \sim X_{k-1} \). This implies that the restriction of \( m \) to \( U \) is a countably additive inner invariant mean. Then \( m(B(G)) > 0 \) by the inner invariance property of \( m \). This is a contradiction, so \( m(B(G)) = 0 \). Therefore, \( G \) is an \([IN]\)-group. (b) \( \Rightarrow \) (c) Let \( H \) be a compact subgroup of \( G \) and \( V \) be a vector group. Then \( G \) is an extension of \( H \) by \( V \).

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with \( m(U) > 0 \) and a sequence \( \{x_n\} \) with \( \{x_n U x_n^{-1}\} \) pairwise disjoint. This contradicts \( m(G) = 1 \). So \( m(B(G)) > 0 \), and hence \( \lambda(B(G)) > 0 \), where \( \lambda \) is the left Haar measure on \( G \). Consequently \( B(G) \) is an open \([FC]\)-subgroup of \( G \). In particular \( G \) in an \([IN]\)-group [15, Corollary 2.2].

That (b) \( \iff \) (c) for connected groups is well known [11, Corollary 2.8]. Also (b) \( \implies \) (a) is clear. \( \square \)

**Proposition 6.2.** Let \( G \) be a locally compact group and \( H \) be a closed normal subgroup of \( G \). If \( G \) is inner amenable, then \( G/H \) is also inner amenable.

**Proof.** Define a map \( \phi: L^\infty(G/H) \to L^\infty(G) \) by \( \phi(f) = f \circ \theta \), where \( \theta \) is the quotient map of \( G \) onto \( G/H \). Then, as is well known (see [25, pp. 66 and 82]), \( \rho \) is a linear isometry from \( L^\infty(G/H) \) into the subspace \( A \) of \( L^\infty(G) \), where

\[
A = \{ f \in L^\infty(G); r_x f = f \text{ for all } x \in G \}.
\]

Furthermore \( \rho(\pi(\hat{x}) f) = (\pi(x) f) \circ \theta \) for each \( x \in G \), where \( \hat{x} = xH \). Let \( m \) be an inner invariant mean on \( L^\infty(G) \). Define \( m'(f) = m(\rho(f)) \), \( f \in L^\infty(G/H) \). Then, as is readily checked, \( m' \) is an inner invariant mean on \( L^\infty(G/H) \). \( \square \)

**References**


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