HOW POROUS IS THE GRAPH OF BROWNIAN MOTION?

J. T. COX AND PHILIP S. GRIFFIN

ABSTRACT. We prove that the graph of Brownian motion is almost surely porous, and determine the Hausdorff dimension of sets with a given porosity index. In particular we show that the porosity index of the graph is \( \gamma_0 \approx 0.6948 \).

1. Introduction and statement of results

Real function theory has seen an increase in interest recently in the study of porous and \( \sigma \)-porous sets. These are sets which are “small” in a certain sense. They were first introduced by Denjoy [4] and later rediscovered by Dolzhenko [5]. Much of the recent interest centers around the work of Zajíček [12]. For further references see [6 and 11].

Since we are primarily interested in the graph of Brownian motion, we will only define porosity for the graph of a real valued function. Given \( f : [0, \infty) \to \mathbb{R} \) let \( G \) be its graph, \( G = \{(t, f(t)) : t \geq 0\} \). Using \( \Lambda \) to denote a square in \( \mathbb{R}^2 \) with sides parallel to the coordinate axes, we define

\[
\Lambda(t)(h) = \{(x, s) : t-h < s < t+h, \ f(t)-h < x < f(t)+h\},
\]

and

\[
\delta_t(h) = \sup\{|\Lambda|^{1/2} : \Lambda \subseteq \Lambda_t(h), \ \Lambda \cap G = \emptyset\},
\]

where \(|\cdot|\) denotes Lebesgue measure. Thus \( \delta_t(h) \) is the supremum of the side lengths of all squares contained in \( \Lambda_t(h) \), with sides parallel to the coordinate axes, which do not intersect \( G \). The function \( f \), or more precisely its graph \( G \), is said to be porous at \( t \) if

\[
\limsup_{h \downarrow 0} \frac{\delta_t(h)}{h} > 0.
\]

It is porous if (1.1) holds for all \( t \geq 0 \). The definition of porosity from the right is obtained by replacing \( \Lambda_t(h) \) and \( \delta_t(h) \) with

\[
\Lambda^+_t(h) = \{(s, x) : t-h < s < t+h, \ f(t)-h < x \leq f(t)+h\}
\]
and
\[ \delta_t^+(h) = \sup\{|A|^{1/2} : A \subseteq \Lambda_t^+(h), \Lambda \cap G = \emptyset\} \]
respectively. The definition of porosity from the left should be clear.

Our original interest in porosity is due to the following question raised by Goffman [7]:

Is the graph of Brownian motion porous with probability one?

More precisely, let \( W(t), t \geq 0 \), be a standard one-dimensional Brownian motion, and let \( \delta_t \) be defined for the function \( f(t) = W(t) \). The question is then whether or not

\[
(1.2) P\left( \limsup_{h \to 0} \frac{\delta_t(h)}{h} > 0 \text{ for all } t \geq 0 \right) = 1.
\]

The analogous question for porosity from the right is whether or not

\[
(1.3) P\left( \limsup_{h \to 0} \frac{\delta_t^+(h)}{h} > 0 \text{ for all } t \geq 0 \right) = 1.
\]

As we will see the analysis of porosity for Brownian motion essentially reduces to the analysis of right porosity, and thus we turn our attention to this first.

We will show that the answer to Goffman's question is yes by showing that (1.3) holds. Our original proof in fact gave the stronger result that there exists a (nonrandom) \( \varepsilon > 0 \) for which

\[
(1.4) P\left( \limsup_{h \to 0} \frac{\delta_t^+(h)}{h} \geq \varepsilon \text{ for all } t \geq 0 \right) = 1.
\]

This then raises the question as to the largest value of \( \varepsilon \) for which (1.4) holds. This value is called the right porosity index of the graph. Analogously the value of the \( \limsup \) in (1.1), with \( \delta_t \) replaced by \( \delta_t^+ \), is called the right porosity index at \( t \).

Before addressing this problem we will make a few simple observations. First, it follows from the definitions that one always has for every \( h > 0 \)

\[
(1.5) \frac{\delta_t^+(h)}{h} \leq 1.
\]

It is not hard to show (see the end of §3) that for every \( t \geq 0 \)

\[
(1.6) P\left( \limsup_{h \to 0} \frac{\delta_t^+(h)}{h} = 1 \right) = 1.
\]

Hence by Fubini

\[
(1.7) P\left( \limsup_{h \to 0} \frac{\delta_t^+(h)}{h} = 1 \text{ for a.e. } t \right) = 1.
\]

Thus the right porosity index at almost all \( t \) is 1 with probability one. However, the value of the right porosity index of the graph is not 1, as can be seen from our first result.
Theorem 1.1. \(P(\inf_{t \geq 0} \limsup_{h \downarrow 0} \delta_t^+(h)/h = \frac{1}{2}) = 1\)

From Theorem 1.1 and (1.7) we see that

\[ A^+ = \left\{ t : \limsup_{h \downarrow 0} \frac{\delta_t^+(h)}{h} < 1 \right\} \]

satisfies

\[ P(A^+ \neq \emptyset) = 1 \text{ and } P(|A|^+ = 0) = 1. \quad (1.8) \]

Thus \(A^+\) is a set of one-dimensional Lebesgue measure zero but it is still quite "large". More precisely, if for any set \(B\) we let

\[ d(B) = \text{Hausdorff dimension of } B, \]

then we are able to show

**Theorem 1.2.** \(P(d(A^+) = 1) = 1\).

Our final result on right porosity concerns the size of the set

\[ A_\gamma^+ = \left\{ t : \limsup_{h \downarrow 0} \frac{\delta_t^+(h)}{h} \leq \gamma \right\} \quad (1.9) \]

where \(0 \leq \gamma < 1\). If \(\gamma < \frac{1}{2}\) then \(A_\gamma^+ = \emptyset\) by Theorem 1.1. If \(\gamma \geq \frac{1}{2}\) we again ask for its Hausdorff dimension. To answer this question, we define \(\alpha(\gamma)\) for \(\gamma \in (0, 1)\) to be the unique solution \(\alpha\) of

\[ \int_1^{(1-\gamma)^{-1}} \frac{t^\alpha - 1}{(t - 1)\frac{3}{2}} \, dt = 2 \left( \frac{1-\gamma}{\gamma} \right)^{\frac{1}{2}}. \quad (1.10) \]

This expression has its origin in the determination of the moments of a certain stopping time (see Lemma 2.1 below). We then have

**Theorem 1.3.** For \(\frac{1}{2} \leq \gamma < 1\)

\[ P(d(A_\gamma^+) = 1 - \alpha(\gamma)) = 1. \]

To state the analogous results to Theorems 1.1–1.3 for porosity, define

\[ A = \left\{ t : \limsup_{h \downarrow 0} \frac{\delta_t(h)}{h} < 1 \right\} \]

and

\[ A_\gamma = \left\{ t : \limsup_{h \downarrow 0} \frac{\delta_t(h)}{h} \leq \gamma \right\}. \]
We then have

**Theorem 1.4.** Let \( \gamma_0 = 0.6948 \) be the unique value of \( \gamma \) for which (1.10) holds with \( \alpha = \frac{1}{2} \). Then

\[
P\left( \inf \limsup_{h \to 0} \frac{\delta_y(h)}{h} = \gamma_0 \right) = 1.
\]

**Theorem 1.5.** \( P(d(A) = 1) = 1 \).

**Theorem 1.6.** For \( \gamma_0 < \gamma < 1 \)

\[
P(d(A_\gamma) = 1 - 2\alpha(\gamma)) = 1.
\]

We are also able to obtain the Hausdorff dimension of the intersection of \( A_\gamma \) and \( A_\gamma^+ \) with an arbitrary analytic set. These results are stated in §2, where we will reformulate our results in a more convenient form (see Theorems 2.1 and 2.2). The proofs are given in §§3 and 4.

In view of (1.6), Theorems 1.1–1.6 can be considered as results concerning points on the graph of Brownian motion which are “less porous” than they should be. The study of such points turns out to be closely related to the study of Brownian slow points (see [1–3, 8–10]). In particular, the methods developed by Davis in [2] for finding slow points, and Perkins in [10] for computing Hausdorff dimensions, play a very important role in our analysis. The reader familiar with their work will note the close correspondence between the statements of our results and those in [10] on which they are modelled.

## 2. Preliminaries

Recall that \( W(s) \) is a standard one-dimensional Brownian motion. For \( \beta > 1 \) and \( \epsilon \in (0, \beta - 1) \) let \( Z(\epsilon, \beta) \) be a random variable with distribution given by

\[
P(Z(\epsilon, \beta) = 0) = \frac{2}{\pi} \arcsin \left( \frac{\epsilon}{\beta - 1} \right)^{1/2},
\]

\[
P(1 + \epsilon \leq Z(\epsilon, \beta) \leq t) = 1 - \frac{2}{\pi} \arcsin \left( \frac{\epsilon}{t - 1} \right)^{1/2}, \quad t \in [1 + \epsilon, \beta].
\]

The way in which this distribution arises is as follows; fix \( h > 0 \) and let

\[
\tau = \inf \{ s \geq (1 + \epsilon)h : W(s) = W(h) \}.
\]

Then the **arcsine law** and scaling imply that

\[
Z(\epsilon, \beta) \overset{d}{=} \frac{1}{h} 1 \left( \frac{\tau}{h} \leq \beta \right),
\]

where we have used \( \overset{d}{=} \) to denote equality in distribution, and \( 1(\cdot) \) for indicator function.

Differentiating in (2.1), one sees that the density of \( Z(\epsilon, \beta) \) on \([1 + \epsilon, \beta]\) is given by

\[
f(t) = \frac{1}{\pi} \left( \frac{\epsilon}{t - 1 - \epsilon} \right)^{1/2} \frac{1}{t - 1}.
\]
Thus for $p \geq 0$
\[
E[Z^p(e, \beta)] = \frac{\epsilon^{1/2}}{\pi} \int_{1+e}^{\beta} \frac{t^p}{(t-1)^{3/2}} \frac{dt}{t-1}.
\]

**Lemma 2.1.** Fix $p \geq 0$ and $\beta > 1$. Then
\[
E[Z^p(e, \beta)] = 1 + \epsilon^{1/2} \frac{1}{\pi} \left[ \int_{1}^{\beta} \frac{t^p - 1}{(t-1)^{3/2}} dt - \frac{2}{(\beta - 1)^{1/2}} \right] + o(\epsilon^{1/2})
\]
as $\epsilon \to 0$.

**Proof.** For any $p \geq 0$ and $\beta > 1$,
\[
E[Z^p(e, \beta)] = \frac{\epsilon^{1/2}}{\pi} \int_{1+e}^{\beta} \frac{t^p - 1}{(t-1)^{3/2}} \frac{dt}{t-1} + \epsilon^{1/2} \int_{1+e}^{\beta} \frac{1}{(t-1)^{3/2}} \frac{dt}{t-1}.
\]

An application of dominated convergence shows that
\[
\int_{1+e}^{\beta} \frac{t^p - 1}{(t-1)^{3/2}} \frac{dt}{t-1} \to \int_{1}^{\beta} \frac{t^p - 1}{(t-1)^{3/2}} dt < \infty
\]
as $\epsilon \to 0$. Thus
\[
(2.2) \quad I = \frac{\epsilon^{1/2}}{\pi} \int_{1}^{\beta} \frac{t^p - 1}{(t-1)^{3/2}} dt + o(\epsilon^{1/2}).
\]
The integral in II can be explicitly evaluated, giving
\[
II = 1 - \frac{2}{\pi} \arcsin \left( \frac{\epsilon}{\beta - 1} \right)^{1/2}
\]
\[
= 1 - \frac{2}{\pi} \left( \frac{\epsilon}{\beta - 1} \right)^{1/2} + o(\epsilon^{1/2}).
\]
Combining (2.2) and (2.3) finishes the proof. \(\square\)

Now fix $\beta > 1$ and define
\[
g(p) = \int_{1}^{\beta} \frac{t^p - 1}{(t-1)^{3/2}} dt, \quad p \geq 0.
\]
Clearly $g$ is continuous, strictly increasing, $g(0) = 0$, and $g(p) \to \infty$ as $p \to \infty$. In particular, for each $\beta \in (1, \infty)$, there exists a unique $p(\beta)$ such that
\[
(2.5) \quad g(p(\beta)) = 2(\beta - 1)^{-1/2}.
\]
The following observations will be used frequently throughout the paper:
(i) $p(\cdot)$ is continuous,
(ii) $p(\cdot)$ is strictly decreasing,
(iii) $p(\beta) \uparrow \infty$ as $\beta \downarrow 1$, $p(\beta) \downarrow 0$ as $\beta \uparrow \infty$,
(iv) $p(2) = 1$,
(v) if $p_1 < p(\beta) < p_2$, then for all $\epsilon > 0$ sufficiently small
\[
E[Z^{p_1}(\epsilon, \beta)] < 1 \quad \text{and} \quad E[Z^{p_2}(\epsilon, \beta)] > 1.
\]
Properties (i)–(iv) are all easy consequences of the definition of \( p(\beta) \), and (v) follows from Lemma 2.1.

Observe that (1.10) is just a reparameterization of (2.5) obtained by setting 
\( \gamma = 1 - \beta^{-1} \). It is more convenient however to work directly with \( p(\beta) \) and so we now reformulate our results in these terms. For \( \beta \in (1, \infty) \) define

\[
B_{\beta}^+ = \left\{ t \in [0, 1] : \limsup_{h \to 0} \frac{\delta_t^+ (h)}{h} \leq 1 - \beta^{-1} \right\},
\]

\[
B_{\beta}^- = \left\{ t \in [0, 1] : \limsup_{h \to 0} \frac{\delta_t^- (h)}{h} \leq 1 - \beta^{-1} \right\}.
\]

We will prove

**Theorem 2.1.** Let \( B \subseteq [0, 1] \) be an analytic set, then

\[
P(d(B \cap B_{\beta}^+) = d(B) - p(\beta)) = 1 \text{ if } p(\beta) \leq d(B),
\]

\[
P(B \cap B_{\beta}^- = \emptyset) = 1 \text{ if } p(\beta) > d(B).
\]

**Theorem 2.2.** Let \( B \subseteq [0, 1] \) be an analytic set, then

\[
P(d(B \cap B_{\beta}^-) = d(B) - 2p(\beta)) = 1 \text{ if } 2p(\beta) \leq d(B),
\]

\[
P(B \cap B_{\beta}^- = \emptyset) = 1 \text{ if } 2p(\beta) > d(B).
\]

In Theorems 2.1 and 2.2 we consider \( t \in [0, 1] \); it follows from the Markov property that the same conclusions hold for \( t \in [n, n+1] \) for any positive integer \( n \), and hence for \( t \in [0, \infty) \). Thus, Theorem 1.1 follows from Theorem 2.1 by using (2.6)(ii) and (iv); Theorem 1.2 follows from Theorem 2.1 by letting \( \beta \to \infty \) and using (2.6)(iii); and Theorem 1.3 is just a special case of Theorem 2.1. Similarly, Theorems 1.4–1.6 follow from Theorem 2.2.

3. **Upper bounds on** \( d(B \cap B_{\beta}^+) \) **and** \( d(B \cap B_{\beta}^-) \)

For \( \eta > 0 \) define

\[
B_{\beta, \eta}^+ = \left\{ t \in [0, 1] : \limsup_{h \to 0} \frac{\delta_t^+ (h)}{h} \leq (1 - \beta^{-1}) \text{ for all } h \in (0, \eta) \right\},
\]

(3.1)

\[
B_{\beta, \eta}^- = \left\{ t \in [0, 1] : \limsup_{h \to 0} \frac{\delta_t^- (h)}{h} \leq (1 - \beta^{-1}) \text{ for all } h \in (0, \eta) \right\},
\]

and

\[
B_{\epsilon, \beta, \eta}^+ = \{ t \in [0, 1] : \forall h \in (0, \eta \beta^{-1}) \exists s \in [(1 + \epsilon)h, \beta h] \text{ with } W(t + s) = W(t) \},
\]

\[
B_{\epsilon, \beta, \eta}^- = \{ t \in [0, 1] : \forall h(0, \eta \beta^{-1}) \exists s \in [(1 + \epsilon)h, \beta h] \text{ with } W(t - s) = W(t) \}.
\]
In order that the definition of $B^{+,\beta,\eta}_{\epsilon}$ make sense we adopt the convention that $W(u) = -\infty$ for $u < 0$. For $\eta_1 < \eta_2$

$$B^{+,\beta}_{\eta_1} \supseteq B^{+,\beta}_{\eta_2},$$

while for $\beta_1 < \beta_2$

$$\lim_{\eta \downarrow 0} B^{+,\beta}_{\beta_1,\eta} \subseteq B^{+,\beta}_{\beta_2,\eta} \subseteq \lim_{\eta \downarrow 0} B^{+,\beta}_{\beta_1,\eta}.$$

The following observation is central to our methods for obtaining bounds on the size of $B^{+,\beta}_{\beta,\eta}$ and hence on $B^{+,\beta}_{\beta}$: if $\beta_1 < \beta_2$, and $\epsilon > 0$ is small enough that $1 - (1 + \epsilon)\beta_2^{-1} > 1 - \beta_1^{-1}$, then

$$B^{+,\beta}_{\beta_1,\eta} \subseteq B^{+,\beta}_{\beta,\eta}.$$

To prove this, suppose $t \notin B^{+,\beta}_{\beta_1,\eta}$, and hence that for some $h \in (0, \eta\beta_2^{-1}]$ we have $W(t + s) \neq W(t)$ for all $s \in [(1 + \epsilon)h, \beta_2 h]$. Then either

$$A = \{(t + s, x) : (1 + \epsilon)h < s < \beta_2 h \text{ and } W(t) - (\beta_2 - (1 + \epsilon))h < x \leq W(t)\}$$

or

$$A = \{(t + s, x) : (1 + \epsilon)h < s < \beta_2 h \text{ and } W(t) < x \leq W(t) + (\beta_2 - (1 + \epsilon))h\}$$

satisfies

$$\Lambda \subseteq A^{+}(\beta_2 h) \setminus G$$

by continuity of the sample paths (G is the graph of $W(s)$). Consequently, $h' = \beta_2 h \in (0, \eta]$ satisfies

$$\frac{\delta^{+}(h')}{h'} = \frac{\delta^{+}(\beta_2 h)}{\beta_2 h} \geq \frac{\beta_2 - (1 + \epsilon)}{\beta_2} > 1 - \beta_1^{-1}.$$  

So $t \notin B^{+,\beta}_{\beta_1,\eta}$, and (3.4) follows.

Similarly, the analogous versions of (3.2)-(3.4) for $B^{-,\beta}_{\beta,\eta}$, $B^{-,\beta}_{\beta,\eta}$, and $B^{-,\beta,\eta}$ are valid. We now give an estimate on the size of $B^{+,\beta,\eta}_{\beta}$. Let

$$\mathcal{F}(t) = \sigma\{W(s) : 0 \leq s \leq t\}.$$

**Proposition 3.1.** Fix $\beta > 1$, $\epsilon \in (0, \beta - 1)$ and $p \geq 0$, and assume that $E[Z^p(\epsilon, \beta)] < 1$. Then for any $\eta > 0$ there exists $C < \infty$ such that for all $v \geq 0$ and all $\zeta$ sufficiently small,

$$P([v, v + \zeta] \cap B^{+,\beta}_{\beta,\eta,\epsilon} \neq 0 | \mathcal{F}(v + \zeta)) \leq C \zeta^p \text{ a.s.}$$

**Proof.** By the Markov property, it suffices to consider the case $v = 0$. Choose $q > p$ and $\beta_1 > \beta$ such that $E[Z^q(\epsilon, \beta_1)] < 1$. Fix $\lambda \in (0, 1)$ such that $\lambda q \geq p$, and assume that

$$\zeta \leq \left(\frac{\beta_1 - \beta}{\beta - 1}\right)^{1/(1-\lambda)} \wedge \frac{1}{2}.$$
Finally let \( r_\zeta = (-2\zeta p \log \zeta)^{1/2} \). Then with \( M(t) = \sup\{|W(s)| : 0 \leq s \leq t\} \) we have
\[
P([0, \zeta] \cap B_{e,\beta,\eta}^+ \neq \emptyset | \mathcal{F}(\zeta))
\leq P(\exists b \in (-\infty, \infty) \forall h \in (\zeta, \eta \beta^{-1}] \exists s \in [(1 + e)h, \beta h] \text{ with } W(s) = b | \mathcal{F}(\zeta))
\]
\[
\leq P(\exists b \in (-\infty, \infty) \forall h \in (2\zeta, \eta \beta^{-1}] \exists s \in [(1 + e)h, \beta h] \text{ with } W(s) = b | \mathcal{F}(\zeta))
\]
\[
= P(\exists b \in (-\infty, \infty) \forall h \in (\zeta, \eta \beta^{-1} - \zeta] \exists s \in [(1 + e)(h + \zeta) - \zeta, \beta(h + \zeta) - \zeta] \text{ with } W(s) = b) \text{ a.s.}
\]
\[
P(M(\zeta) \geq r_\zeta)
+ P(\exists b, |b| \leq r_\zeta \text{ such that } \forall h \in (\zeta, \eta \beta^{-1} - \zeta],
\exists s \in [(1 + e)(h + \zeta) - \zeta, \beta(h + \zeta) - \zeta] \text{ with } W(s) = b)
= I + II.
\]

We will estimate I and II separately.

The first term is simple:
\[
I \leq 4P(W(\zeta) \geq r_\zeta) = 4P(W(1) \geq \zeta^{-1/2}r_\zeta)
\leq 2(-\pi p \log \zeta)^{-1/2} \exp(p \log \zeta) \leq C \zeta^p
\]
for some \( C \).

To bound the second term we use scaling to obtain,
\[
II \leq P(\forall h \in [\zeta^k, \eta \beta^{-1} - \zeta] \exists s \in [(1 + e)(h + \zeta) - \zeta, \beta(h + \zeta) - \zeta] \text{ with } |W(s)| \leq r_\zeta)
\]
\[
= P(\forall u \in [1, \zeta^{-\lambda}(\eta \beta^{-1} - \zeta)] \exists s \in [(1 + e)u + e\zeta^{1-\lambda}, \beta u + (\beta - 1)\zeta^{1-\lambda}] \text{ with } |W(s)| \leq e_\zeta)
\]
\[
\leq P(\forall u \in [1, \zeta^{-\lambda}(\eta \beta^{-1} - \zeta)] \exists s \in [(1 + e)u, \beta_1 u] \text{ with } |W(s)| \leq e_\zeta)
\]
\[
= P(\Gamma_{\zeta})
\]
say, where \( e_\zeta \overset{\text{def}}{=} \zeta^{-1/2}r_\zeta \rightarrow 0 \) as \( \zeta \downarrow 0 \). To estimate this last quantity we introduce stopping times \( \tau_k \), which are finite a.s., defined by
\[
\tau_{-1} = 1, \quad \tau_k = \inf\{s \geq (1 + e)\tau_{k-1} : |W(s)| \leq e_\zeta\}, \quad k \geq 0.
\]

Also define
\[
V_k(e, \beta_1) = \frac{\tau_{k-1}}{\tau_{k-1}} \left( \frac{\tau_k}{\tau_{k-1}} \leq \beta_1 \right), \quad k \geq 0,
\]
\[
V^*(e, \beta_1) = \sup_{k \geq 0} \prod_{m=0}^k V_m(e, \beta_1).
\]
Observe that

\[(3.6) \quad \Gamma_\zeta \subset \{V^*(\epsilon, \beta_1) \geq \zeta^{-\lambda}(\eta \beta^{-1} - \zeta)\}.\]

Now if \( u \geq 1 \) and \(|x| \leq \epsilon \zeta \), then for any \( t \in [1 + \epsilon, \beta_1] \) and any \( k \geq 0 \)

\[
P(V_{k+1}(\epsilon, \beta_1) \in [1 + \epsilon, t] | \tau_k = u, W(\tau_k) = x)
\]

\[
= P(\exists s \in [(1 + \epsilon)u, tu] \text{ with } |W(s)| \leq \epsilon \zeta |W(u) = x)
\]

\[
= P(\exists s \in [(1 + \epsilon), t] \text{ with } |W(s)| \leq \epsilon \zeta u^{-\frac{1}{2}} |W(1) = xu^{-\frac{1}{2}})
\]

\[
\to P(Z(\epsilon, \beta_1) \in [1 + \epsilon, t])
\]

uniformly in \( u, x \), and \( k \) as \( \zeta \downarrow 0 \). Since the random variables are bounded, this means that for all \( \alpha \geq 0 \), all \( k \geq 0 \) and all \( \delta > 0 \)

\[E[V^\alpha_{k+1}(\epsilon, \beta_1) | \mathcal{F}(\tau_k)] \leq E[Z^\alpha(\epsilon, \beta_1)] + \delta \quad \text{a.s.}\]

provided \( \zeta \) is sufficiently small. In particular since \( E[Z^q(\epsilon, \beta_1)] < 1 \), there exists an \( r < 1 \) such that for all \( \zeta \) sufficiently small and all \( k \geq 0 \)

\[E[V^q_{k+1}(\epsilon, \beta_1) | \mathcal{F}(\tau_k)] \leq r \quad \text{a.s.}\]

Thus by the strong Markov property, for any \( j \geq 1 \) and \( \zeta \) small

\[E \left[ \prod_{m=0}^{j} V_m^q(\epsilon, \beta_1) \right] = E E \left[ \prod_{m=0}^{j} V_m^q(\epsilon, \beta_1) | \mathcal{F}(\tau_{j-1}) \right] \leq r E \left[ \prod_{m=0}^{j-1} V_m^q(\epsilon, \beta_1) \right].\]

Hence, since \( V_0(\epsilon, \beta_1) \leq \beta_1 \), we have for all small \( \zeta \)

\[E[V^*(\epsilon, \beta_1)^q] \leq \sum_{k=0}^{\infty} E \left[ \prod_{m=0}^{k} V_m^q(\epsilon, \beta_1) \right] \leq \beta_1^q \sum_{k=0}^{\infty} r^k < \infty.\]

Thus by (3.6)

\[P(\Gamma_\zeta) \leq \frac{E[V^*(\epsilon, \beta_1)^q]}{\zeta^{-\lambda q}(\eta \beta^{-1} - \zeta)^q} \leq C \zeta^p\]

for some constant \( C \). \( \Box \)

We note that by integration in (3.5), for all \( v \geq 0 \) and all \( \zeta \) sufficiently small

\[P([v, \tau_\zeta] \cap B_{\epsilon, \beta, \eta} \neq \emptyset) \leq C \zeta^p,\]

and by considering the Brownian motion \( \widetilde{W}(s) = W(1) - W(1 - s) \), that

\[P([v, \tau_\zeta] \cap B_{\epsilon, \beta, \eta} \neq \emptyset) \leq C \zeta^p.\]

With Proposition 3.1 at hand, the proof of Theorem 3.1 is now similar to that of the upper bound in Theorem 4 of Perkins [10].

**Theorem 3.1.** Assume \( B \subseteq [0, 1] \). Then

\[P(d(B \cap B^+_{\beta_1}) \leq d(B) - p(\beta_1)) = 1, \quad d(B) \geq p(\beta_1);\]

\[P(B \cap B^+_{\beta_1} = \emptyset) = 1, \quad d(B) < p(\beta_1);\]

\[d(B) = p(\beta_1);\]
and

(3.8)

\[ P(d(B \cap B_{\beta_1}^+ \cap B_{\beta_2}^-) \leq d(B) - p(\beta_1) - p(\beta_2)) = 1, \quad d(B) \geq p(\beta_1) + p(\beta_2); \]

\[ P(B \cap B_{\beta_1}^+ \cap B_{\beta_2}^- = \emptyset) = 1, \quad d(B) < p(\beta_1) + p(\beta_2). \]

\textbf{Proof.} We start with the proof of (3.8). For \( i = 1, 2 \) fix \( p_i < p(\beta_i) \), and if \( d(B) < p(\beta_1) + p(\beta_2) \) let the \( p_i \) also satisfy \( p_1 + p_2 > d(B) \). By (2.6) we can find \( \xi_i > \beta_i \) so that \( p_i < p(\xi_i) < p(\beta_i) \) for \( i = 1, 2 \). We can also choose \( \varepsilon > 0 \) sufficiently small such that

\[ 1 - (1 + \varepsilon)\xi_i^{-1} > 1 - \beta_i^{-1} \]

and

(3.9)

\[ E[Z^\varphi(e, \xi_i)] < 1. \]

It follows from (3.3) and (3.4) that

(3.11)

\[ B_{\beta_i}^+ \subseteq \lim_{\eta \to \infty} B_{\xi_i, \eta}^+ \]

with the analogous result holding if \( + \) is replaced by \( - \).

Next, fix \( \lambda > d(B) \) (if \( d(B) < p(\beta_1) + p(\beta_2) \) let \( \lambda = p_1 + p_2 \)). For \( n \) sufficiently large we can choose a cover \( \{S_i\} \) of \( B \) such that

(3.12)

\[ \rho(S_i) \leq n^{-1} \forall i, \quad \text{and} \quad \sum_{i=1}^{\infty} \rho(S_i)^{\lambda} \leq n^{-1}, \]

where \( \rho(S) = \sup\{|x - y| : x, y \in S\} \) is the diameter of \( S \).

Now consider the cover of \( B \cap B_{\xi_1, \eta}^+ \cap B_{\xi_2, \eta}^- \) given by \( \{\tilde{S}_i\} \), where

\[ \tilde{S}_i = \begin{cases} S_i, & \text{if } S_i \cap B_{\xi_1, \eta}^+ \cap B_{\xi_2, \eta}^- \neq \emptyset, \\ \emptyset, & \text{otherwise}. \end{cases} \]

Note that

\[ E \left[ \sum_{i=1}^{\infty} \rho(\tilde{S}_i)^{\lambda-p_1-p_2} \right] = \sum_{i=1}^{\infty} \rho(S_i)^{\lambda-p_1-p_2} P(S_i \cap B_{\xi_1, \eta}^+ \cap B_{\xi_2, \eta}^-, \eta \neq \emptyset). \]

Since \( S_i \subseteq [v_i, v_i + s_i] \) for some \( v_i \), where \( s_i = \rho(S_i) \), we have for sufficiently large \( n \)

\[ P(S_i \cap B_{\xi_1, \eta}^+ \cap B_{\xi_2, \eta}^-, \eta \neq \emptyset) \]

\[ \leq E[1(S_i \cap B_{\xi_1, \eta}^+, \eta \neq \emptyset) 1(S_i \cap B_{\xi_2, \eta}^-, \eta \neq \emptyset) | \mathcal{F}(v_i + s_i)] \]

\[ = E[1(S_i \cap B_{\xi_2, \eta}^-, \eta \neq \emptyset) P(S_i \cap B_{\xi_1, \eta}^+, \eta \neq \emptyset | \mathcal{F}(v_i + s_i))] \]

\[ \leq C \rho(S_i)^{p_1} P(S_i \cap B_{\xi_2, \eta}^-, \eta \neq \emptyset) \]

\[ \leq C^2 \rho(S_i)^{p_1+p_2} \]
by Proposition 3.1 and the remarks following it. Hence by (3.12) and Fatou

\begin{equation}
\liminf_{n \to \infty} \sum_{i=1}^{\infty} \rho(\tilde{S}_{i})^{\lambda-p_1-p_2} = 0 \quad \text{a.s.}
\end{equation}

We now consider the cases \( d(B) \geq p(\beta_1) + p(\beta_2) \) and \( d(B) < p(\beta_1) + p(\beta_2) \) separately. In the former case, since \( \lambda > d(B) \) was arbitrary, (3.13) implies that for any \( \eta > 0 \)

\[ P(d(B \cap B^+_{\varepsilon, \xi_1, \eta} \cap B^-_{\varepsilon, \xi_2, \eta}) \leq d(B) - p_1 - p_2) = 1. \]

By letting \( \eta \downarrow 0 \) and noting (3.11), this gives

\[ P(d(B \cap B^+_{\varepsilon, \xi_1, \eta} \cap B^-_{\varepsilon, \xi_2, \eta} \neq \emptyset) = 1. \]

Since \( p_1 < p(\beta_1) \) are arbitrary, this proves the first part of (3.8). If \( d(B) < p(\beta_1) + p(\beta_2) \), then since we chose \( \lambda = p_1 + p_2 \), (3.13) implies for all \( \eta > 0 \)

\[ P(B \cap B^+_{\varepsilon, \xi_1, \eta} \cap B^-_{\varepsilon, \xi_2, \eta} \neq \emptyset) = 0. \]

Letting \( \eta \downarrow 0 \) completes the proof of (3.8).

The proof of (3.7) is similar, but easier. Fix \( p_1 < p(\beta_1) \), and if \( d(B) < p(\beta_1) \) let \( p_1 \) also satisfy \( p_1 > d(B) \). Choose \( \xi_1 > \beta_1 \) so that \( p_1 < p(\xi_1) < p(\beta_1) \), and choose \( \varepsilon > 0 \) sufficiently small such that

\begin{equation}
1 - (1 + \varepsilon)\xi_1^{-1} > 1 - \beta_1^{-1}
\end{equation}

and

\begin{equation}
E[Z^B_{\varepsilon, \xi_1}] < 1.
\end{equation}

Next, fix \( \lambda > d(B) \) (if \( d(B) < p(\beta_1) \) let \( \lambda = p_1 \)). For \( n \) sufficiently large we can choose a cover \( \{S_i\} \) of \( B \) satisfying (3.12). Define \( \{\tilde{S}_i\} \) by

\[ \tilde{S}_i = \begin{cases} 
S_i, & \text{if } S_i \cap B^+_{\varepsilon, \xi_1, \eta} \neq \emptyset, \\
\emptyset, & \text{otherwise}.
\end{cases} \]

By the remarks following Proposition 3.1 we have

\[ P(S_i \cap B^+_{\varepsilon, \xi_1, \eta} \neq \emptyset) \leq C \rho(S_i)^{\rho_1}. \]

Thus

\[ E \left[ \sum_{i=1}^{\infty} \rho(\tilde{S}_i)^{\lambda-p_1} \right] \to 0, \]

and so by Fatou

\begin{equation}
\liminf_{n \to \infty} \sum_{i=1}^{\infty} \rho(\tilde{S}_i)^{\lambda-p_1} = 0 \quad \text{a.s.}
\end{equation}

The remainder of the argument is as before. \( \square \)

Remark. The estimate (3.5) can be used to give a proof (though not necessarily the most elementary one) of (1.6). Fix \( t \geq 0 \), and note that for any \( \beta > 1 \) and
\( \varepsilon \in (0, \beta - 1) \) we can find \( p > 0 \) such that \( E[Z^p(\varepsilon, \beta)] < 1 \). By Proposition 3.1, for any \( \eta > 0 \) there exists a finite constant \( C \) such that for all sufficiently small \( \zeta \)
\[
P([t, t + \zeta] \cap B^+_{\varepsilon, \beta, \eta} \neq \emptyset) \leq C\zeta^p.
\]
This implies
\[
P(t \in B^+_{\varepsilon, \beta, \eta}) = 0,
\]
and consequently we have
\[
P\left( t \in \bigcup_{\eta > 0} B^+_{\varepsilon, \beta, \eta} \right) = 0.
\]
This means that a.s. for every \( \eta > 0 \) there exists \( h \in (0, \eta] \) with \( \delta^+_t(h) \geq h(1 - (1 + \varepsilon)\beta^{-1}) \). Thus
\[
P\left( \limsup_{h \downarrow 0} \frac{\delta^+_t(h)}{h} \geq 1 - (1 + \varepsilon)\beta^{-1} \right) = 1.
\]
Letting \( \beta \uparrow \infty \) completes the proof of (1.6).

4. Lower Bounds on \( d(B \cap B^+_{\beta}) \) and \( d(B \cap B_{\beta}) \)

For \( \beta > 1, \varepsilon \in (0, \beta - 1) \) and \( \lambda > 0 \), define
\[
\begin{align*}
\tau_0(\varepsilon) &= \inf\{s \geq 1 : W(s) = W(0)\}, \quad X_0(\varepsilon, \beta, \lambda) = \tau_0(\varepsilon)1(\tau_0(\varepsilon) \leq \beta), \\
C_k(\varepsilon, \lambda) &= \{u, v \in [\tau_{k-1}(\varepsilon), (1 + \varepsilon)\tau_{k-1}(\varepsilon)] \\
& \quad \text{with } W(u) = W(0) + \lambda \tau_{k-1}^{1/2}(\varepsilon) \text{ and } W(v) = W(0) - \lambda \tau_{k-1}^{1/2}(\varepsilon)\}, \\
Y_k(\varepsilon, \beta, \lambda) &= \frac{\tau_k(\varepsilon)}{\tau_{k-1}(\varepsilon)}1\left( \frac{\tau_k(\varepsilon)}{\tau_{k-1}(\varepsilon)} \leq \beta \right) \cap C_k(\varepsilon, \lambda), \\
X_k(\varepsilon, \beta, \lambda) &= X_0(\varepsilon, \beta, \lambda) \prod_{m=1}^k Y_m(\varepsilon, \beta, \lambda), \\
X^*(\varepsilon, \beta, \lambda) &= \sup_{k \geq 0} X_k(\varepsilon, \beta, \lambda).
\end{align*}
\]
First observe that \( \tau_k(\varepsilon) < \infty \) a.s. for all \( k \). Next, by the scaling and strong Markov properties, \( Y_k(\varepsilon, \beta, \lambda), k \geq 1 \) are i.i.d. with common distribution
\[
P(Y_1(\varepsilon, \beta, \lambda) \in [1 + \varepsilon, t]) = P(\exists u, v \in [1, 1 + \varepsilon] \text{ with } W(u) = W(1) + \lambda \text{ and } W(v) = W(1) - \lambda, \\
\text{and } \exists s \in [(1 + \varepsilon), t] \text{ with } W(s) = W(1))
\]
for $1 + \varepsilon \leq t \leq \beta$, with all remaining mass at the origin. In particular, if $\Rightarrow$ denotes weak convergence, then

$$Y_1(\varepsilon, \beta, \lambda) \Rightarrow Z(\varepsilon, \beta) \text{ as } \lambda \downarrow 0.$$ 

Thus for all $\alpha \geq 0$

$$E[Y_1^\alpha(\varepsilon, \beta, \lambda)] \Rightarrow E[Z^\alpha(\varepsilon, \beta)].$$

Also by the scaling and strong Markov properties $X_0(\varepsilon, \beta, \lambda)$ is independent of $\{Y_k(\varepsilon, \beta, \lambda) ; k \geq 1\}$, and hence $X_k(\varepsilon, \beta, \lambda)$ is a product of independent random variables.

In what follows we will often consider two independent Brownian motions $W_1$ and $W_2$ defined on the same probability space $\Omega$. We denote the corresponding random variables in (4.2) by $Y_k(i; \varepsilon, \beta, \lambda)$, $X_k(i; \varepsilon, \beta, \lambda)$, etc. for $i = 1, 2$. We will use the notation $a_n \approx b_n$ to mean $a_n/b_n$ is bounded above and below by finite, nonzero constants which are independent of $n$.

**Lemma 4.1.** Fix $\delta > 0$, $\beta_1 > 1$, $\beta_2 > 1$, and assume that $p(\beta_1) + p(\beta_2) < \delta$. Then for all $\varepsilon > 0$ sufficiently small there exists a $\lambda > 0$ and a $p \in (0, \delta)$ such that for all $b \geq \delta$,

$$E[(X^*(1 ; \varepsilon, \beta_1, \lambda) \wedge X^*(2 ; \varepsilon, \beta_2, \lambda) \wedge n) ^ b] \approx n^{b-p},$$

where $a \wedge b = \min(a, b)$.

**Proof.** Let $q_1 > p(\beta_1)$ and $q_2 > p(\beta_2)$ satisfy $q_1 + q_2 < \delta$. Then by (2.6) and (4.3) for all $\varepsilon > 0$ sufficiently small we can find $\lambda > 0$ so that $E[Y_1^q(i; \varepsilon, \beta_i, \lambda)] > 1$ for $i = 1, 2$. So by reducing $q_1$ and $q_2$ we can find $p_1 > 0$ and $p_2 > 0$ satisfying $p_1 + p_2 < \delta$ and

$$E[Y_1^{p_1}(i; \varepsilon, \beta_1, \lambda)] = 1.$$ 

Thus, recalling that $X_k(i; \varepsilon, \beta_i, \lambda)$ is a product of independent random variables, it follows that $X_k^{p_1}(i; \varepsilon, \beta_i, \lambda)$ is a nonnegative martingale. Hence $X_k^{p_1}(i; \varepsilon, \beta_i, \lambda)$ converges a.s., necessarily to zero, as $k \to \infty$. Next, for $r > 1$ set

$$T_r(i) = \inf\{k : X^{p_1}(i; \varepsilon, \beta_i, \lambda) > r\}.$$ 

Since

$$X_{T_r(i) \wedge k}^{p_1}(i; \varepsilon, \beta_i, \lambda) \leq \beta_i r,$$

dominated convergence implies

$$E[X_0^{p_1}(i; \varepsilon, \beta_i, \lambda)] = E[X_{T_r(i) \wedge k}^{p_1}(i; \varepsilon, \beta_i, \lambda)]$$

$$\Rightarrow E[X_{T_r(i)}^{p_1}(i; \varepsilon, \beta_i, \lambda); T_r(i) < \infty]$$

as $k \to \infty$. Observe that on $\{T_r(i) < \infty\} = \{X^*(i; \varepsilon, \beta_i, \lambda)^{p_1} > r\}$ we have

$$r \leq X_{T_r(i)}^{p_1}(i; \varepsilon, \beta_i, \lambda) \leq \beta_i r.$$
Thus
\[
\mu_i(\beta_i r)^{-1} \leq P(X^*(i; \varepsilon, \beta_i, \lambda)^p_i > r) \leq \mu_i r^{-1}
\]
where
\[
\mu_i = E[X^0_i(i; \varepsilon, \beta_i, \lambda)] > 0.
\]
Now let \( p = p_1 + p_2 \), and assume that \( b \geq \delta \). Then since \( p < \delta \)
\[
E[(X^*(1; \varepsilon, \beta_1, \lambda) \Lambda X^*(2; \varepsilon, \beta_2, \lambda) \Lambda n)^b] = \int_0^n b r^{b-1} P(X^*(1; \varepsilon, \beta_1, \lambda) > r) P(X^*(2; \varepsilon, \beta_2, \lambda) > r) dr
\]
\[
\approx n^{b-p}.
\]
We will need to consider slightly smaller sets than \( B^+_\beta \) and \( B^-_\beta \) defined as
follows: for \( a \geq 1 \) let
\[
\Lambda^+_t(h) = \{(s, x) : t \leq s \leq t + h, \ f(t) - ah \leq x \leq f(t) + ah\},
\]
\[
\delta^+_t(h) = \sup\{|A|^{1/2} : A \subseteq \Lambda^+_t(h), \ A \cap G = \emptyset\},
\]
\[
B^+_\beta = \left\{ t \in [0, 1] : \limsup_{h \to 0} \frac{\delta^+_t(h)}{h} \leq 1 - \beta^{-1} \right\},
\]
with analogous definitions if \( + \) is replaced by \( - \). Observe that since \( a \geq 1 \),
\(
\delta^+_t(h) \geq \delta^-_t(h) \) and \( \delta^-_t(h) \geq \delta^-_t(h) \), hence
\[
(a)B^+_\beta \subseteq B^-_\beta \quad \text{and} \quad (a)B^-_\beta \subseteq B^-_\beta.
\]
For \( i = 1, 2 \), let \( \Lambda^+_i(i; h) \), \( \delta^+_i(i; h) \), \( B^+_\beta(i) \) and \( B^-_\beta(i) \) be the quantities analogous to \( \Lambda^+_i(h) \), \( \delta^+_i(h) \), \( B^+_\beta \) and \( B^-_\beta \) defined relative to \( W_i \), and similarly with \( + \) replaced by \( - \). Define
\[
\mathcal{I}_i(t) = \sigma\{W_i(s) : 0 \leq s \leq t\}, \quad \mathcal{F}(t) = \sigma\{\mathcal{I}_1(t) \cup \mathcal{F}_2(t)\}.
\]
Given a subset \( \Gamma \) of \([0, 1] \times \Omega\), let \( \Gamma(\omega) = \{t \in [0, 1] : (t, \omega) \in \Gamma\} \). In what follows we will often write \( \Gamma \) for \( \Gamma(\omega) \) when the meaning is clear. The following proposition is based on Proposition 14 in [10].

**Proposition 4.1.** Let \( \Gamma \) be a \( \mathcal{G}(t) \) progressively measurable subset of \([0, 1] \times \Omega\) such that \( \Gamma \) is closed a.s. Then for any \( a \geq 1 \), and \( \xi_1, \xi_2 > 1 \)
\[
P(\{d(\Gamma) > p(\xi_1) + p(\xi_2)\} \cap \{\Gamma \cap (a B^+_\xi_1(1) \cap a B^+_\xi_2(2)) \neq \emptyset\}) = 0.
\]

**Proof.** Since the proof of this result is rather lengthy, we divide it into six steps. In the first two steps certain parameters are fixed and various sequences of random variables are defined. Steps 3 and 4 are concerned with some purely geometric arguments used to describe certain events in terms of these random variables. In Step 5 the problem is reduced to equation (4.15), which is then verified in Step 6.
Step 1. We may assume \( p(\xi_1) + p(\xi_2) < 1 \), else the result is trivial. Fix \( \delta \in (p(\xi_1) + p(\xi_2), 1) \) and let \( \beta_i \in (1, \xi_i) \) for \( i = 1, 2 \) satisfy
\[
(4.5) \quad p(\beta_1) + p(\beta_2) < \delta.
\]
Choose \( \epsilon > 0 \), \( \lambda > 0 \) and \( \rho \in (0, \delta) \), so that the conclusion of Lemma 4.1 holds, and for \( i = 1, 2 \)
\[
(4.6) \quad (1 + \epsilon) \beta_i \leq \xi_i.
\]
Let \( \beta = \beta_1 \lor \beta_2 \) and \( \beta' = \beta_1 \land \beta_2 \), and choose \( M \) so that
\[
(4.7) \quad M \geq \frac{a \beta^2(1 + \epsilon)^2}{\beta'(1 + \epsilon) - 1}.
\]
Finally, let
\[
(4.8) \quad \Delta \leq \beta \lambda^2 M^{-2}.
\]
The parameters \( \delta, \beta_1, \beta_2, \rho, \epsilon, \lambda, M \) and \( \Delta \) have now been fixed.

Step 2. For \( n \geq 1 \) (an integer) and \( i = 1, 2 \) define
\[
\tau_0(i) = \inf \{ s \geq n^{-1} : W_i(s) = W_i(0) \}, \quad X_0(i) = \tau_0(i) 1(\tau_0(i) \leq \beta_i),
\]
and for \( k \geq 1 \)
\[
\tau_k(i) = \inf \{ s \geq (1 + \epsilon)\tau_{k-1}(i) : W_i(s) = W_i(0) \},
\]
\[
C_k(i) = \{ \exists u, v \in [\tau_{k-1}(i), (1 + \epsilon)\tau_{k-1}(i)] \}
\]
\[
\quad \text{with } W_i(u) = W_i(0) + \lambda \tau_{k-1}^{1/2}(i) \text{ and } W_i(v) = W_i(0) - \lambda \tau_{k-1}^{1/2}(i),
\]
\[
Y_k(i) = \frac{\tau_k(i)}{\tau_{k-1}(i)} 1 \left( \left\{ \frac{\tau_k(i)}{\tau_{k-1}(i)} \leq \beta_i \right\} \cap C_k(i) \right),
\]
\[
X_k(i) = X_0(i) \prod_{m=1}^k Y_m(i), \quad X^*(i) = \sup_{k \geq 0} X_k(i), \quad X^* = X^*(1) \land X^*(2).
\]
Then by scaling
\[
(4.9) \quad nX^*(i) \overset{d}{=} X^*(i; \epsilon, \beta_i, \lambda), \quad nX^* \overset{d}{=} X^*(1; \epsilon, \beta_1, \lambda) \land X^*(2; \epsilon, \beta_2, \lambda).
\]

Step 3. For any \( \eta \leq \Delta \)
\[
(4.10) \quad \{ \beta X^* > \eta \} \subseteq \{ \forall h \in [\beta n^{-1}, \beta^{-2} \eta] \exists u(i), v(i) \in [h, h(1 + \epsilon)\beta_i] \}
\]
\[
\quad \text{with } W_i(u(i)) = W_i(0) + Mh \quad \text{and} \quad W_i(v(i)) = W_i(0) - Mh, \quad i = 1, 2 \}.
\]
To see this, suppose \( \beta X^* > \eta \), and \( h \in [\beta n^{-1}, \beta^{-2} \eta] \). Then for \( i = 1, 2 \) we can find a \( k = k(i) \) such that \( X_k(i) > \beta h \). But this means that for some \( m < k \),
\[
(4.11) \quad h \leq \tau_m(i) \leq \beta h
\]
and $C_{m+1}(i)$ occurs. In particular there exist

$$u(i), v(i) \in [\tau_m(i), \tau_m(i)(1 + \epsilon)] \subseteq [h, h(1 + \epsilon)\beta]$$

such that

$$W_i'(u(i)) = W(0) + x(i)x/2,$$
$$W_i'(v(i)) = W(0) - x(i)x/2.$$ 

But by (4.8) and (4.11), since $h \leq \beta^{-2}\eta \leq \beta^{-2}\Delta$

$$\lambda\tau_m(i)^{1/2} \geq \lambda h^{1/2} \geq M(\Delta\beta^{-1})^{1/2}h^{1/2} \geq Mh^{1/2} \geq Mh$$

since $\beta > 1$. Thus (4.10) holds. The reason for using $\beta X^*$ rather than $X^*$ on the left-side of (4.10) is that clearly $\{X^* \leq u\} \in \mathcal{F}(\beta u)$, and so $\beta X^*$ is a stopping time.

**Step 4.** Let

$$D_{n, \eta} = \{t \in \Gamma : \forall h \in [\beta n^{-1}, \beta^{-2}\eta] \exists u(i), v(i) \in [h, h(1 + \epsilon)\beta_i],$$

$$\text{with } W_i'(t + u(i)) = W_i'(t) + Mh$$
$$\text{and } W_i'(t + v(i)) = W_i'(t) - Mh, \quad i = 1, 2\},$$

and let $\Theta_t$ be the usual shift operator. Then for every $\omega$

$$\{t \in \Gamma : \beta X^* \circ \Theta_t > \eta\} \subseteq D_{n, \eta},$$

(4.12)

$$\bigcup_{n=1}^{\infty} D_{n, \eta} \subseteq \Gamma \cap \{B_{s_1}^+(1) \cap B_{s_2}^+(2)\}.$$  

Assertion (4.12) follows immediately from (4.10). To see (4.13), fix $\eta > 0$ and suppose that $t \in \bigcap_{n=1}^{\infty} D_{n, \eta}$. Let $h \in (0, \beta^{-2}\eta]$, and assume that for either $i = 1$ or $2$

$$\Lambda \subseteq a\Lambda^+_i(i; h)$$

where

$$\Lambda = (t, W_i'(t)) + ([s_1, s_2] \times [x_1, x_2])$$

and

$$s_2 - s_1 = x_2 - x_1 = h(1 - ((1 + \epsilon)\beta_i)^{-1}).$$

We will show that the graph of $W_i'$ intersects $\Lambda$, and hence

$$\frac{a\delta^+_i(i; h)}{h} \leq 1 - \frac{1}{(1 + \epsilon)\beta_i} \leq 1 - \frac{1}{\xi_i},$$

which implies $t \in \Gamma \cap \{B_{s_1}^+(1) \cap B_{s_2}^+(2)\}$. To do this, observe that

$$s_2 \geq \left(1 - \frac{1}{(1 + \epsilon)\beta_i}\right)h, \quad s_1 \leq \frac{1}{(1 + \epsilon)\beta_i}h,$$

and

$$\frac{1}{(1 + \epsilon)\beta_i}s_2 \geq s_1.$$
This last fact follows from
\[
\frac{s_2}{(1 + \varepsilon)\beta_i} - s_1 = \left(1 - \frac{1}{(1 + \varepsilon)\beta_i}\right) \left(\frac{h}{(1 + \varepsilon)\beta_i} - s_1\right) \geq 0.
\]

Now let \( h' = ((1 + \varepsilon)\beta_i)^{-1}\) and note that \( s_1 \leq h' < s_2 \leq h \). Since \( t \in \bigcap_{n=1}^{\infty} D_{n, \eta} \) there must exist \( u(i), v(i) \in [h', (1 + \varepsilon)\beta_i] \subseteq [s_1, s_2] \) with
\[
W_i(t + u(i)) = W_i(t) + Mh', \quad W_i(t + v(i)) = W_i(t) - Mh'.
\]
By the definition of \( M \) and the previous inequalities we see that \( Mh' \geq ah \), and thus path continuity implies the graph of \( W_i \) must intersect \( \Lambda \). This establishes (4.13).

**Step 5.** Let
\[
H_\eta = \inf \left\{ \sum_{i=1}^{k} \rho(S_i)^{\delta} : \Gamma \subseteq \bigcup_{i=1}^{k} S_i, S_i \text{ are closed intervals} \right\}
\]
with \( \rho(S_i) \leq \eta, k \geq 1 \), and
\[
E = \{ H_\eta \to \infty \text{ as } \eta \downarrow 0 \}.
\]
Thus
\[
(4.14) \quad E \supseteq \{ d(\Gamma) > \delta \}.
\]
Finally let
\[
G_{n, \eta} = \{ D_{n, \eta} \neq \emptyset \}.
\]
In the next step we will show that
\[
(4.15) \quad P \left( E \setminus \bigcup_{n}^{\infty} G_{n, \eta} \right) = 0.
\]
Now clearly \( \bigcup_{n \geq 0} \bigcap_{n}^{\infty} G_{n, \eta} \subseteq \{ \bigcup_{n \geq 0} \bigcap_{n}^{\infty} D_{n, \eta} \neq \emptyset \} \), and since a.s. \( D_{n, \eta} \) is compact and \( D_{n+1, \eta} \subseteq D_{n, \eta} \)
\[
P \left( \bigcup_{n \geq 0}^{\infty} G_{n, \eta} \setminus \left\{ \bigcup_{n \geq 0}^{\infty} D_{n, \eta} \neq \emptyset \right\} \right) = 0.
\]
Thus by (4.13), (4.14) and (4.15)
\[
P(\{ d(\Gamma) > \delta \} \setminus \{ \Gamma \cap aB_{\xi_1}(1) \cap aB_{\xi_2}(2) \neq \emptyset \}) = 0.
\]
Letting \( \delta \downarrow p(\xi_1) + p(\xi_2) \) then completes the proof of the proposition. Thus it remains to prove (4.15).

**Step 6.** Define
\[
\zeta_1 = \inf\{ t \geq 0 : t \in \Gamma \} \wedge 2, \quad \sigma_1 = \zeta_1 + ((\beta X^* \circ \Theta_{\zeta_1}) \wedge 1),
\]
and for \( k \geq 2 \)

\[
\zeta_k = \inf\{t \geq \sigma_{k-1} : t \in \Gamma\} \land 2, \quad \sigma_k = \zeta_k + ((\beta X^* \circ \Theta_{\zeta_k}) \land 1)
\]

where \( \inf(\emptyset) = \infty \). Then \( \zeta_k \) and \( \sigma_k \) are stopping times since \( \Gamma \) is progressively measurable and \( \beta X^* \) is a stopping time. Observe also that if \( \zeta_k \leq 1 \), then \( \zeta_k \in \Gamma \) a.s. since \( \Gamma \) is closed a.s. Let

\[
N_1 = \sup\{k : \zeta_k \leq 1\},
\]

where \( \sup(\emptyset) = 0 \). Then

\[
\Gamma \subseteq \bigcup_{k=1}^{N_1} [\zeta_k, \sigma_k] \quad \text{a.s.}
\]

Thus by (4.12)

\[
P(E \cap G_n^c) \leq P \left( E, \max_{1 \leq k \leq N_1} (\sigma_k - \zeta_k) \leq \eta \right)
\]

\[
\leq P \left( E, \sum_{k=1}^{N_1} (\sigma_k - \zeta_k)^\delta \geq H_\eta \right)
\]

\[
\leq P(E, H_\eta \leq x) + P \left( \sum_{k=1}^{N_1} (\sigma_k - \zeta_k)^\delta \geq x \right)
\]

for any \( x > 0 \).

We now estimate the last term above. Since \( (\sigma_k - \zeta_k)^\delta = \beta X^* \land 1 \), and \( \zeta_k \) is independent of \( \sigma_k - \zeta_k \), we have for any \( b > 0 \)

\[
E \left[ \sum_{k=1}^{N_1} (\sigma_k - \zeta_k)^b \right] = E \left[ \sum_{k=1}^{\infty} 1(\zeta_k \leq 1)(\sigma_k - \zeta_k)^b \right]
\]

\[
= \sum_{k=1}^{\infty} P(\zeta_k \leq 1)E[(\sigma_k - \zeta_k)^b] = E[N_1]E[(\beta X^* \land 1)^b].
\]

With \( b = \delta \) this implies

\[
E \left[ \sum_{k=1}^{N_1} (\sigma_k - \zeta_k)^\delta \right] = E[N_1]E[(\beta X^* \land 1)^\delta].
\]

With \( b = 1 \) and the fact that \( 2 \geq \sum_{k=1}^{N_1} (\sigma_k - \zeta_k) \) we get

\[
E[N_1] \leq \frac{2}{E[\beta X^* \land 1]}.
\]

Thus

\[
E \left[ \sum_{k=1}^{N_1} (\sigma_k - \zeta_k)^\delta \right] \leq \frac{2E[(\beta X^* \land 1)^\delta]}{E[\beta X^* \land 1]}.
\]
Now by (4.5), (4.9), and Lemma 4.1, there exists \( p \in (0, \delta) \) such that for \( b \geq \delta \),
\[
E[(\beta X^* \wedge 1)^b] = n^{-b} E[(X^*(1; \varepsilon, \beta_1, \lambda) \wedge X^*(2; \varepsilon, \beta_2, \lambda) \wedge n)^b] \approx n^{-p} .
\]
This implies that there exists a finite constant \( c \) independent of \( n \) such that
\[
E \left[ \sum_{k=1}^{N_1} (\sigma_k - \zeta_k)^\delta \right] \leq c ,
\]
and thus by Chebyshev
\[
P(E \cap G^{c\cdot}_n, \eta) \leq P(E, H_\eta \leq x) + c/\varepsilon .
\]

For any \( r > 0 \) we can choose \( x \) sufficiently large and \( \eta_0 \) sufficiently small so that
\[
P(E \cap G^{c\cdot}_n, \eta_0) < r
\]
for all \( n \). Now let \( G_\eta = \cap_n G_{n, \eta} \). Since \( G_{n+1, \eta} \subseteq G_{n, \eta} \) we have
\[
P(E \cap G^{c\cdot}_{\eta_0}) < r .
\]

Equation (4.15) now follows from the fact that \( G_\eta \uparrow \) as \( \eta \downarrow \). \( \square \)

**Remark.** Let \( C \subseteq [0, 1] \) be closed. By taking \( \Gamma = C \times \Omega \) in Proposition 4.1 we have for every \( a \geq 1 \)
\[
P(C \cap a B^{+}_{\xi_1}(1) \cap a B^{+}_{\xi_2}(2) \neq \emptyset) = 1
\]
provided \( p(\xi_1) + p(\xi_2) < d(C) \). This will be used in the next result.

**Theorem 4.1.** For every analytic set \( B \subseteq [0, 1] \) and every \( a \geq 1 \)
\[
P(d(B \cap a B^{+}_\beta) \geq d(B) - p(\beta)) = 1,
\]
\[
P(d(B \cap a B^{-}_\beta) \geq d(B) - p(\beta)) = 1.
\]

**Proof.** It suffices to prove the \( d B^{+}_\beta \) case since we obtain the \( d B^{-}_\beta \) case by considering \( \tilde{W}(t) = W(1) - W(1 - t) \). We may assume that \( p(\beta) < d(B) \), else there is nothing to prove. Since \( B \) is analytic, for any \( \varepsilon > 0 \) there exists a compact set \( C \subseteq B \) such that \( d(C) > d(B) - \varepsilon \). Assume that \( \varepsilon \) is sufficiently small that \( d(C) > p(\beta) \). Then for any \( \beta_1 \) satisfying \( p(\beta_1) < d(C) - p(\beta) \) we have by the remark above
\[
P(C \cap a B^{+}_{\beta_1}(1) \cap a B^{+}_\beta(2) \neq \emptyset) = 1 .
\]
Thus
\[
P(C \cap a B^{+}_{\beta_1}(1) \cap a B^{+}_\beta(2) \neq \emptyset \mid W_2) = 1 \quad \text{a.s.}
\]

Since for a given \( W_2 \), the random set \( C \cap a B^{+}_\beta(2) \) may be considered fixed under \( P(\cdot \mid W_2) \), and \( a B^{+}_{\beta_1}(1) \subseteq B^{+}_\beta(1) \), we can apply (3.7) with \( B = C \cap a B^{+}_\beta(2) \) to obtain
\[
P(d(C \cap a B^{+}_\beta(2)) \geq p(\beta_1) \mid W_2) = 1 \quad \text{a.s.}
\]
Thus
\[ P(d(C \cap aB^+_\beta(2)) \geq p(\beta_1)) = 1 \]
and hence
\[ P(d(B \cap aB^+_\beta(2)) \geq p(\beta_1)) = 1. \]
Since \( p \) is continuous, \( \beta_1 \) is arbitrary subject to the requirement that \( p(\beta_1) < d(C) - p(\beta) \), and \( \epsilon > 0 \) is arbitrary, we are done. \( \square \)

Setting \( a = 1 \) in (4.17) together with (3.7) then gives Theorem 2.1. Next we turn to Theorem 2.2.

**Theorem 4.2.** For every analytic set \( B \subseteq [0, 1] \) and every \( a \geq 1 \)
\[ P(d(B \cap aB^+_\beta \cap aB^-_{\beta_2}(2)) \geq d(B) - p(\beta_1) - p(\beta_2)) = 1. \]

**Proof.** We may assume \( p(\beta_1) + p(\beta_2) < d(B) \). Fix \( \epsilon > 0 \) sufficiently small that \( p(\beta_1) + p(\beta_2) < d(B) - \epsilon \). Choose \( C \subseteq B \) compact such that \( d(C) > d(B) - \epsilon \), and let \( \beta_3 \) satisfy
\[ p(\beta_1) + p(\beta_2) + p(\beta_3) < d(C). \]

Arguing as in Theorem 4.1, it suffices to show
\[ P(C \cap aB^+_\beta(1) \cap aB^-_{\beta_2}(1) \cap aB^+_\beta_3(2) \neq \emptyset | W) = 1 \quad \text{a.s.} \]

(4.19)
\[ p(\beta_1) + p(\beta_2) + p(\beta_3) < d(C). \]

(4.20)
\[ P(C \cap aB^+_\beta(1) \cap aB^-_{\beta_2}(1) \cap aB^+_\beta_3(2) \neq \emptyset) = 1 \]

where
\[ aB^-_{\beta_2, \eta}(1) = \left\{ t \in [0, 1] : \frac{\delta^-(1; h)}{h} \leq (1 - \beta_2)^{-1} \text{ for all } h \in (0, \eta) \right\}. \]

Since \( aB^-_{\beta_2, \eta}(1) \subseteq aB^+_\beta_2(1) \), (4.20) then follows.

In order to prove (4.21) all we need to do is show that
\[ P(d(C \cap aB^-_{\beta_2, \eta}(1)) > p(\beta_1) + p(\beta_3)) = 1 \]

and apply Proposition 4.1 (\( \Gamma \) defined by \( \Gamma(\omega) = C \cap aB^-_{\beta_2, \eta}(1) \) is a.s. closed and progressively measurable). To do this choose \( \beta < \beta_2 \) such that \( p(\beta_1) + p(\beta) + p(\beta_3) < d(C) \). By Theorem 4.1, since \( d(C) - p(\beta) > p(\beta_1) + p(\beta_3) \),
\[ P(d(C \cap aB^-_{\beta_2}(1)) > p(\beta_1) + p(\beta_3)) = 1. \]

But \( \lim_{\eta \downarrow 0} aB^-_{\beta_2, \eta}(1) \supseteq aB^-_{\beta_2}(1) \), and hence
\[ \lim_{\eta \downarrow 0} d(C \cap aB^-_{\beta_2, \eta}(1)) \geq d(C \cap aB^-_{\beta_2}(1)) > p(\beta_1) + p(\beta_3) \quad \text{a.s.} \]

Thus (4.22) holds. \( \square \)

Combining the results in Theorem 3.1 and Theorem 4.2 with (4.4) we see that for every analytic set \( B \subseteq [0, 1] \) and every \( a \geq 1 \), \( B \cap aB^+_\beta \cap aB^-_\beta = \emptyset \) a.s. if \( 2p(\beta) > d(B) \), while if \( 2p(\beta) \leq d(B) \) then
\[ P(d(B \cap aB^+_\beta \cap aB^-_\beta) = d(B) - 2p(\beta)) = 1. \]
Thus to complete the proof of Theorem 2.2 it suffices to prove

**Proposition 4.2.** If \( a \geq 2\beta(\beta - 1)^{-1} \) then for every \( \omega \)

\[
a B^+ \cap a B^- \subseteq B \subseteq B^+_t \cap B^-_t.
\]

**Proof.** Clearly \( B_t \subseteq B^+_t \cap B^-_t \). To prove the other inclusion we argue by contradiction. Thus assume that \( t \notin (a B^+_t \cap a B^-_t) \). Since \( a \geq 1 \), \( t \notin (B^+_t \cap B^-_t) \). Hence there exists a sequence \( h_n \downarrow 0 \) and a sequence of squares \( A_n \) such that

\[
\Lambda_n \subseteq \Lambda_t(h_n) \backslash G,
\]

\[
\Lambda_n \cap \Lambda^+_t(h_n) \neq \emptyset, \quad \Lambda_n \cap \Lambda^-_t(h_n) \neq \emptyset,
\]

\[
|\Lambda_n|^{1/2} h_n^{-1} \geq \nu > 1 - \beta^{-1}.
\]

By taking a further subsequence if necessary we may assume that at least half of \( \Lambda_n \) intersects \( \Lambda^+_t(h_n) \), i.e.

\[
|\Lambda_n \cap \Lambda^+_t(h_n)|^{1/2} h_n^{-1} \geq \nu / \sqrt{2},
\]

(if not, argue using \( \Lambda^-_t(h_n) \) in place of \( \Lambda^+_t(h_n) \) in what follows). But then by the definition of \( a \),

\[
a \delta^+_t \left( \frac{\nu h_n}{2} \right) \geq \frac{\nu h_n}{2},
\]

which contradicts \( t \in a B^+_t \). \( \Box \)

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**References**


**Department of Mathematics, Syracuse University, Syracuse, New York 13244-1150**

**E-mail address** (J. T. Cox): jtcx@rodan.acs.syre.edu