Abstract. We study certain dense hereditary *-subalgebras of $\sigma$-unital $C^*$-algebras and their relations with the Pedersen ideals. The quasi-multipliers of the dense hereditary *-subalgebras are also studied.

1. Introduction

Let $A$ be a $C^*$-algebra and $K(A)$ its Pedersen's ideal. When $A$ is commutative, that is, $A = C_0(A)$, the algebra of all complex valued continuous functions which vanish at infinity on some locally compact Hausdorff space $X$, then $K(A) = C_00(X)$, the algebra of all complex valued continuous functions with compact support. In [15], we define a dense hereditary *-subalgebra $A_0$ (we used the notation $C_00(A)$ there) of a $\sigma$-unital $C^*$-algebra which satisfies:

(i) For every $a$ in $(A_0)$, there is a $b$ in $(A_0)$ such that $[a] \leq b$, where $[a]$ is the range projection of $a$ in $A^{**}$.
(ii) If $A$ is nonunital, $A_0 \neq A$.
(iii) When $A = C_0(X)$, $A_0 = C_00(X)$.

Naturally, we may view $A_0$ as a noncommutative analogue of $C_00(X)$. In fact the algebra $A_0$ plays an important role in [15]. In this paper we shall study the relation between $A_0$ and $K(A)$. We also study the quasi-multipliers of $A_0$. In the view of [11], where Lazer and Taylor studied the multipliers of $K(A)$ as a noncommutative analogue of (unbounded) continuous functions on locally compact Hausdorff space $X$, the quasi-multipliers of $A_0$ is another noncommutative analogue of $C(X)$. The reason our attention is focused on the quasi-multipliers of $A_0$ and not on the multipliers of $A_0$ is that the set of multipliers of $A_0$ may not contain $A$ and is not closed under a natural topology.

We denote the quasi-multipliers of $A_0$ by $QM(A_0)$. In §2, we give some basic concepts and facts related to quasi-multipliers of $A_0$. In §3, we study the order structure $QM(A_0)$. We also show that

$$QM(A_0) = LM(A_0) + RM(A_0)$$

(a similar equation for $A$ has been studied in [16, 3, 13, 14]). In §4, we...
prove an extension theorem in the sense of Tietze. We also give a version of the Dauns-Hofmann theorem for \( QM(A_{00}) \). In §5, we study the dual and bidual spaces of \( QM(A_{00}) \). We find that \( QM(A_{00})'' \), the bidual of \( QM(A_{00}) \), is isomorphic to the quasi-multipliers of the support algebra of \( M_0(A) \), the hereditary \( C^* \)-subalgebra of \( A'' \) generated by \( A \). In §6, we study the problem when \( A_{00} = K(A) \). Finally, in §7, we consider the uniqueness of \( A_{00} \) for certain \( C^* \)-algebras.

We shall be utilizing the following notations throughout this paper. Suppose that \( A \) is a \( C^* \)-algebra. Then \( K(A) \) denotes the Pedersen's ideal (for a definition see [17 or 18, 5.6]), and \( M(A) \), \( LM(A) \), \( RM(A) \), and \( QM(A) \) denote the multipliers, left multipliers, right multipliers, and quasi-multipliers of \( A \), respectively (see [18, 3.12]). For the element \( a \) in the \( C^* \)-algebra \( A \), \([a]\) shall denote the range projection of \( a \) in the enveloping \( W^* \)-algebra \( A'' \). Any other unexplained notation may be found in [18 or 4].

2. Preliminaries

2.1. Let \( A \) be a \( \sigma \)-unital \( C^* \)-algebra. Then \( A \) has a strictly positive element \( e \). Let \( f_n(t) \) be continuous functions satisfying

(i) \( 0 \leq f_n(t) \leq 1 \);

(ii) \( f_n(t) = 0 \) if and only if \( 0 < t < 1/2n \);

(iii) \( f_n(t) = 1 \) if \( t \geq 1/n \).

Define \( e_n = f_n(e) \). Then \( \{e_n\} \) forms an approximate identity for \( A \). Moreover, \( e_{n+1}e_n = e_ne_{n+1} = e_n \) for all \( n \). Let \( \chi_n \) be the characteristic function of the set \( (1/2n, ||e||) \). Then \( p_n = \chi_n(e) \) is an open projection of \( A \) such that \( [\epsilon_n] = p_n \) and \( e_n \leq p_n \leq e_{n+1} \).

2.2. **Definition.** Let \( A \) and \( p_n \) be as in 2.1. Denote the hereditary \( C^* \)-subalgebra \( p_nA''p_n \cap A \) by \( A_n \). We call \( \bigcup_{n=1}^\infty A_n \) a support algebra of \( A \) and denote it by \( A_{00} \) (or \( A_{00}(e) \), or \( A_{00}((e_n)) \)).

2.3. By [15, 1.1], \( A_{00} \) is a norm dense, hereditary *-subalgebra of \( A \) contained in \( K(A) \). Since \( e \notin A_{00} \), if \( A \) is not unital, then \( A_{00} \neq A \). Moreover, for every \( a \in (A_{00})_+ \), there is an \( n \) such that \( [a] \leq e_n \). Thus, as in [15], we regard \( A_{00} \) as a noncommutative analogue of \( C_{00}(X) \).

2.4. **Example.** Let \( X \) be a locally compact, \( \sigma \)-compact Hausdorff space and let \( A = C_0(X) \). (\( \sigma \)-compact means \( X = \bigcup_{n=1}^\infty X_n \), where each \( X_n \) is compact.) Then for any strictly positive element \( e \), \( A_{00}(e) = C_{00}(X) \).

2.5. **Example.** Let \( H \) be a separable Hilbert space and let \( A = K \), the compact operators on \( H \). Let \( \{H_n\} \) be an increasing sequence of finite-dimensional subspaces of \( H \) such that \( \bigcup_{n=1}^\infty H_n \) is dense in \( H \). Denote by \( M_n \) the set of bounded linear operators on \( H_n \). Then \( \bigcup_n M_n \) is a support algebra for \( A = K \). We shall see in §7 that, up to isomorphisms, \( \bigcup_n M_n \) is the only support algebra for \( K \).
2.6. **Lemma.** Suppose that $A$ is a $C^*$-algebra. Let $a, p \in A_+$ and $p \leq a \leq 1$. If $p$ is a projection, the $ap = pa = p$.

2.7. **Lemma.** Suppose that $a_n \in A_+$ and $p_n$ are open projections of $A$. If $\{a_n\}$ forms an approximate identity for $A$ and $a_n \leq p_n \leq a_{n+1}$ for each $n$, then there is a support algebra $A_{00}$ of $A$ such that

\[ A_{00} = p_n A^* p_n \cap A. \]

2.8. By 2.7, we may define $A_{00}$ by an approximate identity $\{e_n\}$ together with open projections $\{p_n\}$ satisfying:

\[ e_n \leq p_n \leq e_{n+1} \quad \text{for all} \ n. \]

If $e_n \leq p_n \leq e_{n+1}$ for each $n$, then $e_{n+1} e_n = e_n e_{n+1} = e_n$. Conversely, if $e_{n+1} e_n = e_n e_{n+1} = e_n$, then $e_{n+1} \geq [e_n]$. Thus we will always assume that every support algebra $A_{00}$ of $A$ is defined by an approximate identity $\{e_n\}$ which satisfies $e_{n+1} e_n = e_n e_{n+1} = e_n$.

We now fix a $\sigma$-unital $C^*$-algebra $A$ and a support algebra $A_{00} = A_{00}(\{e_n\})$.

2.9. **Definitions.** A linear map $\rho: A_{00} \to A_{00}$ is called a left, respectively right, multiplier if $\rho(ab) = \rho(a)b$, respectively $\rho(ab) = a\rho(b)$. A multiplier is a pair $(\rho_1, \rho_2)$ consisting of a right multiplier $\rho_1$ and a left multiplier $\rho_2$ such that $\rho_1(a)b = a\rho_2(b)$ for all $a, b \in A_{00}$. A quasimultiplier is a bilinear map $\rho: A_{00} \times A_{00} \to A_{00}$ such that for each fixed $a \in A_{00}$ the map $\rho(a, \cdot)$ is a left multiplier and the map $\rho(\cdot, a)$ is a right multiplier. We denote by $M(A_{00})$, $LM(A_{00})$, $RM(A_{00})$, and $QM(A_{00})$ the sets of multipliers, left multipliers, right multipliers, and quasimultipliers of $A_{00}$, respectively.

2.10. Suppose that $\rho \in QM(A_{00})$, and $a$ and $b \in A_{00}$. Then we denote the element $\rho(a, b)$ by $a \cdot \rho \cdot b$. If $\rho \in LM(A_{00})$, we denote $\rho(a)$ by $\rho \cdot a$ and if $\rho \in RM(A_{00})$, we denote $\rho(a)$ by $a \cdot \rho$. If $z = (\rho_1, \rho_2) \in M(A_{00})$, we denote $\rho_1(a)$ by $a \cdot z$ and $\rho_2(a)$ by $z \cdot a$.

2.11. For $a, b \in A_{00}$, we have the following seminorms:

(i) \[ z \to ||a \cdot z|| + ||z \cdot a||, \quad z \in M(A_{00}); \]

(ii) \[ z \to ||z \cdot a||, \quad z \in LM(A_{00}); \]

(iii) \[ z \to ||a \cdot z||, \quad z \in RM(A_{00}); \]

(iv) \[ z \to ||a \cdot z \cdot b||, \quad z \in QM(A_{00}). \]

We define $(A_{00})^*$, $L^*A_{00}$, $R^*A_{00}$, and $Q^*A_{00}$-topologies on $M(A_{00})$, $LM(A_{00})$, $RM(A_{00})$, and $QM(A_{00})$ to be those locally convex topologies generated by the seminorms (i), (ii), (iii), and (iv) (for all $a, b \in A_{00}$), respectively.

2.12. **Proposition.** $QM(A_{00})$ is a locally convex complete topological vector space under the $Q^*A_{00}$-topology.

2.13. We define the following subsets of $QM(A_{00})$:
$QM_f(A_{00}) = \{ \rho \in QM(A_{00}) : \text{for each } k, \text{ there exist } N(\rho, k) \text{ such that } \\
\rho(e_n, e_k) = \rho(e_m, e_k) \text{ if } n, m > N(\rho, k) \}$,

$QM_r(A_{00}) = \{ \rho \in QM(A_{00}) : \text{for each } k, \text{ there exists } N(\rho, k) \text{ such that } \\
\rho(e_k, e_n) = \rho(e_k, e_m) \text{ if } n, m > N(\rho, k) \}$,

$QM_d(A_{00}) = QM_f(A_{00}) \cap QM_r(A_{00})$, and

$QM^b(A_{00})$ is the subset of those elements in $QM(A_{00})$ such that

$$\sup\{\|a \cdot \rho \cdot b\| : a, b \in A_{00}, \|a\| \leq 1, \|b\| \leq 1\} < \infty.$$

2.14. Theorem. There are bijective correspondences between

(i) $QM_f(A_{00})$ and $LM(A_{00})$;
(ii) $QM_r(A_{00})$ and $RM(A_{00})$;
(iii) $QM_d(A_{00})$ and $M(A_{00})$;
(iv) $QM^b(A_{00})$ and $QM(A)$.

2.15. We shall use notations $LM(A_{00})$, $RM(A_{00})$, $M(A_{00})$, and $QM(A)$ instead of $QM_f(A_{00})$, $QM_r(A_{00})$, $QM_d(A_{00})$, and $QM^b(A_{00})$. Thus

$M(A_{00}) \subset LM(A_{00}) \subset QM(A_{00})$, 

$LM(A_{00}) \cap RM(A_{00}) = M(A_{00})$, 

and

$A_{00} \subset A \subset QM(A) \subset QM(A_{00})$.

2.16. Lemma. If $A$ is not unital, then

$QM(A_{00}) \neq QM^b(A_{00}) (= QM(A))$.

Proof. We may assume that $e_n - e_{n-1} \neq 0$ for all $n$. Define

$$z = \sum_{n=1}^{\infty} n(e_n - e_{n-1}),$$

where the convergence is in $Q - A_{00}$-topology. Clearly $z \in QM(A_{00})$, but $z \notin QM^b(A_{00})$.

2.17. We notice that, in general, $A \not\subset M(A_{00})$ and $M(A_{00})$ is not complete under $A_{00}$-topology. These are the reasons why we choose $QM(A_{00})$ and not $M(A_{00})$ as our main subject.

2.18. Proposition. $A_{00}$ is $L$-$A_{00}$-dense (respectively, $R$-$A_{00}$-dense, $Q$-$A_{00}$-dense, and $A_{00}$-dense) in $LM(A_{00})$ (respectively in $RM(A_{00})$, $QM(A_{00})$, and $M(A_{00})$).

2.19. We now define an operation "·" on some of the elements of $QM(A_{00})$. If $\rho \in QM(A_{00})$, $y \in LM(A_{00})$, and $z \in RM(A_{00})$, we denote by $\rho \cdot y$ the element $\rho(\cdot, y(\cdot))$ and $z \cdot \rho$ the element $\rho(z(\cdot), \cdot)$. It is easy to see that "·" is the "natural" extension of the multiplication on $M(A)$.
2.20. Let \( \rho \in QM(A_{00}) \). The involution \( \rho^* \) of \( \rho \) is a quasi-multiplier defined by \( \rho^*(a, b) = [\rho(b^*, a^*)]^* \). It is easy to see that the involution is conjugate linear and \( Q-A_{00} \)-continuous. Moreover the involution is the extension of the original involution on \( QM(A) \). Thus

\[
LM(A_{00})^* = RM(A_{00}).
\]

An element is called selfadjoint if \( \rho = \rho^* \). We denote by \( QM(A_{00})_{sa} \) the set of selfadjoint elements.

2.21. Example. Let \( X \) be a locally compact, \( \sigma \)-compact Hausdorff space, and let \( B \) be a unital \( C^* \)-algebra. Denote by \( A \) the \( C^* \)-algebra of all the continuous mappings from \( X \) into \( B \) vanishing at infinity. One of the support algebras (in fact, it is the only one) \( A_{00} \) is the set of all continuous mappings with compact supports. One can check that \( QM(A_{00}) \) is the set of all continuous mappings from \( X \) into \( B \).

Throughout §§3–7, \( A \) will denote a \( \sigma \)-unital \( C^* \)-algebra, and \( A_{00} \) one of its support algebras. \( e \), \( e_n \), and \( A_n \) will be the same as in 2.1.

3. Decompositions

3.1. Definition. We say that an element \( z \in QM(A_{00}) \) is positive, denoted by \( z \geq 0 \), if \( a^*za \geq 0 \) for all \( a \in A_{00} \). We let \( QM(A_{00})_+ \) denote the set of all positive elements in \( QM(A_{00}) \).

Suppose that \( y \) and \( z \in QM(A_{00}) \). We say that \( z \geq y \) (or \( y \leq z \)), if \( z - y \geq 0 \).

3.2. Corollary. The set \( QM(A_{00})_+ \) is a \( Q-A_{00} \)-closed real convex cone and \( QM(A_{00})_+ \cap (-QM(A_{00})_+) = \{0\} \).

3.3. Proposition. Let \( z \in QM(A_{00}) \). Then

(i) If \( -y \leq z \leq y \) for some \( y \in QM(A)_+ \), then \( z \in QM(A) \).

(ii) If \( -a \leq z \leq a \) for some \( a \in A^+ \), then \( z \in A \).

(iii) If \( z \in LM(A_{00}) \) and there is an element \( a \in A^+ \) such that \( z^*z \leq a \), then \( z \in A \).

Proof. (i) Since \( y - z \geq 0 \), \( a^*(-y)a \leq a^*za \leq a^*ya \) for all \( a \in A_{00} \). Therefore \( a^*za \leq a^*ya \). It follows that \( z \in QM^b(A_{00}) = QM(A) \).

(ii) By (i), \( z \in QM(A) \). Then by [1, Proposition 4.5], \( z \in A \).

(iii) For every \( b \in A_{00} \), we have \( b^*zb \leq b^*ab \). Thus \( \|zb\| \leq \|a^{1/2}b\| \). Hence \( z \in QM(A) \cap LM(A_{00}) \). It follows from [1, Proposition 4.5] that \( z \) is in \( A \).

3.4. Let \( LM(A_{00}, AA_{00}) \) denote the set of those linear mappings \( \rho \) from \( A_{00} \) into \( AA_{00} \) satisfying \( \rho(xy) = \rho(x)y \) for all \( x, y \in A_{00} \). As in §2, we can view \( LM(A_{00}, AA_{00}) \) as a subset of \( QM(A_{00}) \). If \( x \in LM(A_{00}, AA_{00}) \), we define \( x^* \cdot x(a, b) = (a \cdot x^*)(x \cdot b) \). Hence \( x^* \cdot x \in QM(A_{00})_+ \).
3.5. **Theorem.** If \( z \in QM(A_{00})_+ \), then there is an \( x \in LM(A_{00}, AA_{00}) \) (\( \subset QM(A_{00}) \)) such that \( x^* \cdot x = z \).

**Proof.** Let \( \alpha_k = \|z\|_{A_k} \). Define \( b_k = (1/\alpha_{k+1}) (1/2)^k (e_k - e_{k-1}) \) for \( k = 1, 2, \ldots \) (where \( e_0 = 0 \)), \( a_k = \sum_{i=1}^k b_i \), and \( b = \sum_{i=0}^\infty b_i \). Let \( z_k = a_k z a_k \), \( k = 1, 2, \ldots. \) Then, if \( k \geq m \)

\[
\|z_k - z_m\| \leq \left\| \sum_{i=m+1}^k b_i z a_k \right\| + \left\| \sum_{j=m+1}^k a_k z b_j \right\|
\]

\[
= \left\| \sum_{i=m+1}^k \sum_{j=1}^k b_i z b_j \right\| + \left\| \sum_{j=m+1}^k \sum_{i=1}^k b_j z b_j \right\|
\]

\[
\leq \sum_{i=m+1}^k \sum_{j=1}^k (1/2)^{i+j} + \sum_{j=m+1}^k \sum_{i=1}^k (1/2)^{i+j}
\]

\[
\leq 1/(2)^{m-1}.
\]

Thus \( z_k \) converges to a positive element \( h \) in \( A \) in norm. It is easy to see that \( e_k h e_k = e_k z_{k+1} e_k \) for every \( k \). Take \( u_n = h^{1/2} (b^2 + 1/n)^{-1} b \). Then, for every \( k \),

\[
\|u_n e_k\|^2 = \|e_k b(b^2 + 1/n)^{-1} h(b^2 + 1/n)^{-1} b e_k\|
\]

\[
= \|b(b^2 + 1/n)^{-1} e_k h e_k (b^2 + 1/n)^{-1} b e_k\|
\]

\[
= \|b(b^2 + 1/n)^{-1} a_{k+1} e_k h e_k a_{k+1} (b^2 + 1/n)^{-1} b e_k\|
\]

\[
\leq \alpha_k \|b(b^2 + 1/n)^{-1} b e_k a_{k+1}\|^2 \leq \alpha_k.
\]

So \( \|u_n e_k\| \) is bounded for every \( k \).

Put \( d_{nm} = (1/n + b^2)^{-1} - (1/n + b^2)^{-1} \). Then, for each \( k \),

\[
\|u_n a_k - u_m a_k\|^2 = \|h^{1/2} d_{nm} b a_k\|^2
\]

\[
= \|b d_{nm} a_k h a_k d_{nm} b\|
\]

\[
\leq \alpha_{k+1} \|b d_{nm} a_k a_{k+1} h a_k d_{nm} b\|
\]

\[
= \alpha_{k+1} \|d_{nm} b a_k (a_{k+1})^{1/2}\|^2.
\]

From spectral theory we see that the sequence \( \{(1/n + b^2)^{-1} b a_k (a_{k+1})^{1/2}\} \) is increasing to an element in \( A \) and by Dini’s theorem it is uniformly convergent to it. Consequently

\[
\|d_{nm} b a_k (a_{k+1})^{1/2}\| \to 0,
\]

so that \( \{u_n a_k\} \) is norm convergent to an element in \( A \) for each \( k \). Since \( \|u_n e_{k+1}\| \) is bounded and \( A_k \supset A_k \), it follows that \( \{u_n y\} \) is norm convergent for every \( y \in A_k \). Thus we have an element \( x \in LM(A_{00}, AA_{00}) \) defined by

\[
x(a) = \lim u_n a \quad \text{for every } a \in A_{00}.
\]
It is easy to check that for every \( k \),
\[
a_{k+1}x^* \cdot a_{k+1} = a_{k+1}z a_{k+1}.
\]
Therefore \( x^* \cdot x = z \).

3.6. The idea of the proof of 3.5 is taken from [3, 4, 9; and 18, 1.44]. The element \( x \) in 3.5 is in \( QM(A_{00}) \) but not in \( QM(A_{00})^+ \). In general, \( x \) may not be taken from \( LM(A_{00}) \).

3.7. Theorem. \( QM(A_{00}) = LM(A_{00}) + RM(A_{00}) \).

Proof. Let \( z \in QM(A_{00}) \). Define
\[
x = \sum_{k=1}^{\infty} e_k z(e_k - e_{k-1})
\]
and
\[
y = \sum_{k=1}^{\infty} (1 - e_k) z(e_k - e_{k-1}).
\]
Both sums converge in \( Q \)-\( A_{00} \)-topology. It is easy to verify that \( x \in LM(A_{00}) \) and \( y \in RM(A_{00}) \). For every \( n \),
\[
e_n(x + y)e_n = \left( \sum_{k=1}^{n-1} e_k z(e_k - e_{k-1}) + e_n^2 z(e_n - e_{n-1})e_n + e_n z(e_n + e_n e_{n+1} - e_n) e_n \right)
\]
\[+ \left( \sum_{k=1}^{n-1} (e_n - e_k) z(e_k - e_{k-1}) + (e_n - e_n^2) z(e_n - e_{n-1})e_n \right)
\]
\[= \left( \sum_{k=1}^{n-1} e_n z e_k - e_{k-1} + e_n z(e_n - e_n) + e_n z(e_n^2 - e_{n-1}) \right)
\]
\[= e_n z e_{n-1} + e_n z(e_n - e_{n-1}) = e_n z e_n.
\]
So \( x + y = z \).

3.8. The problem when \( QM(A) = LM(A) + RM(A) \) had been studied in [16, 3, 13, 14]. In general, \( QM(A) \neq LM(A) + RM(A) \).

4. The Tietze theorem and Dauns-Hofmann theorem

This section is inspired by [11]. Our results are similar to the corresponding ones in [11].

4.1. Let \( B \) be a \( \sigma \)-unital \( C^* \)-algebra and let \( \phi \) be a \( * \)-homomorphism from \( A \) onto \( B \). Then \( B_{00} = \phi(A_{00}) \) is a support algebra of \( B \) and \( \phi \) can be extended to a linear map \( \hat{\phi} \) from \( LM(A_{00}) \) into \( LM(B_{00}) \) as follows:

(i) \( \hat{\phi}(z) \cdot \phi(a) = \phi(z \cdot a) \)

for \( z \in LM(A_{00}) \) and \( a \in A_{00} \). We can further extend \( \hat{\phi} \) from \( QM(A_{00}) \) into \( QM(B_0) \) by

(ii) \( \phi(a) \cdot \hat{\phi}(z) \cdot \phi(b) = \phi(a \cdot z \cdot b) \)
for \( z \in QM(A_{00}) \) and \( a, b \in A_{00} \). It can be verified that if \( z \in QM(A_{00}) \), \( x \in LM(A_{00}) \), \( y \in RM(A_{00}) \), and \( a \in A_{00} \), then

(iii) \( \phi(a) \cdot \tilde{\phi}(y) = \phi(a \cdot y) \);
(iv) \( \tilde{\phi}(y \cdot z) = \phi(y) \cdot \tilde{\phi}(z) \);
(v) \( \tilde{\phi}(z \cdot x) = \phi(z) \cdot \tilde{\phi}(x) \);
(vi) \( \tilde{\phi}(z)^* = \phi(z^*) \) and \( \phi(z) \geq 0 \) if \( z \in QM(A_{00})_+ \).

4.2. Proposition. The extension \( \tilde{\phi} \) is continuous when \( QM(A_{00}) \) is considered with \( Q-A_{00}-\text{topology} \) and \( QM(B_{00}) \) with \( Q-B_{00}-\text{topology} \).

4.3. Next we shall show that the extension \( \tilde{\phi} \) is surjective. In view of 2.20, the following theorem can be regarded as a noncommutative extension of Tietze’s theorem. The same results for bounded multipliers \( M(A) \) and bounded quasi-multipliers \( QM(A) \) can be found in [9, 3]. A similar result for (unbounded) multipliers of \( K(A) \) can be found in [11].

4.4. Theorem. Let \( \phi \) be a homomorphism from \( A \) onto \( B \) and \( B_{00} = \phi(A_{00}) \). Then

(i) \( \tilde{\phi}(QM(A_{00})) = QM(B_{00}) \);
(ii) \( \tilde{\phi}(LM(A_{00})) = LM(B_{00}) \);
(iii) \( \tilde{\phi}(RM(A_{00})) = RM(B_{00}) \);
(iv) \( \tilde{\phi}(M(A_{00})) = M(B_{00}) \).

Proof. (i) We shall show that \( \tilde{\phi} \) is surjective. Let \( z \in QM(B_{00}) \) and \( z_k = e^{-k} \tilde{\phi}(e_k) \), where \( e_k = \phi(e_k) \), \( k = 1, 2, \ldots \). Suppose that \( y_k \in A_{00} \) such that \( \phi(y_k) = z_k \). Let \( z_1 = y_1 \),

\[
\begin{align*}
z_{k+1} &= y_{k+1} - e_k y_{k+1} e_k + z_k, \\
&\quad k = 1, 2, \ldots.
\end{align*}
\]

Then \( z_{k+1} \in A_{00} \); moreover,

\[
\phi(z_{k+1}) = \bar{z}_{k+1} - \bar{e}_k \bar{z}_{k+1} \bar{e}_k + \bar{z}_k = z_{k+1}.
\]

If \( k > m \), then

\[
e_m(z_{k+1} - z_k) e_m = e_m y_{k+1} e_m - e_m e_k y_{k+1} e_k e_m + e_m z_k e_m - e_m z_k e_m.
\]

Thus, if \( k, k' > m \),

\[
e_m(z_k - z_{k'}) e_m = 0.
\]

So \( \{z_k\} \) is a \( Q-A_{00} \)-Cauchy sequence. Suppose that \( z = \lim z_k \). Then, by the continuity of \( \tilde{\phi} \) (4.2),

\[
\tilde{\phi}(z) = \lim \phi(z_k) = \lim \bar{z}_k = \bar{z}.
\]

Then \( \tilde{\phi} \) is onto.

(ii) Let \( x \in LM(A_{00}) \) and \( x_k = \bar{x} \bar{e}_k \), \( k = 1, 2, \ldots \). Suppose that \( a_k \in A_{00} \) such that \( \phi(a_k) = \bar{x}_k \). Define \( x_1 = a_1 \) and \( x_{k+1} = a_{k+1} - a_{k+1} \cdot e_k + x_k \),
k = 1, 2, . . . Then \( \phi(x_{k+1}) = \overline{x}_{k+1}, \) \( k = 1, 2, \ldots \) As in (i), \( \{x_{k+1}\} \) is an \( L-A_{00} \)-Cauchy sequence, hence a \( Q-A_{00} \)-Cauchy sequence. Let \( x = \lim x_k \). Then \( \phi(x) = x \). To show that \( x \in LM(A_{00}) \), take \( a \in A_n \). Then

\[
x_{k+1}a - xka = x_{k+1}e_{n+1}a - xke_{n+1}a
= (x_{k+1} - x_k)e_{n+1}a = 0
\]

if \( k > n + 1 \). So \( x_k a = x_{k+2} a \) for every \( k > n + 1 \). Thus \( x \cdot a \in A_{00} \). We conclude that \( x \) is in \( LM(A_{00}) \).

We omit the proofs for (iii) and (iv).

4.5. Let \( z \in QM(A_{00}) \) and \( a \in A_{00} \). Then \( z \cdot a, a \cdot z \in QM(A_{00}) \). In fact, \( a \cdot z \in LM(A_{00}) \), while \( z \cdot a \in RM(A_{00}) \). The center of \( QM(A_{00}) \) is the set \( Z = \{z \in QM(A_{00}) : a \cdot z = z \cdot a \ \text{for all} \ a \in A_{00}\} \).

4.6. Proposition. \( Z \subset M(A_{00}) \). Moreover, \( Z \) is the center of \( M(A_{00}) \).

Proof. Suppose that \( z \in Z \). Then for every \( k \), if \( n, m > k \),

\[
e_nze_k = e_n^{1/2} z e_k^{1/2} = e_k^{1/2} z e_k^{1/2} = e_m z e_k.
\]

Thus \( z \in QM(A_{00}) = LM(A_{00}) \). Similarly, \( z \in RM(A_{00}) \), so \( z \in M(A_{00}) \).

Let \( y \in M(A_{00}) \). Then

\[
z \cdot y \cdot a = (y \cdot a) \cdot z = y \cdot z \cdot a \quad \text{for every} \ a \in A_{00}.
\]

Hence \( z \cdot y = y \cdot z \). \( Z \) is in the center of \( M(A_{00}) \). The center of \( M(A_{00}) \) contained in \( Z \) is trivial.

4.7. Lemma. Let \( z \in Z \). Then for each \( f \in P(A) \), the pure state space of \( A \), \( f(z) = \lim f(e_n z e_n) \) exists. Moreover, the function \( f \to f(z) \) is a weak*-continuous function on \( P(A) \).

Proof. Let \( f \) be in \( P(A) \), let \( \pi_f \) be the corresponding irreducible representation of \( A \), and let \( H \) be the associated Hilbert space. Suppose that \( z_n = z|_{A_n} \). Then \( z_n \) is in the center of \( M(A_n) \). We may assume that \( A_n \not\subseteq \ker \pi_f \). Then \( (\pi_f|_{A_n}, \pi_f(A_n)H) \) is an irreducible representation of \( A_n \). Let \( q_n \) be the projection corresponding to \( H_n \), the closure of \( \pi_f(A_n)H \). Then

\[
\pi_f(z_n)|_{H_n} = \lambda_n q_n \quad \text{for some scalar} \ \lambda_n.
\]

Since \( \pi_f(z_{n+1})|_{H_n} = \pi_f(z_n)|_{H_n} \), \( \lambda_{n+1} = \lambda_n \) for each \( n \). Thus \( \pi_f(z) \) is a scalar multiple of the identity. Moreover, \( \pi_f(z) = f(z) \cdot \text{id}_H \).

Next we shall show that \( f \to f(z) \) is continuous. Let \( f_0 \in P(A) \). There is \( k_0 \) such that \( 1 \geq f_0(e_{k_0}) > 1/2 \). Let \( V_0 = \{f \in P(A) : |f(e_{k_0}) - f_0(e_{k_0})| < 1/4\} \).

Then for every \( f \in V_0 \), \( f(e_{k_0}) > 1/4 \).

Let \( \pi_f \) be the associated irreducible representation and \( H_f \) the associated Hilbert space. Then, since \( \pi_f(z^* z) \) is a scalar, for every unit vector \( \xi \in H_f \),

\[
\langle \pi_f(z^* z)\xi, \xi \rangle = f(z^* z).
\]
Suppose that $f(a) = \langle \pi_f(a)\xi_f, \xi_f \rangle$ for every $a \in A$. Then
\[
\begin{align*}
    f(z^*z) &= 1/f(e_k)^2 \langle \pi_f(z^*z)e_k\xi_f, e_k\xi_f \rangle \\
    &\leq 1/f(e_k)^2 \|e_kz^*ze_k\| \\
    &\leq 16\|e_kz^*ze_k\|
\end{align*}
\]
for every $f \in V_0$.

Let $M = \max\{1, 16\|e_kz^*ze_k\|\}$. For $\varepsilon > 0$, choose $k \geq k_0$ such that $1 \geq f_0(e_k) > 1 - \varepsilon^2/8M$. Denote
\[
V = V_0 \cap \{ f \in P(A) : |f(e_k) - f_0(e_k)| < \varepsilon^2/8M, |f(e_kz) - f_0(e_kz)| < \varepsilon/4 \}.
\]
So for every $f \in V$, $|f(z^*z)| < M$ and $|f(1-e_k)| < \varepsilon^2/4M$. Hence, if $f \in V$,
\[
|f(z) - f_0(z)| \leq |f(z) - f(e_kz)| + |f(e_kz) - f_0(e_kz)| + |f_0(e_kz) - f_0(z)| \\
\leq |f((1-e_k)z)| + \varepsilon/4 + |f_0((1-e_k)z)| \\
\leq f((1-e_k)^{1/2}f(z^*z)^{1/2} + f_0((1-e_k)^2)^{1/2}f_0(z^*z)^{1/2} + \varepsilon/4 \\
\leq f((1-e_k)^{1/2}M^{1/2} + f_0((1-e_k)^{1/2}M^{1/2} + \varepsilon/4 \\
< \varepsilon/2 + \varepsilon/8 + \varepsilon/4 < \varepsilon.
\]

4.8. The idea of the proof of 4.7 was taken from [11, 5.41]. However, the proof of [11, 5.41] is not complete. (The number $M$ there depends on the choice of $a$ and $a$ depends on $\varepsilon$, so $M$ depends on $\varepsilon$.) Nevertheless, the proof could be easily completed. The same result as [11, 5.41] is not true for $QM(A_{00})$, as we shall see in 4.14.

4.9. In the proof of 4.7, we see that if $\pi_{f_1}$ and $\pi_{f_2}$ are equivalent, then $f_1(z) = f_2(z)$ for $z \in Z$. Thus every $z \in Z$ defines a continuous function $z$ on $\hat{A}$ by $\hat{z}(\pi_f) = f(z)$.

4.10. Theorem. The mapping $z \rightarrow \hat{z}$ is a *-isomorphism of $Z$ onto $C(\hat{A})$. Moreover, the mapping is bicontinuous when $Z$ is considered with the $A_{00}$-topology and $C(\hat{A})$ with the compact open topology.

Proof. Clearly, $z \rightarrow \hat{z}$ is a *-homomorphism. If $\hat{z}_1 = \hat{z}_2$ for $z_1, z_2 \in Z$, then $\pi(z_1) = \pi(z_2)$ for every $\pi \in \hat{A}$. Thus $z_1 = z_2$. Hence the mapping is one-to-one.

Suppose that $f \in C(\hat{A})$. For every $k$, by [11, 5.39], $\{ \pi \in \hat{A} : \pi(e_{k+1}) \neq 0 \}$ is contained in a compact subset of $\hat{A}$. Thus $\hat{A}_k$ is contained in a compact subset of $A$. Thus $f|_{\hat{A}_k}$ is bounded and by the Dauns-Hofmann theorem (we use the version [18, 4.4.6]), for every $a \in A_k$, there is $\rho(a) \in A_k \subset A_{00}$ such that $\pi(\rho(a)) = f(\pi)\pi(a)$ for $\pi \in \hat{A}_k$. 
Hence, the above equality holds for all $\pi \in \hat{A}$, and $\rho$ defines a linear map from $A_{00}$ into $A_{00}$. Let $a, b \in A_{00}$. We have
\[ \pi(a \rho(b)) = f(\pi)\pi(a)\pi(b) = \pi(\rho(a)b) \]
for all $\pi \in \hat{A}$. Thus $z = (\rho, \rho) \in M(A_{00}) \subset QM(A_{00})$ and, clearly, $z \in Z$. It is then easy to see that $\hat{z}(\pi) = f(\pi)$ for each $\pi \in \hat{A}$. Thus the mapping is surjective.

The proof of the bicontinuity is essentially the same as the proof of [11, 5.44] with the obvious minor modifications.

4.11. Corollary. Let $f \in C(\hat{A})$. Then, for any $z \in QM(A_{00})$, there is $y \in QM(A_{00})$ such that $\pi(y) = f(\pi)\pi(z)$ for all $\pi \in \hat{A}$.

4.12. By [18, 4.417], we may replace $\hat{A}$ by $Prim(A)$ in 4.10 and 4.11.

4.13. We shall denote $FQM(A_{00}) = \{z \in QM(A_{00}) : f(z) = \lim f(e_n z e_n) \text{ exists for each } f \in P(A)\}$. Clearly, $FQM(A_{00})$ is a $*$-invariant linear space containing $QM(A)$.

4.14. Theorem. (i) If $z \in FQM(A_{00})$, then $\hat{\pi}(z) \in QM(\pi(A))$ for every $\pi \in \hat{A}$.

(ii) If $C^b(\hat{A}) \neq C(\hat{A})$, then $FQM(A_{00}) \neq QM(A)$.

(iii) $FQM(A_{00}) = QM(A_{00})$ if and only if $\pi(A)$ is unital for each $\pi \in \hat{A}$.

Proof. (i) We may assume that $z = z^*$. Let $\pi \in \hat{A}$, $H$ be the associated Hilbert space, and $\xi$ be a unit vector in $H$.

Since $\langle \pi(e_n z e_n)\xi, \xi \rangle$ converges, we may assume that there is a positive number $M_\xi$ such that
\[ |\langle \pi(e_n z e_n)\xi, \xi \rangle| \leq M_\xi \quad \text{for all } n. \]
Hence
\[ |\langle \pi(e_n z e_n)\xi, \xi \rangle| \leq M_\xi \quad \text{for all } n. \]
So
\[ \| (e_n z e_n)^{1/2} \xi \| \leq M_\xi \quad \text{for all } n. \]
by the uniform boundedness theorem, $\{(e_n z e_n)^{1/2}\}$ is bounded. Hence $\{(e_n z e_n)_+\}$ is bounded. Similarly, $\{(e_n z e_n)^-\}$ is bounded, thus $\{(e_n z e_n)\}$ is bounded. This implies that $\hat{\pi}(z) \in QM(\pi(A))$.

(ii) If $C^b(A) \neq C(A)$, then, by Theorem 4.10, there is $z \in Z \subset QM(A_{00})$ such that $z$ is not bounded. Thus $z \notin QM(A)$. However, $z \in FQM(A_{00})$.

(iii) Suppose that $\pi \in \hat{A}$ and $\pi(A)$ has no unit. By taking a subsequence if necessary, we may assume that
\[ \pi(e_{nm}) - \pi(e_{n-1}) \neq 0. \]
Thus there are $\xi_k \in H$ such that $\|\xi_k\| = 1$, and $\xi_k \perp \xi_j$ if $k \neq j$; and
\[ \| (\pi(e_{2k+2}) - \pi(e_{2k}))^{1/2} \xi_k \| = a_k > 0 \]
and
\[ [\pi(e_{2k+2}) - \pi(e_{2k})]x_m = 0 \quad \text{if } m \neq k \]
for every \( k \). Define
\[ y = \sum_k (k+1)(2^{k+1}/a_k)(e_{2k+2} - e_{2k}) \cdot \]
Then it is easy to see that \( y \in M(A_{00}) \subset QM(A_{00}) \). Let \( \xi = \sum_{k=1}^{\infty} (1/2)^{k/2} x_k \); then \( \|\xi\| = 1 \). So \( f(\cdot) = \langle \cdot, \xi \rangle \) is a pure state of \( A \). But
\[ f(e_{2k+2}ye_{2k+2}) \geq k \cdot \]
So \( y \in FQM(A_{00}) \).
Conversely, if \( \pi(A) \) is unital for each \( \pi \in \hat{A} \), then \( \pi(QM(A_{00})) = QM(\pi(A)) \).
The conclusion is obvious.

5. Duals and biduals

In this section, we shall study \( QM(A_{00})' \), the dual of \( QM(A_{00}) \) (the latter being considered with the \( Q\)-\( A_{00} \)-topology), and \( QM(A_{00})'' \), the bidual of \( QM(A_{00}) \).

5.1. Theorem. \( QM(A_{00})' = \{f(a \cdot b) : a, b \in A_{00}, f \in A^*, \text{and } \|f\| \leq 1\} \).
Proof. For \( a, b \in A_{00} \), denote
\[ U_{a,b} = \{z \in QM(A_{00}) : \|azb\| \leq 1\} \cdot \]
Then \( \{U_{a,b}\} \) forms a neighborhood base at 0. Let
\[ U_{a,b}^0 = \{f \in QM(A_{00})' : |f(z)| < 1 \text{ if } z \in U_{a,b}\} \cdot \]
Then
\[ QM(A_{00})' = \bigcup\{U_{a,b}^0 : a, b \in A_{00}\} \cdot \]
Suppose that \( f \in U_{a,b}^0 \); then \( |f(z)| < 1 \) for each \( z \in U_{a,b} \), or, equivalently,
\[ |f(z)| < \|azb\| \quad \text{for each } z \in QM(A_{00}) \cdot \]
Define a linear functional \( g \) on the normed linear space \( \{azb : z \in QM(A_{00})\} \) of \( A \) by \( g(azb) = f(z) \). Then \( g \) is well defined and \( |g(azb)| < \|azb\| \). By the Hahn-Banach theorem, we can assume that \( g \) is in \( A^* \) and \( \|g\| < 1 \). Thus
\[ U_{a,b}^0 \subseteq \{f(a \cdot b) : f \in A^*, \|f\| \leq 1\} \cdot \]
This completes the proof.

5.2. Let \( g \in A_n^* \) and \( p_n = [e_n] \). For every \( a \in A \), define \( f(a) = g(p_n ap_n) \).
Then \( f \in A^* \) and \( \|f\| = \|g\| \). Moreover,
\[ f(e_{nm+1}ae_{n+1}) = g(p_n e_{n+1}ae_{n+1}p_n) = g(p_n ap_n) = f(a) \quad \text{for every } a \in A \cdot \]
Define $\hat{f}(z) = (e_{n+1}z e_{n+1})$; then $\hat{f} \in QM(A_{00})'$. We denote by $L_n$ the set
$$\{f : f(a) = g(p_n a p_n), \; g \in A_n^*, \; \text{for every} \; a \in A\}.$$  
Then $L_n \subseteq QM(A_{00})'$. If $g \in QM(A_{00})'$, by Theorem 5.1, $g(\cdot) = f(a \cdot b)$ for some $a, b \in A_n$ and some $n$. Clearly $g(p_n \cdot p_n) = g$, so $g \in L_n$.

5.3. Corollary. $QM(A_{00})' = \bigcup_{n=1}^{\infty} L_n$.

5.4. By 5.2 we can identify $L_n$ with $A_n^*$.

5.5. Proposition. Let $f$ be a positive $Q \cdot A_{00}$-continuous functional on $QM(A_{00})$. Then there is a positive functional $g \in (A_n^*)_+$ and $n$ such that
$$f(z) = g(e_{n+1}z e_{n+1}) \; \text{for all} \; z \in QM(A_{00}).$$

Proof. It is an immediate consequence of 5.3.

5.6. Proposition. $QM(A_{00})'$ is the linear span of its positive cone.

Proof. Since $L_n$ ($= A_n^*$) is the linear span of its positive cone, by 5.3 $QM(A_{00})'$ is the linear span of its positive cone.

5.7. We shall denote by $M_0(A)$ the norm closure of $\bigcup_{n=1}^{\infty} A_n^{**}$ (cf. [15]). Then $\bigcup_{n=1}^{\infty} A_n^{**} = \bigcup_{n=1}^{\infty} p_n A_n^{**} p_n$ is a support algebra of $M_0(A)$, where $p_n = [e_n]$.

5.8. Let $QM(A_{00})''$ be the bidual of $QM(A_{00})$. The "strong" topology on $QM(A_{00})''$ is the locally convex topology generated by seminorms
$$\|F\|_{a,b} = \sup\{|F(f)| : f \in U_{a,b}^0\},$$
where $F \in QM(A_{00})''$, $a, b \in A_{00}$, and $U_{a,b}^0$ as in 5.1.

5.9. Theorem. $QM(A_{00})''$ is isomorphic to $QM(\bigcup_{n=1}^{\infty} A_n^{**})$ as topological vector spaces, the former is considered with "strong" topology and the latter is considered with $Q \cdot \bigcup_{n=1}^{\infty} A_n^{**}$-topology.

Proof. Let $L_n$ be the same as in 5.2. There is a natural isometry from $L_n$ onto $A_n^*$. We may identify $L_n$ with $A_n^*$.

Let $F \in QM(A_{00})''$. Define $F_n = F|_{L_n} (= F|_{A_n^*})$. So there is $z_n(F) \in A_n^{**}$ such that
$$F_n(f) = z_n(F)(f) \; \text{for all} \; f \in A_n^*.$$  
We define a map $\Phi$ from $QM(A_{00})''$ into $QM(\bigcup_{n=1}^{\infty} A_n^{**})$ as follows:
$$\Phi : F \rightarrow \rho_F, \; \text{where} \; \rho_F(a, b) = az_n(F)b$$
for all $a, b \in A_n^{**}$, $n = 1, 2, \ldots$. Since $F_{n+1}|_{A_n^*} = F_n$, $\rho_F$ is well defined and $\rho_F$ is in $QM(\bigcup_{n=1}^{\infty} A_n^{**})$. Clearly $\Phi$ is a linear map.

If $\rho_F = 0$, then $F_n(f) = 0$ for all $f \in A_n^{**}$ and all $n$. So $F = 0$. Hence $\Phi$ is one-to-one.

Take $z \in QM(\bigcup_{n=1}^{\infty} A_n^{**})$. Then $p_n z p_n \in A_n^{**}$. For each $f \in A_n^*$ ($= L_n$) define
$$F_z(f) = f(p_n z p_n) \; \text{for} \; f \in A_n^* \; (= L_n).$$
Thus we define an element \( F_z \) in \( QM(A_{00})'' \). It is easy to see that \( \Phi(F_z) = z \). Hence \( \Phi \) is onto.

Now suppose that \( F_\alpha, F \in QM(A_{00})'' \) such that \( F_\alpha \to F \) in the “strong” topology.

Let \( U_n^0 = \{ f \in QM(A_{00})' : |f(z)| < 1 \text{ if } \|e_{n+1}ze_{n+1}\| \leq 1 \} \). Then
\[
\sup\{|F_n(f) - F(f)| : f \in U_n^0 \} \to 0.
\]
If \( f \in A_\alpha^* (= L_n) \) and \( \|f\| \leq 1 \), then
\[
|f(z)| = |f(p_ne_{n+1}ze_{n+1}p_n)| \leq \|p_ne_{n+1}ze_{n+1}p_n\| \leq \|e_{n+1}ze_{n+1}\|.
\]
Hence \( f \in U_n^0 \). Thus,
\[
\|p_n(\rho_{F_\alpha} - \rho_F)p_n\| = \sup\{|f(p_ne_n(z_{\alpha}(F)) - z_n(F)p_n)| : f \in A_\alpha^*, \|f\| \leq 1 \}
\leq \sup\{|F_\alpha(f) - F(f)| : f \in L_n, \|f\| \leq 1 \}
\leq \sup\{|F_\alpha(f) - F(f)| : f \in U_n^0 \} \to 0.
\]
Hence \( \rho_{F_\alpha} \to \rho_F \) in \( Q - \bigcup_{n=1}^\infty A_n^{**} \)-topology.

Conversely, suppose that \( \rho_{F_\alpha} \to \rho_F \) in \( Q - \bigcup_{n=1}^\infty A_n^{**} \)-topology. For each \( n \), by 5.1,
\[
U_n^0 \subset \{ f(e_{n+1}e_{n+1}) : f \in A^*, \|f\| \leq 1 \}.
\]
Thus
\[
U_n^0 \subset \{ f \in L_n : \|f\| < 1 \}.
\]
Hence
\[
\|p_n(\rho_{F_\alpha} - \rho_F)p_n\| = \sup\{|f(p_ne_n(z_{\alpha}(F)) - z_n(F)p_n)| : f \in L_n, \|f\| \leq 1 \}
\geq \sup\{|f(F_\alpha) - f(F)| : f \in U_n^0 \}.
\]
Thus \( \|p_n(\rho_{F_\alpha} - \rho_F)p_n\| \to 0 \) implies
\[
\sup\{|f(F_\alpha) - f(F)| : f \in U_n^0 \} \to 0.
\]
So \( \Phi \) is bicontinuous.

5.10. Example. Let \( K \) be the \( C^* \)-algebra of all compact operators on a separable Hilbert space. Let \( A_{00} = \bigcup_{n=1}^\infty M_n \) be a support algebra of \( K \), where each \( M_n \) is isomorphic to the \( n \times n \) matrix algebra. Since \( M_n^{**} = M_n \), \( M_0(A) = A \).

Hence \( QM(\bigcup_{n=1}^\infty M_n^{**}) = QM(A_{00}) \). By 5.9, \( QM(A_{00})'' = QM(A_{00}) \).

5.11. Proposition. Every \( \sigma \)-unital dual \( C^* \)-algebra has reflexive quasi-multipliers.

Proof. Let \( e \) be a strictly positive element of \( A \). By [4, 4.7.20], every nonzero point of \( \text{Sp}(e) \) is isolated. So we may assume that \( e_n \) are projections. Consequently, \( A_n = e_neA_n \) and are unital dual \( C^* \)-algebras. Thus \( A_n \) are finite dimensional. This implies that \( A_n^{**} = A_n \). Hence \( M_0(A) = A \). By 5.9, \( QM(A_{00})'' = QM(A_{00}) \).
6. PSEUDO-COMMUTATIVE $C^*$-ALGEBRAS

In §3, we showed that $QM(A_{00}) = LM(A_{00}) + RM(A_{00})$. We now consider the problem when $QM(A_{00}) = M(A_{00})$. It turns out that the problem is equivalent to the problem when $K(A) = A_{00}$.

6.1. Theorem. Let $A$ be a $\sigma$-unital $C^*$-algebra and $A_{00}({\{e_n}\})$ a support algebra of $A$. Then the following are equivalent:

(i) $M(A_{00}) = QM(A_{00})$.

(ii) For every $n$, there is an integer $N(n) < n$ such that $e_n a = e_n a e_{N(n)}$ for all $a \in A$.

Proof. (i) $\Rightarrow$ (ii). Since $M(A_{00}) = QM(A_{00})$, $A \subset M(A_{00})$. So for every $a \in A$, $e_n a \in A_{00}$, that is, $e_n a \in A_k$ for some $k$. Thus $e_n a = e_n a e_{k+1}$. If (i) does not imply (ii), there are $a_k \in A$ such that
\[
x_k = e_n a_k (e_{n+1} - e_n) \neq 0
\]
for some subsequence $\{n_k\}$. We may assume that $\|x_k\| = 1$ for all $k$. Define $z = \sum_{k=1}^{\infty} (1/2)^k x_k$. Then $z \in A \subset QM(A_{00})$. But
\[
e_n z = e_{n+1} \left( \sum_{k=1}^{\infty} (1/2)^k \right) = \sum_{k=1}^{\infty} (1/2)^k x_k = z \notin A_{00}.
\]
Hence $z \notin M(A_{00})$, a contradiction.

(ii) $\Rightarrow$ (i) For fixed $n$,
\[
(a e_n)^* = e_n a^* = e_n a e_{N(n)} \quad \text{for all } a \in A.
\]
So $ae_n = e_{N(n)} a e_n$.

Suppose that $z \in QM(A_{00})$. For fixed $k$,
\[
e_n z e_k = e_{n+1} e_n z e_k e_{k+1} = e_{n+1} e_n e_{N(k+1)} z e_k
\]
\[
= e_{N(k+1)} z e_k \quad \text{if } n > N(k+1).
\]
Thus $z \in QM(A_{00})$. Similarly, $z \in QM(A_{00})$, so $z \in M(A_{00})$.

6.2. Definition. A $\sigma$-unital $C^*$-algebra $A$ (without unit) is called pseudo-commutative if $A$ satisfies (i) or (ii) in 6.1.

6.3. Proposition. Suppose that $A$ is a pseudo-commutative $C^*$-algebra (without identity). Then the following are true:

(i) The Pedersen ideal $K(A)$ is a support algebra of $A$.

(ii) $M(A) = QM(A)$.

(iii) The spectrum $\hat{A}$ of $A$ is not compact.

(iv) For every irreducible representation $\pi$ of $A$, $\pi(A)$ has a unit.

Proof. (i) By (ii) of 6.1, $A_{00}$ is a dense ideal of $A$. Since $K(A) \subset A_{00}$, we conclude that $K(A) = A_{00}$.
(ii) Suppose that $z \in QM(A)$. Then $z \in M(A_{00})$. For every $a \in A$,
$$e_n a e_n z \in A_{00} \subset A.$$  
Since $z$ is bounded and $\|e_n a e_n - a\| \to 0$, we conclude that $az \in A$. Similarly $za \in A$. So $z \in M(A)$.

(iii) If $\hat{A}$ is compact, by [11, 10.8], $A$ is a PCS-algebra, that is, $M(A) = \Gamma(K(A))$. It follows from (i) that $\Gamma(K(A)) = M(A_{00})$. Hence $M(A) = M(A_{00}) = QM(A_{00})$. However, by Lemma 2.16, if $A$ is not unital, $QM(A_{00}) \neq QM(A)$. A contradiction.

(iv) By [11, 10.4], $\pi(A)$ is a PCS-algebra, so, as in (iii), $QM(\pi(A)) = QM(\pi(A_{00}))$. By Lemma 2.16, it happens only when $\pi(A)$ has a unit.

The following lemma is taken from [11, 10.7] but in a slightly different setting. The terminology follows from [11].

6.4. Lemma (cf. [11, 10.7]). Let $A$ be a C*-algebra and let $\{x_n\}$ be an orthogonal sequence in $(K(A))^+$ (that is, $x_n x_m = 0$, if $n \neq m$) such that the sequence of partial sum $\{\sum_{k=1}^{\infty} x_k\}$ is K-Cauchy. Let $a \in K(A)$, $S$ be a subset of $\hat{A}$, and let $\{\alpha_n\}$ be the sequence defined by
$$\alpha_n = \sup\{\|\pi(a)\| : \pi \in S \text{ and } \|\pi(x_n)\| > \|x_n\|_S / 2\},$$
where $\|x_n\|_S = \sup\{\|\pi(x_n)\| : \pi \in S\}$. If $\|x_n\|_S \to \infty$, then $\alpha_n \to 0$.

Proof. The proof is the same as the proof of [11, 10.7]. We only need to change $\hat{A}$ and $\|x_n\|_S$ into $S$ and $\|x_n\|_S$, respectively.

6.5. Theorem. Suppose that $A$ is a $\sigma$-unital C*-algebra. Then $A$ is pseudo-commutative if and only if one of its support algebras $A_{00} = K(A)$.

Proof. Let $A_{00} = A_{00}(\{e_n\})$. For every $n$, denote
$$F_n = \{\pi \in \hat{A} : \|\pi(e_n)\| \geq 1/n + 1\}.$$
We claim that there is a $b_n \in A_{00}$ such that
$$\pi(b_n) = 1 \quad \text{for each } \pi \in F_n.$$
If not, by taking a subsequence if necessary, we may assume that there are $\pi_k \in F_n$ such that
$$\pi_k(e_k - e_{k-1}) \neq 0.$$

Let $x_k = \beta_k(e_{2k} - e_{2k-1})$, where $\beta_k = k \cdot \max(1, 1/\|\pi_k(e_{2k} - e_{2k-1})\|)$, $k = 1, 2, \ldots$. Then $x_k x_m = 0$ if $n \neq m$ and $\sum_{k=1}^{\infty} x_k$ is $A_{00}$-Cauchy. By letting $a = e_n$, and $S = F_n$ in Lemma 6.4, we have $\|x_k F_n\| \to \infty$ as $k \to \infty$, hence $\|\pi_k(e_n)\| \to 0$ as $k \to \infty$. This contradicts the fact $\|\pi(e_n)\| \geq 1/n + 1$ for all $\pi \in F_n$. So we complete the proof of the claim.

Now let $a_1 = b_1$. Then $a_1 \in A_{00}$, so $a_1 \in A_{N(1)}$ for some $N(1)$. Suppose that $a_1, a_2, \ldots, a_k$ have been chosen from $A_{00}$, and assume that $a_k \in A_{N(k)}$. Then
$$a_k e_{N(k+1)} = e_{N(k)+1} a_k = a_k.$$
So

\{ \pi \in \hat{A} : \pi(a_k) \neq 0 \} \subset \{ \pi \in \hat{A} : \| \pi(e_{N(k)+1}) \| \geq 1 \}
\subset F_{N(k)+1}.

We choose \( a_{k+1} = b_{N(k)+1} \). Thus \( \pi(a_{k+1}) = 1 \) for all \( \pi \in \{ \pi \in \hat{A} : \pi(a_k) \neq 0 \} \).

Hence \( a_{k+1}a_k = a_ka_{k+1} = a_k \). For every \( a \in A \),

\[ \pi(a_k a) = \pi(a_k) \pi(a) = 0 \quad \text{if} \quad \pi(a_k) = 0. \]

Thus

\[ \pi(e_k a) = \pi(e_k) \pi(a) \pi(a_{k+1}) \]

for all \( \pi \in \hat{A} \). We conclude that

\[ a_k a = a_k a a_{k+1} \quad \text{for all} \quad a \in A \quad \text{and} \quad k. \]

Clearly \( \{a_k\} \) forms an approximate identity for \( A \). By 6.1 we conclude that \( A \) is pseudo-commutative.

The converse is (i) of 6.3.

6.6. Theorem. Let \( A \) be a pseudo-commutative \( C^* \)-algebra. Then \( K(A) \) is the only support algebra of \( A \).

Proof. By the proof of 6.5, there is an approximate identity \( \{a_n\} \) satisfying

\[ a_{k+1}a_k = a_ka_{k+1} = a_k \quad \text{for each} \quad k \quad \text{and} \quad a_k a = a_k a a_{k+1} \quad \text{for every} \quad a \in A. \]

Moreover, there are compact subsets \( F_n \) of \( A \) such that \( F_n \subset F_{n+1}, \bigcup_{n=1}^{\infty} F_n = \hat{A} \), and

\[ \pi(a_n) = \begin{cases} 1 & \text{for all} \quad \pi \in F_n, \\ 0 & \text{if} \quad \pi \in \hat{A} \setminus F_{n+1}. \end{cases} \]

Since \( a_k a = a_k a a_{k+1} \) for every \( a \in A \), \( A_00(\{a_k\}) \) is an ideal. So \( A_00(\{a_n\}) = K(A) \).

Now suppose that \( A_00 = A_00(\{e_n\}) \) is any support algebra of \( A \). For every \( n \), there is \( k(n) \) such that

\[ \| e_{k(n)} a_n - a_n \| < 1/2. \]

Hence

\[ \| \pi(e_{k(n)}) - 1 \| < 1/2 \quad \text{for all} \quad \pi \in F_n. \]

Thus \( \pi(A_{k(n)}) = \pi(A) \) for all \( \pi \in F_n \). Since \( \pi(a_{n-1}) = 0 \) for \( \pi \in \hat{A} \setminus F_n \), we conclude that \( e_{k(n)} \geq a_{n-1} \) for every \( n \). Hence

\[ A_00 \supseteq A_00(\{a_n\}) = K(A). \]

This completes the proof.

6.7. Definition. An approximate identity \( \{e_n\} \) of \( A \) is said to be central if \( e_n a = ae_n \) for all \( a \in A \) and all \( n \).
6.8. **Theorem.** Suppose that $A$ is a $\sigma$-unital $C^*$-algebra such that $\text{Prim}(A)$ is a Hausdorff space. Then $A$ is pseudo-commutative if and only if $A$ has a central approximate identity $\{e_n\}$ satisfying $e_{n+1}e_n = e_ne_{n+1} = e_n$ for all $n$.

**Proof.** Suppose that $A$ is pseudo-commutative. Let

$$T_n = \{ \pi \in \text{Prim}(A) : \|\pi(e_n)\| \geq 1/n \},$$
$$O_n = \{ \pi \in \text{Prim}(A) : \|\pi(e_n)\| > 1/n + 1 \},$$
and

$$F_n = \{ \pi \in \text{Prim}(A) : \|\pi(e_n)\| \geq 1/n + 1 \}.$$

by [18, 4.43 and 4.45], $T_n$ and $F_n$ are closed and compact and $O_n$ is open. The element $b_n$ in 6.5 satisfies $\pi(b_n) = 1$ for all $\pi \in F_n$. Since $\text{Prim}(A)$ is a locally compact Hausdorff space, there is $f \in C(\text{Prim}(A))$ such that $0 \leq f \leq 1$, $f|_{T_n} = 1$, and $f|_{\text{Prim}(A) \setminus O_n} = 0$. By the Dauns-Hofmann theorem (cf. [6, Theorem 3]), there is $x_n \in A_+$ such that

$$\pi(x_n) = f(\pi)b_n \quad \text{for all } \pi \in \text{Prim}(A).$$

Notice that $T_n \subset O_n \subset F_n$; we have

$$\pi(x_n) = f(\pi) \quad \text{for all } \pi \in \text{Prim}(A).$$

Hence $x_n$ is in the center of $A$. Moreover, $\{x_n\}$ forms an approximate identity for $A$ satisfying

$$x_{n+1}x_n = x_nx_{n+1} = x_n \quad \text{for all } n.$$

The converse follows from (ii) of 6.1.

6.9. **Proposition.** Every homomorphic image of a pseudo-commutative $C^*$-algebra $A$ is pseudo-commutative.

**Proof.** Let $\phi$ be a homomorphism of $A$, $B = \phi(A)$, and $B_{00} = \phi(A_{00})$. Clearly, by (ii) of 6.1, for every $n$, $\phi(e_n)\phi(a) = \phi(e_n)\phi(a)\phi(e_{N(n)})$ for every $a \in A$. Thus $B$ is also a pseudo-commutative $C^*$-algebra.

6.10. **Theorem.** Suppose that $A$ is a $\sigma$-unital $C^*$-algebra with continuous trace. Then $A$ is pseudo-commutative if and only if $A$ is a locally trivial continuous field of matrix algebras.

**Proof.** Assume that $A$ is a pseudo-commutative $C^*$-algebra. Since $A$ has continuous trace, $\hat{A}$ is a locally compact Hausdorff space. Fix $\pi \in A$. Let $F$ be a compact (hence closed) neighborhood of $\pi$. Let $I = \{ a : a \in A, \pi(a) = 0 \}$, and $\phi$ be the canonical homomorphism from $A$ onto $A/I$. So $\phi(A)^*$ is compact. By the argument used in (iii) of 6.2 and 6.9, $\phi(A)$ has an identity. Thus, $\phi(A_n) = \phi(A)$ for some $n$. Let $a \in A_n$ such that $\pi(a_n) = 1$. Then $\pi(a_n) = 1$ for all $\pi \in F$. Since $A_n \subset K(A)$, $\text{Tr}(\pi(a_n))$ is continuous. So $\text{Tr}(\pi(a))$ is a constant in some neighborhood of $\pi$. This implies that $A$ is locally homogeneous of finite rank. By [7, Theorem 3.2], $A$ is a locally trivial continuous field of matrix algebras.
Now we assume that $A$ is a locally trivial continuous field of matrix algebras and $\{e_n\}$ is as usual. Denote

$$F_n = \{\pi \in \hat{A} : \pi(e_n) \geq 1/2n\}.$$

Then $F_n$ is compact. For each point $\pi \in F_n$, there is a neighborhood $U_\pi$ such that $A$ is trivial on $\overline{U}_\pi$, where $\overline{U}_\pi$ is the closure of $U_\pi$. We assume $U_\pi$ is compact. Thus there is an $a_\pi \in A_00(\{e_n\})$ such that $\rho(a_\pi) = 1$ for all $\rho \in \overline{U}_\pi$. Since $F_n$ is compact, we may assume that there are $\pi_1, \pi_2, \ldots, \pi_k$, such that $\bigcup_{i=1}^{k} U_{\pi_i} \supset F_n$. There is $m$, such that

$$\|\pi(e_m) - 1\| < 1/2 \quad \text{for } i = 1, 2, \ldots, k.$$

So

$$\|\pi(e_m) - 1\| < 1/2 \quad \text{for all } \pi \in F_n.$$

Thus $\pi(A_m) = \pi(A)$ for each $\pi \in F_n$. Hence $\pi(e_{m+1}) = 1$ for each $F_n$. Now we can use the argument in 6.8 to construct a central approximate identity $\{a_n\}$ satisfying $a_{n+1}a_n = a_n a_{n+1} = e_n$. It follows then from 6.8 that $A$ is pseudo-commutative.

6.11. Examples. Clearly every $\sigma$-unital commutative $C^*$-algebra is pseudo-commutative.

Let $X$ be a locally compact and $\sigma$-compact Hausdorff space, and let $B$ be a unital $C^*$-algebra. Let $A$ be $C_0(X, B)$, the set of continuous mappings from $X$ into $B$ vanishing at infinity. It is easy to check that $A$ has a central approximate identity $\{e_n\}$ such that $e_{n+1}e_n = e_ne_{n+1} = e_n$. So $A$ is pseudo-commutative.

7. Singly supported $C^*$-algebras

7.1. We see from 6.7 that a pseudo-commutative $C^*$-algebra has a unique support algebra. It is evident that this may not be true for other $C^*$-algebras. But must every two support algebras of a given $C^*$-algebra be $*$-isomorphic?

7.2. Definition. We say that a $\sigma$-unital $C^*$-algebra is singly supported if every two support algebras are $*$-isomorphic.

7.3. Corollary. Every pseudo-commutative $C^*$-algebra is singly supported.

7.4. Theorem. Let $A$ be a $C^*$-algebra with approximate identities $\{e_n\}$ and $\{p_n\}$. Suppose that $e_n$ and $p_n$ are projections and

$$A_00 = \bigcup_{n=1}^{\infty} e_n A e_n, \quad A'_00 = \bigcup_{n=1}^{\infty} p_n A p_n.$$

Then there is a unitary $u \in M(A)$ (the multiplier algebra of $A$) such that $u^* A_00 u = A'_00$. 
Proof. We claim that there are subsequences \( \{ e_{n(k)} \} \) of \( \{ e_n \} \) and \( \{ p_{m(i)} \} \) of \( \{ p_n \} \), elements \( \{ f_k \} \), \( \{ f'_k \} \), \( \{ q_k \} \), \( \{ q'_k \} \), \( \{ v_k \} \), and \( \{ w_k \} \) in \( A \), and unitary elements \( \{ u_k \} \) and \( \{ \bar{u}_k \} \) in \( M(A) \) satisfying the following:

(i) \( f_k, f'_k, q_k, q'_k \) are projections in \( A \), where \( f_k, q'_k \in A_{00} \) and \( q_k, f'_k \in A'_{00} \).
(ii) \( f_if_j = 0, f_i f'_j = 0, q_i q_j = 0, \) and \( q_i q'_j = 0 \) if \( i \neq j \).
(iii) \( q'_i f_i = f'_i q'_i = 0 \) and \( q_i f'_i = f'_i q_i = 0 \) for all \( i \) and \( k \).
(iv) \( e_1 = f_1 \) and \( \sum_{i=1}^{k} f_i + \sum_{i=1}^{k} q'_i = e_{n(k)} \).
(v) \( p_{mk} = \sum_{i=1}^{k} q_i + \sum_{i=1}^{k} f'_i \).
(vi) \( u_k e_{n(k)} u_k = \sum_{i=1}^{k} q_i + \sum_{i=1}^{k} f'_i \) and \( u_k^* p_{m(k)} u_k = \sum_{i=1}^{k} f_i + \sum_{i=1}^{k} q'_i \).
(vii) \( v_k^* v_k = f_k, v_k v_k^* = f'_k, u_k^* w_k = q'_k, \) and \( w_k w_k^* = q_k \).

We shall use induction.

Since \( A_{00} \) is dense in \( A \), there is a selfadjoint element \( a \in A'_{00} \) such that \( \| a - e_1 \| < 1/8 \). We may assume that \( a \in p_n A p_n \) for some \( n(1) \). By [5, Lemma A.8.1], there is a projection \( f'_1 \in p_{n(1)} A p_{n(1)} \) such that

\[ \| f'_1 - e_1 \| < 1/4. \]

By [5, Lemmas A.8.1 and A.8.3], there is \( v_1 \in A \) such that \( \| v_1 - e_1 \| < 1/2 \), \( v_1^* v_1 = e_1 \), and \( v_1 v_1^* = f'_1 \), and there is a unitary element \( u_1 \in M(A) \) such that \( u_1^* e_1 u_1 = f'_1 \) and \( u_1^* f'_1 u_1 = e_1 \).

Let \( q_1 = p_{n(1)} - f'_1 \). Then \( u_1^* q_1 u_1 \in (1 - e_1)A(1 - e_1) = (1 - f_1)A(1 - f_1) \).

Since \( (1 - e_1)A_{00}(1 - e_1) \) is dense in \( (1 - e_1)A(1 - e_1) \), by the above argument there is a projection \( q'_1 \in (1 - e_1)A_{00}(1 - e_1) \) such that

\[ \| q'_1 - u_1^* q_1 u_1 \| < 1/4. \]

By [5, Lemmas A.8.1 and A.8.3], there is a \( w'_1 \in (1 - e_1)A(1 - e_1) \) such that \( (w'_1)^* (w'_1) = q'_1 \), \( w'_1 w'_1^* = u_1^* q_1 u_1 \), and

\[ \| w'_1 - q'_1 \| < 1/2. \]

Moreover there is a unitary \( u' \) in \( (1 - e_1)M(A)(1 - e_1) \) such that \( u'(w'_1)^* u' = u_1^* q_1 u_1 \) and

\[ (u')^* (u_1^* q_1 u_1)(u') = q'_1. \]

Let \( w_1 = u_1 w'_1 \) and \( \bar{u}_1 = (1 - f'_1) u_1 u' + f'_1 u_1 \). Then \( w^* w_1 = q' \), \( (w_1)(w_1)^* = q'_1 \), and \( \bar{u}_1 \) is a unitary in \( M(A) \) such that

\[ \bar{u}_1^* p_{n(1)} \bar{u}_1 = e_1 + q'_1 = f_1 + q'_1. \]

Now we assume that we have chosen \( e_{n(i)} \), \( p_{m(i)} \), \( f_i \), \( f'_i \), \( q_i \), \( q'_i \), \( v_i \), \( w_i \), \( u_i \), \( \bar{u}_i \), \( i = 1, 2, \ldots, k \). Suppose that \( q'_k \in e_{n(k+1)} A e_{n(k+1)} \) and let

\[ f_{k+1} = e_{n(k+1)} - \left( \sum_{i=1}^{k} f_i \sum_{i=1}^{k} q_i' \right) \].

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Then \( \overline{u}_k f_{k+1} \overline{u}_k^* \in (1 - p_{n(k)})A(1 - p_{n(k)}) \). Since \( (1 - p_{n(k)})A_{00}(1 - p_{n(k)}) \) is dense in \( (1 - p_{n(k)})A(1 - p_{n(k)}) \), there is a projection \( f'_{k+1} \in (1 - p_{n(k)})A'_{00}(1 - p_{n(k)}) \) (\( \subset A'_{00} \)) such that
\[
\| f'_{k+1} - \overline{u}_k f_{k+1} \overline{u}_k^* \| < 1/4.
\]

By [5, Lemmas A.8.1 and A.8.3], there is \( v'_{k+1} \in (1 - p_{n(k)})A'_{00}(1 - p_{n(k)}) \) such that
\[
(v'_{k+1})^*(v'_{k+1}) = f'_{k+1}, \quad (v'_{k+1})(v'_{k+1})^* = \overline{u}_k f_{k+1} \overline{u}_k^*,
\]
and a unitary \( u'_1 \in (1 - p_{n(k)})M(A)(1 - p_{n(k)}) \) such that
\[
(u'_1)(u'_1)^* = \overline{u}_k f_{k+1} \overline{u}_k^*.
\]

Define \( v_{k+1} = v'_{k+1} \overline{u}_k \) and
\[
u_{k+1} = (u'_1)^* \overline{u}_k \left( 1 - \sum_{i=1}^{k} f_i - \sum_{i=1}^{k} q'_i \right) + \overline{u}_k \left( \sum_{i=1}^{k} f_i + \sum_{i=1}^{k} q'_i \right).
\]

Then \( v_{k+1}^* v_{k+1} = f_{k+1} \), \( v_{k+1} v_{k+1}^* = f_{k+1} \), and
\[
u_{k+1}^* e_{n(k+1)}^* v_{k+1}^* = \sum_{i=1}^{k} q_i + \sum_{i=1}^{k+1} f_i.
\]

Let
\[
q_{k+1} = p_{m(k+1)} - \left( \sum_{i=1}^{k} q_i + \sum_{i=1}^{k+1} f_i \right)
= p_{m(k+1)} - u_{k+1} e_{n(k+1)}^* u_{k+1}^*.
\]

Then
\[
u_{k+1}^* q_{k+1}^* u_{k+1} \in (1 - e_{n(k+1)})A(1 - e_{n(k+1)}).
\]

Since \( (1 - e_{n(k+1)})A_{00}(1 - e_{n(k+1)}) \) is dense in \( (1 - e_{n(k+1)})A(1 - e_{n(k+1)}) \), there is a projection \( q'_{k+1} \in (1 - e_{n(k+1)})A_{00}(1 - e_{n(k+1)}) \) (\( \subset A'_{00} \)) such that
\[
\| q'_{k+1} - \nu_{k+1}^* q_{k+1} u_{k+1} \| < 1/4.
\]

By [5, Lemmas A.8.1 and A.8.3], there is a \( w'_{k+1} \in (1 - e_{n(k+1)})A(1 - e_{n(k+1)}) \) such that \( (w'_{k+1})^*(w'_{k+1}) = q'_{k+1} \), \( (w'_{k+1})(w'_{k+1})^* = u_{k+1}^* q_{k+1} u_{k+1} \), and
\[
\| w_{k+1}^* - q'_{k+1} \| < 1/2.
\]

Moreover, there is a unitary \( u'_2 \) in \( (1 - e_{n(k+1)})M(A)(1 - e_{n(k+1)}) \) such that
\[
(u'_2)^* q_{k+1}^* (u'_2) = u_{k+1}^* q_{k+1} u_{k+1}^* u_{k+1}
\]
and
\[
(u'_2)^* (u_{k+1}^* q_{k+1} u_{k+1})(u'_2) = q'_{k+1}.
\]
Define \( w_{k+1} = u_{k+1}w_{k+1}^\prime \) and
\[
\overline{u}_{k+1} = (1 - u_{k+1}e_{n(k+1)}u_{k+1}^\prime)u_{k+1}u_{k+1}^\prime + u_{k+1}e_{n(k+1)}u_{k+1}^\prime.
\]
Then \( w_{k+1}^*w_{k+1} = q_{k+1} \), \( w_{k+1}w_{k+1}^* = q_{k+1} \), and
\[
\overline{u}_{k+1}^*p_{m(k+1)} u_{k+1} = \sum_{i=1}^{k+1} f_i' + \sum_{i=1}^{k+1} q'_i.
\]
This completes the induction.

Now we define
\[
u = \sum_{k=1}^{\infty} v_k + \sum_{k=1}^{\infty} w_k.
\]
It is easily checked that \( v \) is a unitary in \( M(A) \) and
\[
u^* e_n^k A e_n^k v = (f_n' + p_{m(k-1)}) A (f_n' + p_{m(k-1)})
\]
if \( k \geq 2 \). Thus
\[
u^* A_{00} v = A'_{00}.
\]

7.5. Let \( A \) be a \( C^* \)-algebra. We denote by \( \text{Aut}(A) \) the automorphism group of \( A \). If \( \nu \) is a unitary in \( M(A) \), we denote the automorphism \( a \to \nu^* a \nu \) by \( \text{aut}(\nu) \).

7.6. Corollary. Let \( A \) be a \( C^* \)-algebra with an approximate identity \( \{e_n\} \) consisting of projections. Define
\[
G = \{ \rho \in \text{Aut}(A): \rho(A_{00}(\{e_n\})) = A_{00}(\{e_n\}) \}.
\]
Then for every \( \phi \in \text{Aut}(A) \) there are a unitary element \( u \in M(A) \) and \( \rho \in G \) such that \( \phi = \text{aut}(u) \circ \rho \).

Proof. Let \( A'_{00} = \phi(A_{00}(\{e_n\})) \). It follows from 7.4 that there is a unitary \( u \in M(A) \) such that
\[
u(A_{00}) u^* = A_{00}.
\]
Thus \( \rho = \text{aut}(u^*) \circ \phi \in G \). Hence \( \phi = \text{aut}(u) \circ \rho \).

7.7. Recall that a \( C^* \)-algebra \( A \) is called scattered if every state of \( A \) is atomic, equivalently, if \( A \) has a composition series with elementary quotients (cf. [9, and 10]).

7.8. Theorem. Every \( \sigma \)-unital scattered \( C^* \)-algebra is singly supported.

Proof. It follows from [13, Lemma 5.1; 5, Lemma 9.4] that \( A \) has a support algebra \( A_{00} = \bigcup_{n=1}^\infty e_n^k A e_n^k \), where the \( e_n^k \) are projections in \( A \). Let \( a \) be any strictly positive element of \( A \) and \( A'_{00} = A_{00}(a) \). By [12], \( \text{Sp}(a) \) is countable. Thus there are \( t_n, 0 < t_n < 1 \), such that \( t_n \downarrow 0 \) and \( \chi_{(t_n, [a])}(a) \) is in \( A \). Let \( p_n = \chi_{(t_n, [a])}(a) \). Then
\[
A'_{00} = \bigcup_{n=1}^\infty p_n A p_n.
\]
By 7.6, \( A_{00} \) and \( A'_{00} \) are isomorphic.
7.9. Let \( A \) be a \( \sigma \)-unital \( C^* \)-algebra and \( e_n, p_n \) be as in 2.1. Let \( B^{**} \) be the enveloping Borel *-algebra of \( A \). We denote the norm closure of \( \bigcup_{n=1}^{\infty} p_n B^{**} p_n \) by \( B_0(A) \). Clearly \( B_0(A) \) is a \( \sigma \)-unital \( C^* \)-algebra. It follows from [15, Theorem 3.7] that \( B_0(A) \) does not depend on the choices of \( \{e_n\} \). We denote the norm closure of \( \bigcup_{n=1}^{\infty} p_n A^{**} p_n \) by \( M_0(A) \). Then \( M_0(A) \) is a \( \sigma \)-unital \( C^* \)-algebra. By [15, Theorem 3.7], \( M_0(A) \) is the hereditary \( C^* \)-subalgebra of \( A^{**} \) generated by \( A \), hence it does not depend on the choices of \( \{e_n\} \).

7.10. **Theorem.** For every \( \sigma \)-unital \( C^* \)-algebra \( A \), \( B_0(A) \) and \( M_0(A) \) are singly supported.

**Proof.** Clearly, \( \bigcup_{n=1}^{\infty} p_n B^{**} p_n \) is a support algebra of \( B_0(A) \). Take any strictly positive element \( x \) of \( B_0(A) \). By [15, Corollary 3.9], for every \( n \), \( \chi_{(1/n, \|x\|)}(x) \in B_0(A) \). Let \( q_n = \chi_{(1/n, \|x\|)}(x) \). Then the support algebra associated with the strictly positive element \( x \) is \( \bigcup_{n=1}^{\infty} q_n B^{**} q_n \). By 7.6, \( B_0(A) \) is singly supported.

The proof for \( M_0(A) \) is similar.

7.11. **Corollary.** Let \( A \) be a \( \sigma \)-unital \( C^* \)-algebra, and let \( A_{00} \) and \( A'_{00} \) be two support algebras of \( A \). Then \( \text{QM}(A_{00})'' \) is isomorphic to \( \text{QM}(A'_{00})'' \).

**Proof.** By 7.10, \( M_0(A) \) is singly supported. Therefore (up to isomorphism) there is only one quasi-multiplier space for supported algebras of \( M_0(A) \). It follows from 5.9 that \( \text{QM}(A_{00})'' \) is isomorphic to \( \text{QM}(A'_{00})'' \).

7.12. The algebras in 7.8 and 7.10 have a rich structure of projections. Projectionless singly supported \( C^* \)-algebras can be found in pseudo-commutative \( C^* \)-algebras. The following is an example of a projectionless singly supported \( C^* \)-algebra which is not pseudo-commutative.

7.13. Let \( B \) be a separable nonelementary simple AF \( C^* \)-algebra with unique trace \( \tau \). Suppose that \( p \) is a nonzero projection of \( B \). Then \( pbp \cong B \) (see [2]). Let \( \sigma \) be a nonzero endomorphism of \( B \), and \( A \) be the set of continuous functions from \([0, 1] \) into \( B \) such that \( f(1) = \sigma(f(0)) \). We assume that \( \sigma(1) = p \neq 0 \). By [2], \( A \) has no nonzero projections. \( A \) is nonunital but is a \( \sigma \)-unital \( C^* \)-algebra. Moreover, \( \text{Prim}(A) \) is homeomorphic to the unit circle. It follows from 6.3 that \( A \) is not pseudo-commutative.

Suppose that \( \sigma(B) = pBp \) for some nonzero projection \( p \) in \( B \). Let

\[
e_n = \begin{cases} 1 & \text{if } 1/n < t \leq 1; \\ p + n(n+1)(t-1/n+1)(1-p) & \text{if } 1/n + 1 \leq t \leq 1/n; \\ p & \text{if } 0 \leq t < 1/n + 1. \\
\end{cases}
\]

Then \( \{e_n\} \) forms an approximate identity for \( A \), and \( e_n e_{n+1} e_n = e_n e_{n+1} = e_n \) for all \( n \).

Let \( A = [e_n] A^{**} [e_n] \cap A \) and \( A_{00} = \bigcup_{n=1}^{\infty} A_n \).
Suppose that \( \{b_n\} \) is another approximate identity for \( A \) satisfying \( b_n b_{n+1} = b_n \) for all \( n \). Define \( A' = [b_n]A^* [b_n] A \) and \( A'_{00} = \bigcup_{n=1}^{\infty} A'_n \). For each \( n \), there is an \( m(n) \) such that \( \|b_m(t)e_n(t) - e_n(t)\| < 1/2 \) for all \( m \geq m(n) \) and \( t \in [0, 1] \). Thus, if \( m \geq m(n) \), \( \|b_m(t) - 1\| < 1/2 \) for all \( t \in [1/n, 1] \) and \( \|b_m(0) - p\| < 1/2 \). So if \( m \geq m(n) \), \( b_m(t) = 1 \) if \( t \in [1/n, 1] \) and \( b_m(0) = p \).

Without loss of generality we may assume that \( b_n(t) = 1 \) if \( t \in [1/n, 1] \) and \( b_n(0) = p \) for all \( n \). For each \( n \), there is a number \( \alpha_n > 0 \) such that \( \|b_n(t) - p\| < 1/4 \) and \( \|b_n(t) - p\| < 1/4 \) for \( 0 \leq t < \alpha_n \). Thus \( \text{Sp}(b_n(t)) \subseteq [0, 1/4] \cup [3/4, 1] \) and \( \text{Sp}(b_{n+1}(t)) \subseteq [0, 1/4] \cup [3/4, 1] \) for all \( 0 \leq t < \alpha_n \).

The characteristic function \( \chi = \chi_{[1/4, 1]} \) is continuous on \( \text{Sp}(b_n(t)) \) and \( \text{Sp}(b_{n+1}(t)) \) for \( 0 \leq t < \alpha_n \), and thus \( q_1 = \chi(b_n) \) and \( q_2 = \chi(b_{n+1}) \) are continuous on \( [0, \alpha_n) \). Moreover,

\[
\|q_1(t) - p\| < 1/2, \quad \|q_2(t) - p\| < 1/2 \quad \text{if} \quad 0 \leq t < \alpha_n.
\]

Clearly,

\[
q_2(t) \geq [b_n(t)] \geq q_1(t).
\]

Since \( \tau(q_2(t)) = \tau(q_1(t)) \) for \( 0 \leq t < \alpha_n \), we conclude that

\[
q_2(t) = [b_n(t)] = q_1(t) \quad \text{for} \quad 0 \leq t < \alpha_n.
\]

Furthermore, since \( b_n \) is increasing,

\[
[b_{n+k}(t)] = [b_n(t)] \quad \text{if} \quad 0 \leq t < \min(\alpha_n, \alpha_{n+k}).
\]

Let \( A_1 \) be the \( C^* \)-algebra \( A|[0, (1/2)\alpha_1] \). Since \( [b_1(t)] = \chi_{[b_1(t)]} \) for \( t \in [0, (1/2)\alpha_1] \),

\[
a_1 = [b_1(t)]_{[0, (1/2)\alpha_1]} \in A_1.
\]

Put \( q(t) = p \) for all \( t \in [0, (1/2)\alpha_1] \). Then \( q(t) \in A_1 \). By [5, Corollary A.8.3], there is a unitary \( u_1 \in M(A_1) \) such that

\[
u_1^*a_1u_1 = a_1 \quad \text{and} \quad u_1a_1u_1^* = q.
\]

Define

\[
u = \begin{cases} 1, & t = 0; \\
u_1(t), & 0 < t \leq (1/2)\alpha_1; \\
u_1(\alpha_1 - t), & (1/2)\alpha_1 < t \leq \alpha_1; \\
1, & \alpha_1 < t \leq 1.
\end{cases}
\]

It is easy to verify that \( u \) is a unitary in \( M(A) \). Moreover, \( ub_nu^* \leq e_N \) and \( u e_nu \leq b_n \), where \( N > n \) and \( 1/N \leq (1.2)\alpha_n \).

We conclude that

\[
u^*A_{00}u = A'_{00}.
\]

So \( A \) is a singly supported \( C^* \)-algebra.

7.14. We denote \( K_0 = \{a \in A_+ : \text{there is a } b \in (A_+) \text{ such that } [a] \leq b\} \).

The following result may help to find a separable \( C^* \)-algebra which is not singly supported.
7.15. Theorem. Let $A$ be a separable $C^*$-algebra with an approximate identity consisting of projections. Suppose that $A$ is singly supported. Then

$$K_0^+ = \{ a \in A_+: a \leq p, \ p \ a \ projection \ in \ A \}.$$ 

Proof. Suppose that $a$ is a nonzero element in $K_0^+$ but no projection in $A$ majorizes $a$. Let $b$ be an element in $(A_+)_1$ such that $0 \leq [a] \leq b \leq 1$. Let $B$ be the norm closure of $(1-b)A(1-b)$ and $a'$ be a strictly positive element of $B$. We may assume that $0 \leq a' \leq 1$. Put $e = a' + b$. Then $e$ is a strictly positive element of $A$. Since $a'[a] = [a]a' = 0$, it follows from Lemma 2.6 that $[a]e = e[a]$. By considering the abelian $C^*$-algebra generated by $e$, $[a]$, and 1, we obtain

$$p_n = \chi_{(1/n, e]}(e) \geq [a].$$

Thus $a \in \bigcup_{n=1}^\infty p_nA^{**}p_n \cap A$. We also notice that $A_{00} = \bigcup_{n=1}^\infty p_nA^{**}p_n \cap A$ is a support algebra of $A$.

Suppose that $A_{00}$ is a support algebra of $A$ associated with an approximate identity $\{e_n\}$ consisting of projections. Since $A$ is singly supported, there is an isometry $\phi$ such that $\phi(A_{00}) = A_{00}$. Thus we may assume that $\phi(a) \leq e_k$ for some $k$. Then $\phi^{-1}(e_k) \geq a$ and $\phi^{-1}(e_k)$ is a projection. A contradiction.

7.16. To conclude the paper, we state the following questions.

1. Is $QM(A_{00})$ the linear span of its positive cone?
2. Is every $\sigma$-unital $C^*$-algebra singly supported?
3. Let $A$ be a $\sigma$-unital $C^*$-algebra. We denote by $s(A)$ the number of nonisomorphic support algebras of $A$. For every $n$, is there a $\sigma$-unital $C^*$-algebra $A$ such that $s(A) = n$?
4. Are the dual $C^*$-algebras the only $C^*$-algebras which have reflexive quasi-multipliers?
5. Does every pseudo-commutative $C^*$-algebra have a central approximate identity?

References


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