TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 325, Number 1, May 1991

HYPERHOLOMORPHIC FUNCTIONS AND SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS IN R^n

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ABSTRACT. Hyperholomorphic functions in R^n with $n \geq 2$ are introduced, extending the hitherto considered hyperholomorphic functions in R^2. A Taylor formula is derived, and with it a unique representation theorem is proved for hyperholomorphic functions that are real analytic at the origin. Hyperanalyticity is seen to be generally a consequence of hyperholomorphy and real analyticity combined. Results for hyperholomorphic functions are applied to gradients of solutions of second order homogeneous partial differential equations with constant coefficients. Polynomial solutions of such a second order equation are obtained by a matrix algorithm. These polynomials are modified and combined to form polynomial bases for real analytic solutions. It is calculated that in such a basis there are $(m+n-3)!(2m+n-2)/m!(n-2)!$ homogeneous polynomials of degree m.

1. HYPERHOLOMORPHIC FUNCTIONS

A hyperholomorphic function is a matrix-valued function F with a domain in R^n whose component functions are (totally) differentiable and satisfy a first order system of partial differential equations. Such a function turns out to be differentiable with respect to certain $n-1$ matrix-valued variables collectively denoted by Z. Z-differentiability of F, as we shall see, gives rise to a Taylor formula for F.

By a *matrix-valued function* we mean a mapping from a domain in a real Euclidean space R^n with $n \geq 2$ to a complex matrix space $C^{k \times s}$ with k, $s \geq 1$. Two matrix-valued functions F and G with a common domain may be added only if their dimensions $k \times s$ are identical, but may be multiplied as long as their dimensions are *compatible*, namely when the number of columns s of F is equal to the number of rows k of G. Although most basic concepts in calculus of scalar-valued functions can be readily extended to matrix-valued functions by "componentwise applications", certain complications arise because of noncommutativity of matrix multiplication.

If a matrix-valued function $F(x)$ is (totally) differentiable, or further belongs to the class C^1 in some domain, namely every component function $f_{ij}(x)$ has

Received by the editors April 25, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35E20; Secondary 35C10.
Key words and phrases. Hyperholomorphic functions, symmetric powers, Taylor formula, Leibniz formula, hyperanalyticity, real analytic solutions, polynomial bases.

continuous first order partial derivatives, then the *differential* of F, that is (df_{ij}), is conveniently expressed as

$$dF = F_{x_1}\,dx_1 + F_{x_2}\,dx_2 + \cdots + F_{x_n}\,dx_n,$$

or

$$dF = (\partial_1 F)\,dx_1 + (\partial_2 F)\,dx_2 + \cdots + (\partial_n F)\,dx_n,$$

where $x = (x_1, x_2, \ldots, x_n)$. If F belongs to C^m with $m \geq 1$, then a higher order partial derivative is denoted by using a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where α_i are integers. Specifically, if the α_i are all nonnegative, then

$$\partial^\alpha F := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n} F;$$

and $\partial^\alpha F$ is a partial derivative of order

$$|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

The order in which the differentiation ∂_i is performed is immaterial as long as $|\alpha| \leq m$. If any α_i is negative, we set $\partial^\alpha F = 0$.

A matrix-valued function F is said to be *M-holomorphic* with $M = (M_1, M_2, \ldots, M_{n-1})$ in some domain if F is differentiable and satisfies in that domain the first order equation

$$F_{x_n} = M_1 F_{x_1} + M_2 F_{x_2} + \cdots + M_{n-1} F_{x_{n-1}}, \tag{1.1}$$

or alternatively,

$$\mathbf{M}F := (M_1\partial_1 + M_2\partial_2 + \cdots + M_{n-1}\partial_{n-1} - I\partial_n)F = 0, \tag{1.2}$$

where the M_i are constant square matrices of identical dimensions. If F is a $k \times 1$ column vector, then M_i are all $k \times k$, so that we have in effect a first order system consisting of k partial differential equations in k unknown scalar-valued functions. If F is $k \times s$, then the M_i are still $k \times k$, and we may regard the s columns of F as sharing the same first order system. Given that the M_i are all $k \times k$, the solution space of (1.1) is quite enormous since the size of F is not restricted columnwise. However, we shall be concerned mostly with a square F or a single-columned F. Among the square solutions of (1.1) the following are basic and serve later as variables of differentiation for other solutions of (1.1):

$$Z_i = x_i I + x_n M_i \quad \text{for } 1 \leq i \leq n-1, \tag{1.3}$$

where I is the $k \times k$ identity matrix. It is easily checked that these $n-1$ matrix-valued functions satisfy equation (1.1), as do their sums and linear combinations. But a product of Z_i, such as $Z_1 Z_2$, turns out to be in general not a solution of (1.1). Fortunately however, we can get around this difficulty by permuting the factors in the product and adding the permuted products to form a new product, which will then satisfy (1.1). Specifically, for $Z = (Z_1, Z_2, \ldots, Z_{n-1})$ and each multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_{n-1})$ we define the *symmetric power* Z^β to be the sum of all possible Z_i products each of

which contains the Z_i factor exactly β_i times. For example, for $n = 3$, we have

$$(Z_1, Z_2)^{(1,1)} = Z_1Z_2 + Z_2Z_1,$$
$$(Z_1, Z_2)^{(2,1)} = Z_1^2Z_2 + Z_1Z_2Z_1 + Z_2Z_1^2,$$

and so on. If any component of β is negative, we simply set $Z^\beta = 0$; if all the components of β are zero, we set $Z^\beta = I$. Otherwise, Z^β will contain $|\beta|!/\beta!$ terms, where $\beta! := \beta_1!\beta_2!\cdots\beta_{n-1}!$.

2. Symmetric powers Z^β

In order to show that symmetric powers Z^β satisfy equation (1.1), we need some basic lemmas.

Lemma 1 (reduction formulas). *Given* $Z = (Z_1, Z_2, \dots, Z_{n-1})$, $\beta = (\beta_1, \beta_2, \dots, \beta_{n-1})$, *and* $e^i = (0, \dots, 0, 1, 0, \dots, 0)$, *the ith unit multi-index, we have*

$$Z^\beta = Z_1Z^{\beta-e^1} + Z_2Z^{\beta-e^2} + \cdots + Z_{n-1}Z^{\beta-e^{n-1}}, \tag{2.1}$$

$$Z^\beta = Z^{\beta-e^1}Z_1 + Z^{\beta-e^2}Z_2 + \cdots + Z^{\beta-e^{n-1}}Z_{n-1}. \tag{2.2}$$

Proof. Both (2.1) and (2.2) are quite obvious since all the terms in Z^β can be classified and grouped according to the first factor, or alternatively the last factor, of each term. For example,

$$\begin{aligned}(Z_1, Z_2)^{(2,1)} &= Z_1^2Z_2 + Z_1Z_2Z_1 + Z_2Z_1^2\\ &= Z_1(Z_1Z_2 + Z_2Z_1) + Z_2Z_1^2\\ &= Z_1(Z_1, Z_2)^{(1,1)} + Z_2(Z_1, Z_2)^{(2,0)}.\end{aligned}$$

We omit a more formal proof.

Lemma 2 (joint commutability formulas). *Given* $Z_i = x_iI + x_nM_i$ *for* $1 \le i \le n-1$, *we have*

$$M_iZ_j + M_jZ_i = Z_jM_i + Z_iM_j, \tag{2.3}$$

$$Z_i\,dZ_j + Z_j\,dZ_i = (dZ_j)Z_i + (dZ_i)Z_j. \tag{2.4}$$

Proof. We show (2.3) first:

$$\begin{aligned}M_iZ_j + M_jZ_i &= M_i(x_jI + x_nM_j) + M_j(x_iI + x_nM_i)\\ &= x_jM_i + x_nM_iM_j + x_iM_j + x_nM_jM_i.\end{aligned}$$

Interchanging the second and the fourth terms, we obtain

$$M_iZ_j + M_jZ_i = (x_jI + x_nM_j)M_i + (x_iI + x_nM_i)M_j.$$

To show (2.4) we write

$$\begin{aligned}Z_i\,dZ_j + Z_j\,dZ_i &= Z_i(dx_jI + dx_nM_j) + Z_j(dx_iI + dx_nM_i)\\ &= dx_jZ_i + dx_nZ_iM_j + dx_iZ_j + dx_nZ_jM_i,\end{aligned}$$

and proceed by applying (2.3). We omit the details.

Lemma 3 (differentiation formulas). *Given Z and β as in Lemma 1, we have*

$$\partial_i Z^\beta = |\beta| Z^{\beta - e^i} \quad \text{for } 1 \le i \le n-1, \tag{2.5}$$

$$\partial_n Z^\beta = |\beta| \sum_{j=1}^{n-1} M_j Z^{\beta - e^j}. \tag{2.6}$$

Proof. Formula (2.5) can be proved by induction on $|\beta|$. For $|\beta| = 1$, $\beta = e^j$ for some $1 \le j \le n-1$, so

$$\partial_i Z^\beta = \partial_i Z_j = \partial_i (x_j I + x_n M_j) = \delta_j^i I,$$

where δ_j^i is a Kronecker delta. On the other hand,

$$|\beta| Z^{\beta - e^i} = |e^j| Z^{e^j - e^i} = \delta_j^i I,$$

where we recall the convention that $Z^\beta = 0$ if any component of β is negative. Thus (2.5) holds for $|\beta| = 1$. Next, assume as our induction hypothesis (2.5) for $|\beta| \le m$ $(m \ge 1)$. Then for $|\beta| = m+1$, we have

$$\begin{aligned}
\partial_i Z^\beta &= \partial_i \left(\sum_{j=1}^{n-1} Z_j Z^{\beta - e^j} \right) \quad \text{(by (2.1))} \\
&= \sum_{j=1}^{n-1} (\delta_j^i Z^{\beta - e^j} + Z_j |\beta - e^j| Z^{\beta - e^j - e^i}) \\
&\qquad\qquad \text{(by the induction hypothesis)} \\
&= Z^{\beta - e^i} + (|\beta| - 1) \sum_{j=1}^{n-1} Z_j Z^{\beta - e^i - e^j} \\
&= Z^{\beta - e^i} + (|\beta| - 1) Z^{\beta - e^i} \quad \text{(again by (2.1))} \\
&= |\beta| Z^{\beta - e^i},
\end{aligned}$$

and (2.5) holds for $|\beta| = m+1$.

We prove (2.6) also by induction on $|\beta|$. For $|\beta| = 1$, $\beta = e^i$ for some $1 \le i \le n-1$, so

$$\partial_n Z^\beta = \partial_n Z_i = \partial_n (x_i I + x_n M_i) = M_i.$$

On the other hand, for $\beta = e^i$ we have

$$|\beta| \left(\sum_{j=1}^{n-1} M_j Z^{\beta - e^j} \right) = \sum_{j=1}^{n-1} M_j Z^{e^i - e^j} = \sum_{j=1}^{n-1} M_j \delta_j^i I = M_i.$$

Hence (2.6) holds for $|\beta| = 1$. Now assume (2.6) for $|\beta| \leq m$. Then for $|\beta| = m+1$ we have

$$\begin{aligned}
\partial_n Z^\beta &= \partial_n \left(\sum_{j=1}^{n-1} Z_j Z^{\beta - e^j} \right) \quad \text{(by (2.1))} \\
&= \sum_{j=1}^{n-1} \left[M_j Z^{\beta - e^j} + Z_j (|\beta| - 1) \sum_{k=1}^{n-1} M_k Z^{\beta - e^j - e^k} \right] \\
&\qquad\qquad \text{(by the induction hypothesis)} \\
&= \sum_{j=1}^{n-1} M_j Z^{\beta - e^j} + (|\beta| - 1) \sum_{j,k=1}^{n-1} Z_j M_k Z^{\beta - e^j - e^k}.
\end{aligned}$$

But $Z_j M_k$ above can be switched to $M_k Z_j$ by the joint commutability formula (2.3). Then with the reduction formula (2.1) applied backwards, we obtain

$$\begin{aligned}
\partial_n Z^\beta &= \sum_{j=1}^{n-1} M_j Z^{\beta - e^j} + (|\beta| - 1) \sum_{k=1}^{n-1} M_k Z^{\beta - e^k} \\
&= |\beta| \sum_{j=1}^{n-1} M_j Z^{\beta - e^j},
\end{aligned}$$

and (2.6) holds for $|\beta| = m+1$. This completes the proof.

We note that (2.5) leads obviously to a kindred formula:

$$\partial^\alpha Z^\beta = (|\beta|!/|\beta - \alpha|!) Z^{\beta - \alpha} \quad \text{for } \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, 0), \tag{2.7}$$

where as before $Z^\beta = 0$ if any component of β is negative.

Theorem 1. *Every symmetric power Z^β satisfies equation* (1.1). *That is,*

$$\mathbf{M} Z^\beta = 0. \tag{2.8}$$

Proof. Again we prove by induction on $|\beta|$. For $|\beta| = 1$, $\beta = e^j$ for some $1 \leq j \leq n-1$, so $Z^\beta = Z_j$, which satisfies (1.1). Now assume the induction

hypothesis that Z^β satisfies (1.1) for $|\beta| \le m$. Then for $|\beta| = m+1$, we have

$$\begin{aligned}
\mathbf{M}Z^\beta &= \left(\sum_{j=1}^{n-1} M_j\partial_j - I\partial_n\right) Z^\beta \\
&= \sum_{j=1}^{n-1} M_j\partial_j \left(\sum_{i=1}^{n-1} Z_i Z^{\beta-e^i}\right) - \partial_n \left(\sum_{i=1}^{n-1} Z_i Z^{\beta-e^i}\right) \\
&= \sum_{i,j=1}^{n-1} M_j(\delta_j^i Z^{\beta-e^i} + Z_i\partial_j Z^{\beta-e^i}) \\
&\quad - \sum_{i=1}^{n-1}(M_i Z^{\beta-e^i} + Z_i\partial_n Z^{\beta-e^i}) \\
&= \sum_{j=1}^{n-1} M_j Z^{\beta-e^j} + \sum_{i,j=1}^{n-1} M_j Z_i|\beta - e^i|Z^{\beta-e^i-e^j} \\
&\quad - \sum_{i=1}^{n-1} M_i Z^{\beta-e^i} - \sum_{i=1}^{n-1} Z_i\partial_n Z^{\beta-e^i} \\
&= \sum_{i,j=1}^{n-1} Z_i M_j|\beta - e^i|Z^{\beta-e^i-e^j} - \sum_{i=1}^{n-1} Z_i\partial_n Z^{\beta-e^i} \quad \text{(by (2.3))} \\
&= \sum_{i=1}^{n-1} Z_i \left(\sum_{j=1}^{n-1} M_j\partial_j Z^{\beta-e^i}\right) - \sum_{i=1}^{n-1} Z_i\partial_n Z^{\beta-e^i} \quad \text{(by (2.3))} \\
&= \sum_{i=1}^{n-1} Z_i \left(\sum_{j=1}^{n-1} M_j\partial_j - I\partial_n\right) Z^{\beta-e^i} \\
&= \sum_{i=1}^{n-1} Z_i \mathbf{M} Z^{\beta-e^i} \\
&= 0 \quad \text{(by the induction hypothesis).}
\end{aligned}$$

This completes the proof.

Delanghe [1] and Hile [3] have noninductive proofs of similar theorems.

3. Z-DIFFERENTIATION

Since symmetric powers Z^β are solutions of (1.1), they are M-holomorphic and can play the role of generators for other M-holomorphic functions. With this in mind we begin by attempting to "differentiate" an M-holomorphic function "with respect to Z". Suppose F is M-holomorphic in R^3, say. Then

$$dF = F_x\,dx + F_y\,dy + F_z\,dz,$$

and if F_z is eliminated with (1.1), then

$$\begin{aligned} dF &= F_x\,dx + F_y\,dy + (M_1F_x + M_2F_y)\,dz \\ &= (I\,dx + M_1\,dz)F_x + (I\,dy + M_2\,dz)F_y \\ &= (dZ_1)F_x + (dZ_2)F_y. \end{aligned}$$

We end up with "partial derivatives" of F with respect to Z_1 and Z_2, which turn out to be equal to F_x and F_y.

More generally, we shall say that a matrix-valued function F is differentiable with respect to Z or *Z-differentiable* with $Z = (Z_1, Z_2, \dots, Z_{n-1})$ if F is differentiable and

$$dF = \sum_{i=1}^{n-1}(dZ_i)H_i$$

for some H_i. Each H_i is referred to as the *partial derivative* of F with respect to Z_i and denoted by D_iF. For convenience we shall also refer to H_i collectively as the *derivative* of F with respect to Z.

Theorem 2. *A matrix-valued function F is M-holomorphic if and only if it is Z-differentiable, where Z is determined from* (1.1) *by* (1.3). *The Z-derivative and x-derivative of F are related by*

$$D_iF = \partial_iF \quad \textit{for } 1 \le i \le n-1. \tag{3.1}$$

Proof. If F is M-holomorphic, then

$$\begin{aligned} dF &= \sum_{i=1}^{n-1}(dx_i)\,\partial_iF \\ &= \sum_{i=1}^{n-1}(dx_i)\partial_iF + dx_n\left(\sum_{i=1}^{n-1}M_i\partial_iF\right) \quad \text{(by (1.1))} \\ &= \sum_{i=1}^{n-1}(dx_iI + dx_nM_i)\partial_iF \\ &= \sum_{i=1}^{n-1}(dZ_i)\partial_iF, \end{aligned}$$

and so F is Z-differentiable, with $D_iF = \partial_iF$ for $1 \le i \le n-1$.

Conversely, if F is Z-differentiable, then

$$\begin{aligned} dF &= \sum_{i=1}^{n-1}(dZ_i)H_i \quad \text{(for some } H_i) \\ &= \sum_{i=1}^{n-1}(I\,dx_i + M_i\,dx_n)H_i \\ &= \sum_{i=1}^{n-1}H_i\,dx_i + \left(\sum_{i=1}^{n-1}M_iH_i\right)dx_n. \end{aligned}$$

Hence $\partial_i F = H_i$ for $1 \le i \le n-1$, and $\partial_n F = \sum_{i=1}^{n-1} M_i H_i$. Eliminating H_i, we obtain

$$\partial_n F = \sum_{i=1}^{n-1} M_i \partial_i F,$$

and so F is M-holomorphic. This completes the proof.

It is significant that if F is M-holomorphic, then the x-differential of F is expressible as the Z-differential of F. That is,

$$dF = \sum_{i=1}^{n} (\partial_i F)\, dx_i = \sum_{i=1}^{n-1} (dZ_i) D_i F. \tag{3.2}$$

Note that dZ_i comes before $D_i F$. However, it turns out that when $F = Z^\beta$, this need not be.

Lemma 4 (collective commutability formulas). *For any multi-index* $\beta = (\beta_1, \beta_2, \dots, \beta_{n-1})$, *we have*

$$dZ^\beta = |\beta| \sum_{i=1}^{n-1} (dZ_i) Z^{\beta - e^i} = |\beta| \sum_{i=1}^{n-1} Z^{\beta - e^i} (dZ_i), \tag{3.3}$$

$$\begin{aligned} d(Z-A)^\beta &= |\beta| \sum_{i=1}^{n-1} (dZ_i)(Z-A)^{\beta - e^i} \\ &= |\beta| \sum_{i=1}^{n-1} (Z-A)^{\beta - e^i} (dZ_i), \end{aligned} \tag{3.4}$$

$$\begin{aligned} d(A-Z)^\beta &= -|\beta| \sum_{i=1}^{n-1} (dZ_i)(A-Z)^{\beta - e^i} \\ &= -|\beta| \sum_{i=1}^{n-1} (A-Z)^{\beta - e^i} (dZ_i), \end{aligned} \tag{3.5}$$

where $A = (A_1, A_2, \dots, A_{n-1})$ *and* $A_i = Z_i(a) = a_i I + a_n M_i$.

Proof. We prove by induction on $|\beta|$. For $|\beta| = 1$, $\beta = e^j$ for some $1 \le j \le n-1$, so

$$\begin{aligned} \sum_{i=1}^{n-1} (dZ_i) Z^{\beta - e^i} &= \sum_{i=1}^{n-1} (dZ_i) Z^{e^j - e^i} = dZ_j \\ &= \sum_{i=1}^{n-1} Z^{e^j - e^i}\, dZ_i = \sum_{i=1}^{n-1} Z^{\beta - e^i} (dZ_i), \end{aligned}$$

and (3.3) holds for $|\beta| = 1$. Now assume (3.3) for $|\beta| \le m$. Then for

$|\beta| = m+1$ we have

$$\begin{aligned}
\sum_{i=1}^{n-1}(dZ_i)Z^{\beta-e^i} &= \sum_{i=1}^{n-1}(dZ_i)\left(\sum_{j=1}^{n-1} Z_j Z^{\beta-e^i-e^j}\right) \quad \text{(by (2.1))} \\
&= \sum_{i,j=1}^{n-1}(dZ_i)Z_j Z^{\beta-e^i-e^j} \\
&= \sum_{i,j=1}^{n-1} Z_j(dZ_i)Z^{\beta-e^i-e^j} \quad \text{(by (2.4))} \\
&= \sum_{j=1}^{n-1} Z_j\left[\sum_{i=1}^{n-1}(dZ_i)Z^{\beta-e^i-e^j}\right] \\
&= \sum_{j=1}^{n-1} Z_j\left(\sum_{i=1}^{n-1} Z^{\beta-e^i-e^j}\, dZ_i\right) \quad \text{(by the induction hypothesis)} \\
&= \sum_{i=1}^{n-1}\left(\sum_{j=1}^{n-1} Z_j Z^{\beta-e^i-e^j}\right) dZ_i \\
&= \sum_{i=1}^{n-1} Z^{\beta-e^i}\, dZ_i \quad \text{(by (2.1))}.
\end{aligned}$$

Hence (3.3) holds for $|\beta| = m+1$.

The proof of (3.4) is technically more complicated, but requires no new ideas, and hence will be omitted. As for (3.5), it can be derived from (3.4) as follows. Since

$$(A-Z)^\beta = (-1)^{|\beta|}(Z-A)^\beta, \tag{3.6}$$

we have in view of the first equality in (3.4)

$$\begin{aligned}
d(A-Z)^\beta &= (-1)^{|\beta|}|\beta|\sum_{i=1}^{n-1}(dZ_i)(Z-A)^{\beta-e^i} \\
&= (-1)^{|\beta|}|\beta|\sum_{i=1}^{n-1}(dZ_i)(-1)^{|\beta|-1}(A-Z)^{\beta-e^i} \\
&= (-1)^{2|\beta|-1}|\beta|\sum_{i=1}^{n-1}(dZ_i)(A-Z)^{\beta-e^i};
\end{aligned}$$

and the first equality of (3.5) is proved. The second equality is proved likewise by combining (3.6) and the second equality of (3.4). This completes the proof of Lemma 4.

If the partial derivatives of F with respect to Z_i are themselves differentiable with respect to Z, then we have the second and possibly higher order partial derivatives of F with respect to Z_i, and we may employ notations such as

$D^\beta F$ since, as we shall see, the order of partial differentiations is immaterial as long as F is sufficiently smooth. If the $D^\beta F$ are continuous functions of x for all $|\beta| = m$, then we say that F is C^m with respect to Z.

Theorem 3. *F is M-holomorphic and C^m with respect to $x = (x_1, x_2, \dots, x_n)$ if and only if F is C^m with respect to $Z = (Z_1, Z_2, \dots, Z_{n-1})$. The Z-derivatives and the x-derivatives of F are related by*

$$D^\beta F = \partial^\beta F \tag{3.7}$$

for any $\beta = (\beta_1, \beta_2, \dots, \beta_{n-1})$ with $|\beta| \le m$, where ∂^β is to be interpreted as ∂^α with $\alpha = (\beta_1, \beta_2, \dots, \beta_{n-1}, 0)$.

Proof. First assume F to be M-holomorphic and C^m with respect to x. By Theorem 2, $D_i F = \partial_i F$ for $1 \le i \le n-1$, but $\partial_i F$ is itself M-holomorphic since $\mathbf{M}\partial_i F = \partial_i \mathbf{M} F = 0$. Hence, again by Theorem 2, $D_i F$ is Z-differentiable and

$$D_j(D_i F) = D_j(\partial_i F) = \partial_j \partial_i F.$$

Likewise,

$$D_i(D_j F) = D_i(\partial_j F) = \partial_i \partial_j F.$$

But $\partial_j \partial_i F = \partial_i \partial_j F$ because F is C^m with respect to x; therefore we may simply write with disregard to the order of differentiations

$$D^{e^i + e^j} F = \partial^{e^i + e^j} F,$$

and more generally $D^\beta F = \partial^\beta F$ for $|\beta| \le m$. Since the $\partial^\beta F$ are continuous, so are the $D^\beta F$ and F is C^m with respect to Z.

Conversely, if F is C^m with respect to Z, then by Theorem 2 F is M-holomorphic and the $D^\beta F$ are continuous for $|\beta| \le m$. But $D^\beta F = \partial^\beta F$, so the $\partial^\alpha F$ are continuous for all $|\alpha| \le m$ with $\alpha_n = 0$. To show the $\partial^\alpha F$ are continuous also for $\alpha_n \ne 0$, we note that since $\partial_n = M_1 \partial_1 + \cdots + M_{n-1}\partial_{n-1}$, such a $\partial^\alpha F$ can be expanded as a sum of several $\partial^\alpha F$ in which $\alpha_n = 0$. This concludes the proof.

In view of the preceding theorem we can say that hyperholomorphy plus C^∞ with respect to x is equivalent to C^∞ with respect to Z. Later we will show in Theorem 5 that hyperholomorphy (M-holomorphy) plus real analyticity (x-analyticity) is equivalent to hyperanalyticity (Z-analyticity).

4. Z-INTEGRATION

We need an integration-by-parts formula for line integrals in order to derive a Taylor formula. The *line integral* along a path of integration (having a continuously varying tangent vector) of a differential form, denoted by

$$\int_C F_1(x)\,dx_1 + F_2(x)\,dx_2 + \cdots + F_n(x)\,dx_n,$$

where the F_i are continuous in a neighborhood of the path C, is defined to be the matrix whose (i, j)-component is given by

$$\int_C f_{ij}^1(x)\,dx_1 + f_{ij}^2(x)\,dx_2 + \cdots + f_{ij}^n(x)\,dx_n,$$

where f_{ij}^k is the (i, j)-component of F_k. If F and G are C^1 matrix-valued functions of compatible dimensions, then we further define

$$\int_C (dF)G := \int_C (\partial_1 F)G\,dx_1 + \cdots + (\partial_n F)G\,dx_n,$$
$$\int_C F(dG) := \int_C (F\partial_1 G)\,dx_1 + \cdots + (F\partial_n G)\,dx_n.$$

Proposition 1 (integration-by-parts formulas). *If F and G are both C^1 in a domain Ω in R^n, and C is a path of integration going from a to b in Ω, then*

$$\int_C (dF)G = F(b)G(b) - F(a)G(a) + \int_C (-F)\,dG, \tag{4.1}$$

$$\int_C F(dG) = F(b)G(b) - F(a)G(a) + \int_C dF(-G). \tag{4.2}$$

Proof. We have

$$F(b)G(b) - F(a)G(a) = \int_C d(FG) = \int_C (dF)G + F(dG).$$

Hence, by transposing and by putting the minus sign in appropriate places, we obtain both formulas.

Theorem 4 (Taylor formula). *If F is M-holomorphic and C^{m+1} with $m \geq 0$ in a domain Ω containing the origin 0 in R^n, then for any x in Ω, we have*

$$F(x) = \sum_{|\beta|\leq m} (Z^\beta/|\beta|!)D^\beta F(0) + \int_0^x \sum_{|\beta|=m+1} d[-(Z-\tilde{Z})^\beta/|\beta|!]D^\beta F(\tilde{x}), \tag{4.3}$$

where the line integral is taken along any path of integration in Ω going from 0 to x, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ is the variable of integration, $\tilde{Z} = Z(\tilde{x}) = (Z_1(\tilde{x}), Z_2(\tilde{x}), \ldots, Z_{n-1}(\tilde{x}))$, and $Z_i(\tilde{x}) = \tilde{x}_i I + \tilde{x}_n M_i$.

Proof. We prove by induction on m. For $m = 0$, (4.3) reduces in two steps to something obvious:

$$F(x) = F(0) + \int_0^x \sum_{i=1}^{n-1} d[-(Z_i - \tilde{Z}_i)]D_i F(\tilde{x}),$$

$$F(x) = F(0) + \int_0^x \sum_{i=1}^{n-1} d\tilde{Z}_i[D_i F(\tilde{x})],$$

$$F(x) = F(0) + \int_0^x dF(\tilde{x}) \quad \text{(by (3.2))}.$$

The last equation holds for any path of integration going from 0 to x, so (4.3) holds for $m = 0$. Now assume as our induction hypothesis:

$$F(x) = \sum_{|\beta| \le m-1} (Z^\beta / |\beta|!) D^\beta F(0) + \int_0^x \sum_{|\beta|=m} d[-(Z - \tilde{Z})^\beta / |\beta|!] D^\beta F(\tilde{x}), \tag{4.4}$$

where the line integral is taken along any path of integration in Ω going from 0 to x. Applying the integration-by-parts formula (4.1) to the line integral above, we obtain

$$\sum_{|\beta|=m} (Z^\beta / |\beta|!) D^\beta F(0) + \int_0^x \sum_{|\beta|=m} [(Z - \tilde{Z})^\beta / |\beta|!] d[D^\beta F(\tilde{x})].$$

Now, the last integrand becomes, after expanding $d[D^\beta F(\tilde{x})]$ by (3.2),

$$\sum_{|\beta|=m} \sum_{i=1}^{n-1} [(Z - \tilde{Z})^\beta |\beta|!] \, d\tilde{Z}_i [D^{\beta+e^i} F(\tilde{x})],$$

which can be rewritten, with $\beta + e^i = \gamma$, as

$$\sum_{|\gamma|=m+1} \sum_{i=1}^{n-1} [(Z - \tilde{Z})^{\gamma - e^i} / |\gamma - e^i|!] \, d\tilde{Z}_i [D^\gamma F(\tilde{x})],$$

which in view of (3.4) is equal to

$$\sum_{|\gamma|=m+1} d[-(Z - \tilde{Z})^\gamma / |\gamma|!][D^\gamma F(\tilde{x})].$$

With this last expression as the integrand of the last line integral, we see that (4.3) holds. This completes the proof.

Note that since $D^\beta F = \partial^\beta F$ according to (3.7), we may also write our Taylor formula more simply as

$$F(x) = \sum_{|\beta| \le m} (Z^\beta / |\beta|!) \partial^\beta F(0) + \int_0^x \sum_{|\beta|=m+1} d[-(Z - \tilde{Z})^\beta / |\beta|!] \partial^\beta F(\tilde{x}), \tag{4.5}$$

where again ∂^β is to be interpretated as ∂^α with $\alpha = (\beta_1, \beta_2, \ldots, \beta_{n-1}, 0)$.

5. Z-ANALYTICITY

Using the Taylor formula we can expand in terms of Z^β any M-holomorphic function which is real analytic at the origin of R^n. We recall that a matrix-valued function F is *real analytic* at the origin if there exists a domain Ω containing 0 such that $F(x) = \sum_{i=0}^\infty F_i(x)$ for any x in Ω, where each $F_i(x)$

is a matrix whose components are ith degree homogeneous polynomials in x or possibly zero polynomials. In order to proceed from Taylor formula to Taylor expansion we need a Leibniz formula for differentiation under integral sign to be applied to the remainder term in the Taylor formula.

First we recall that for a line integral of scalar-valued differential form we have the following Leibniz formula:

$$\partial_i \int_0^x \sum_{j=1}^n f_j(x, \tilde{x})\, d\tilde{x}_j = \int_0^x \sum_{j=1}^n \partial_i f_j(x, \tilde{x})\, d\tilde{x}_j + f_i(x, x) \tag{5.1}$$

provided that f_j and $\partial_i f_j$ are all continuous in $\Omega \times \Omega$ for some Ω containing the origin 0 in R^n, and that the line integral is independent of the paths of integration in Ω going from 0 to x. The differentiation ∂_i is with respect to the ith variable x_i, and the $\tilde{x}_j$ denote the variables of integration.

We also recall a Leibniz formula for differentiation of the product of scalar-valued functions $u(x)$ and $v(x)$ defined in a domain in R^n:

$$\partial^\alpha(uv) = \sum_{\gamma \le \alpha} C_\gamma^\alpha (\partial^\gamma u)(\partial^{\alpha-\gamma} v), \tag{5.2}$$

where α is a multi-index of dimension n and C_γ^α is the product of binomial coefficients $C_{\gamma_i}^{\alpha_i}$ with $1 \le i \le n$. The Leibniz formula stated below relies on both Leibniz formulas above.

Proposition 2 (Leibniz formula). *Let F and G be matrix-valued functions of class C^m in $\Omega \times \Omega$, where Ω is a domain containing the origin of R^n, and let H be defined for any x in Ω by*

$$H(x) = \int_0^x F(x, \tilde{x})\, dG(x, \tilde{x}),$$

assuming that F and G are such that the line integral is independent of the paths of integration in Ω going from 0 to x. Then

$$\begin{aligned} \partial_i H(x) = &\int_0^x \partial_i F(x, \tilde{x})\, dG(x, \tilde{x}) + \int_0^x F(x, \tilde{x})\, d\partial_i G(x, \tilde{x}) \\ &+ F(x, x)\tilde{\partial}_i G(x, x), \end{aligned} \tag{5.3}$$

where $\tilde{\partial}_i$ denotes differentiation with respect to $\tilde{x}_i$, and

$$\tilde{\partial}_i G(x, x) = \tilde{\partial}_i G(x, \tilde{x})|_{\tilde{x}=x}.$$

If further (as happens often)

$$\partial^\alpha F(x, \tilde{x}) \tilde{\partial}^\gamma G(x, \tilde{x})|_{\tilde{x}=x} = 0 \tag{5.4}$$

for all α and γ with $|\alpha| \le m$ and $|\gamma| \le m$, then we have for any α with $|\alpha| \le m$

$$\partial^\alpha H(x) = \int_0^x \sum_{\gamma \le \alpha} C_\gamma^\alpha \partial^\gamma F(x, \tilde{x})\, d\partial^{\alpha-\gamma} G(x, \tilde{x}), \tag{5.5}$$

where $\gamma \le \alpha$ means $\gamma_i \le \alpha_i$ for all $i = 1, 2, \ldots, n$.

Proof. We prove (5.3) first. From

$$H(x) = \int_0^x F(x, \tilde{x}) \sum_{j=1}^n \tilde{\partial}_j G(x, \tilde{x})\, d\tilde{x}_j$$

we obtain in view of (5.1)

$$\partial_i H(x) = \int_0^x \partial_i \left[F(x, \tilde{x}) \sum_{j=1}^n \tilde{\partial}_j G(x, \tilde{x}) \right] d\tilde{x}_j + F(x, x)\tilde{\partial}_i G(x, x),$$

but the integrand in the above line integral is equal to

$$\left[\partial_i F(x, \tilde{x}) \sum_{j=1}^n \tilde{\partial}_j G(x, \tilde{x}) + F(x, \tilde{x}) \sum_{j=1}^n \partial_i \tilde{\partial}_j G(x, \tilde{x}) \right] d\tilde{x}_j$$
$$= \partial_i F(x, \tilde{x})\, dG(x, \tilde{x}) + F(x, \tilde{x})\, d\partial_i G(x, \tilde{x});$$

and (5.3) is proved. If now (5.4) holds, then

$$\partial_i H(x) = \int_0^x [\partial_i F(x, \tilde{x})\, dG(x, \tilde{x}) + F(x, \tilde{x})\, d\partial_i G(x, \tilde{x})].$$

Thus, superficially, ∂_i acts on $F(x, \tilde{x})\, dG(x, \tilde{x})$ as if differentiating the formal product $[F(x, \tilde{x})d][G(x, \tilde{x})]$. Hence in view of formula (5.2) for differentiation of the product, we have

$$\partial^\alpha H(x) = \int_0^x \sum_{\gamma \le \alpha} C_\gamma^\alpha [\partial^\gamma F(x, \tilde{x})\, d\partial^{\alpha-\gamma} G(x, \tilde{x})].$$

This completes the proof.

Quite analogously to (5.5) we also have

$$\partial^\alpha \int_0^x [dF(x, \tilde{x})] G(x, \tilde{x}) = \int_0^x \sum_{\gamma \le \alpha} C_\gamma^\alpha [d\partial^\gamma F(x, \tilde{x})] \partial^{\alpha-\gamma} G(x, \tilde{x}), \tag{5.6}$$

assuming that the line integral is independent of the paths of integration going from 0 to x, and that (5.4) holds.

Using the Taylor formula (4.5) aided by the Leibniz formula (5.6), we can obtain a Z-expansion of an M-holomorphic function which is real analytic at the origin of R^n. Evidently, Z-analyticity is a consequence of Z-differentiability and real analyticity. Against this general background we have special situations in which hyperanalyticity is a consequence of hyperholomorphy and something less than real analyticity, such as C^1 or C^2 (see [3, 5]). We can anticipate here a general inquiry into types of hyperholomorphy under which hyperanalyticity is assured with minimal C^m. The following theorem, in a slightly different form, was proved by Hile [3] with difficult computation, that is, without use of the Taylor formula.

Theorem 5 (Taylor expansion). *Let F be real analytic at the origin 0 of R^n and suppose Ω is the domain containing 0 in which the power series in x of F is convergent. If F is M-holomorphic in Ω, then for any x in Ω we have*

$$F(x) = \sum_{\beta} [Z^{\beta}/|\beta|!]\partial^{\beta} F(0), \tag{5.7}$$

where $\sum_{\beta} = \sum_{|\beta|=0}^{\infty}$ and $\partial^{\beta} = \partial^{\alpha}$ with $\alpha = (\beta_1, \beta_2, \dots, \beta_{n-1}, 0)$. In other words, an M-holomorphic real analytic F is Z-analytic. Conversely, if F is Z-analytic, that is

$$F(x) = \sum_{\beta} [Z^{\beta}/|\beta|!]C_{\beta}, \tag{5.8}$$

where C_{β} are constant matrices of the same dimensions as F such that the series converges in some neighborhood Ω of the origin, then F is M-holomorphic and real analytic.

Proof. F being real analytic at 0 with Ω as given, we have for any x in Ω

$$F(x) = \sum_{i=0}^{\infty} F_i(x), \tag{5.9}$$

where $F_i(x)$ is a matrix consisting of ith degree homogeneous polynomials in x and possibly also zero polynomials. On the other hand, in view of the Taylor formula (4.5) we also have

$$\begin{aligned} F(x) = &\sum_{|\beta|\le m} [Z^{\beta}/|\beta|!]\partial^{\beta} F(0) \\ &+ \int_0^x \sum_{|\beta|=m+1} d[-(Z-\tilde{Z})^{\beta}/|\beta|!]\partial^{\beta} F(\tilde{x}) \end{aligned} \tag{5.10}$$

for all $m \ge 0$, where we used the fact that F is infinitely many times differentiable. Comparing (5.9) and (5.10), we see that it suffices to show

$$F_i(x) = \sum_{|\beta|=i} [Z^{\beta}/|\beta|!]\partial^{\beta} F(0) \quad \text{for all } i \ge 0. \tag{5.11}$$

We do this inductively. For $i = 0$, we have $F_0(x) = F(0)$ from (5.9) since $F_0(x)$ is just the constant part of $F(x)$, which is $F(0)$. On the other hand we have

$$\sum_{|\beta|=0} [Z^{\beta}/|\beta|!]\partial^{\beta} F(0) = F(0).$$

Hence (5.11) holds for $i = 0$. Now we assume (5.11) for $i \le k-1$ $(k \ge 1)$ and show (5.11) holds for $i = k$. Equating (5.9) and (5.10) and cancelling out the beginning portions by the induction hypothesis, we obtain

$$\begin{aligned} F_k(x) + \sum_{i=k+1}^{\infty} F_i(x) = &\sum_{|\beta|=k} [Z^{\beta}/|\beta|!]\partial^{\beta} F(0) \\ &+ \int_0^x \sum_{|\beta|=k+1} d[-(Z-\tilde{Z})^{\beta}/|\beta|!]\partial^{\beta} F(\tilde{x}). \end{aligned}$$

Applying ∂^α with $|\alpha| = k$ and using the Leibniz formula (5.6), we obtain

$$(5.12)\qquad \partial^\alpha F_k(x) + \sum_{i=k+1}^{\infty} \partial^\alpha F_i(x) = \sum_{|\beta|=k} [\partial^\alpha Z^\beta / |\beta|!]\partial^\beta F(0)$$
$$+ \int_0^x \sum_{|\beta|=k+1} \sum_{\gamma \le \alpha} C_\gamma^\alpha d[-\partial^\gamma (Z - \tilde{Z})^\beta / |\beta|!]\partial^{\alpha-\gamma}(\partial^\beta F)(\tilde{x}).$$

Note however the expression $(\partial^\beta F)(\tilde{x})$, which perhaps should be rewritten here as $\tilde{\partial}^\beta F(\tilde{x})$, does not involve x, and hence $\partial^{\alpha-\gamma}[(\partial^\beta F)(\tilde{x})] = 0$ unless $\alpha = \gamma$, in which case $\partial^{\alpha-\gamma}[(\partial^\beta F)(\tilde{x})] = (\partial^\beta F)(\tilde{x})$, and the above line integral reduces to

$$\int_0^x \sum_{|\beta|=k+1} d[-\partial^\alpha (Z - \tilde{Z})^\beta / |\beta|!]\partial^\beta F(\tilde{x}).$$

Now if we let $x = 0$ in (5.12), the line integral vanishes and so do the $\partial^\alpha F_i(x)$ for all $i \ge k + 1 = |\alpha| + 1$ consisting of homogeneous polynomials of degree at least 1. And so (5.12) reduces to

$$\partial^\alpha F_k(x) = \partial^\alpha \left[\sum_{|\beta|=k} (Z^\beta / |\beta|!)\partial^\beta F(0) \right].$$

Thus, $F_k(x)$ and $\sum_{|\beta|=k}(Z^\beta / |\beta|!)\partial^\beta F(0)$ have identical kth order partial derivatives, and hence their respective components can differ by at most a polynomial of degree less than k. But since these components are all homogeneous polynomials of degree k, they must in fact be equal, and (5.11) holds for $i = k$. This concludes the proof of the first half of the theorem. As for the converse, if (5.8) holds, then F is M-holomorphic since by Theorem 1 each Z^β is. Also F is real analytic at the origin since each component of F is the sum of a convergent power series in x. This concludes the proof of Theorem 5.

It will be appropriate here to make sure that the representation of a Z-analytic function in terms of Z^β is unique. For this it is quite sufficient to prove

Theorem 6. *The Z^β are linearly independent. That is, if*

$$(5.13)\qquad \sum_{|\beta|=m} Z^\beta C_\beta = 0$$

in a neighborhood of the origin, where C_β are constant matrices of identical dimensions (not necessarily square) compatible with Z^β, then $C_\beta = 0$ for all $|\beta| = m$.

Proof. We prove by induction on $|\beta|$. If $|\beta| = 1$, then (5.13) reduces to $\sum_{i=1}^{n-1} Z_i C_i = 0$, or $\sum_{i=1}^{n-1}(x_i I + x_n M_i)C_i = 0$. Letting $x_1 \ne 0$, $x_2 = x_3 = \cdots = x_n = 0$, we see that $C_1 = 0$, and likewise $C_2 = C_3 = \cdots = C_{n-1} = 0$. Next

assume as our induction hypothesis that for $|\beta| = m$ (5.13) implies $C_\beta = 0$. Let $|\beta| = m+1$ in (5.13). Then

$$\partial_1 \left(\sum_{|\beta|=m+1} Z^\beta C_\beta \right) = |\beta| \sum_{\substack{|\beta|=m+1 \\ \beta_1 \geq 1}} Z^{\beta - e^1} C_\beta = 0.$$

Now by the induction hypothesis $C_\beta = 0$ for all β with $\beta_1 \geq 1$. That is, $C_\beta = 0$ unless $\beta_1 = 0$. Applying ∂_2 instead of ∂_1, we see $C_\beta = 0$ unless $\beta_2 = 0$. Continuing thus, we see $C_\beta = 0$ unless $\beta_1 = \beta_2 = \cdots = \beta_{n-1} = 0$, which does not happen since $|\beta| = m+1 \geq 1$. This completes the proof.

Theorem 7 (unique representation). *The totality of real analytic solutions at the origin of the first order system* (1.1) *is given by the convergent series* (5.8), *where the* C_β *are constant matrices of the same dimensions as* F. *The representation* (5.8) *is unique, and in fact* $C_\beta = \partial^\beta F(0)$.

Proof. By the first half of Theorem 5 every real analytic solution at the origin of (1.1) is of the form (5.8), and the second half of Theorem 5 asserts that a convergent series of the form (5.8) is a real analytic solution at the origin of (1.1). Thus (5.8) gives the totality of real analytic solutions at the origin of (1.1), and in view of Theorem 6 we must have $C_\beta = \partial^\beta F(0)$. That is, the representation (5.8) is unique. This completes the proof.

Our unique representation theorem is flexible in that the number of columns in a solution matrix F is not restricted. In particular, we can say that each real analytic column-vector solution of (1.1) is given uniquely by a convergent series

$$\mathbf{f}(x) = \sum_\beta (Z^\beta / |\beta|!) \mathbf{c}_\beta,$$

where the $\mathbf{c}_\beta$ are constant column vectors, and in fact $\mathbf{c}_\beta = \partial^\beta \mathbf{f}(0)$.

6. Second order equations

We consider a second order equation in R^n of the form

$$Lu := \sum_{i \leq j=1}^{n} a_{ij} \partial_i \partial_j u = 0, \tag{6.1}$$

where the complex constant coefficients a_{ij} have been so arranged that $a_{ij} = 0$ for $i > j$ and $a_{nn} = -1$. If u is a C^2 solution of (6.1), then its gradient ∇u expressed as an $n \times 1$ column vector $\mathbf{f}$ with components $f_i = \partial_i u$ is C^1 and satisfies the following first order matrix equation, which amounts to a rewriting of (6.1):

$$\begin{aligned} (a_{11}, a_{12}, \ldots, a_{1n})\partial_1 \mathbf{f} + (0, a_{22}, a_{23}, \ldots, a_{2n})\partial_2 \mathbf{f} \\ + \cdots + (0, \ldots, 0, a_{n-1,n-1}, a_{n-1,n})\partial_{n-1} \mathbf{f} \\ + (0, \ldots, 0, -1)\partial_n \mathbf{f} = 0. \end{aligned} \tag{6.2}$$

Additionally, $\mathbf{f}$ being a gradient of a C^2 function, its components satisfy a number of so-called compatibility equations, such as $\partial_1 f_2 = \partial_2 f_1$. These equations can be combined with (6.2) and grouped into one primary and $n-2$ secondary first order systems. In order to describe these first order systems we need to introduce several $n \times n$ square matrices which will appear as coefficients in the systems.

First, we introduce the *associated matrices* M_i of the second order operator L by:

(6.3) Each M_i for $1 \le i \le n-1$ is an $n \times n$ matrix whose entries are all 0 except the ith entry in the last column, which shall be 1, and the bottom row, which shall be $(0, \dots, 0, a_{ii}, a_{i,i+1}, \dots, a_{i,n})$.

These associated matrices M_i (examples are shown in §9) use up the coefficients from (6.2) and also the coefficients from some of the compatibility equations. The matrices M_i will appear as coefficients in the primary system (see Theorem 8 below).

Next, we introduce an array of almost-null or totally-null $n \times n$ matrices denoted by N_i^k. The totally-null matrices are needed to simplify statements and proofs of theorems. The square matrices N_i^k with $1 \le i \le n-1$ will appear as coefficients in the kth secondary system, where $k = 2, 3, \dots, n-1$. The nonzero entries in N_i^k come solely from the compatibility equations relating $\partial_k \mathbf{f}$ to $\partial_i \mathbf{f}$ for $i < k$. Details are not important at this point, but this is the suitable place to lay down the formal definitions for N_i^k.

(6.4) Each N_i^k for $2 \le k \le n-1$ and $1 \le i \le n-1$ is an $n \times n$ matrix whose entries are all 0 except that if

(6.4a) $i \le k-1$, then the (i, k)-entry shall be 1,

(6.4b) $i = k$, then the first $k-1$ diagonal entries shall be -1,

(6.4c) $i \ge k+1$, then no entries shall be an exception.

We are ready to convert equation (6.1) to a family of $n-1$ first order systems.

Theorem 8 (conversion theorem). *Given L in* (6.1), *let*

$$\mathbf{M} = \sum_{i=1}^{n-1} M_i \partial_i - I \partial_n, \tag{6.5}$$

$$\mathbf{N}^k = \sum_{i=1}^{n-1} N_i^k \partial_i \quad \textit{for } 2 \le k \le n-1, \tag{6.6}$$

where M_i and N_i^k are as defined in (6.3) *and* (6.4). *Then the gradient of a*

C^2 solution of (6.1) *satisfies the following first order systems*:

$$\mathbf{Mf} = 0, \tag{6.7}$$

$$\mathbf{N}^k\mathbf{f} = 0, \qquad 2 \le k \le n-1. \tag{6.8}$$

Conversely, if $\mathbf{f}$ *is a* C^1 $n \times 1$ *column-vector-valued function satisfying* (6.7) *and* (6.8) *in a simply-connected domain, then there exists a* C^2 *scalar-valued function* u *such that* $Lu = 0$ *and* $\nabla u = \mathbf{f}$.

Proof. Spelling out systems (6.7) and (6.8) according to (6.3) and (6.4), we see that all (except one) individual scalar equations carried by (6.7) and (6.8) are just the compatibility equations for components of the gradient of a C^2 solution of (6.1), the only exception being the last equation carried by (6.7), which can be recognized as (6.2). The converse is assured by the fact that if the components of a C^1 $\mathbf{f}$ are compatible as stipulated by (6.8) and part of (6.7) in a simply-connected domain, then $\mathbf{f}$ is known to be *exact* in the sense that there exists a C^2 scalar-valued function u such that $\nabla u = \mathbf{f}$, and the fact that $Lu = 0$ because ∇u satisfies the last equation carried by (6.7), which we recall is just a restatement of (6.1). This concludes the proof.

According to Theorem 8 the gradient ∇u of a C^2 solution of (6.1) is M-holomorphic with $M = (M_1, M_2, \dots, M_{n-1})$, where the M_i are associated matrices of L. Unfortunately, the converse is not true. That is, an $n \times 1$ M-holomorphic function $\mathbf{f}$ is not necessarily the gradient of any solution of (6.1) unless $\mathbf{f}$ happens to satisfy (6.8) as well. In other words, a column of an $n \times n$ M-holomorphic function is not always the gradient of a solution of (6.1), and therefore if we are interested in the solutions of (6.1), then some and possibly all columns of an $n \times n$ M-holomorphic function are going to be useless. For example, if we take the M-holomorphic functions Z^β with Z ultimately derived from L, then the columns of Z^β may or may not be gradients of solutions of (6.1).

Surprisingly however, it turns out that the last column of Z^β always satisfies (6.8) and hence is actually the gradient of some solution of (6.1).

Theorem 9. *Let* $Z = (Z_1, Z_2, \dots, Z_{n-1})$ *and* $Z_i = x_i I + x_n M_i$, *where the* M_i *are associated matrices of* L *as defined in* (6.3). *Then*

$$\mathbf{M}(Z^\beta E_n) = 0, \tag{6.9}$$

$$\mathbf{N}^k(Z^\beta E_n) = 0, \qquad 2 \le k \le n-1, \tag{6.10}$$

where $\mathbf{M}$ *and* $\mathbf{N}^k$ *are as defined in* (6.5) *and* (6.6), *and* E_n *is the nth unit column vector.*

Furthermore, we have

$$L(Z^\beta E_n) = 0. \tag{6.11}$$

That is, every entry of the last column of Z^β *satisfies* (6.1).

Proof. Equation (6.9) follows from equation (2.8) of Theorem 1 with a specific M arising from L of (6.1). To prove equation (6.10) we note

$$\begin{aligned}\mathbf{N}^k(Z^\beta E_n) &= \left(\sum_{i=1}^{n-1} N_i^k \partial_i\right)(Z^\beta E_n)\\ &= |\beta| \sum_{i=1}^{n-1} N_i^k Z^{\beta-e^i} E_n \quad \text{(by (2.5))}.\end{aligned}$$

Therefore, it suffices to show

$$\sum_{i=1}^{n-1} N_i^k Z^{\beta-e^i} E_n = 0. \tag{6.12}$$

We prove (6.12) by induction on $|\beta|$. For $|\beta| = 1$, $Z^\beta = Z^{e^j}$ for $1 \le j \le n-1$, and

$$\sum_{i=1}^{n-1} N_i^k Z^{e^j-e^i} E_n = N_j^k E_n = 0 \quad \text{(by (6.4))},$$

so (6.12) holds for $|\beta| = 1$. Now assume (6.12) holds for $|\beta| \le m$. Then for $|\beta| = m+1$, we have

$$\begin{aligned}\sum_{i=1}^{n-1} N_i^k Z^{\beta-e^i} E_n &= \sum_{i,j=1}^{n-1} N_i^k Z_j Z^{\beta-e^i-e^j} E_n \quad \text{(by (2.1))}\\ &= \sum_{i,j=1}^{n-1} N_i^k (x_j I + x_n M_j) Z^{\beta-e^i-e^j} E_n\\ &= \sum_{j=1}^{n-1} x_j \sum_{i=1}^{n-1} N_i^k Z^{\beta-e^j-e^i} E_n + x_n \sum_{i,j=1}^{n-1} N_i^k M_j Z^{\beta-e^i-e^j} E_n.\end{aligned}$$

The first summation above reduces to 0 by the induction hypothesis since $|\beta - e^j| = m$, therefore it remains to show

$$\sum_{i,j=1}^{n-1} N_i^k M_j Z^{\beta-e^i-e^j} E_n = 0. \tag{6.13}$$

Now it can be checked that $N_i^k M_j$ vanish for all $1 \le i \le n-1$ and $1 \le j \le n-1$ except

$$N_i^k M_j = \begin{cases} -N_j^n & \text{for } j \le k-1,\ i = k,\\ N_i^n & \text{for } j = k,\ i \le k-1,\end{cases} \tag{6.14}$$

where N_j^n consists of 0's, except at (j, n), and ditto for N_i^n, so

$$\begin{aligned}\sum_{i,j=1}^{n-1} N_i^k M_j Z^{\beta-e^i-e^j} E_n \\ &= \sum_{j=1}^{k-1}(-N_j^n) Z^{\beta-e^k-e^j} E_n + \sum_{i=1}^{k-1} N_i^n Z^{\beta-e^i-e^k} E_n \\ &= \sum_{i=1}^{k-1}(-N_i^n + N_i^n) Z^{\beta-e^i-e^k} E_n = 0.\end{aligned}$$

This concludes the proof of (6.10). As for (6.11), since $Z^\beta E_n$ satisfies (6.9) and (6.10), there exists by Theorem 8 a C^2 scalar-valued function u such that $Lu = 0$ and $\nabla u = Z^\beta E_n$. Therefore,

$$L(Z^\beta E_n) = L(\nabla u) = \nabla(Lu) = 0,$$

where the commuting of L and ∇ is justified by the fact that u, being actually a polynomial in this case, is certainly C^3. This completes the proof of Theorem 9.

Since according to (6.9) and (6.10) each $Z^\beta E_n$ represents the gradient of a (polynomial) solution of (6.1), one might hope to represent the gradient of any real analytic solution of (6.1) by infinite series in $Z^\beta E_n$. Unfortunately, the $Z^\beta E_n$ alone turn out to be insufficient for the purpose. This is revealed by the next theorem.

Theorem 10. *If u is a real analytic solution of* (6.1) *at the origin, then*

$$\nabla u = \sum_\beta (Z^\beta / |\beta|!) \partial^\beta (\nabla u)_{x=0}. \tag{6.15}$$

This representation of ∇u in terms of columns of Z^β is unique.

Proof. Since u is a real analytic solution of (6.1), by Theorem 8 its gradient ∇u is M-holomorphic with M as defined in (6.3), and ∇u is still real analytic. Consequently, by Theorem 7 ∇u is uniquely represented in terms of Z^β as in (6.15). This concludes the proof.

According to Theorem 10 then we do not have enough gradients in $Z^\beta E_n$ alone to generate the gradients of all the real analytic solutions of (6.1). But we also recall from (6.11) that the n polynomials in $Z^\beta E_n$ are all solutions of (6.1). Hence we can bypass the gradients altogether and try directly to represent all the real analytic solutions of (6.1) in terms of polynomial solutions found in $Z^\beta E_n$. Interestingly enough, this time it turns out that we have too many polynomials and we must, in order to arrive at a unique representation, make suitable choices among these polynomials.

7. Polynomial solutions

We now examine the polynomial solutions of (6.1) that appear in the last column of Z^β. Let p_i^β denote the ith polynomial in the last column of Z^β,

$$p_i^\beta = E_i' Z^\beta E_n, \tag{7.1}$$

where E_i' denotes the ith unit row vector and E_n the nth unit column vector. Now, in general, a polynomial $p(x)$ can be characterized by its derivatives at the origin, $\partial^\alpha p(0)$ for all α, but if $p(x)$ happens to be a solution of (6.1), then it can be sufficiently characterized by $\partial^\alpha p(0)$ with $0 \le \alpha_n \le 1$ since (6.1) allows us to depose $\partial_n^2 p(x)$ by

$$\partial_n^2 p(x) = (L + \partial_n^2) p(x), \tag{7.2}$$

where we note $L + \partial_n^2$ contains ∂_n to at most the first order.

Lemma 5. *For all* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ *with* $0 \le \alpha_n \le 1$ *and all* $\beta = (\beta_1, \beta_2, \dots, \beta_{n-1})$ *with* $|\alpha| = |\beta|$, *we have*

$$\partial^\alpha Z^\beta = |\beta|! \left(\delta_\beta^\alpha I + \sum_{j=1}^{n-1} \delta_{\beta - e^j + e^n}^\alpha M_j \right), \tag{7.3}$$

where δ *is the Kronecker delta, and* α *and* β *are considered equal when their respective components are equal and* $\alpha_n = 0$.

Proof. We proceed in two cases.

Case I. $\alpha_n = 0$. In this case we have by (2.7)

$$\partial^\alpha Z^\beta = |\beta|! Z^{\beta - \alpha},$$

but since $|\alpha| = |\beta|$, either $\alpha = \beta$, in which case we have $Z^{\beta-\alpha} = I$, or $\alpha \ne \beta$, in which case we have $\alpha_j > \beta_j$ for some j and $Z^{\beta-\alpha} = 0$. Combining both situations, we have

$$\partial^\alpha Z^\beta = |\beta|! \delta_\beta^\alpha I. \tag{7.4}$$

Case II. $\alpha_n = 1$. In this case a moment of reflection shows that $\partial^\alpha Z^\beta = 0$ unless $\alpha = \beta - e^j + e^n$ for some $1 \le j \le n-1$. But if so, then

$$\partial^\alpha Z^\beta = \partial_n \partial^{\beta - e^j} Z^\beta = \partial_n |\beta|! Z_j = |\beta|! M_j.$$

Hence we may write

$$\partial^\alpha Z^\beta = |\beta|! \sum_{j=1}^{n-1} \delta_{\beta - e^j + e^n}^\alpha M_j, \tag{7.5}$$

where we note $\delta_{\beta - e^j + e^n}^\alpha = 1$ at most once as j runs through 1 to $n-1$.

Combining (7.4) and (7.5), we obtain (7.3). This completes the proof.

Lemma 6. *For all* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ *with* $0 \le \alpha_n \le 1$ *and all* $\beta = (\beta_1, \beta_2, \dots, \beta_{n-1})$, *we have*

$$\partial^\alpha Z^\beta(0) = |\beta|! \left(\delta^\alpha_\beta I + \sum_{j=1}^{n-1} \delta^\alpha_{\beta - e^j + e^n} M_j \right). \tag{7.6}$$

Proof. If $|\alpha| = |\beta|$, this lemma says nothing new from (7.3) since $\partial^\alpha Z^\beta$ is a constant matrix and $\partial^\alpha Z^\beta = \partial^\alpha Z^\beta(0)$. Now if $|\alpha| < |\beta|$, then $\partial^\alpha Z^\beta$ consists of homogeneous polynomials of degree $|\beta| - |\alpha|$, hence setting $x = 0$ will annihilate $\partial^\alpha Z^\beta$, that is $\partial^\alpha Z^\beta(0) = 0$; on the other hand all the Kronecker deltas will be zero since $|\alpha| \ne |\beta| = |\beta - e^j + e^n|$. Hence (7.6) holds. Finally if $|\alpha| > |\beta|$, then either some $\alpha_j > \beta_j$ for $1 \le j \le n-1$ or else $\alpha_j = \beta_j$ for $1 \le j \le n-1$ and $\alpha_n = 1$. In the former case we have obviously $\partial^\alpha Z^\beta = 0$, and in the latter case we have $\partial^\alpha Z^\beta = \partial_n \partial^\beta Z^\beta = \partial_n |\beta|! I = 0$. So in both cases $\partial^\alpha Z^\beta(0) = 0$. On the other hand, again all the Kronecker deltas will be zero since $|\alpha| \ne |\beta| = |\beta - e^j + e^n|$. Hence (7.6) holds.

Theorem 11. *For all* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ *with* $0 \le \alpha_n \le 1$ *and all* $\beta = (\beta_1, \beta_2, \dots, \beta_{n-1})$, *we have*

$$\partial^\alpha p^\beta_i(0) = |\beta|! \left[\delta^\alpha_\beta \delta^i_n + \sum_{j=1}^{n-1} \delta^\alpha_{\beta - e^j + e^n} (\delta^i_j + \delta^i_n a_{jn}) \right], \tag{7.7}$$

where the a_{jn} *are the coefficients appearing in* (6.1).

Proof. In view of (7.1) and (7.6), we have

$$\begin{aligned} \partial^\alpha p^\beta_i(0) &= E'_i(\partial^\alpha Z^\beta)(0) E_n \\ &= |\beta|! \left(\delta^\alpha_\beta E'_i E_n + \sum_{j=1}^{n-1} \delta^\alpha_{\beta - e^j + e^n} E'_i M_j E_n \right) \\ &= |\beta|! \left[\delta^\alpha_\beta \delta^i_n + \sum_{j=1}^{n-1} \delta^\alpha_{\beta - e^j + e^n} (\delta^i_j + \delta^i_n a_{jn}) \right], \end{aligned}$$

where we recall that the column vector $M_j E_n$ has 1 as the jth entry and a_{jn} as the nth entry according to (6.3). This completes the proof.

Formula (7.7) gives a complete characterization of each polynomial solution p^β_i, but it also suggests that perhaps p^β_i can be modified somewhat to give simpler derivative data. Accordingly, we shall propose a *modified polynomial basis* for solutions of (6.1), by which we mean a collection of polynomial solutions q_α indexed by $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $0 \le \alpha_n \le 1$ such that their derivatives at the origin satisfy

$$\partial^\gamma q_\alpha(0) = \delta^\gamma_\alpha c_\alpha, \qquad 0 \le \alpha_n \le 1, \ 0 \le \gamma_n \le 1, \tag{7.8}$$

where each c_α is a nonzero complex number. If further we let $\hat{q}_\alpha = q_\alpha / c_\alpha$, then $\hat{q}_\alpha$ will form a *thoroughly* modified polynomial basis. Postponing the actual construction of such a basis, we first justify the use of the term *basis*.

Theorem 12. *Let $\hat{q}_\alpha$ with α subject to $0 \le \alpha_n \le 1$ be polynomial solutions of* (6.1) *satisfying*

$$\partial^\gamma \hat{q}_\alpha(0) = \delta_\alpha^\gamma \quad \textit{for all } \gamma \textit{ with } 0 \le \gamma_n \le 1. \tag{7.9}$$

Then any solution u of (6.1) *which is real analytic at the origin can be expressed uniquely in terms of $\hat{q}_\alpha$ in some neighborhood of the origin as*

$$u = \sum_\alpha \hat{q}_\alpha \partial^\alpha u(0), \qquad 0 \le \alpha_n \le 1. \tag{7.10}$$

Proof. Let u be a real analytic solution of (6.1) so that

$$u(x) = \sum_\alpha x^\alpha a_\alpha, \qquad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \tag{7.11}$$

in some neighborhood of the origin. If we let

$$u_m(x) = \sum_{|\alpha|=m} x^\alpha a_\alpha, \tag{7.12}$$

then

$$u(x) = \sum_{m=0}^{\infty} u_m(x). \tag{7.13}$$

First, we want to ascertain that each u_m is actually a solution of (6.1). Now since $Lu = 0$,

$$\sum_{m=0}^{\infty} L(u_m) = 0, \tag{7.14}$$

whence follows $L(u_m) = 0$ for every m because a power series in x such as (7.14) is 0 if and only if all the coefficients are zero including those for $L(u_m)$.

Next, we have only to show

$$u_m = \sum_{|\alpha|=m} \hat{q}_\alpha \partial^\alpha u(0), \qquad 0 \le \alpha_n \le 1. \tag{7.15}$$

But since both sides are solutions of (6.1) and are homogeneous polynomials of degree m, we need only compare their ∂^γ-derivatives at the origin for γ subject to $0 \le \gamma_n \le 1$ and $|\gamma| = m$. Now,

$$\begin{aligned} \partial^\gamma \left[\sum_{|\alpha|=m} \hat{q}_\alpha(x) \partial^\alpha u(0) \right]_{x=0} &= \sum_{|\alpha|=m} (\partial^\gamma \hat{q}_\alpha)(0) \partial^\alpha u(0) \\ &= \sum_{|\alpha|=m} \delta_\alpha^\gamma \partial^\alpha u(0) = \partial^\gamma u(0) = \partial^\gamma u_m(0). \end{aligned}$$

Thus (7.15) is valid. Substituting (7.15) in (7.13), we obtain (7.10). The uniqueness of representation (7.10) follows from the fact that $\sum_{|\alpha|=m} \hat{q}_\alpha b_\alpha = 0$

implies $b_\alpha = 0$, which is an easy consequence of (7.9). This concludes the proof of Theorem 12.

Theorem 13. *If p is an mth degree homogeneous polynomial solution of* (6.1) *and $\hat{q}_\alpha$ are as defined in Theorem* 12, *then p is uniquely represented as a linear combination of $\hat{q}_\alpha$. In fact,*

$$p = \sum_{|\alpha|=m} \hat{q}_\alpha \partial^\alpha p(0). \tag{7.16}$$

The totality of $\hat{q}_\alpha$ with $|\alpha| = m$ has the cardinality given by the formula:

$$\begin{aligned} &\operatorname{Card}\{\hat{q}_\alpha : |\alpha| = m,\ 0 \le \alpha_n \le 1\} \\ &\quad = (m+n-3)!(2m+n-2)/m!(n-2)!. \end{aligned} \tag{7.17}$$

Proof. p being a polynomial is real analytic, and so by (7.10) is uniquely represented as $p = \sum_{|\alpha|=0}^{\infty} \hat{q}_\alpha \partial^\alpha p(0)$, but $\partial^\alpha p(0) = 0$ unless $|\alpha| = m$ because p is homogeneous of degree m; hence (7.16) follows. To determine the cardinality of the set of all $\hat{q}_\alpha$ with $|\alpha| = m$, we consider

$$\begin{aligned} &\operatorname{Card}\{\alpha : |\alpha| = m,\ \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n),\ 0 \le \alpha_n \le 1\} \\ &\quad = \operatorname{Card}\{\alpha : |\alpha| = m,\ \alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n-1}, 0)\} \\ &\qquad + \operatorname{Card}\{\alpha : |\alpha| = m,\ \alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n-1}, 1)\} \\ &\quad = C_m^{m+n-2} + C_{m-1}^{m-1+n-2}, \end{aligned}$$

where we used the well-known combinatorial formula that gives the total number of ways in which $m \ge 1$ identical apples can be distributed among $k \ge 1$ people as $C_m^{m+k-1} = (m+k-1)!/m!(k-1)!$. Now, simple calculations show

$$C_m^{m+n-2} + C_{m-1}^{m+n-3} = (m+n-3)!(2m+n-2)/m!(n-2)!.$$

This completes the proof.

8. Polynomial bases

We will now construct a modified polynomial basis for solutions of (6.1) by using only some of the polynomials q_i^β that appear in the last column of Z^β. Such a construction not only shows the existence of a modified polynomial basis but also demonstrates an overabundance of polynomial solutions in $Z^\beta E_n$. As it turns out, we need only the first and the last polynomials, p_1^β and p_n^β, to construct a modified polynomial basis. And as we shall see later, p_1^β and p_n^β themselves also form a polynomial basis, though not a modified one unless all the coefficients a_{kn} for $1 \le k \le n-1$ happen to vanish, as they do in the case of Laplace or wave equations.

Theorem 14. *Let the polynomials p_i^β with $\beta = (\beta_1, \beta_2, \dots, \beta_{n-1})$ and $1 \le i \le n$ be as defined in* (7.1), *and let q_α with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ subject to*

$0 \le \alpha_n \le 1$ *be polynomials given by*

$$q_\alpha = p_n^\alpha - \sum_{k=1}^{n-1} a_{kn} p_1^{\alpha - e^k + e^1} \quad \text{for } \alpha_n = 0, \tag{8.1}$$

$$q_\alpha = p_1^{\alpha - e^n + e^1} \quad \text{for } \alpha_n = 1. \tag{8.2}$$

Then these q_α *satisfy* (7.8) *and therefore form a modified polynomial basis.*

Needless to say, p_n^α *with* $\alpha_n = 0$ *means* p_n^β *with* $\beta = (\alpha_1, \alpha_2, \dots, \alpha_{n-1})$, *and* p_i^α *is undefined unless* $\alpha_n = 0$.

Proof. We will show that for all α and γ subject to $0 \le \alpha_n \le 1$ and $0 \le \gamma_n \le 1$, we have

$$\partial^\gamma q_\alpha(0) = \delta_\alpha^\gamma |\alpha|!. \tag{8.3}$$

First, if $\alpha_n = 0$, then by (8.1) we have

$$\partial^\gamma q_\alpha(0) = \partial^\gamma p_n^\alpha(0) - \sum_{k=1}^{n-1} a_{kn} \partial^\gamma p_1^{\alpha - e^k + e^1}(0). \tag{8.4}$$

Working on these two terms separately with formula (7.7), we have

$$\begin{aligned} \partial^\gamma p_n^\alpha(0) &= |\alpha|! \left[\delta_\alpha^\gamma \delta_n^n + \sum_{j=1}^{n-1} \delta_{\alpha - e^j + e^n}^\gamma (\delta_j^n + \delta_n^n a_{jn}) \right] \\ &= |\alpha|! \left(\delta_\alpha^\gamma + \sum_{j=1}^{n-1} \delta_{\alpha - e^j + e^n}^\gamma a_{jn} \right), \end{aligned} \tag{8.5}$$

$$\begin{aligned} &\sum_{k=1}^{n-1} a_{kn} \partial^\gamma p_1^{\alpha - e^k + e^1}(0) \\ &\quad = |\alpha|! \sum_{k=1}^{n-1} a_{kn} \left[\delta_{\alpha - e^k + e^1}^\gamma \delta_n^1 + \sum_{j=1}^{n-1} \delta_{\alpha - e^k + e^1 - e^j + e^n}^\gamma (\delta_j^1 + \delta_n^1 a_{jn}) \right] \\ &\quad = |\alpha|! \sum_{k=1}^{n-1} a_{kn} (\delta_{\alpha - e^k + e^n}^\gamma). \end{aligned} \tag{8.6}$$

Subtracting (8.6) from (8.5), we obtain (8.3).

Next, if $\alpha_n = 1$, then by (8.2) and (7.7), we have

$$\begin{aligned} \partial^\gamma q_\alpha(0) &= \partial^\gamma p_1^{\alpha - e^n + e^1}(0) \\ &= |\alpha|! \left[\delta_{\alpha - e^n + e^1}^\gamma \delta_n^1 + \sum_{j=1}^{n-1} \delta_{\alpha - e^n + e^1 - e^j + e^n}^\gamma (\delta_j^1 + \delta_n^1 a_{jn}) \right] \\ &= |\alpha|! \left(\sum_{j=1}^{n-1} \delta_{\alpha + e^1 - e^j}^\gamma \delta_j^1 \right) = |\alpha|! \delta_\alpha^\gamma, \end{aligned}$$

and again (8.3) is obtained. This completes the proof.

Corollary 14. *If* $a_{kn} = 0$ *in* (6.1) *for all* $1 \le k \le n-1$, *then a modified polynomial basis is formed* (*without modification*) *by*

$$p_n^\beta = E_n' Z^\beta E_n \quad \text{for all } \beta = (\beta_1, \beta_2, \dots, \beta_{n-1}), \tag{8.7}$$

and

$$p_1^\beta = E_1' Z^\beta E_n \quad \text{for all } \beta = (\beta_1, \beta_2, \dots, \beta_{n-1}) \text{ with } \beta_1 \ge 1. \tag{8.8}$$

In fact, a thoroughly modified polynomial basis is formed by

$$p_n^\beta / |\beta|! = \hat{q}_\beta \quad \text{for all } \beta, \tag{8.9}$$

and

$$p_1^\beta / |\beta|! = \hat{q}_{\beta - e^1 + e^n} \quad \text{for all } \beta \text{ with } \beta_1 \ge 1, \tag{8.10}$$

where $\hat{q}_\beta$ *means* $\hat{q}_\alpha$ *with* $\alpha = (\beta_1, \beta_2, \dots, \beta_{n-1}, 0)$, *and* $\hat{q}_\alpha$ *are as defined in Theorem* 12.

Proof. If $a_{kn} = 0$ for all $1 \le k \le n-1$, then (8.1) and (8.2) reduce to

$$q_\alpha = p_n^\alpha \quad \text{for } \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \text{ with } \alpha_n = 0, \tag{8.11}$$

$$q_\alpha = p_1^{\alpha - e^n + e^1} \quad \text{for } \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \text{ with } \alpha_n = 1, \tag{8.12}$$

which together form a modified polynomial basis for solutions of (6.1). Now, using $(n-1)$-dimensional indices β instead, we can rewrite (8.11) and (8.12) above as

$$p_n^\beta = q_\beta \quad \text{for all } \beta, \tag{8.13}$$

$$p_1^\beta = q_{\beta - e^1 + e^n} \quad \text{for all } \beta \text{ with } \beta_1 \ge 1. \tag{8.14}$$

This shows that polynomials (8.7) and (8.8) form a modified polynomial basis with appropriate basis-element labels given by (8.13) and (8.14). In view of (8.3) if we divide (8.13) and (8.14) by $|\beta|!$, we obtain the thoroughly modified polynomial basis (8.9) and (8.10). This completes the proof.

According to (8.8), in taking p_1^β to form a modified polynomial basis we must make sure that $\beta_1 \ge 1$, whereas in taking p_n^β we need no such precaution. However, it turns out $p_1^\beta = 0$ whenever $\beta_1 = 0$, so the precaution is almost unnecessary.

Lemma 7. *For any* $\beta = (\beta_1, \beta_2, \dots, \beta_{n-1})$ *with* $\beta_1 = 0$, *we have*

$$p_1^\beta := E_1' Z^\beta E_n = 0. \tag{8.15}$$

Proof. We prove by induction on $|\beta|$. For $|\beta| = 1$, we have $Z^\beta = Z_j$ with $2 \le j \le n-1$ since $\beta_1 = 0$. Now then in view of (6.3) we have

$$p_1^\beta = E_1' Z_j E_n = x_j E_1' I E_n + x_n E_1' M_j E_n = 0.$$

Hence (8.15) holds for $|\beta| = 1$. Now assume (8.15) for $|\beta| = m$. Then for $|\beta| = m+1$ with $\beta_1 = 0$ we have in view of the reduction formula (2.1)

$$p_1^\beta = E_1' Z^\beta E_n = \sum_{j=2}^{n-1} (E_1' Z_j)(Z^{\beta - e^j} E_n),$$

but the row vector $E_1' Z_j$ is equal to $(x_j, 0, 0, \dots, 0)$ for $j \geq 2$, and the column vector $Z^{\beta - e^j} E_n$ has a 0 top entry by the induction hypothesis; hence the product $(E_1' Z_j)(Z^{\beta - e^j} E_n) = 0$ for all $j \geq 2$, and (8.15) holds. This completes the proof.

Theorem 15 (special representation). *If in* (6.1) *the coefficients* a_{kn} *vanish for all* $1 \leq k \leq n-1$, *then the general real analytic solution at the origin of* (6.1) *is given by the convergent series*

$$u = \sum_\beta \mathbf{c}_\beta Z^\beta E_n, \tag{8.16}$$

where the $\mathbf{c}_\beta$ *are* $1 \times n$ hollow *row vectors* (*all entries except the first and the last are necessarily zero*). *The representation* (8.16) *is unique provided we choose the first entry of* $\mathbf{c}_\beta$ *to be zero whenever* $\beta_1 = 0$. *We have in fact*

$$\mathbf{c}_\beta E_1 = \partial^{\beta - e^1 + e^n} u(0)/|\beta|!, \tag{8.17}$$

$$\mathbf{c}_\beta E_n = \partial^\beta u(0)/|\beta|!, \tag{8.18}$$

where we recall $\partial^\gamma u = 0$ *if any component of* γ *is negative.*

Proof. First, any convergent series of the form (8.16) is a real analytic solution of (6.1) since all polynomials in $Z^\beta E_n$ are solutions of (6.1).

Conversely, if u is a real analytic solution of (6.1), then in view of (7.10) we can write

$$u = \sum_{\alpha_n = 0} \hat{q}_\alpha \partial^\alpha u(0) + \sum_{\alpha_n = 1} \hat{q}_\alpha \partial^\alpha u(0),$$

which can be rewritten as

$$u = \sum_\beta \hat{q}_\beta \partial^\beta u(0) + \sum_{\beta_1 \geq 1} \hat{q}_{\beta - e^1 + e^n} \partial^{\beta - e^1 + e^n} u(0).$$

Referring to (8.9) and (8.10), and by (7.1), we have

$$\begin{aligned} u &= \sum_\beta (p_n^\beta / |\beta|!) \partial^\beta u(0) + \sum_{\beta_1 \geq 1} (p_1^\beta / |\beta|!) \partial^{\beta - e^1 + e^n} u(0) \\ &= \sum_\beta [(\partial^\beta u(0)/|\beta|!) E_n' + (\partial^{\beta - e^1 + e^n} u(0)/|\beta|!) E_1'] Z^\beta E_n. \end{aligned}$$

The coefficient of $Z^\beta E_n$ above amounts to a row vector $\mathbf{c}_\beta$ satisfying (8.17) and (8.18). Hence we have shown (8.16).

The representation (8.16) is unique since the polynomials $E_1' Z^\beta E_n$ and $E_n' Z^\beta E_n$ form a modified polynomial basis if we discard all the zero polynomials $E_1' Z^\beta E_n$ when $\beta_1 = 0$, which we did in essence by choosing the first entry of $\mathbf{c}_\beta$ to be zero whenever $\beta_1 = 0$. This completes the proof.

In the preceding theorem we assumed $a_{kn} = 0$ for all $1 \le k \le n-1$. But even if we do not make such an assumption, the top and the bottom polynomials of $Z^\beta E_n$ (discarding certain zero polynomials of course) still form a polynomial basis albeit a nonmodified one.

Theorem 16 (general representation). *The general real analytic solution at the origin of* (6.1) *is given by the convergent series*

$$u = \sum_\beta \mathbf{c}_\beta Z^\beta E_n, \tag{8.19}$$

where the $\mathbf{c}_\beta$ *are* $1 \times n$ hollow *row vectors* (*all entries except the first and the last are necessarily zero*). *The representation* (8.19) *is unique provided we choose the first entry of* $\mathbf{c}_\beta$ *to be zero whenever* $\beta_1 = 0$.

Proof. According to (8.1) and (8.2) the modified polynomial basis elements q_α with $|\alpha| = m$ are generated by p_1^β (with $\beta_1 \ge 1$) and p_n^β where $|\beta| = m$, and according to Theorem 13 these polynomials q_α form a basis for the mth degree homogeneous polynomial solution space of (6.1). Therefore, the polynomials p_1^β (with $\beta_1 \ge 1$) and p_n^β where $|\beta| = m$ will also form a basis if they are exactly as numerous as q_α. But since we already know from (7.17) exactly how many q_α there are with $|\alpha| = m$, we need only calculate

$$\begin{aligned}
&\operatorname{Card}\{\beta : |\beta| = m,\ \beta_1 \ge 1\} + \operatorname{Card}\{\beta : |\beta| = m\} \\
&\quad = 2\operatorname{Card}\{\beta : |\beta| = m\} - \operatorname{Card}\{\beta : |\beta| = m,\ \beta_1 = 0\} \\
&\quad = 2C_m^{m+n-2} - C_m^{m+n-3} = (m+n-3)!(2m+n-2)/m!(n-2)!.
\end{aligned}$$

This last expression agrees with that in (7.17), and so the polynomials p_1^β (with zero polynomials discarded) and p_n^β form a polynomial basis for solutions of (6.1); and as in Theorem 15 the unique representation of a real analytic solution at the origin is given by (8.19). This concludes the proof.

9. Algorithm and examples

According to Theorem 16, all the basic polynomial solutions of (6.1) are to be found in $Z^\beta E_n$, or $E_1' Z^\beta E_n$ and $E_n' Z^\beta E_n$ to be specific. Since the construction of the matrix Z^β involves the coefficients a_{ij} in (6.1) in a definite systematic way, we have in effect an elaborate matrix algorithm built on a_{ij} and $x = (x_1, x_2, \dots, x_n)$ for obtaining all the polynomial solutions of (6.1). We present some examples of this algorithm for $n = 3$. We begin with the second

order equation

$$Lu := a_{11}u_{xx} + a_{12}u_{xy} + a_{13}u_{xz} + a_{22}u_{yy} + a_{23}u_{yz} - u_{zz} = 0. \tag{9.1}$$

The associated matrices of L, according to (6.3), are

$$M_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & a_{22} & a_{23} \end{pmatrix}.$$

If we specialize (9.1) to the Laplace equation, then

$$a_{11} = a_{22} = -1 \quad \text{and} \quad a_{12} = a_{13} = a_{23} = 0,$$

so that from $Z_1 = xI + zM_1$ and $Z_2 = yI + zM_2$ of (1.3) we have

$$Z_1 = \begin{pmatrix} x & 0 & z \\ 0 & x & 0 \\ -z & 0 & x \end{pmatrix}, \qquad Z_2 = \begin{pmatrix} y & 0 & 0 \\ 0 & y & z \\ 0 & -z & y \end{pmatrix}.$$

Using the reduction formula (2.1) as a recursive formula, we build up Z^β from Z_1 and Z_2. Thus,

$$Z^{(0,0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$Z^{(1,0)} = \begin{pmatrix} x & 0 & z \\ 0 & x & 0 \\ -z & 0 & x \end{pmatrix},$$

$$Z^{(0,1)} = \begin{pmatrix} y & 0 & 0 \\ 0 & y & z \\ 0 & -z & y \end{pmatrix},$$

$$Z^{(2,0)} = \begin{pmatrix} x^2 - z^2 & 0 & 2xz \\ 0 & x^2 & 0 \\ -2xz & 0 & x^2 - z^2 \end{pmatrix},$$

$$Z^{(1,1)} = \begin{pmatrix} 2xy & -z^2 & 2yz \\ -z^2 & 2xy & 2xz \\ -2yz & -2xz & 2xy \end{pmatrix},$$

$$Z^{(0,2)} = \begin{pmatrix} y^2 & 0 & 0 \\ 0 & y^2 - z^2 & 2yz \\ 0 & -2yz & y^2 - z^2 \end{pmatrix},$$

$$Z^{(3,0)} = \begin{pmatrix} x^3 - 3xz^2 & 0 & 3x^2z - z^3 \\ 0 & x^3 & 0 \\ z^3 - 3x^2z & 0 & x^3 - 3xz^2 \end{pmatrix},$$

$$Z^{(2,1)} = \begin{pmatrix} 3x^2y - 3yz^2 & -3xz^2 & 6xyz \\ -3xz^2 & 3x^2y & 3x^2z - z^3 \\ -6xyz & z^3 - 3x^2z & 3x^2y - 3yz^2 \end{pmatrix},$$

$$Z^{(1,2)} = \begin{pmatrix} 3xy^2 & -3yz^2 & 3y^2z - z^3 \\ -3yz^2 & 3xy^2 - 3xz^2 & 6xyz \\ -3y^2z + z^3 & -6xyz & 3xy^2 - 3xz^2 \end{pmatrix},$$

$$Z^{(0,3)} = \begin{pmatrix} y^3 & 0 & 0 \\ 0 & y^3 - 3yz^2 & 3y^2z - z^3 \\ 0 & -3y^2z + z^3 & y^3 - 3yz^2 \end{pmatrix},$$

and so on. From the upper and lower right corners of these matrices we obtain the basis polynomials for the Laplace equation. Letting $|\beta| = m$, we list these polynomials according to their degree m:

$$\begin{array}{ll} m = 0 & 0, 1; \\ m = 1 & z, x; 0, y; \\ m = 2 & 2xz, x^2 - z^2; 2yz, 2xy; 0, y^2 - z^2; \\ m = 3 & 3x^2z - z^3, x^3 - 3xz^2; 6xyz, 3x^2y - 3yz^2; \\ & 3y^2z - z^3, 3xy^2 - 3xz^2; 0, y^3 - 3yz^2; \end{array}$$

and so on. We retained the zero polynomials for the time being for orderliness of presentation. Note that the number of basis polynomials for each degree m checks out with the formula (7.17), by which we are to have for $n = 3$

$$(m + n - 3)!(2m + n - 2)/m!(n - 2)! = 2m + 1$$

homogeneous polynomials of degree m. Note also that because $a_{13} = a_{23} = 0$ in the Laplace equation, the above polynomials form a modified basis (recall Corollary 14). Dividing by $|\beta|!$, we obtain a thoroughly modified basis:

$$\begin{aligned} &1; x, y, z; xy, xz, yz, \tfrac{1}{2}(x^2 - z^2), \tfrac{1}{2}(y^2 - z^2); \\ &\tfrac{1}{6}(x^3 - 3xz^2), \tfrac{1}{2}(x^2y - yz^2), \tfrac{1}{6}(3x^2z - z^3), \tfrac{1}{2}(xy^2 - xz^2), \\ &\tfrac{1}{6}(y^3 - 3yz^2), \tfrac{1}{6}(3y^2z - z^3), xyz; \ldots . \end{aligned}$$

Similarly, if we consider the wave equation $u_{xx} + u_{yy} - u_{zz} = 0$, with z identified as the time variable, then we have

$$a_{11} = a_{22} = 1 \quad \text{and} \quad a_{12} = a_{13} = a_{23} = 0,$$

and the corresponding Z_1 and Z_2 look exactly like those for the Laplace equation except that no negative signs appear in the entries. Proceeding as before,

we obtain a thoroughly modified basis that looks like that for the Laplace equation except that all the negative signs in the polynomials are replaced by positive signs.

The present paper is a partial generalization of an earlier paper [6]. If we choose $n = 2$, as in [6], then Z consists of just one component $Z_1 = xI + yM_1$, a 2×2 matrix, and the symmetric power Z^β reduces to an ordinary power Z^m, whose last column provides the two polynomial solutions of degree m. Thus, for the wave equation we would have $Z = \left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$, and $Z^m E_2$ will provide all the polynomial solutions which form a modified basis. With routine calculations we find these polynomials to be:

$$\begin{aligned} &1;\, x,\, y;\, x^2+y^2,\, 2xy;\, x^3+3xy^2,\, 3x^2y+y^3;\\ &x^4+6x^2y^2+y^4,\, 4x^3y+4xy^3;\\ &x^5+10x^3y^2+5xy^4,\, 5x^4y+10x^2y^3+y^5;\, \ldots. \end{aligned}$$

There are exactly two polynomials for each degree $m \geq 1$, as confirmed also by formula (7.17) for $n = 2$:

$$(m+n-3)!(2m+n-2)/m!(n-2)! = 2.$$

The same results were obtained in [6] by means of hyperconjugation.

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