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MINIMAL SUBMANIFOLDS OF $\mathbf{E}^{2n+1}$ ARISING FROM DEGENERATE $SO(3)$ ORBITS ON THE GRASSMANNIAN

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ABSTRACT. We give new examples of minimal submanifolds of $\mathbf{E}^{2n+1}$ characterised by having their Gauss map's image lie in degenerate $SO(3)$ orbits of $G_{p,2n+1}$, the Grassmannian of p-planes in $\mathbf{E}^{2n+1}$ (where the action on $G_{p,2n+1}$ is induced from the irreducible $SO(3)$ action on $\mathbf{R}^{2n+1}$). These submanifolds are all given explicitly in terms of holomorphic data and are linearly full in $\mathbf{E}^{2n+1}$.

INTRODUCTION

This paper is an example of a new technique for studying minimal submanifolds. The technique may be regarded as generalizing the use of calibrations. A calibration on a Riemannian manifold is a closed unit comass p-form that determines a subset of the Grassmann bundle (called a face) having a minimizing property; any submanifold of the original manifold whose Gauss map's image is contained entirely in a face is area minimizing in its holomology class (see [HaL]). Here we consider other subsets of the Grassmannian, not determined by calibrations but still having a minimality (but perhaps not minimizing) property. We will call these subsets m-subsets. (General questions regarding faces and m-subsets of the Grassmannian will be discussed in [L].)

The m-subsets in this paper are degenerate $SO(3)$ orbits for the $SO(3)$ actions on $G_{p,2n+1}$, the Grassmannian of p-planes in $\mathbf{E}^{2n+1}$ (Euclidean $2n+1$ space), induced from the irreducible representation of $SO(3)$ on $\mathbf{R}^{2n+1}$. Since $SO(3)$ preserves no nontrivial forms on $\mathbf{R}^{2n+1}$ under this action, the subsets cannot arise from calibrations. If one considers the action as an almost faithful $SU(2)$ action instead, then the analogous two-dimensional orbits on $G_{2,4}$ are just the complex Grassmannians, $G^{\mathbf{C}}_{1,2}$ (which are calibrated by the Kahler form) and one recovers the classical fact that complex submanifolds of $\mathbf{C}^2$ when considered as real submanifolds of $\mathbf{E}^4$ are minimal (in fact area minimizing). The solution minimal submanifolds to the examples of this paper share a trait with their classical counterparts, in that they may be written explicitly in terms

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of holomorphic data. However, in the classical case the holomorphic functions are defined as graphs in $\mathbf{C}^2 = \mathbf{E}^4$ and in our case the functions are naturally defined on an auxiliary manifold and need to be transported to $\mathbf{E}^{2n+1}$ in order to obtain the solution submanifolds. The solutions turn out to be ruled nontrivially by $(p-2)$-planes and the base submanifold of the rulings is the transported complex curve. The simplest of these solutions for minimal 3-folds in $\mathbf{R}^5$ (the transported zero function) already gives an interesting submanifold, the cone over the so-called real Veronese surface in S^4 (as described in [M] and [HsL]).

In §1, we set up the principal coframe bundles we work on and describe the $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ action on $\mathbf{R}^{2n+1}$ explicitly. In §2, we prove the two-dimensional $SO(3)$ orbits are involutive m-subsets and set up the equations for the minimal submanifolds. We also show that no three-dimensional orbits are involutive m-subsets by proving the stronger statement that there are no three-dimensional involutive m-subsets of any $G_{p,n}$. In §3 we introduce complex notation, describe the auxiliary manifold on which we solve for the complex curve, and prove the existence of solutions depending on the complex curve. §4 begins the computation to describe the solutions explicitly in coordinates on $\mathbf{E}^{2n+1}$ which is finished in §5 for 3-folds in $\mathbf{E}^5$.

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1. The $\mu(SU(2))$ coframe bundle and its structure equations

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mu\colon G \to SO(N)$ be a representation of G in $SO(N)$. Let u_0 be a global framing of $T\mathbf{E}^N$, i.e. $u_0|_x\colon T_x\mathbf{E}^N \to \mathbf{R}^N$ is a linear isomorphism for all $x \in \mathbf{E}^N$. Let $\mathscr{F}_{\mu(G)}(\mathbf{E}^N)$, or simply $\mathscr{F}$, denote the space of pairs (x, u) where $x \in \mathbf{E}^N$ and $u = g^{-1}u_0$ where $g \in \mu(G)$. $\mathscr{F}$ is called the $\mu(G)$-principal coframe bundle (G acts on $\mathscr{F}$ by $L_g(x, u) = (x, gu)$). It comes equipped with a canonical left invariant $\mathbf{R}^N$ valued one-form, defined by $\omega_{(x,u)} := g^{-1}dx$, where $u = g^{-1}u_0$, and a $\mu(\mathfrak{g})$-valued connection form $\phi_{(x,u)} := g^{-1}dg$. We have the structure equations

$$d\omega = -\phi \wedge \omega, \qquad d\phi = -\phi \wedge \phi. \tag{1.1}$$

In this paper we will take $G = SO(3) = SU(2)/\mathbf{Z}_2$, and $\mu\colon SO(3) \to SO(2n+1)$ will be the unique irreducible representation of $SO(3)$ on $\mathbf{R}^{2n+1}$. For geometric reasons that will become clear later we take advantage of the isomorphism $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ and work with $\mathfrak{su}(2)$. If we write the Maurer-Cartan form of $\mathfrak{su}(2)$ as

$$\psi = \frac{1}{2}\begin{pmatrix} i\rho & -\bar{\pi} \\ \pi & -i\rho \end{pmatrix}, \qquad \rho \in \mathbf{R}, \quad \pi = \alpha + i\beta \in \mathbf{C}, \tag{1.2}$$

and take a basis of $\mathbf{R}^{2n+1}$ such that $\mu(\mathfrak{su}(2))$ has the following Maurer-Cartan form

$$(1.3)\qquad \Psi = \begin{pmatrix} 0 & -c_1^t\widehat{\Pi} & 0 & \cdots & \cdots & \cdots & 0 \\ c_1\widehat{\Pi} & J & -c_2{}^t\Pi & 0 & \cdots & \cdots & 0 \\ 0 & c_2\Pi & 2J & -c_3{}^t\Pi & 0 & \cdots & 0 \\ \vdots & 0 & c_3\Pi & 3J & -c_4{}^t\Pi & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & c_{n-1}\Pi & (n-1)J & -c_n{}^t\Pi \\ 0 & 0 & \cdots & \cdots & 0 & c_n\Pi & nJ \end{pmatrix},$$

where

$$\widehat{\Pi} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \qquad \Pi = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix}$$

and

$$c_1 = \sqrt{\binom{n+1}{2}}, \qquad c_k = \sqrt{\frac{1}{2}\left(\binom{n+1}{2} - \binom{k}{2}\right)}, \qquad 2 \le k \le n,$$

then the entries of Ψ have the same structure equations as those of ψ, namely,

$$(1.4)\qquad d\rho = \frac{i}{2}\pi \wedge \bar{\pi}, \quad d\pi = i\rho \wedge \pi.$$

This basis is natural in the sense that the truly natural basis on $\mathbf{C}^{2n+1}$ of increasing weights along the diagonal (see (5.1)) is not convenient since the standard $\mathfrak{so}(2n+1)$ is skew symmetric, so we pair the weights $k\rho$ and $-k\rho$ in 2×2 blocks arranged in increasing order along the diagonal. This and requiring the same structure equations of the standard basis of $\mathfrak{su}(2)$ determine the basis.

2. Description of the m-subset orbits

Let $\mathscr{F}_{SO(N)}$ denote the bundle of oriented orthonormal coframes of $\mathbf{E}^N$ and consider $\mathscr{F}_{\mu(SO(3))}$ as a submanifold. On $\mathscr{F}_{SO(N)}$ we have the semibasic forms ω^i, and the connection forms ϑ_s^r where we may write

$$\vartheta = \begin{pmatrix} \alpha_j^i & \phi_a^i \\ \phi_j^a & \beta_b^a \end{pmatrix}, \qquad \beta_b^a = -\beta_a^b, \ \alpha_j^i = -\alpha_i^j, \ \phi_a^i = {}^t\phi_i^a,$$
$$1 \le i, j \le p, \ 1 \le a, b \le N-p,$$

where the ϕ_j^a are semibasic for the projection $\mathscr{F}_{SO(N)} \to G(p, \mathbf{E}^N)$ and $G(p, \mathbf{E}^N)$ denotes the Grassmann bundle over $\mathbf{E}^N$ whose fibers are isomorphic to $G_{p,N}$. Since $\mathbf{E}^N$ is flat, we will identify each fiber with $G_{p,N}$ and work there. Given an embedding $f\colon M \to \mathbf{E}^N$ let $\hat{f}$ be a lift of f to $\mathscr{F}_{SO(N)}$. We have $\hat{f}^*\phi_i^a = h_{ij}^a \hat{f}^*\omega^j$, where h_{ij}^a are the coefficients of $II_f \in N_{f(M)} \otimes \mathrm{Sym}^2(T^*(f(M)))$, the

second fundamental form (where $N_{f(M)}$ is the induced normal bundle). The image of II_f under the map

$$\text{trace}\colon N_{f(M)} \otimes \mathrm{Sym}^2(T^*(f(M))) \to N_{f(M)} \tag{2.1}$$

is called the mean curvature vector. $f(M)$ is said to be minimal if the mean curvature vector is zero, i.e. if $\Sigma_i h^a_{ii} = 0$ for all $n-p \le a \le n$. Notice that when restricted to $\mathscr{F}_{\mu(SO(3))}$, $\vartheta = \Psi$.

There are two types of $SO(3)$ of orbits on $G_{p,2n+1}$; the generic three-dimensional ones, and the special two-dimensional ones which occur when a maximal torus (an S^1) acts trivially on the orbit. Note that none of these orbits could be the face of a calibration. This is because any such calibration would have to be invariant under the $SO(3)$ action on $\Lambda^p\mathbf{R}^{2n+1}$, but this action has no fixed elements.

Theorem 2.1. *The special $SO(3)$ orbits are m-subsets of $G_{p,2n+1}$, i.e. any $f\colon M \to \mathbf{E}^n$ such that the image of the Gauss map $\gamma_f\colon M \to G_{p,2n+1}$ is contained in a special orbit Σ is minimal.*

Proof. (Case p is odd.) To have an S^1 acting trivially on an orbit implies that we can choose bases such that

$$\Sigma = \mathrm{orb}\{dx^0 \wedge dx^{2i_1-1} \wedge dx^{2i_1} \wedge dx^{2i_2-1} \wedge dx^{2i_2} \wedge \cdots \wedge dx^{2i_q-1} \wedge dx^{2i_q}\},$$

where $(i_1, \dots, i_q) \subset (1, \dots, n)$ and $1 \le i_1 < \cdots < i_q \le n$ and Ψ of (1.3) is the Maurer-Cartan form. In fact take $\Sigma =$ orbit of $dx^0 \wedge \cdots \wedge dx^{p-1}$, the other cases are similar.

We are looking for maps f such that

$$\tilde{f}^*(\omega^0 \wedge \cdots \wedge \omega^{p-1}) \neq 0, \quad \tilde{f}^*(\omega^a) = 0, \qquad p \le \alpha \le 2n,$$

where $\tilde{f}$ is a lift of f to $\mathscr{F}$.

In the language of Exterior Differential Systems (see [BCG^3] or the appendix to [B2]) we look for integral manifolds of the differential ideal $\mathscr{I} = \{\omega^{p+1}, \dots, \omega^{2n}\}$, with independence condition $\Omega = \omega^0 \wedge \cdots \wedge \omega^{p-1}$ (where $\{\omega^{p+1}, \dots, \omega^{2n}\}$ means the differential ideal generated by $\omega^{p+1}, \dots, \omega^{2n}$).

Since exterior differentiation commutes with pullback, on a solution we will have $\tilde{f}^*(d\omega^\alpha) = 0$, or, on the lift of a solution to $\mathscr{F}$, we need $d\omega^\alpha \equiv 0 \bmod \mathscr{I}$. So on any integral element (i.e. any p-plane that is a candidate to being a tangent plane to a solution) we must have $d\omega^\alpha \equiv 0 \bmod \mathscr{I}$. All these are identities except $(q = (p+1)/2)$,

$$d\begin{pmatrix} \omega^{p+1} \\ \omega^{p+2} \end{pmatrix} \equiv -c_q \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \wedge \begin{pmatrix} \omega^{p-1} \\ \omega^p \end{pmatrix} \mod \mathscr{I}. \tag{2.2}$$

So on an integral element we must have

$$\alpha = a\omega^{p-1} + b\omega^p, \quad \beta = b\omega^{p-1} - a\omega^p$$

for some constants a and b. This implies $h^{p+3}, \dots, h^{2n} = 0$, and

$$h^{p+1} = \begin{pmatrix} a & b & 0 & \cdots & 0 \\ b & -a & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad h^{p+2} = \begin{pmatrix} -b & a & 0 & \cdots & 0 \\ a & b & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

so any solution must be minimal.

Case p is even: Here we may reduce to the case $\Sigma = \operatorname{orb}\{dx^{2n-p} \wedge \cdots \wedge dx^{2n}\}$, so $\mathscr{I} = \{\omega^0, \dots, \omega^{2n-p-1}\}$ and on integral elements we need

$$d\omega^0 \equiv \cdots \equiv d\omega^{2n-p+1} \equiv 0 \mod \mathscr{I}.$$

All these are satisfied already except $(q = \frac{2n-p}{2})$,

$$d\begin{pmatrix} \omega^{2n-p-1} \\ \omega^{2n-p} \end{pmatrix} \equiv -c_q \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \wedge \begin{pmatrix} \omega^{2n-p+1} \\ \omega^{2n-p+2} \end{pmatrix} \mod \mathscr{I}. \tag{2.3}$$

So on an integral element we must have

$$\alpha = a\omega^{2n-p+1} + b\omega^{2n-p+2}, \quad \beta = b\omega^{2n-p+1} - a\omega^{2n-p+2}.$$

This implies that $h^0, \dots, h^{2n-p-3} = 0$, and

$$h^{2n-p-2} = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ & & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & a & b \\ 0 & \cdots & 0 & b & -a \end{pmatrix}, \quad h^{2n-p-1} = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & \ddots & & \cdots & \vdots \\ & & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & -b & -a \\ 0 & \cdots & 0 & -a & b \end{pmatrix};$$

so again any solution must be minimal.

Remark. One may think of these $SO(3)$ orbits as "twisted" faces, in that the tangent space (tableau) associated to these orbits is the same as that of the $\mathbf{CP}^1$ face corresponding to the Kahler form on the $\mathbf{C}$ factor of $\mathbf{R}^{2n+1} \cong \mathbf{C}^1 \oplus \mathbf{R}^{2n-1}$ wedged with the dual of a $(p-2)$-plane. The difference is that the $(p-2)$-planes associated to the $SO(3)$ orbits move, while the plane in the $\mathbf{CP}^1$ face is fixed.

Remark. (2.2) and (2.3) may be compared with the equations to get a complex curve in $\mathbf{C}^2 \simeq \mathbf{R}^4$ using the same techniques as this paper. One works on $\mathscr{F}_{R(SU(2))}$, where by $R(SU(2))$ we mean $SU(2)$ embedded as a subgroup of $SO(4)$ using the identification $\mathbf{C}^2 \simeq \mathbf{R}^4$. The reader may wish to carry out this computation while following the computations of this paper to compare with the results presented here. The structure equations on $\mathscr{F}_{R(SU(2))}$ are

$$d\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} = -\begin{pmatrix} 0 & -\rho & -\alpha & -\beta \\ \rho & 0 & \beta & -\alpha \\ \alpha & -\beta & 0 & \rho \\ \beta & \alpha & -\rho & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix}. \tag{2.4}$$

For the differential system one takes $\mathscr{I} = \{\omega^3, \omega^4\}$ and $\Omega = \omega^1 \wedge \omega^2$.

Remark. In §3 we will see that there are abundant solution minimal submanifolds in each dimension which have an open subset of the m-subset Σ as the image of their Gauss map. There exist faces having only trivial smooth solutions (see Proposition 8.7 of [HM] where a face which is an S^3 has no $\mathbf{C}^2$ solutions) and faces having only solution submanifolds whose Gauss map lies in a proper m-subset of the given one. In this paper we restrict attention to m-subsets that have solution submanifolds whose Gauss map's image does not lie in any proper sub-m-subset. In fact, we restrict our attention to *involutive m-subsets* so named because their associated EDS for solution submanifolds is involutive. To dispense with the three-dimensional orbits we prove the more general

Theorem 2.2. *No three-dimensional submanifold Σ of any $G_{p,n}$ is an involutive m-subset.*

Remark. In the proof and lemmas that follow we will use facts and terminology from EDS, so the reader not familiar with the language of EDS may wish to skip to Proposition 2.6. (What follows will not be used elsewhere in this paper.) However, the reader should know that most of the results of this paper were originally found using techniques from EDS and it was possible to reformulate them with just passing reference to EDS only because the relevant systems turned out to be equivalent to the system associated to a familiar PDE system, that of the Cauchy Riemann equations.

Proof. Let Σ be an involutive m-subset and let $f\colon M \to \mathbf{E}^n$ be such that $\gamma_f(M)$ is an open subset of Σ. Using the identification $T_\xi G_{p,n} \cong \xi^\perp \otimes \xi^*$ we have $T_\xi\Sigma \subset \xi^\perp \otimes \xi^*$. Let $T_\xi^{(1)}\Sigma$ denote $(T_\xi\Sigma \otimes \xi^*) \cap (\xi^\perp \otimes \mathrm{Sym}^2\xi^*)$. $T_\xi^{(1)}\Sigma$ may be thought of as the space of possible second fundamental forms for solutions f. More precisely, the tableau of the differential system for lifts to $\mathscr{F}_{SO(n)}$ of maps f such that $\gamma_j(M)^{\text{open}} \subset \Sigma$ is $T_\xi\Sigma \subset \xi^\perp \otimes \xi^*$. Thus $T_\xi^{(1)}\Sigma$ is the space of integral elements of $T_\xi\Sigma$. For example, the lower-left-hand $p \times (2n+1-p)$ block of (1.3) is the associated tableau for maps having images in special $SO(3)$ orbits. (The identification is possible because $G_{p,n}$ factors through the projection $\mathscr{F} \to \mathbf{E}^n$.) Given such an f, by the discussion above (2.1), we see that $II_f|_x$ is well defined as an element of $T_\xi^{(1)}\Sigma$ for each $x \in M$. Thus to guarantee that every potential second fundamental form arising from a Gauss map to Σ is traceless we need $T_\xi^{(1)}\Sigma$ to be in the kernel of the map (2.1) which we now consider as trace: $T_\xi^{(1)}G_{p,n} \to \xi^\perp$. We will call tableau A such that $A^{(1)} \subset \mathrm{Ker(trace)}$ 'm-tableau'.

There are three possible forms an involutive 3-dimensional tableau A could take, which are distinguished by the possibilities for the reduced characters. Namely we could have

Case 1. $s_1' = 3$, $s_2' = 0$,

Case 2. $s_1' = 2$, $s_2' = 1$, $s_3' = 0$, or
Case 3. $s_1' = s_2' = s_3' = 1$, $s_4' = 0$.
The following three lemmas dispose of these cases while proving more general statements hinting at the severe restrictions placed on involutive m-tableau. (A partial classification of all involutive m-tableaux is given in [L].)

Lemma 2.3. *Any involutive tableau with* $s_1' = 2l - 1$, $s_2' = 0$, $l \in \mathbf{Z}^+$, *cannot be an* m*-tableau.*

Proof. Using Guillemin normal form as described in Chapter 4 of [BCG^3], we may write the nonzero part of the tableau as

$$(\pi_i^a) = (C_1\pi, C_2\pi, \dots, C_n\pi),$$

where

$$\pi = \begin{pmatrix} \pi^1 \\ \vdots \\ \pi^{2l-1} \end{pmatrix}$$

are independent one-forms, and $C_i \in M_{s\times s}(\mathbf{R})$, are matrices satisfying $C_iC_j = C_jC_i$, $1 \le i, j \le n$, $C_1 = \mathrm{Id}$.

Integral elements are given by writing the π_i^a as linear combinations of a basis $\omega^1, \dots, \omega^n$ of $\mathscr{J}/\mathscr{I}$, $\pi_i^a = p_{ij}^a\omega^j$ where $p_{ij}^a = p_{ji}^a$. In this case we may arbitrarily specify the $2l-1$ constants p_{11}^a which determine the other p_{ij}^a by

$$p_{ij}^a = (C_i)_b^a (C_j)_c^b p_{11}^c.$$

The minimality condition is that $\sum_{i=1}^n p_{ii}^a = 0$ for all $1 \le a \le 2l-1$. We have

$$\sum_{i=1}^n p_{ii}^a = (C_i)_b^a (C_i)_c^b p_{11}^c,$$

which must equal 0 for all choices for p_{11}^c, i.e. we need $\sum_{i=1}^n (C_i)^2 = 0$ or

$$\mathrm{Id} + \sum_{\sigma=2}^n (C_\sigma)^2 = 0. \tag{2.5}$$

Now since $C_iC_j = C_jC_i$, they share the same eigenspaces. In particular, since s is odd they share a common eigenline along with the $(C_i)^2$'s and that line must have a nonnegative eigenvalue for each $(C_i)^2$. Thus the sum of these will not cancel the $+1$ eigenvalue of Id in that direction, so (2.5) is not possible.

Lemma 2.4. *Any involutive tableau with* $s_1' = l$, $s_2' = 1$, $l \ge 2$, *cannot be an* m*-tableau.*

Proof. The nonzero part of any such tableau can be written as

$$(C_1\Pi + E_1\pi_2^1, \dots, C_n\Pi + E_n\pi_2^1),$$

where

$$\Pi = \begin{pmatrix} \pi_1^1 \\ \vdots \\ \pi_1^l \end{pmatrix}, \quad C_j \in M_{l\times l}, \quad C_1 = \mathrm{Id}, \quad C_2 = \begin{pmatrix} 0 & \cdots & 0 \\ a_1^2 & \cdots & a_n^2 \\ & \vdots & \\ a_1^l & \cdots & a_n^l \end{pmatrix},$$

$$E_j \in M_{l\times 1}, \quad E_1 = (0), \quad E_2 = \begin{pmatrix} 1 \\ e_2^2 \\ \vdots \\ e_2^l \end{pmatrix},$$

and $\pi_1^1, \dots, \pi_1^l$, and π_2^1 are independent one-forms. We may assume that

$$E_2 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

by a $GL(s, \mathbf{R})$ change of coordinates in the $\mathscr{S}$ variables θ^a.

Certain commutation relations hold among the C_j's and the E_j's which we will exploit to prove the lemma. In determining integral elements, one may freely specify p_{11}^q, $1 \le q \le l$, p_{12}^1, and p_{22}^1 and these determine the integral element by

$$\begin{pmatrix} p_{1j}^1 \\ \vdots \\ p_{1j}^l \end{pmatrix} = C_j \begin{pmatrix} p_{11}^1 \\ \vdots \\ p_{11}^l \end{pmatrix} + E_j p_{21}^1$$

and

$$\begin{pmatrix} p_{ij}^1 \\ \vdots \\ p_{ij}^l \end{pmatrix} = C_j \begin{pmatrix} p_{1i}^1 \\ \vdots \\ p_{1i}^l \end{pmatrix} + E_j p_{2i}^1, \tag{2.5}$$

where p_{2i}^1 is gotten using

$$\begin{pmatrix} p_{i2}^1 \\ \vdots \\ p_{i2}^l \end{pmatrix} = C_i \begin{pmatrix} p_{12}^1 \\ \vdots \\ p_{12}^l \end{pmatrix} + E_i p_{22}^1 = C_i \left(C_2 \begin{pmatrix} p_{11}^1 \\ \vdots \\ p_{11}^l \end{pmatrix} + E_2 p_{21}^1 \right) + E_i p_{22}^1. \tag{2.6}$$

Requiring $p_{ij}^a = p_{ji}^a$ gives compatibility conditions

$$C_j(C_i\Pi + E_i p_{21}^1) + E_j p_{2i}^1 = C_i(C_j\Pi + E_j p_{21}^1) + E_i p_{2j}^1,$$

where by (2.6)

$$p_{2i}^1 = (C_i)_k^1 p_{11}^k + e_2^1 p_{21}^1 + e_i^1 p_{22}^1.$$

Comparing the p^1_{22} terms in (2.6), we have $E_j(e^1_i p^1_{22}) = E_i(e^1_j p^1_{22})$, which implies the relation

$$\begin{pmatrix} e^1_j e^1_i \\ \vdots \\ e^l_j e^1_i \end{pmatrix} = \begin{pmatrix} e^1_i e^1_j \\ \vdots \\ e^l_i e^1_j \end{pmatrix}.$$

Since

$$E_2 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

we have $e^r_i = 0$, $2 \le r \le l$. Hence π^2_1 occurs only in the top row, so the p^1_{22} coefficient in the expansion of the first entry of the mean curvature vector is $1 + (e^1_3)^2 + \cdots + (e^1_3)^2$ which is never zero.

Lemma 2.5. *Any involutive tableau of any dimensions, with* $s'_1 = s'_2 = \cdots = s'_l = 1$, *for any* $l \in \mathbf{Z}^+$, *cannot be an m-tableau.*

Proof. We may write the nonzero part of such a tableau as

$$(\pi_1, \pi_2, \dots, \pi_l, *, \dots, *),$$

where $\pi_1, \dots, \pi_l$ are independent and $*$'s are linear combinations of the π_r's. One may arbitrarily specify $p^1_{11} \cdots p^1_{ll}$ and the entries $*$ will only add positive numbers to the sum $\sum_{i=1}^n p^1_{ii}$ (e.g. if the entry in the $(1, m)$th slot is $\lambda^k \pi_k$ then $p^1_{mm} = (\lambda^k)^2 p^1_{kk}$).

Proposition 2.6. *Any submanifold of* $\mathbf{E}^{2n+1}$ *whose Gauss map's image is an open subset of any* $\mu(SO(3))$ *orbit is linearly full in* $\mathbf{E}^{2n+1}$.

Proof. This is clear as the linear span of the $\mu(SO(3))$ orbit of any p-plane is all of $\mathbf{R}^{2n+1}$ as the representation μ is irreducible on $\mathbf{R}^{2n+1}$.

3. Introduction of complex notation and the auxiliary manifold

Notice that in all cases the equations have the same symbol (in the sense of EDS, see [BCG^3]) as the Cauchy-Riemann equations. This suggests introducing complex notation. Assume p is odd $(q = (p-1)/2)$, the even-dimensional cases are similar. Let

$$\begin{aligned} \eta^r &= \omega^{2r-1} + i\omega^{2r}, \qquad 1 \le r \le q, \\ \theta^a &= \omega^{2(a+q)-1} + i\omega^{2(a+q)}, \qquad 1 \le a \le n-q, \end{aligned} \tag{3.1}$$

$\pi = \alpha + i\beta$ and $\bar\eta^r = \omega^{2r-1} - i\omega^{2r}$ etc. ....

We have

$$\begin{aligned} &\mathscr{I} = \{\theta^a, \bar\theta^a\}, \qquad \Omega = \omega^0 \wedge \eta^1 \wedge \bar\eta^1 \wedge \cdots \wedge \bar\eta^q, \\ &d\theta^2 \equiv \cdots \equiv d\theta^{n-q} \equiv 0 \mod \mathscr{I}, \\ &d\theta^1 \equiv -c_q \pi \wedge \eta^q \mod \mathscr{I} \end{aligned} \tag{3.2}$$

and similarly for the conjugate equations.

Notice that the equations do not involve $\omega^0, \eta^1, \ldots, \eta^{q-1}\bar{\eta}^1, \ldots, \bar{\eta}^{q-1}$ and ρ. In the language of EDS, the lines these forms are dual to are called the Cauchy characteristics. In general, a differential system defined on some manifold M may actually be defined on a quotient manifold of M by quotienting out the Cauchy characteristics. Call our quotient manifold N. The differential system on $\mathscr{F}$ descends to a well-defined system on N, and although the individual forms θ^a, η^q, π, $\bar{\theta}^a$, $\bar{\eta}^q$, and $\bar{\pi}$ themselves do not descend, in our case the line each spans gives a well-defined line in each fiber of T^*N. The collection of these lines spans T^*N. Note that span $\{\theta^a, \eta^q, \pi\}$ defines an almost complex structure on N by specifying the $(1,0)$ subspace of T^*N. Since

$$d\eta^q \equiv d\theta^a \equiv d\pi \equiv 0 \quad \mathrm{mod}\{\theta^a, \eta^q, \pi\},$$

by the complex Frobenius theorem we see that this almost complex structure is integrable, giving N the structure of a complex manifold. Moreover, the differential system $\widetilde{\mathscr{I}}$ on N induced from $\mathscr{I}$ on $\mathscr{F}$ is a Cauchy-Riemann system, i.e. solutions to $\widetilde{\mathscr{I}}$ are complex curves in N.

Geometrically, N is the space of rulings of $\mathbf{E}^{2n+1}$ by certain $(p-2)$-planes and a complex curve in N specifies the $(p-2)$-planes whose union fills out the minimal submanifold in $\mathbf{E}^{2n+1}$. Since the minimal submanifolds depend on solutions to the Cauchy-Riemann equations and since we know how to solve the Cauchy-Riemann equations explicitly, one might hope to give an explicit formula for the minimal submanifolds. We begin the general construction in the codimension two case and carry it out explicitly for the case of three manifolds in $\mathbf{E}^5$, indicating how to write out the general cases.

4. Coordinates on $\mathscr{F}$

To get an explicit formula for solutions we need to put coordinates on $\mathscr{F}$, and to do this we take advantage of the geometry of the differential system by using coordinates that reflect the projection onto N. In fact we go further and notice that the natural projection $\mathscr{F} \to S^2 = SU(2)/U(1)$ factors through N. On S^2 we have a standard metric $\hat{\pi} \circ \bar{\hat{\pi}}$ where

$$\hat{\pi} = \frac{2dz}{1+|z|^2}$$

and π is the pullback of $\hat{\pi}$ to $\mathscr{F}$. So take

$$\pi = \frac{2e^{is}\,dz}{1+|z|^2} \tag{4.1}$$

where z is a complex variable and s is a real variable. The e^{is} is inserted to measure rotation in the fiber. The structure equation $d\pi = i\rho \wedge \pi$ and the reality of ρ require that we must have

$$\rho = ds + \frac{i(z\,d\bar{z} - \bar{z}\,dz)}{1+|z|^2}. \tag{4.2}$$

Let $\theta = \theta^1$; recall that by (1.1) (taking $\phi = \Psi$ of (1.3)) we have

$$d\theta = -\sqrt{\frac{n}{2}}\pi \wedge \eta^{n-1} + in\rho \wedge \theta. \tag{4.3}$$

Notice that

$$d(e^{-nis}(1+|z|^2)^n \theta \wedge dz) = 0;$$

so by the complex Pfaff-Darboux theorem we may write

$$\theta = \frac{e^{nis}(dw - y\,dz)}{(1+|z|^2)^n} \tag{4.4}$$

where w is a well-defined function on $\mathscr{F}$ modulo dz. w may also be considered as a function defined on N (again modulo dz) and, on N it is a holomorphic function (by the complex Pfaff-Darboux theorem). Similarly y, considered as a function defined on N modulo dz is also holomorphic because $\theta \wedge d\theta$ is the lift of a well-defined $(3, 0)$-form on N. To find a coordinate expression for η^{n-1} we compute $d\theta$ in coordinates and compare it with the structure equation expression (4.3), substituting in the coordinate expressions for π, θ and ρ. We get

$$\frac{2\sqrt{\frac{n}{2}}e^{is}}{1+|z|^2}\,dz \wedge \eta^{n-1} = \frac{e^{nis}}{(1+|z|^2)^n}\,dz \wedge \left(\frac{2n\bar{z}}{1+|z|^2}dw - dy\right) \tag{4.5}$$

so

$$\eta^{n-1} = \frac{e^{(n-1)is}}{\sqrt{\frac{n}{2}}(1+|z|^2)^{n-1}}\left(\frac{n\bar{z}}{1+|z|^2}\,dw - \frac{dy}{2} + p_{n-1}\,dz\right) \tag{4.6}$$

where p_{n-1} is a complex valued function defined modulo dz.

We could continue by computing $d\eta^{n-1}$ two ways to get an expression for η^{n-2} etc.... , but instead we now specialize to the case $n = 2$, i.e. $\mathbf{E}^5$. Let $\eta = \eta^n$, $p = p_{n-1}$. Equations (1.1) and (1.3) tell us

$$d\eta = -\sqrt{3}\pi \wedge \omega^0 + i\rho \wedge \eta + \bar{\pi} \wedge \theta,$$

and (4.6) specializes to

$$\eta = \frac{e^{is}}{(1+|z|^2)}\left(\frac{2\bar{z}}{1+|z|^2}\,dw - \frac{dy}{2} + p\,dz\right).$$

To simplify the calculations, let

$$\lambda = \frac{2\bar{z}w}{1+|z|^2} - \frac{y}{2};$$

then

$$d\lambda = \frac{2\bar{z}\,dw}{1+|z|^2} - \frac{dy}{2} + \frac{2w\,d\bar{z}}{(1+|z|^2)^2} - \frac{2\bar{z}^2 w\,dz}{(1+|z|^2)^2},$$

and we may write

$$\eta = \frac{e^{is}}{(1+|z|^2)}\left(d\lambda - \frac{2w\,d\bar{z}}{(1+|z|^2)^2} + q\,dz\right). \tag{4.7}$$

Comparing our two expressions for $d\eta$ we get

$$2\sqrt{3}\omega^0 = \frac{2\bar{z}\,d\lambda}{(1+|z|^2)} - \frac{4\lambda\,d\bar{z}}{(1+|z|^2)^2} + dq + m\,dz \tag{4.8}$$

for some function m defined modulo dz. To determine m and q we differentiate further. From (1.1) and (1.3) we have

$$d\omega^0 = \frac{\sqrt{3}}{2}(\bar{\pi}\wedge\eta + \pi\wedge\bar{\eta}).$$

Comparing with $d\omega^0$ in coordinates gives

$$dm \equiv -\frac{2\bar{z}^2\,d\lambda}{(1+|z|^2)^2} + \frac{8\bar{z}\lambda\,d\bar{z}}{(1+|z|^2)^3} + \frac{6(q-\bar{q})\,d\bar{z}}{(1+|z|^2)^2} - \frac{6\,d\bar{\lambda}}{(1+|z|^2)^2} \bmod\{dz\}.$$

Taking

$$m = -\frac{2\bar{z}^2\lambda}{(1+|z|^2)^2} - \frac{6\bar{\lambda}}{(1+|z|^2)^2}$$

implies that

$$q - \bar{q} = \frac{2(z\bar{\lambda} - \bar{z}\lambda)}{(1+|z|^2)}$$

and the reader may check that everything works out. Setting $t = \mathrm{Re}(q)$ so that $q = t + \frac{i\,\mathrm{Im}(z\bar{\lambda})}{(1+|z|^2)}$ gives

$$\begin{aligned} \eta &= \frac{e^{is}}{1+|z|^2}\left(d\lambda - \frac{2w\,d\bar{z}}{(1+|z|^2)^2} + \left(t + \frac{2i\,\mathrm{Im}(z\bar{\lambda})}{(1+|z|^2)}\right)dz\right), \\ 2\sqrt{3}\omega^0 &= dt + \frac{2}{(1+|z|^2)}\,\mathrm{Re}\left(\bar{z}\,d\lambda - \frac{5\bar{\lambda} + \bar{z}^2\lambda}{(1+|z|^2)}\,dz\right). \end{aligned} \tag{4.9}$$

Finally we need to express $g \in SU(2)$ in terms of the coordinates $(s, t, z, \bar{z}, w, \bar{w}, y, \bar{y})$ on $\mathscr{F}$. Write

$$g = \begin{pmatrix} a & -\bar{b} \\ b & -\bar{a} \end{pmatrix}, \quad a\bar{a} + b\bar{b} = 1.$$

Recall that (1.2) says

$$g^{-1}dg = \psi = \frac{1}{2}\begin{pmatrix} i\rho & -\bar{\pi} \\ \pi & -i\rho \end{pmatrix} \quad \text{or} \quad dg = g\psi.$$

Comparing the first row of each side gives

$$da = \tfrac{1}{2}(ia\rho - \bar{b}\pi), \quad d\bar{b} = -\tfrac{1}{2}(-a\bar{\pi} + i\bar{b}\rho).$$

We conclude that

$$a = e^{is/2}(1+|z|^2)^{-1/2}, \quad \bar{b} = \bar{z}e^{-is/2}(1+|z|^2)^{-1/2}.$$

5. The explicit formula for 3-folds in $\mathbf{R}^5$

To compute $\mu(g)$ for $g = \left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$, we compute the induced action on $\mathbf{C}^{2n+1} = \mathrm{Sym}^{2n}(\mathbf{C}^2)$ and then take a real subspace (Alternatively we could compute the map $SU(2) \to SO(3)$ (the Hopf map) and then compute the $SO(3)$ action on $\mathbf{R}^{2n+1} = \mathscr{H}_3^n$: the harmonic polynomials of degree n on $\mathbf{R}^3$.)

The induced action of $Y \in \mathfrak{su}(2)$ on $x_1^k x_2^l$ (where $k + l = n$ and we write $x_1 x_2$ for $x_1 \circ x_2$ etc.... ,) is

$$Y(x_1^k x_2^l) = k(Yx_1)x_1^{k-1}x_2^l + l x_1^k (Yx_2) x_2^{l-1}.$$

We want a basis of $\mathbf{C}^5 = \mathrm{Sym}^4(\mathbf{C}^2)$ such that we have the Maurer-Cartan form

$$\Psi_{\mathbf{C}} = \begin{pmatrix} 2i\rho & -\bar{\pi} & & & \\ \pi & i\rho & -\sqrt{\tfrac{3}{2}}\bar{\pi} & & \\ & \sqrt{\tfrac{3}{2}}\pi & 0 & -\sqrt{\tfrac{3}{2}}\bar{\pi} & \\ & & \sqrt{\tfrac{3}{2}}\pi & -i\rho & -\bar{\pi} \\ & & & \pi & -2i\rho \end{pmatrix}. \tag{5.1}$$

(This is the natural basis of decreasing weights along the diagonal and structure equations (1.4).)

Taking

$$Y = \frac{1}{2}\begin{pmatrix} i\rho & -\bar{\pi} \\ \pi & -i\rho \end{pmatrix},$$

the reader may check that the basis of $\mathrm{Sym}^4(\mathbf{C}^2)$ that gives (5.1) is

$$v_1 = x_1^4, \quad v_2 = 2x_1^3 x_2, \quad v_3 = \sqrt{6}x_1^2 x_2^2, \quad v_4 = 2x_1 x_2^3, \quad v_5 = x_2^4,$$

and the corresponding real basis having (1.2) as its Maurer-Cartan form is

$$\begin{gathered} e_0 = v_3, \quad e_1 = \frac{1}{\sqrt{2}}(-v_2 + v_4), \quad e_2 = -\frac{i}{\sqrt{2}}(v_2 + v_4), \\ e_3 = \frac{1}{\sqrt{2}}(v_1 + v_5), \quad e_4 = \frac{i}{\sqrt{2}}(v_1 - v_5). \end{gathered} \tag{5.2}$$

We actually do not need to work with the real form directly, as we may use the complex notation (3.1) throughout. So we compute the map $\mu_{\mathbf{C}}\colon SU(2) \to GL(5, \mathbf{C})$ instead.

Let

$$g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \qquad a\bar{a} + b\bar{b} = 1.$$

We have

$$\mu_{\mathbf{C}}(g)(x_1^k x_2^l) = (gx_1)^k (gx_2)^l,$$

so in the basis $v_1, \dots, v_5$ we get

(5.3)

$$\mu_{\mathbf{C}}(g) = \begin{pmatrix} a^4 & -2a^3\bar{b} & \sqrt{6}a^2\bar{b}^2 & -2a\bar{b}^3 & \bar{b}^4 \\ 2a^3b & a^2(|a|^2-3|b|^2) & \sqrt{6}a\bar{b}(|b|^2-|a|^2) & \bar{b}^2(3|a|^2-|b|^2) & -2\bar{b}^3\bar{a} \\ \sqrt{6}a^2b^2 & \sqrt{6}ab(|a|^2-|b|^2) & 1-6|a|^2|b|^2 & \sqrt{6}\bar{a}\bar{b}(|b|^2-|a|^2) & \sqrt{6}\bar{a}^2\bar{b}^2 \\ 2ab^3 & b^2(3|a|^2-|b|^2) & \sqrt{6}\bar{a}b(|a|^2-|b|^2) & \bar{a}^2(|a|^2-3|b|^2) & -2\bar{a}^3\bar{b} \\ b^4 & 2\bar{a}b^3 & \sqrt{6}\bar{a}^2b^2 & 2\bar{a}^3b & \bar{a}^4 \end{pmatrix}.$$

Let $\mathbf{V} = \operatorname{span}_{\mathbf{R}}\{v_1, \dots, v_5\}$, let $T\colon \mathbf{V} \to \mathbf{R}^5$ denote the linear map extending (5.2), and let $\mathscr{F}_{\mathbf{V}}$ denote the principal coframe bundle associated to $\mu_{\mathbf{C}}(SU(2))$. We have the following commutative diagram (where we slightly abuse notation by also calling the induced map on coframe bundles T):

$$\begin{tikzcd} \mathscr{F} \arrow[r, "T^{-1}"] \arrow[d, "\mu(g)u_0"'] & \mathscr{F}_{\mathbf{V}} \arrow[d, "\mu_{\mathbf{C}}(g)u_0"] \\ \mathbf{R}^5 \arrow[r, "T^{-1}"'] & \mathbf{V} \end{tikzcd}.$$

Hence $\mu(g) = T_{\mu_{\mathbf{C}}}(g)T^{-1}$. Recalling that

$$d\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \mu(g)\begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix},$$

we may write

$$T^{-1}d\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \mu_{\mathbf{C}}(g)T^{-1}\begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix};$$

i.e.,

$$\begin{pmatrix} \frac{1}{\sqrt{2}}(dx^3 - i\,dx^4) \\ -\frac{1}{\sqrt{2}}(dx^1 - i\,dx^2) \\ dx^0 \\ \frac{1}{\sqrt{2}}(dx^1 + i\,dx^2) \\ \frac{1}{\sqrt{2}}(dx^3 + i\,dx^4) \end{pmatrix} = \mu_{\mathbf{C}}(g)\begin{pmatrix} \frac{1}{\sqrt{2}}\bar{\theta} \\ -\frac{1}{\sqrt{2}}\bar{\eta} \\ \omega^0 \\ \frac{1}{\sqrt{2}}\eta \\ \frac{1}{\sqrt{2}}\theta \end{pmatrix}.$$

Thus on our adapted lift of $\mathbf{R}^5$ to $\mathscr{F}$ we have

(5.5)
$$\begin{aligned}
dx^0 &= 2\sqrt{3}\,\mathrm{Re}(\bar{a}^2\bar{b}^2\theta) + 2\sqrt{3}(|b|^2 - |a|^2)\,\mathrm{Re}(\bar{a}\bar{b}\eta) + (1 - 6|a|^2|b|^2)\omega^0,\\
dx^1 + idx^2 &= 2ab^3\bar{\theta} - b^2(3|a|^2 - |b|^2)\bar{\eta} + 2\sqrt{3}\bar{a}b(|a|^2 - |b|^2)\omega^0\\
&\quad + \bar{a}^2(|a|^2 - 3|b|^2)\eta - 2\bar{a}^3\bar{b}\theta,\\
dx^3 + idx^4 &= b^4\bar{\theta} - 2\bar{a}b^3\bar{\eta} + 2\sqrt{3}\bar{a}^2b^2\omega^0 + 2\bar{a}^3b\eta + \bar{a}^4\theta.
\end{aligned}$$

Substituting in (4.9) and (4.10) and reducing modulo $\{dz, d\bar{z}\}$ gives

$$\begin{aligned}
dx^0 &\equiv 2\sqrt{3}\,\mathrm{Re}\left(\frac{\bar{z}^2}{(1+|z|^2)^4}\,dw + \frac{\bar{z}(|z|^2-1)}{(1+|z|^2)^3}\,d\lambda\right)\\
&\quad + \left(1 - \frac{6|z|^2}{(1+|z|^2)^2}\right)\frac{1}{2\sqrt{3}}\left(dt + \frac{2\,\mathrm{Re}(\bar{z}\,d\lambda)}{(1+|z|^2)}\right) \quad \mathrm{mod}\{dz, d\bar{z}\},\\
dx^1 + idx^2 &\equiv \frac{2z^3\,d\bar{w}}{(1+|z|^2)^4} + \frac{z^2(|z|^2-3)\,d\bar{\lambda}}{(1+|z|^2)^3} + \frac{z(1-|z|^2)}{(1+|z|^2)^2}\left(dt + \frac{2\,\mathrm{Re}(\bar{z}\,d\lambda)}{(1+|z|^2)}\right)\\
&\quad + \frac{(1-3|z|^2)\,d\lambda}{(1+|z|^2)^3} - \frac{2\bar{z}\,dw}{(1+|z|^2)^4} \quad \mathrm{mod}\{dz, d\bar{z}\},\\
dx^3 + idx^4 &\equiv \frac{z^4\,d\bar{w}}{(1+|z|^2)^4} - \frac{2z^3\,d\bar{\lambda}}{(1+|z|^2)^3} + \frac{z^2}{(1+|z|^2)^2}\left(dt + \frac{2\,\mathrm{Re}(\bar{z}\,d\lambda)}{(1+|z|^2)}\right)\\
&\quad + \frac{2z\,d\lambda}{(1+|z|^2)^3} + \frac{dw}{(1+|z|^2)^4} \quad \mathrm{mod}\{dz, d\bar{z}\}.
\end{aligned}$$

After a trivial integration, and checking that none of the dx^i contain terms involving only functions of z times dz or $d\bar{z}$, we have the following

Theorem 5.1. *Given a holomorphic function of one variable $h(z)$, setting $w = h(z)$, $y = \frac{dh}{dz}$ one obtains a minimal 3-manifold in $\mathbf{R}^5$ given in coordinates by*

(5.6)
$$\begin{aligned}
x^0 &= \frac{1-4|z|^2+|z|^4}{2\sqrt{3}(1+|z|^2)^2}t + \frac{2(-2+2|z|^2+|z|^4)}{\sqrt{3}(1+|z|^2)^4}\,\mathrm{Re}(\bar{z}^2w)\\
&\quad - \frac{-5+2|z|^2+|z|^4}{2\sqrt{3}(1+|z|^2)^3}\,\mathrm{Re}(\bar{z}y),\\
x^1 + ix^2 &= \frac{z(1-|z|^2)}{(1+|z|^2)^2}t - \frac{2\bar{z}|z|^2(2+|z|^2)}{(1+|z|^2)^4}w - \frac{2z^3}{(1+|z|^2)^4}\bar{w}\\
&\quad - \frac{1-2|z|^2-|z|^4}{2(1+|z|^2)^3}y + \frac{z^2}{(1+|z|^2)^3}\bar{y},\\
x^3 + ix^4 &= \frac{z^2}{(1+|z|^2)^2}t + \frac{1+4|z|^2+2|z|^4}{(1+|z|^2)^4}w - \frac{z^4}{(1+|z|^2)^4}\bar{w}\\
&\quad - \frac{z(|z|^2+2)}{2(1+|z|^2)^3}y + \frac{z^3}{2(1+|z|^2)^3}\bar{y}.
\end{aligned}$$

Remark. These are not all the solutions; only solutions having invertible Gauss maps are obtained. For example 3-planes are not among the above solutions. One could derive a more general formula, but it would involve integrals.

Finally we give a description of the simplest solution for 3-folds in $\mathbf{E}^5$ $(h(z) \equiv 0)$.

Consider $\mathbf{R}^5$ as the traceless symmetric 3×3 matrices. The cone over the real Veronese, C_V, is the set of rank one matrices whose repeated eigenvalue is positive (see [M] or [HsL, Example 1.4]). Let $\operatorname{opp} C_V$ denote the cone over the opposite Veronese (rank one with repeated eigenvalue negative, see [M]). In [M], the cone over the real Veronese is shown to be twisted calibrated. This essentially means that it is minimizing with respect to other submanifolds of $\mathbf{R}^5 - \operatorname{opp} C_V$ whose orientation bundle is the restriction of the trivial line bundle on $\mathbf{R}^5 - \operatorname{opp} C_V$. Among these are all "nearby" manifolds, so in particular the cone over the real Veronese is stable.

Proposition 5.2. *The minimal submanifold given by $w(z) \equiv 0$ is the cone over the real Veronese, C_V.*

Proof. As shown in [M] and [HsL], the cone over the real Veronese is the $\mu(SO(3))$ orbit of a weight zero line which is $\operatorname{diag}(t/\sqrt{3}, t/\sqrt{3}, -2t/\sqrt{3})$ in the matrix model, or the x^0 axis in our model, i.e.

$$C_V = \left\{ \mu(g) \begin{pmatrix} t \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \middle| g \in SU(2) \right\} = \left\{ T\mu_{\mathbf{C}}(g)T^{-1} \begin{pmatrix} t \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \middle| g \in SU(2) \right\}.$$

Using complex notation we have (where the $*$'s consist of redundant information)

$$T\mu_{\mathbf{C}}(g)T^{-1} \begin{pmatrix} t \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = T\mu_{\mathbf{C}}(g) \begin{pmatrix} 0 \\ 0 \\ t \\ 0 \\ 0 \end{pmatrix}$$

$$= T \begin{pmatrix} * \\ * \\ (1 - 6|a|^2|b|^2)t \\ \sqrt{6}\bar{a}b(|a|^2 - |b|^2)t \\ \sqrt{6}\bar{a}^2b^2t \end{pmatrix} = T \begin{pmatrix} * \\ * \\ t(1 - \frac{6|z|^2}{(1+|z|^2)^2}) \\ \frac{\sqrt{6}z(1-|z|^2)t}{(1+|z|^2)^2} \\ \frac{\sqrt{6}z^2t}{(1+|z|^2)^2} \end{pmatrix}$$

which comparing with (5.6) we see is the solution $w(z) \equiv 0$ (after scaling t by $\frac{1}{2\sqrt{3}}$).

Remark. One can easily see that, in general, the zero solutions for 3-folds in $\mathbf{R}^{2n+1}$ are always the orbit of the weight zero line. More generally, the zero

solution to any case will be the orbit of a $(p-2)$-plane which is preserved by a maximal torus (an S^1).

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