ALTERNATING SEQUENCES AND INDUCED OPERATORS

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Abstract. We show that when a positive $L_p$ contraction is equipped with a norming function having full support, then it is related in a natural way to an operator on any other $L_p$ space, $1 < p < \infty$. This construction is used to generalize a theorem of Rota concerning the convergence of alternating sequences.

1. Introduction

Let $L_p$ be the usual Banach space of complex-valued functions. Denote by $L_p^+$ the class of $L_p$ functions taking nonnegative values. An $L_p$ operator $T$ is positive if $TL_p^+ \subseteq L_p^+$. It is a contraction if $\|Tf\|_p \leq \|f\|_p$ for every $f \in L_p$. We say $u$ is semi-invariant for a positive $L_p$ contraction $T$ if both $u$ and $Tu$ have full support and $\|Tu\|_p = \|u\|_p$.

(1.1) Theorem. Suppose $1 < p < \infty$ and $1 < r < \infty$. If $T$ is a positive $L_p$ contraction with a semi-invariant function $u$, then the formula

$$T_rf = (Tu)^{p/r-1}T(u^{1-p/r}f),$$

where $f \in L_r$, defines a positive $L_r$ contraction. This operator is independent of the choice of semi-invariant function. We call $T_r$ the $L_r$ operator induced by $T$.

We apply this notion of induced operators to the question of convergence of alternating sequences. For simplicity of notation, the following theorem is stated for $L_p^+$ only. The analogous result is proved for all of $L_p$. $T^*$ denotes the adjoint of $T$; it is an operator on $L_q$ where $q = p(p - 1)^{-1}$. Whenever $u$ is semi-invariant for an $L_p$ operator $T$, then $(Tu)^{p-1}$ is semi-invariant for $T^*$.

(1.2) Theorem. Suppose $1 < p < \infty$ and $1 < r < \infty$. Let $(T_n)_{n=1}^\infty$ be a sequence of positive $L_p$ contradictions with semi-invariant functions defined over a $\sigma$-finite
Lebesgue space. Then
\[(T_1^*), \ldots, (T_n^*), (T_1 \cdots T_n f)^{p/r}\]
converges a.e. for every \( f \in L^+_p \).

This theorem generalizes Rota’s theorem of the alternating procedure [Rt]. We say an operator is bistochastic if \( T1 = T^*1 = 1 \), where 1 is the function taking the value 1 everywhere.

(1.3) **Theorem (Rota).** If \( (T_n)_{n=1}^\infty \) is a sequence of positive bistochastic operators over a probability space, then
\[(1.4) T_1^* \cdots T_n^* T_n \cdots T_1 f \]
converges a.e. for every \( f \in L_p, \) where \( 1 < p < \infty \).

A positive bistochastic operator is a contraction of every \( L_p \), where \( 1 \leq p \leq \infty \); thus the expression (1.4) is well defined for every \( p \). A positive \( L_p \) contraction with a semi-invariant function does not necessarily have this property, but we may use the operator induced by \( T^* \) to define a “pseudo-adjoint” of \( T \) which operates on \( L_p \).

In the finite measure case, 1 is semi-invariant for any bistochastic operator and for its adjoint. Furthermore, \( T_r^* = T^* \) for any \( r, \) \( 1 < r < \infty \). Thus, Rota’s theorem is a consequence of (1.1) with \( r = p \).

2. Preliminaries

(2.1) **Definitions.** For any \( \sigma \)-finite measure space \((X, \mathcal{F}, \mu)\), let \( \mathcal{M}(d\mu) \) be the vector space of \( \mathcal{F} \)-measurable complex-valued functions defined on \( X \). Let \( \mathcal{M}^+(d\mu) \) be the class of functions in \( \mathcal{M}(d\mu) \) whose ranges are subsets of \( \mathbb{R}^+ = [0, \infty) \). Let \( \overline{\mathcal{M}}^+(d\mu) \) be the set of \( \mathcal{F} \)-measurable functions on \( X \) with values in the extended nonnegative reals, \( [0, \infty] \).

The usual Banach space of functions in \( \mathcal{M}(d\mu) \) for which \( \int_X |f|^p d\mu < \infty \) is denoted by \( L_p(d\mu) \), where \( 1 \leq p < \infty \), while \( L_\infty(d\mu) \) denotes the space of essentially bounded functions \( \mathcal{M}(d\mu) \). We also use \( L^+_p(d\mu) = L_p(d\mu) \cap \mathcal{M}^+(d\mu) \). All of the relations between the functions in these classes are in the \( \mu \)-a.e. sense, even when this is not made explicit. With the convention \( 0 \cdot \infty = 0 \), functions in \( \overline{\mathcal{M}}^+(d\mu) \) may be multiplied pointwise.

Let \((Y, \mathcal{G}, \nu)\) be another \( \sigma \)-finite measure space. Consider the class of all mappings
\[ T : \overline{\mathcal{M}}^+(d\mu) \rightarrow \overline{\mathcal{M}}^+(d\nu) \]
which satisfy the following two conditions:

(2.2) \( T \) is “positive-linear”; that is, if \( \alpha, \beta \in \mathbb{R}^+ \) and \( f, g \in \overline{\mathcal{M}}^+(d\mu) \), then
\[ T(\alpha f + \beta g) = \alpha Tf + \beta Tg. \]
(2.3) T is “order-continuous” in the sense that \( T f_n \uparrow T f \) \( \nu \)-a.e. whenever \( f_n \uparrow f \) \( \mu \)-a.e. (the arrows indicate monotone nondecreasing pointwise convergence in \( \mathbb{R}^+ \)).

If \( T \) is such a mapping, then its restriction to \( \mathcal{M}^+(d\mu) \) need not be extendable linearly to \( \mathcal{M}(d\mu) \). Thus, these mappings should not necessarily be associated with the usual class of linear operators. Nonetheless, it is convenient to make the following definition.

(2.4) Definition A mapping satisfying (2.2) and (2.3) will be called a positive operator on \( \mathcal{M}^+(d\mu) \) (or from \( \mathcal{M}^+(d\mu) \) to \( \mathcal{M}^+(d\nu) \)).

(2.5) Lemma. Given a positive operator \( T : \mathcal{M}^+(d\mu) \to \mathcal{M}^+(d\nu) \) there exists a unique positive operator \( T^* : \mathcal{M}^+(d\nu) \to \mathcal{M}^+(d\mu) \) such that

\[
\int_X f T^* g \, d\mu = \int_Y T f \cdot g \, d\nu
\]

for every \( f \in \mathcal{M}^+(d\mu) \) and \( g \in \mathcal{M}^+(d\nu) \).

Proof. Given \( g \in \mathcal{M}^+(d\nu) \), the mapping

\[
f \in \mathcal{M}^+(d\mu) \mapsto \int_Y T f \cdot g \, d\nu \in \mathbb{R}^+
\]

is integration with respect to some measure on \( (X, \mathcal{F}) \) which is absolutely continuous with respect to \( \mu \). This measure may be represented as \( \rho \, d\mu \) for some \( \rho \in \mathcal{M}^+(d\mu) \). Define \( T^* \) by \( T^* g = \rho \). \( \square \)

(2.6) Definition. The operator \( T^* \) defined above is called the adjoint of \( T \).

If \( T : L^p(d\mu) \to L^p(d\nu) \) is a positive operator in the usual sense, then its restriction to \( L^p(d\mu) \) can be extended to a positive operator on \( \mathcal{M}^+(d\mu) \), which will also be called \( T \). It is unique by the requirement that it satisfy (2.3). If a positive operator on \( \mathcal{M}^+(d\mu) \) in the sense of (2.4) can be obtained in this way, then we will call it a positive \( L^p \) operator on \( \mathcal{M}^+(d\mu) \). The following definition states this in a different way.

(2.7) Definition. A positive operator \( T \) on \( \mathcal{M}^+(d\mu) \) is said to be a positive \( L^p \) operator if

\[
\|T\|^p_p = \sup \left\{ \int (T f)^p \, d\nu \left| f \in \mathcal{M}^+(d\mu) \text{ and } \int f^p \, d\mu \leq 1 \right. \right\}
\]

is finite. If, furthermore, \( \|T\|_p \leq 1 \), then \( T \) is called a positive \( L^p \) contraction.

Throughout this paper, whenever a number \( p \) with \( 1 < p < \infty \) is understood, then \( q \) denotes the adjoint index; that is, the number \( p(p - 1)^{-1} \). Note that \( T \) is a positive \( L^p \) operator if and only if \( T^* \) is a positive \( L^q \) operator. In this case, the definition of the adjoint operator agrees with the usual definition in the Banach space sense.

The following theorem is a standard result. Under the hypothesis one easily shows that the operator is a contraction of both \( L_1 \) and \( L_\infty \). The conclusion then follows by the Riesz convexity theorem.
(2.8) **Theorem.** Let $T$ be a positive operator such that $T1 \leq 1$ and $T^*1 \leq 1$. Then $T$ is a positive $L_p$ contraction for all $p$, $1 \leq p \leq \infty$.

(2.9) **Definition.** If $T$ is a positive $L_p$ operator and $u \in L_p$ is a function satisfying $\|Tu\|_p = \|T\|_p \|u\|_p$, we say that $u$ is a norming function for $T$. We say that $u$ is semi-invariant for $T$ if $\|Tu\|_p = \|u\|_p$ and both $u$ and $Tu$ are strictly positive a.e. A semi-invariant function for a contraction is clearly a norming function.

(2.10) **Lemma.** If $u$ is a norming function for a positive $L_p$ operator $T$, then

$$\|T^*(Tu)^{p-1}\|_p = \|T\|_p \|u\|_p^{p-1}.$$  

Consequently, if $u$ is semi-invariant for a positive contraction $T$, then $(Tu)^{p-1}$ is semi-invariant for $T^*$.

**Proof.**

$$\|Tu\|_p^p = \int (Tu)(Tu)^{p-1} d\nu = \int uT^*(Tu)^{p-1} d\mu$$

$$\leq \|u\|_p \|T^*(Tu)^{p-1}\|_q \leq \|u\|_p \|T^*\|_q \|(Tu)^{p-1}\|_q$$

$$= \|u\|_p \|T\|_p \|Tu\|_p^{p-1} = \|Tu\|_p^p,$$

where the first inequality follows from Hölder's inequality. Thus, we have equality in Hölder's inequality, and so $T^*(Tu)^{p-1}$ is a constant multiple of $u^{p-1}$. □

(2.11) **Definition.** Suppose $T$ is a positive operator on $\mathcal{M}^+(d\mu)$. A set $E \in \mathcal{F}$ is called a reducing set for $T$ if $T(\chi_E) \cdot T(1 - \chi_E) = 0$, where $\chi_E$ is the characteristic function of the set $E$.

(2.12) **Lemma.** The support of a norming function is a reducing set.

**Proof.** Let $u$ be a norming function for $T$, and $E$ be the support of $u$. Then

$$\int (Tu)^{p-1}T(1 - \chi_E) d\nu = \int T^*(Tu)^{p-1}(1 - \chi_E) d\mu$$

$$= \|T\|_p^p \int u^{p-1}(1 - \chi_E) d\mu = 0.$$

Hence $(Tu)^{p-1}T(1 - \chi_E) = 0$, and so $(Tu)T(1 - \chi_E) = 0$. Now approximate $1/\chi_E$ from below by simple functions. Conclude by (2.3) and positivity that $T(\chi_E)T(1 - \chi_E) = 0$. □

The following lemma concerning functions of a real variable is needed. Observe that the conclusion of the lemma remains valid if we replace $\theta$ in the hypothesis by any differentiable function which is strictly monotone almost everywhere.

(2.13) **Lemma.** Let $\phi, \theta : \mathbb{R}^+ \to \mathbb{R}^+$ be measurable functions satisfying

$$\int_0^\infty \phi(t) dt = \int_0^\infty \theta(t) dt < \infty,$$

(2.14)

$$\int_0^\alpha \phi(t) dt \leq \int_0^\alpha \theta(t) dt.$$
and

\[(2.15) \quad \int_0^\infty t^{\alpha} \phi(t) \, dt = \int_0^\infty t^{\theta} \theta(t) \, dt\]

for every \( \alpha \geq 0 \) and some \( r > 0 \). Then \( \phi = \theta \) a.e.

**Proof.**

\[
\int_0^\infty t^{\alpha} \phi(t) \, dt = \int_0^\infty r s^{r-1} \left( \int_s^\infty \phi(t) \, dt \right) \, ds \\
\geq \int_0^\infty r s^{r-1} \left( \int_s^\infty \theta(t) \, dt \right) \, ds = \int_0^\infty t^{\theta} \theta(t) \, dt.
\]

By (2.15), we have equality. Thus, the set of points at which inequality (2.14) is strict has measure zero. Since

\[
\int_0^\alpha \phi(t) \, dt = \int_0^\alpha \theta(t) \, dt
\]

for a.a. \( \alpha \), and \( \phi \) and \( \theta \) are positive functions, it follows that \( \phi = \theta \) a.e., as desired. \( \square \)

(2.16) **Definition.** A point transformation \( \tau: X \rightarrow X \) is called an automorphism if it is invertible and both \( \tau \) and \( \tau^{-1} \) are measurable and nonsingular. An automorphism induces two measures, \( \mu \circ \tau^{-1} \) and \( \mu \circ \tau \), both absolutely continuous with respect to \( \mu \). Let \( \rho \) denote the Radon-Nikodým derivative of \( \mu \circ \tau^{-1} \) with respect to \( \mu \). If \( 1 \leq p < \infty \), then define \( Q: L_p \rightarrow L_p \) by

\[Qf = \rho^{1/p}(f \circ \tau^{-1})\]

for \( f \in L_p \). We call \( Q \) the \( L_p \) isometry induced by \( \tau \).

(2.17) **Lemma.** If \( Q \) is the \( L_p \) isometry induced by an automorphism \( \tau \), then \( Q^{-1} \) is the \( L_p \) isometry induced by \( \tau^{-1} \) and \( Q^* \) is the \( L_q \)-isometry induced by \( \tau^{-1} \).

**Proof.** This follows immediately from the definitions if one observes that when \( \rho \) is the Radon-Nikodým derivatives of \( \mu \circ \tau^{-1} \) with respect to \( \mu \), then the Radon-Nikodým derivatives of \( \mu \circ \tau \) with respect to \( \mu \) is \( 1/(\rho \circ \tau) \). \( \square \)

(2.18) **Definition.** Suppose \( 1 \leq p < \infty \) and \( 1 \leq r < \infty \). Define \( \psi_{p,r}: L_p \rightarrow L_r \) by means of the equation

\[ [\psi_{p,r}(f)](x) = \text{sign}(f(x)) |f(x)|^{p/r}, \]

where \( \text{sign}(z) \) is the complex number of unit modulus having the same argument as \( z \). When \( p \) and \( r \) are understood, we refer to this embedding simply as \( \psi \). Usually \( f^* \) is used to represent \( \psi_{p,q}f \). Perhaps the most important property of \( \psi_{p,r} \) is that when \( f \in L_p \), then \( \|\psi_{p,r}f\|_r = \|f\|_p^{p/r} \).
(2.19) **Lemma.** Let $1 < p < \infty$ and $1 < r < \infty$. Suppose $Q_p$ and $Q_r$ are, respectively, the $L_p$ and $L_r$ isometries induced by an automorphism $\tau$. If $\psi = \psi_{p,r}$ and $f \in L_p$, then

$$Q_r \psi f = \psi Q_p f.$$ 

**Proof.**

$$Q_r \psi f = \rho^{1/r} \left| \text{sign}(f) \right| f^{p/r} \circ \tau^{-1}$$

$$= \text{sign}(f \circ \tau^{-1}) \rho^{1/r} \left| f \circ \tau^{-1} \right|^{p/r}$$

$$= \text{sign} \left[ \rho^{1/p} (f \circ \tau^{-1}) \right] \rho^{1/p} (f \circ \tau^{-1})^{p/r}$$

$$= \psi Q_p f. \quad \square$$

(2.20) **Definition.** When $(X, \mathcal{F}, \mu)$ is a measure space and $\mathcal{F}'$ is a sub-$\sigma$-algebra of $\mathcal{F}$, then $E(\cdot | \mathcal{F}')$ denotes the conditional expectation operator with respect to $\mathcal{F}'$. We adopt the convention that $E(f | \mathcal{F}')$ is 0 on any atom of $\mathcal{F}'$ of infinite measure.

(2.21) **Theorem (Martingale convergence theorem for finite $\sigma$-algebras).** Let $(X, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space. For each $k \geq 1$, suppose $\mathcal{F}_k$ is a finite sub-$\sigma$-algebra of $\mathcal{F}$ and $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$. Let $\mathcal{F}_\infty = \sigma(\bigcup_{k=1}^{\infty} \mathcal{F}_k)$, the smallest $\sigma$-algebra containing the algebra $\bigcup_{k=1}^{\infty} \mathcal{F}_k$. Suppose $1 \leq p < \infty$ and $f \in L_p(d\mu)$. Let $f_k = E(f | \mathcal{F}_k)$ for $1 \leq k \leq \infty$. Then $f_k \to f$ a.e. and in $L_p$ norm.

If $p > 1$, then the $f_k$'s have a maximal function; more precisely, there is a function $g \in L_p^+$ with $|f_k| \leq g$ for every $k \geq 1$, and $\|g\|_p \leq q\|f\|_p$.

**Proof.** See any reference on martingales, e.g. [S, pp. 89-94].

(2.22) **Lemma.** Let $(\mathcal{G}_k)_{k=1}^{\infty}$ be as in the previous theorem and suppose $(\mathcal{H}_k)_{k=1}^{\infty}$ is another monotone sequence of finite sub-$\sigma$-algebras of $\mathcal{F}$. Let

$$\mathcal{H}_\infty = \sigma \left( \bigcup_{k=1}^{\infty} \mathcal{H}_k \right).$$

Let $f \in L_p^+(d\mu)$, where $1 < p < \infty$, and $f_k = E(f | \mathcal{G}_k)$. Then

$$E(f_k^p | \mathcal{H}_k) \to E(f_\infty^p | \mathcal{H}_\infty)$$

a.e. and in $L_1$ norm.

**Proof.** Let $\phi_k = f_k^p$ for each $k \geq 1$. Then $g = \sup f_k \in L_p$ by the martingale convergence theorem. Thus $0 \leq \phi_k \leq \theta = g^p \in L_1$, and $\phi_k \to \phi_\infty$ a.e. The proof is then completed by the following more general lemma.

(2.23) **Lemma.** Let $0 \leq \phi_k \leq \theta \in L_1$ for $k \geq 1$, and let $\phi_k \to \phi_\infty$ a.e. Then $E(\phi_k | \mathcal{G}_k) \to E(\phi_\infty | \mathcal{G}_\infty)$ a.e. and in $L_1$ norm.

**Proof.** Let

$$\xi_k = \inf_{n \geq k} \phi_n \quad \text{and} \quad \eta_k = \sup_{n \geq k} \phi_n.$$
Then \((\eta_k - \xi_k) \downarrow 0\) a.e. and in \(L_1\) norm, by the dominated convergence theorem. We have, for any \(n \geq k\),
\[
E(\xi_k | \mathcal{F}_n) \leq E(\xi_n | \mathcal{F}_n) \leq E(\phi_n | \mathcal{F}_n) \\
\leq E(\eta_n | \mathcal{F}_n) \leq E(\eta_k | \mathcal{F}_n).
\]

If \(n \to \infty\) with \(k\) fixed, then
\[
E(\xi_k | \mathcal{F}_\infty) \leq \lim E(\phi_n | \mathcal{F}_n) \leq \lim E(\phi_n | \mathcal{F}_n) \leq E(\eta_k | \mathcal{F}_\infty).
\]

Thus
\[
\| \lim E(\phi_n | \mathcal{F}_n) - \lim E(\phi_n | \mathcal{F}_n) \|_1 \\
\leq \| E(\eta_k | \mathcal{F}_\infty) - E(\xi_k | \mathcal{F}_\infty) \|_1 \leq \| \eta_k - \xi_k \|_1
\]
which can be made arbitrarily small. This completes the proof. \(\square\)

We will need the following four lemmas from [AS2], where they are numbered (2.2), (2.3), (2.5), and (2.8) respectively. \(L_p\) always refers to the case \(1 < p < \infty\) over a \(\sigma\)-finite measure space.

(2.24) Lemma. Let \(f_k \in L_p\) for every \(k, 1 \leq k \leq n\). If \(V: L_p \to L_p\) is a positive bounded linear operator, then
\[
\max_{1 \leq k \leq n} |Vf_k| \leq V \left( \max_{1 \leq k \leq n} |f_k| \right)
\]
and, consequently,
\[
\left\| \max_{1 \leq k \leq n} |Vf_k| \right\|_p \leq \|V\| \cdot \left\| \max_{1 \leq k \leq n} |f_k| \right\|_p.
\]

(2.25) Lemma. For each \(\varepsilon > 0\) there is a \(\delta > 0\) such that if \(E: L_p \to L_p\) is a conditional expectation operator, \(f \in L_p\), and
\[
\|f\|_p - \|Ef\|_p < \delta \|f\|_p,
\]
then
\[
\|f - Ef\|_p < \varepsilon \|f\|_p.
\]

(2.26) Lemma. Let \(f_{km} \in L_p\) for every \(m \geq 0\) and every \(k, 1 \leq k \leq n\). If
\[
\lim_{m \to 0} \|f_{km} - f_m\|_p = 0 \text{ for each } k,
\]
then
\[
\lim_{m \to 0} \left\| \max_{1 \leq k \leq n} |f_{km}| - \max_{1 \leq k \leq n} |f_k| \right\|_p = 0.
\]

(2.27) Lemma. Let \(\langle f_n \rangle_{n=0}^\infty\) be a sequence of functions in \(L_p\) such that \((\sup_{n \geq 0} |f_n|) \in L_p\). Then \(\langle f_n \rangle_{n=0}^\infty\) converges a.e. if and only if
\[
\lim_{n \to 0} \left\| \sup_{k \geq n} |f_k - f_n| \right\|_p = 0.
\]

The following are analogous to Lemmas (2.6) and (2.7) in [AS2]. The first one follows from a result of Mazur [M], since the mapping \(\psi_{p,r}\) may be regarded as a composition of his map \(F\) from \(L_1\) to \(L_p\) and his map \(G\) from \(L_p\) to \(L_1\), both uniformly continuous on the unit ball.
Lemma (Uniform continuity of $\psi_{r,r}$). Let $1 \leq p < \infty$ and $1 \leq r < \infty$. Given $\varepsilon > 0$ and $M > 0$, there is a $\delta > 0$ depending only on $\varepsilon$, $M$, $p$, and $r$ such that $\|\psi f - \psi g\|_r < \varepsilon$ whenever $\|f\|_p \leq M$, $\|g\|_p \leq M$, and $\|f - g\|_p < \delta$.

Lemma. Given $\varepsilon > 0$ and $M > 0$, there is a $\delta > 0$ depending only on $\varepsilon$, $M$, $p$, and $r$ such that if $(f_k)_{k=0}^\infty$ is a sequence in $L_p$ with $\|\sup_{k>0} |f_k|\|_p \leq M$ and $\|\sup_{k>0} |f_k - f_0|\|_p < \delta$, then

$$\left\| \sup_{k>0} |\psi f_k - \psi f_0| \right\|_r < \varepsilon.$$ 

Proof. Let $\delta$ be as given in the uniform continuity of $\psi$ corresponding to $\varepsilon/2$, $M$, $p$, and $r$. Let $n \geq 1$ be given. Fix a partition $\{A_1, \ldots, A_n\}$ of $X$ such that

$$\max_{0 \leq k \leq n} |\psi f_k - \psi f_0| = \sum_{m=1}^n |\psi f_m - \psi f_0| \chi_{A_m}.$$ 

Let $f = \sum_{m=1}^n f_m \chi_{A_m}$, so that

$$\max_{0 \leq k \leq n} |\psi f_k - \psi f_0| = |\psi f - \psi f_0|.$$ 

We have $\|f\|_p \leq M$, $\|f_0\|_p \leq M$, and $\|f - f_0\| \leq \|\sup_{k>0} |f_k - f_0|\|_p$. Therefore, if this last norm is less than $\delta$, the uniform continuity of $\psi$ implies that $\|\psi f - \psi f_0\|_r < \varepsilon/2$. This completes the proof.

We also need the following, which is an immediate consequence of

$$\|T_n f_n - T f\|_p \leq \|T_n\| \cdot \|f_n - f\|_p + \|T_n f - T f\|_p.$$ 

Lemma. Suppose $(T_n)_{n=1}^\infty$ and $T$ are $L_p$ contractions and

$$\lim_{n \geq 1} \|T_n f - T f\|_p = 0$$ 

whenever $f \in L_p$. If $f_n \to f$ in $L_p$ norm, then

$$\lim_{n \geq 1} \|T_n f_n - T f\|_p = 0.$$ 

3. Induced operators

In this section, we will be interested primarily in positive $L_p$ operators with strictly positive norming functions. We begin, however, with two more general lemmas.

Lemma. Let $T$ be a positive operator on $\mathbb{M}^+(d\mu)$. Suppose $u \in \mathbb{M}^+(d\mu)$ is strictly positive. If there is a $\lambda \in \mathbb{R}^+$ such that

$$T^* (Tw)^{p-1} \leq \lambda^p u^{p-1},$$ 

then $T$ is a positive $L_p$ operator with $\|T\|_p \leq \lambda$.

Remarks. In the Borel case, this follows from a result in [AS1] concerning dilations. The general case was considered in [K1]. We have included the following short proof to make this paper more self-contained.
Proof. If \( \lambda = 0 \), it is easy to see that \( T = 0 \), since \( \int (Tu)^{p-1}(Tf) \, d\mu = 0 \) for every \( f \in \mathcal{M}^+(d\mu) \).

Suppose \( \lambda > 0 \) and let \( v = Tu \). Because of (3.2), the \( \sigma \)-finiteness of \( \mu \) and the fact that \( u \) is finite a.e., one argues that \( v \) is finite a.e. (The proof is essentially contained in [AS1, p. 391].)

Let \( d\mu' = u^p \, d\mu \) and \( dv' = (v/\lambda)^p \, dv \). Define an operator \( R: \mathcal{M}^+(d\mu') \to \mathcal{M}^+(dv') \) by \( Rf = \chi_G \frac{1}{\lambda} T(uf) \) for \( f \in \mathcal{M}^+(d\mu') \), where \( G \) is the support of \( v \). This is clearly a positive operator in the sense of (2.4). A routine computation shows that the adjoint, \( R^*: \mathcal{M}^+(dv') \to \mathcal{M}^+(d\mu') \), is given by

\[
R^* g = \frac{1}{\lambda^p u^{p-1}} T^*(v^{p-1} g)
\]

for \( g \in \mathcal{M}^+(dv') \). Thus \( R1 \leq 1 \) and \( R^* 1 \leq 1 \), so by Theorem (2.8), \( R \) is an \( L_p \) contraction. This means that if \( f \in \mathcal{M}^+(d\mu') \), then

\[
\int (Rf)^p \, dv' \leq \int f^p \, d\mu'.
\]

If \( f \in \mathcal{M}^+(d\mu) \), then \( f = uf \) for some \( \tilde{f} \in \mathcal{M}^+(d\mu') \). Hence

\[
\int (Tf)^p \, dv = \int [T(uf)]^p \, dv = \lambda^p \int (R\tilde{f})^p \, dv' \\
\leq \lambda^p \int \tilde{f}^p \, d\mu' = \lambda^p \int f^p \, d\mu.
\]

This shows that \( T \) is an \( L_p \) operator with \( \|T\|_p \leq \lambda \). \( \square \)

(3.4) Lemma. Let \( T \) be a positive operator on \( \mathcal{M}^+(d\mu) \). Suppose \( u \in \mathcal{M}^+(d\mu) \) is strictly positive, and that there is a \( \lambda \in \mathbb{R}^+ \) such that

\[
T^*(Tu)^{p-1} \leq \lambda^p u^{p-1}.
\]

Let \( v = Tu \) and let \( G \) be the support of \( v \). Let \( r \) be any exponent, \( 1 < r < \infty \). Then

\[
Sf = \chi_G \left( \frac{u}{\lambda} \right)^{p/r-1} T(\lambda u^{1-p/r} f),
\]

for \( f \in \mathcal{M}^+(d\mu) \), defines a positive \( L_r \) operator \( S: \mathcal{M}^+(d\mu) \to \mathcal{M}^+(dv) \) with \( \|S\|_r \leq \lambda \).

Proof. \( S^*: \mathcal{M}^+(dv) \to \mathcal{M}^+(d\mu) \) is easily calculated; one sees that for \( g \in \mathcal{M}^+(dv) \),

\[
S^* g = (\lambda u)^{1-p/r} T^*(v^{p/r-1} \chi_G g).
\]

Let \( \tilde{u} = u^{p/r} \). Then \( \tilde{u} \) is strictly positive a.e., and \( S^*(Su)^{r-1} \leq \lambda' \tilde{u}^{r-1} \). Thus, Lemma (3.1) completes the proof. \( \square \)

(3.5) Lemma. Suppose \( u_1 \) and \( u_2 \) are strictly positive norming functions for a positive \( L_p \) operator \( T \) on \( \mathcal{M}^+(d\mu) \). For any \( \alpha \in \mathbb{R}^+ \), the set

\[
E_\alpha = \left\{ x \in X \middle| \frac{u_2(x)}{u_1(x)} > \alpha \right\}
\]

is a reducing set for \( T \).
Proof. As in the proof of Lemma (3.1), let $d\mu' = u_1'\,d\mu$ and $d\nu' = (v_1/\lambda)^p\,dv$, where $v_1 = Tu_1$ and $\lambda = \|T\|_p$. Observe that even if $v_1$ is not strictly positive a.e., its support is equal to the support of $v_2 = Tu_2$ a.e. Without loss of generality then, we may replace the set $Y$ with this common support. Define $R: \mathcal{M}^+(d\mu') \to \mathcal{M}^+(d\nu')$ for $f \in \mathcal{M}^+(d\mu')$ by $Rf = (u_1f)/v_1$.

$R1 = R^*1 = 1$, so $R$ is an $L_p$ contraction. $1$ is a norming function for $R$; we now show that $u = u_2/u_1$ is another. One may verify that $R^*(Ru)^{p-1} = u^{p-1}$, from which $\|Ru\|_p = \|u\|_p$ easily follows. Let $v = Ru$.

Let $\alpha \geq 0$ be arbitrary. Let $u_\alpha = u \land \alpha$, the function $u$ truncated at the value $\alpha$. Observe that $E_\alpha$ is the support of $u - u_\alpha$. Also note that $Ru_\alpha \leq v_\alpha = v \land \alpha$, hence

$$
(3.6) \quad \int u_\alpha\,d\mu' = \int Ru_\alpha\,d\nu' \leq \int v_\alpha\,d\nu'.
$$

Let $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ be the distribution of $u$; that is, $\phi(t) = \mu'\{x: u(x) > t\}$. Let $\theta$ be the distribution of $v$, similarly defined with respect to $\nu'$. Inequality (3.6) has the equivalent form

$$
(3.7) \quad \int_0^\alpha \phi(t)\,dt \leq \int_0^\alpha \theta(t)\,dt.
$$

Since $\|u\|_p = \|v\|_p$, we have

$$
(3.8) \quad \int_0^\infty t^{p-1}\phi(t)\,dt = \int_0^\infty t^{p-1}\theta(t)\,dt.
$$

Finally, $u \in L_1(d\mu')$, since $p > 1$ and $\mu'$ is a finite measure. Since $\|u\|_1 = \|v\|_1$, we have

$$
(3.9) \quad \int_0^\infty \phi(t)\,dt = \int_0^\infty \theta(t)\,dt < \infty.
$$

Conditions (3.7)–(3.9) allow us to invoke Lemma (2.13) and conclude that $\phi = \theta$ a.e. in Lebesgue measure. Since

$$
\|u - u_\alpha\|_p^p = p \int_0^{\infty} t^{p-1}\phi(t)\,dt,
$$

we have

$$
\|u - u_\alpha\|_p = \|v - v_\alpha\|_p \leq \|R(u - u_\alpha)\|_p,
$$

where the inequality follows because $Ru_\alpha \leq v_\alpha$. As $R$ is a contraction, we conclude that the norms are in fact equal. Thus, $u - u_\alpha$ is a norming function. By Lemma (2.12), then, its support is a reducing set for $R$. It easily follows that $E_\alpha$ also reduces $T$. \(\square\)

(3.10) Remarks. One may replace the “less than” in the definition of $E_\alpha$ by any other inequality, simply by considering complements or reversing the roles of $u_1$ and $u_2$. The complement of a reducing set is a reducing set; it is also easy
to show that the intersection of reducing sets is a reducing set. In fact, the class
of reducing sets of a bounded \( L_p \) operator is a sub-\( \sigma \)-algebra of the underlying
measure space. This is shown in [K2], which also includes a different proof of
the above lemma.

(3.11) Theorem. Suppose \( T \) is a positive \( L_p \) operator on \( M^+(d\mu) \), and \( u_1 \) and
\( u_2 \) are strictly positive normalizing functions for \( T \). Let \( v_i = T u_i \) for \( i = 1, 2 \) and
let \( G \) be the support of the \( v_i \)'s. Let \( 1 < r < \infty \), and define positive operators
\( S_1 \) and \( S_2 \) on \( M^+(d\mu) \) by

\[
S_i f = \chi_G \|T\|_p^{1-p/r} v_i^{p/r-1} T( u_i^{1-p/r} f)
\]

for \( f \in M^+(d\mu) \) and \( i = 1, 2 \). Then \( S_1 f = S_2 f \) a.e. for every \( f \in M^+(d\mu) \).

Proof. By (2.3), it suffices to consider \( f \in L^+(d\mu) \).

Let \( s = p/r - 1 \). If \( s = 0 \), there is nothing to prove. Otherwise, let \( \varepsilon > 0 \)
be given, and choose a positive integer \( N > 1/\varepsilon \).

For each \( n \geq 1 \), let

\[
E_n = \left\{ x \in X \mid \frac{N+n-1}{N} \leq \frac{u_2(x)}{u_1(x)} \leq \frac{N+n}{N} \right\}
\]

and

\[
E_{-n} = \left\{ x \in X \mid \frac{N+n-1}{N} \leq \frac{u_1(x)}{u_2(x)} \leq \frac{N+n}{N} \right\}.
\]

Also, let \( E_0 \) be the set of points in \( X \) where \( u_1(x) = u_2(x) \). Then \( \{E_n \mid n \in \mathbb{Z}\} \)
is a partition of \( X \) into reducing sets.

Let \( f \in M^+(d\mu) \) be given and let \( f_n = f \chi_{E_n} \) for every \( n \in \mathbb{Z} \). The \( f_n \)'s
have disjoint support, as do the functions \( T(u_1^{-s} f_n) \) and \( T(u_2^{-s} f_n) \).

Now suppose \( n \geq 1 \) and \( s > 0 \). Since \( T \) is positive, we have

\[
\left( \frac{N}{N+n} \right)^s T \left( \frac{f_n}{u_1} \right) \leq T \left( \frac{f_n}{u_2} \right) \leq \left( \frac{N}{N+n-1} \right)^s T \left( \frac{f_n}{u_1} \right).
\]

Let \( u_{in} = u_i \chi_{E_n} \) and \( v_{in} = T(u_{in}) \) for every \( n \in \mathbb{Z} \) and \( i = 1, 2 \). The
functions \( T(u_1^{-s} f_n) \) and \( v_{mj} \) will have disjoint supports unless \( m = n \); thus
\( S_i f_n \) depends only on \( T(u_i^{-s} f_n) \) and \( v_{n1}^s \). We have

\[
\left( \frac{N+n-1}{N} \right)^s v_{n1}^s \leq v_{n2}^s \leq \left( \frac{N+n}{N} \right)^s v_{n1}^s.
\]

Therefore,

\[
\left( \frac{N+n-1}{N+n} \right)^s S_1 f_n \leq S_2 f_n \leq \left( \frac{N+n}{N+n-1} \right)^s S_1 F_N.
\]

If \( (S_1 f_n)(x) = 0 \), then \( (S_2 f_n)(x) \) must be zero as well. Otherwise,

\[
\left| \left( \frac{(S_2 f_n)(x)}{(S_1 f_n)(x)} \right)^{1/s} - 1 \right| \leq \frac{1}{N+n-1} < \varepsilon.
\]
If $s < 0$, then the order of the terms in (3.14) is reversed, but (3.15) remains valid.

If $n \leq -1$, the argument is symmetric, with the conclusion

$$\left| \frac{(S_1 f_n)(x)}{(S_2 f_n)(x)} \right|^{1/s} - 1 \leq \frac{1}{N + n - 1} < \varepsilon.$$ 

It is clear that $S_1 f_0 = S_2 f_0$. Since $\varepsilon > 0$ is arbitrary, we conclude that $S_1 f_n = S_2 f_n$ a.e. for each $n \in \mathbb{Z}$. Thus $S_1 f = S_2 f$ a.e., as desired. □

(3.16) Theorem. Suppose $1 < p < \infty$ and $1 < r < \infty$. Let $T$ be a positive $L_p$ operator with a strictly positive norming function $u$. Let $v = Tu$ and let $G$ be the support of $v$. Then

$$T_r f = \chi_g\|T\|^{1-p/r}v^{p/r-1}T(u^{1-p/r}f),$$

for $f \in \mathcal{M}^+(d\mu)$, defines a positive $L_r$ operator $T_r : \mathcal{M}^+(d\mu) \to \mathcal{M}^+(dv)$ such that $\|T_r\|_r = \|T\|_p$. This operator, called the $L_r$ operator induced by $T$, is independent of the choice of $u$.

Proof. Whether $T$ is given as an $L_p$ operator in the Banach space sense or in the sense of Definition (2.4), it is clear that $T_r$ is a positive operator in the sense of (2.4). Lemmas (2.10) and (3.4) combine to show that $T_r$ is in fact an $L_r$ operator with norm bounded by $\|T\|_p$. To see that this norm is actually achieved, let $f = u^{p/r}$. Theorem (3.11) demonstrates that $T_r$ does not depend on the choice of norming function. □

(3.17) Corollary. Suppose $T$ is an $L_p$ contraction with a semi-invariant function where $1 < p < \infty$. For every $r$, $1 < r < \infty$,

$$T_r f = v^{p/r-1}T(u^{1-p/r}f)$$

defines a positive contraction of $L_r$.

(3.18) Remarks. If $T$ is an $L_p$ isometry induced by an automorphism $\tau$ (as in (2.16)), then $T_r$ is simply the $L_r$ isometry induced by $\tau$. When the underlying space has finite measure, we may take $u = 1$ and $v = \rho^{1/p}$. The general $\sigma$-finite case is not much harder to check.

A larger and more important class of operators has the form $EQE$, where $Q$ is an $L_p$ isometry induced by an automorphism and $E$ is a conditional expectation operator of finite rank. Such operators where crucial to the proof of the pointwise ergodic theorem for positive $L_p$ contractions (see [A]). Thus, the following lemma is of some general interest as well as being necessary for §5 of this paper.

(3.19) Lemma. Suppose $1 < p < \infty$, $1 < r < \infty$, and that $Q_p$ and $Q_r$ are, respectively, the $L_p$ and $L_r$ isometries induced by an automorphism $\tau$ over a measure space $(X, \mathcal{F}, \mu)$. Let $\mathcal{F}$ be a sub-$\sigma$-algebra of $\mathcal{F}$ and let $\mu$ be the
restriction of \( \mu \) to \( \mathcal{F} \). Let \( E \) be conditional expectation with respect to \( \mathcal{F} \) and suppose

\[
T : L_p(X, \mathcal{F}, \mu) \to L_p(X, \mathcal{F}, \mu)
\]

is given by \( T = EQ_pE \). If \( T \) has a semi-invariant function \( u \), then \( T_r = EQ_rE \).

**Proof.** Let \( v = Tu \). For \( f \in L_r(X, \mathcal{F}, \mu) \), we have

\[
T_rf = v^{p/r-1}T(u^{1-p/r}f) \\
= v^{p/r-1}E(\rho^{1/p}(u \circ \tau^{-1})^{1-p/r}(Ef) \circ \tau^{-1}) \\
= v^{p/r-1}E[\rho^{1/p}(u \circ \tau^{-1})^{1-p/r}(f \circ \tau^{-1})]
\]

where the third line follows because \( f \) is already \( \mathcal{F} \)-measurable. Because \( \|v\|_p = \|u\|_p \), \( Q \) is an isometry and \( p > 1 \), we conclude that \( Q_pu \) must already be \( \mathcal{F} \)-measurable, lest some norm be lost in taking the conditional expectation. Thus \( v = \rho^{1/p}(u \circ \tau^{-1}) \) and

\[
T_rf = E[v^{p/r-1}\rho^{1/p}(u \circ \tau^{-1})^{1-p/r}(f \circ \tau^{-1})] \\
= E[(\rho^{1/p})^{p/r-1}(u \circ \tau^{-1})^{p/r-1}\rho^{1/p}(u \circ \tau^{-1})^{1-p/r}(f \circ \tau^{-1})] \\
= E[\rho^{1/r}(f \circ \tau^{-1})] = EQ_rf = EQ_tE. \quad \Box
\]

**(3.20) Lemma.** Let \( 1 < p < \infty \), \( 1 < r < \infty \), and let \( Q \) be the \( L_p \) isometry induced by an automorphism \( \tau \). Let \( T = EQE \) for some conditional expectation operator \( E \). If \( T \) has a semi-invariant function and \( R = R_r \) is the \( L_r \) isometry induced by \( \tau^{-1} \), then \( (T^*)_r = ERE \).

**Proof.** \( (T^*)_r = (EQ^*E)_r = (ER_qE)_r = ERE \). We have used the self-adjointness of \( E \) and Lemmas (3.19) and (2.17) for the fact that \( Q^* \) is the \( L_r \) isometry induced by \( \tau^{-1} \). \( \Box \)

## 4. Finite-dimensional approximation

In [AK], it was shown that all positive contractions over the unit interval are induced by a point mapping of some type, followed by a conditional expectation. For positive contractions with semi-invariant functions, the argument is easier and does not require the underlying space to be interval. However, we will want to extract a point mapping from a set mapping, so we will require our measure spaces to be Lebesgue spaces. That is, a measure space \((X, \mathcal{F}, \mu)\) where \( X \) is a complete metric space and \( \mathcal{F} \) is the Borel \( \sigma \)-algebra. We allow the space to have \( \sigma \)-finite measure. Since a separable metric space is second countable, the \( \sigma \)-algebra of measurable sets in a Lebesgue space can always be generated by a countable algebra of sets.

The details of the construction give us a family of finite-dimensional operators \((T^*_n)_{n=1}^{\infty}\) (these are ordinary superscripts, not powers), each with a semi-invariant function \( u_n \), where \( u_n \to u \) a.e. Furthermore, these operators have
the property that \( (T^n)_n \rightarrow T_r f \) a.e. and in \( L_r \) norm for every \( f \in L_r \). These finite-dimensional approximations to the induced operator provide the key to the proof of the Theorem (1.2).

(4.1) Definitions. Let \( X = (X, \mathcal{T}, \mu) \) be a \( \sigma \)-finite Lebesgue space and suppose \( T: L_p(d\mu) \rightarrow L_p(d\mu) \) has a semi-invariant function \( u \). Let \( I = (I, \mathcal{B}, m) \) be the usual Lebesgue space of the unit interval. Let \( W = (W, \mathcal{F}, \omega) = X \times I \).

Let \( \mathcal{I} = \{ F \times I | F \in \mathcal{F} \} \), the “vertical” sub-\( \sigma \)-algebra of \( \mathcal{F} \), and let \( v \) be the \( \mathcal{I} \)-measurable function given by \( v(x, y) = (Tu)(x) \) for every \( y \) in the unit interval.

Suppose \( (\mathcal{F}_n)_{n=1}^\infty \) is an increasing sequence of finite sub-\( \sigma \)-algebras of \( \mathcal{F} \) such that \( \sigma(\bigcup_{n=1}^\infty \mathcal{F}_n) = \mathcal{F} \). That is, \( \mathcal{F} \) is the smallest \( \sigma \)-algebra containing all the \( \mathcal{F}_n \)’s. Let \( \mathcal{I}_n = \{ F \times I | F \in \mathcal{F}_n \} \).

For each \( n \geq 1 \), fix an enumeration \( \{ F_{n,i} \}_{i=1}^{k_n} \) of the atoms of \( \mathcal{F}_n \). Let \( \gamma_{n,0} = 0 \), and for each \( i, 1 \leq i \leq k_n \), let

\[
\gamma_{n,i} = T \left( u \sum_{j=1}^{i} \chi_{F_{n,j}} \right),
\]

and

\[
H_{n,i} = \left\{ (x, y) \in W \left| \frac{\gamma_{n,i-1}(x)}{(Tu)(x)} < y \leq \frac{\gamma_{n,i}(x)}{(Tu)(x)} \right. \right\}.
\]

Let \( \mathcal{H} \) be the finite sub-\( \sigma \)-algebra of \( \mathcal{F} \) generated by the partition \( \{ H_{n,i} \}_{i=1}^{k_n} \) of \( W \). Let \( \Pi_n \) be the set mapping from \( \mathcal{F}_n \) to \( \mathcal{H} \) determined by \( \Pi_n F_{n,i} = H_{n,i} \) for each \( i, 1 \leq i \leq n \).

(4.2) Lemma. There is a point mapping \( \pi: W \rightarrow X \) such that \( \pi^{-1} F_{n,i} = H_{n,i} \) for every \( n \geq 1 \) and every \( i, 1 \leq i \leq k_n \).

Proof. The family of set mappings \( \Pi_n \) determines a unique set mapping of the algebra \( \bigcup_{n=1}^\infty \mathcal{F}_n \), because of \( \mathcal{F}_n \)'s form a monotone sequence. This mapping preserves unions and complements, and it extends to a homomorphism of the measure algebras of \( (X, \mathcal{F}) \) and \( (W, \mathcal{H}) \). Since the sets underlying both spaces are complete metric spaces, there is a point mapping \( \pi \) defined from almost all of \( W \) onto almost all of \( X \) which induces the set mapping (see [Ry, p. 329]). Thus if \( \Pi F = H \), then \( \pi^{-1} F = H \). Since \( \Pi F_{n,i} = \Pi_n F_{n,i} \), the desired result follows. \( \Box \)

(4.3) Lemma. For every \( F \in \mathcal{F} \), \( \int_F u^p \, d\mu = \int_{\pi^{-1} F} v^p \, d\omega \).
Proof. If $F = F_{n,i} \in \mathcal{F}$ for some $n \geq 1$ and some $i$, $1 \leq i \leq k_n$, then
\[
\int_{\pi^{-1}F} v^p \, d\omega = \int_{\chi} (Tu)^p \left[ \gamma_{n,i} - \gamma_{n,i-1} \right] \, d\mu = \int_{\chi} (Tu)^{p-1} T(u\chi_F) \, d\mu = \int_{F} uT^*(Tu)^{p-1} \, d\mu = \int_{F} u^p \, d\mu.
\]

The lemma is true for a generating subalgebra of $\mathcal{F}$. The proof is easily completed.  □

(4.4) Lemma. Suppose $\phi$ is an $\mathcal{F}$-measurable function and $\theta$ is a $\mathcal{H}$-measurable function with $\phi > 0 \mu$-a.e. and $\theta > 0 \omega$-a.e. such that
\[
\frac{\phi}{\theta} = 1
\]
for every $F \in \mathcal{F}$. Then, if $1 < p < \infty$,
\[
Sf = \left( \frac{\phi}{\theta \circ \pi} \right) (f \circ \pi),
\]
for $f \in L_p(d\mu)$, defines an isometry $S: L_p(d\mu) \rightarrow L_p(d\omega)$.

Proof. First suppose $f = \phi^{1/p} \chi_F$ for some $F \in \mathcal{F}$. Then
\[
\|Sf\|_p = \int_{\frac{\theta}{\phi \circ \pi} (\phi^{1/p} \chi_F)^p \circ \pi} d\omega = \int_{\pi^{-1}F} \theta \, d\omega = \int_{F} \phi \, d\mu = \|f\|_p^p.
\]
In the general case, approximate $f \phi^{-1/p}$ by $\mathcal{F}$-simple functions.  □

This isometry yields a result analogous to the theorem of Akcoglu and Koop [AK].

(4.5) Theorem. Define $Q: L_p(d\mu) \rightarrow L_p(d\omega)$ by
\[
Q = \frac{\theta}{\phi \circ \pi} (f \circ \pi),
\]
for $f \in L_p(d\mu)$.

If $\omega'$ is $\omega$ restricted to $\mathcal{F}$, and we identify $X$ with $(W, \mathcal{F}, \omega')$, then $Tf = E(\mathcal{F} \circ \mathcal{F})$ for every $F \in L_p(d\mu)$.

Proof. By the two previous lemmas, we see that $Q$ is an isometry of the indicated spaces. Suppose $f = u\chi_F$ for some $F \in \mathcal{F}$. Then
\[
[E(\mathcal{F} \circ \mathcal{F})](x) = \int_{0}^{1} (Qf)(x, y) \, dy = \int_{0}^{1} v(x, y) \chi_{\pi^{-1}F}(x, y) \, dy = (Tu)(x) \left( \frac{T(u\chi_F)(x)}{(Tu)(x)} \right) = (Tf)(x).
\]
For a general $\mathcal{F}$-measurable $f$, approximate $fu^{-1}$ by $\mathcal{F}$-simple functions. □

(4.6) Lemma. Suppose $\mathcal{F}$ is a finite sub-σ-algebra of $\mathcal{F}$ and $\bar{\mu}$ is the restriction of $\mu$ to $\mathcal{F}$. If $\bar{\nu} \in L^p(d\omega)$ satisfies $\bar{\nu} > 0$ a.e., then there is a unique $\mathcal{F}$-measurable function $\bar{u}$ such that, for every $F \in \mathcal{F}$,

$$\int_F \bar{u}^p \, d\bar{\mu} = \int_{\pi^{-1}F} \bar{\nu}^p \, d\omega.$$

Furthermore, the mapping

$$f \in L_p(X, \mathcal{F}, \bar{\mu}) \mapsto \frac{\bar{\nu}}{\bar{u} \circ \pi} (f \circ \pi) \in L_p(d\omega)$$

is an isometry which transforms $\bar{u}$ to $\bar{\nu}$.

Proof. Let $\{F_i\}_{i=1}^k$ be an enumeration of the atoms of $\mathcal{F}$. Let $H_i = \pi^{-1}F_i$ for each $i$, $1 \leq i \leq k$. Then

$$\bar{u} = \sum_{i=1}^k \left( \frac{1}{\mu F_i} \int_{H_i} \bar{\nu}^p \, d\omega \right)^{1/p} \chi_{F_i}.$$

If $f = \sum_{i=1}^k c_i \chi_{F_i} \in L_p(X, \mathcal{F}, \bar{\mu})$, then

$$\left\| \frac{\bar{\nu}}{\bar{u} \circ \pi} (f \circ \pi) \right\|^p_p = \sum_{i=1}^k c_i^p \int_{\pi^{-1}F_i} \frac{\bar{\nu}^p}{(\bar{u} \circ \pi)^p} \, d\omega = \sum_{i=1}^k c_i^p \mu F_i = \|f\|^p_p,$

as desired. □

(4.7) Theorem. For each $n \geq 1$, let $v_n = E(v|\mathcal{F}_n)$. Let $u_n$ be the corresponding $\mathcal{F}_n$-measurable functions as given by Lemma (4.6). Then $u_n \to u$ $\mu$-a.e.

Proof. Let $u_n = \sum_{i=1}^k u_{n,i} \chi_{F_{n,i}}$. Then

$$u_{n,i}^p = \frac{1}{\mu F_{n,i}} \int_{H_{n,i}} v_n^p \, d\omega = \left( \frac{1}{\mu F_{n,i}} \int_{H_{n,i}} u^p \, d\mu \right)^{(\omega H_{n,i})^{-1}} \int_{H_{n,i}} v_n^p \, d\omega \left( (\omega H_{n,i})^{-1} \int_{H_{n,i}} v_n^p \, d\omega \right)^{-1}.$$

Thus

$$u_{n}^p \circ \pi = \frac{E(u_{n}^p|\mathcal{H}_n)}{E(u^p|\mathcal{H}_n)}[E(u^p|\mathcal{H}_n) \circ \pi].$$

By the martingale convergence theorem, with $p = 1$, we have $E(u_{n}^p|\mathcal{H}_n) \to u^p$ $\mu$-a.e., and $E(u_{n}^p|\mathcal{H}_n) \to E(u^p|\mathcal{H})$ $\omega$-a.e. By Lemma (2.22), we also have $E(u^p|\mathcal{H}_n) \to E(u^p|\mathcal{H})$ $\omega$-a.e. Therefore

$$u_{n}^p \circ \pi \to E(u^p|\mathcal{H}) \circ \pi,$$

and so $u_{n}^p \to u^p$ $\mu$-a.e., by the martingale convergence theorem. This completes the proof. □
(4.8) Definition. For each \( n \geq 1 \), let \( u_n \) and \( v_n \) be as defined in the hypothesis of the previous lemma. Define

\[
Q^n : L_p(X, \mathcal{F}_n, \mu_n) \to L_p(d\omega),
\]
where \( \mu_n \) is the restriction of \( \mu \) to \( \mathcal{F}_n \), by

\[
Q^n f = \frac{v_n}{u_n \circ \pi} (f \circ \pi)
\]
for \( f \in L_p(X, \mathcal{F}_n, \mu_n) \).

By Lemma (4.6), this is an isometry. If \( \omega_n \) is the restriction of \( \omega \) to \( \mathcal{F}_n \), and we make the obvious identification of \( (W, \mathcal{F}_n, \omega_n) \) with \( (X, \mathcal{F}_n, \mu_n) \), then define \( T^n : L_p(d\mu) \to L_p(X, \mathcal{F}_n, \mu_n) \) by

\[
T^n f = E(Q^n E(f|\mathcal{F}_n)|\mathcal{F}_n)
\]
for \( f \in L_p(X) \). Each \( T^n \) is a positive contraction, and it is easy to see that if \( f \in L_p(d\mu) \), then \( T^n f \to Tf \) \( \mu \)-a.e.

Observe that \( u_n \) is a semi-invariant function for each \( T^n \); the reason is that \( v_n \) is already \( \mathcal{F}_n \)-measurable. (In fact, it is easy to see that \( u_n \) is the only normalized semi-invariant function for \( T^n \).) Thus, the induced operator \( (T^n)_r \) is defined for any \( r, 1 < r < \infty \). For brevity, denote it \( R_n \).

(4.9) Theorem. \( \|R_n f - T_r f\|_r \to 0 \) as \( n \to \infty \), for every \( f \in L_r(d\mu) \).

Proof. If \( f \) is \( \mathcal{F}_n \)-measurable, then

\[
R_n f = v_n^{p/r-1} T^n (u_n^{1-p/r} f)
\]

\[
= v_n^{p/r-1} E \left[ \frac{v_n}{u_n \circ \pi} (u_n^{1-p/r} \circ \pi)(f \circ \pi)|\mathcal{F}_n \right]
\]

\[
= E \left[ \left( \frac{v_n}{u_n \circ \pi} \right)^{p/r} (f \circ \pi)|\mathcal{F}_n \right],
\]

since \( v_n \) is \( \mathcal{F}_n \)-measurable. Whether or not \( f \) is \( \mathcal{F}_n \)-measurable, define

\[
\phi_n = \left( \frac{v_n}{u_n \circ \pi} \right)^{p/r} [E(f|\mathcal{F}_n) \circ \pi].
\]

Then \( R_n f = E(\phi_n|\mathcal{F}_n) \) for any \( f \in L_r(d\mu) \). Similarly, if \( \phi = (v/u \circ \pi)^{p/r} (f \circ \pi) \), then \( T_r f = E(\phi|\mathcal{F}_n) \).

Clearly \( \phi_n \to \phi \) a.e.; if we can show that \( \|\phi_n\|_r \to \|\phi\|_r \), we may conclude that \( \phi_n \to \phi \) in \( L_r \) norm (see [Ry, p. 118]):

\[
\|\phi_n\|_r^r = \|Q^n([E(f|\mathcal{F}_n)]^{p/r})\|_p^p
\]

\[
= \|[E(f|\mathcal{F}_n)]^{p/r}\|_p^p = \|E(f|\mathcal{F}_n)|_r^r \to \|f\|_r^r.
\]

as \( E(f|\mathcal{F}_n) \) is an \( L_r \) martingale. The second line follows because \( Q^n \) is an isometry. Also \( \|\phi\|_r = \|f\|_r \) by a similar calculation. This tells us that

\[
\|\phi_n - \phi\|_r \to 0.
\]
To conclude the proof, observe that
\[ \|R_n f - T_r f\|_r \leq \|E(\phi_n \mathcal{F}_n) - E(\phi_n \mathcal{F})\|_r + \|E(\phi_n \mathcal{F}) - E(\phi \mathcal{F})\|_r. \]
The first term tends to zero by the martingale convergence theorem and the second term is dominated by \( \|\phi_n - \phi\|_r \). □

5. Convergence of the alternating sequence

This section is in many ways analogous to §§3 and 4 of [AS2], and so the reader will often be referred there for details. Where we follow [AS2] closely, every effort is made to keep the notation consistent.

(5.1) Definitions. Suppose \( 1 < p < \infty \), \( 1 < r < \infty \), and let \( \psi = \psi_{p,r} \). Let \( \langle T_n \rangle_{n=1}^{\infty} \) be a sequence of positive linear contractions with semi-invariant functions operating on the \( L_p \) space of a \( \sigma \)-infinite Lebesgue space. Call such a sequence of operators a norming sequence. Call a norming sequence special if all operators are finite dimensional.

Let \( V_0 \) and \( U_0 \) be the identities on \( L_p \) and \( L_r \) respectively, and make the following definitions for each \( n \geq 1 \):

\[ V_n = T_n \cdots T_1, \quad U_n = (T_n^*)_r \cdots (T_1^*)_r. \]

For a given \( f \in L_p \) and an \( n \geq 0 \), let \( g_n = U_n \psi (V_n f) \). Observe that \( g_0 = \psi f \) and that \( \|g_0\|_r = \|f\|^{p/r}_p \).

We say that Estimate A holds for a norming sequence \( \langle T_n \rangle_{n=1}^{\infty} \) if
\[ \sup_{n \geq 0} |g_n| \leq (q\|f\|_p)^{p/r}_r \quad (= q^{p/r}_p \|g_0\|_r) \]
for every \( f \in L_p \).

We say that Estimate B holds for a norming sequence \( \langle T_n \rangle_{n=1}^{\infty} \) if for every \( \epsilon > 0 \) there is a \( \delta > 0 \), depending only on \( \epsilon \), \( p \), and \( r \), such that
\[ \sup_{n \geq 0} |g_n - g_0| < \epsilon |f|^{p/r}_p \quad (= \epsilon \|g_0\|_r) \]
whenever \( f \in L_p \) is such that
\[ \|f\|_p - \lim_{n \to \infty} \|V_n f\|_p < \delta \|f\|_p. \]

Given a norming sequence \( \langle T_n \rangle_{n=1}^{\infty} \), a fixed \( n \geq 1 \), and a function \( f \in L_p \), let \( \tilde{f} = V_n f \). For every \( k \geq 1 \), let \( \tilde{T}_k = T_{n+k} \). Let \( \tilde{V}_0 \) and \( \tilde{U}_0 \) be the identities on \( L_p \) and \( L_r \) respectively. For each \( k \geq 1 \), let
\[ \tilde{V}_k = \tilde{T}_k \cdots \tilde{T}_1, \quad \tilde{U}_k = (\tilde{T}_k^*)_r \cdots (\tilde{T}_1^*)_r, \]
and for each \( k > 0 \), let \( \tilde{g}_k = \tilde{U}_k \psi (\tilde{V}_k \tilde{f}) \). Observe that \( g_{n+k} = U_n \tilde{g}_k \) for every \( k \geq 0 \).
(5.2) **Theorem.** Suppose Estimates A and B are satisfied for every norming sequence. Then given a norming sequence \( (T_n)_{n=1}^{\infty} \) and an \( f \in L_p \), \( (g_n)_{n=0}^{\infty} \) converges a.e.

**Proof.** Because of Estimate A, \( (\sup_{n \geq 0} |g_n|) \in L_r \). Therefore, by Lemma (2.27), it suffices to show that

\[
\lim_{n \to 0} \left\| \sup_{k \geq n} |g_{n+k} - g_n| \right\|_r = 0.
\]

Let \( \beta = \lim_{n \to 0} \|V_n f\|_p \) and distinguish two cases:

**Case 1:** \( \beta = 0 \). Given \( \epsilon > 0 \), find \( n_0 \geq 1 \) such that

\[
\|V_{n_0} f\|_p < \frac{1}{q} \left( \frac{\epsilon}{2} \right)^{1/p}.
\]

Fix \( n \geq n_0 \) and define \( \tilde{f} \) and \( \tilde{g}_k \) as above. Observe that \( \|\tilde{f}\|_p \leq \|V_{n_0} f\|_p \). We have

\[
\left\| \sup_{k \geq 0} |g_{n+k} - g_n| \right\|_r \leq \left\| \sup_{k \geq 0} |\tilde{g}_k - \tilde{g}_0| \right\|_r \leq 2 \|\tilde{g}_k\|_r \leq 2(q \|\tilde{f}\|_p)^{p/r} < \epsilon,
\]

where the first inequality follows from Lemma (2.24) and the third follows from Estimate A for the sequence \( (\tilde{T}_k)_{k=1}^{\infty} \).

**Case 2:** \( \beta > 0 \). Given \( \epsilon > 0 \), choose \( \delta > 0 \) as given by Estimate B, corresponding to \( \epsilon/(\|f\|_p)^{p/r} \). Choose \( n_0 \geq 1 \) such that \( \|V_{n_0} f\|_p < (1 + \delta)\beta \). Fix \( n \geq n_0 \) and define \( \tilde{f} \) and \( \tilde{g}_k \) as above. Observe that \( \beta \leq \|\tilde{f}\|_p \), since the \( \|V_n f\|_p \)'s form a monotone sequence. We have

\[
\|\tilde{f}\|_p - \lim_{k \to 0} \|V_k \tilde{f}\|_p = \|V_n f\|_p - \beta < (1 + \delta)\beta - \beta \leq \delta \|\tilde{f}\|_p.
\]

Now apply Estimate B for \( (\tilde{T}_k)_{k=1}^{\infty} \); we conclude

\[
\left\| \sup_{k \geq 0} |g_{n+k} - g_k| \right\|_r \leq \left\| \sup_{k \geq 0} |\tilde{g}_k - \tilde{g}_0| \right\|_r \leq \left( \frac{\epsilon}{\|f\|_p^{1/r}} \right) \|\tilde{f}\|_p^{p/r} \leq \epsilon,
\]

where the first inequality follows as in Case 1. \( \Box \)

(5.3) **Lemma.** If Estimate A holds for every special norming sequence, then it holds for every norming sequence.

**Proof.** Suppose \( (T_n)_{n=1}^{\infty} \) is a uniform norming sequence for which Estimate A fails. Then there is a function \( f \in L_p \) and an \( n \geq 1 \) such that

\[
\left\| \max_{0 \leq k \leq n} |g_k| \right\|_r > q^{p/r} \|g_0\|_r.
\]
Suppose \((\mathcal{F}_m)_{m=1}^\infty\) is a monotone sequence of finite sub-\(\sigma\)-algebras of \(\mathcal{F}\) with \(\mathcal{F} = \sigma(\bigcup_{m=1}^\infty \mathcal{F}_m)\), the smallest \(\sigma\)-algebra containing the algebra \(\bigcup_{m=1}^\infty \mathcal{F}_m\). For each \(k\) and \(m\), \(1 \leq k \leq n\) and \(m \geq 1\), let \(T_k^m\) be the finite-dimensional operator as defined in (4.8). Let \(f^m = E(f|\mathcal{F}_m)\).

Let \(m \geq 1\) be arbitrary. Let \(V_0^m\) and \(U_0^m\) be \(E(\cdot|\mathcal{F}_m)\) operating on \(L_p\) and \(L_r\) respectively. For each \(k\), \(1 \leq k \leq n\), let
\[
V_k^m = T_1^m \cdots T_k^m, \quad U_k^m = ((T_1^*)^m)_r \cdots ((T_k^*)^m)_r.
\]

For \(f \in L_p\), \(m \geq 1\), and each \(k\), \(0 \leq k \leq n\), let
\[
g_{km} = U_k^m \psi(V_k^m f) = U_k^m \psi(V_k^m f^m).
\]

By the martingale convergence theorem, \(\lim_{m \to \infty} \|f - f^m\|_p = 0\). We will show that
\[
(5.4) \quad \lim_{m \to \infty} \|g_{km} - g_k\|_r = 0
\]
as well. Therefore, by applying Lemma (2.26),
\[
\lim_{m \to \infty} \max_{0 \leq k \leq n} \|g_{km} - g_k\|_r = 0.
\]
Thus, for a suitably large integer \(m_0\),
\[
\left\| \max_{0 \leq k \leq n} |g_{km}| - \max_{0 \leq k \leq n} |g_k| \right\|_r > (q\|f^m\|_p)^{p/r},
\]
since the same inequality holds for \(f\) and the \(g_k\)'s. Because \(\mathcal{F}_{m_0}\) is finite, the operators \((T_1^{m_0}, \ldots, T_n^{m_0})\) are essentially finite dimensional. Therefore, they form the initial portion of a special norming sequence for which Estimate A fails, contradicting the hypothesis of the lemma.

To prove (5.4), we first prove
\[
(5.5) \quad \lim_{m \to \infty} \|V_k^m f - V_k f\|_p = 0
\]
for every \(k\), \(0 \leq k \leq n\). When \(k = 0\), this is simply the martingale convergence theorem. For the inductive step, observe that
\[
\|V_{k+1}^m f - V_{k} f\|_p = \|T_{k+1}^m V_k^m f - T_{k+1}^m V_k f\|_p,
\]
where \(\lim_{m \to \infty} \|V_k^m f - V_k f\|_p = 0\) by the inductive hypothesis. We apply Theorem (4.9) with \(r = p\) and Lemma (2.30) to conclude that
\[
\lim_{m \to \infty} \|V_{k+1}^m f - V_{k+1} f\|_p = 0,
\]
completing the induction.

Because of the uniform continuity of \(\psi\),
\[
\lim_{m \to \infty} \|\psi V_k^m f - \psi V_k f\|_p = 0
\]
for each \(k\), \(0 \leq k \leq n\).
We now perform another induction similar to the proof of (5.5) to show that when \( g \in L_r \),

\[
\lim_{m \geq 1} \| U_k^m g - U_k g \|_r = 0,
\]

for each \( k, \ 0 \leq k \leq n \). This completes the proof. \( \square \)

(5.6) **Lemma.** Suppose that for every \( \xi > 0 \), there is an \( \eta > 0 \) depending only on \( \xi, p, \) and \( r \) such that

\[
\max_{0 \leq k \leq n} |g'_k - g'_0| < \xi \| f' \|_p^{p/r}
\]

whenever \( (T_n')_{n=1}^{\infty} \) is a special norming sequence, \( n \geq 1 \), and \( f' \in L_p \) is such that \( \| f' \|_p - \| V_n' f' \|_p < \eta \| f' \|_p \), where \( V_n' \) and \( g_n' \) are defined exactly as \( V_n \) and \( g_n \) in (5.1), relative to \( (T_n')_{n=1}^{\infty} \). Then Estimate B holds for every norming sequence.

**Proof.** Let \( (T_n')_{n=1}^{\infty} \) be a norming sequence and suppose \( \xi > 0 \) is given. Choose \( \eta > 0 \) from the hypothesis of the lemma, corresponding to \( \xi/2 \). If Estimate B fails for \( (T_n)_{n=1}^{\infty} \), then there is a function \( f \in L_p \) with \( \| f \|_p - \| V_n f \|_p < \eta \| f \|_p \), but for which

\[
\max_{0 \leq k \leq n} |g'_k - g'_0| > \xi \| f \|_p^{p/r}.
\]

As in the proof of the previous lemma, we approximate the operators \( T_k \) with the operators \( T^m_k \) from (4.8). Define \( g_{km} \) as before, for each \( m \geq 1 \) and each \( k, \ 0 \leq k \leq n \), and let \( h_k = g_k - g_0 \) and \( h_{km} = g_{km} - g_{0m} \) for the same set of indices. Then

\[
\| h_{km} - h_k \|_r \leq \| g_{km} - g_k \|_r + \| g_{0m} - g_0 \|_r,
\]

and we have seen that both of these terms tend to zero as \( m \) increases. Thus \( \lim_{m \geq 1} \| h_{km} - h_k \|_r = 0 \), and we may apply Lemma (2.26) to conclude

\[
\lim_{m \geq 1} \max_{0 \leq k \leq n} |g_{km} - g_{0m}| - \max_{0 \leq k \leq n} |g_k - g_0|_r = 0.
\]

At the same time, we have

\[
\lim_{m \geq 1} \| f - f^m \|_p = 0 \quad \text{and} \quad \lim_{m \geq 1} \| V_n f - V_n f^m \|_p = 0.
\]

Thus, we may choose an \( m_0 \) sufficiently large that we maintain the relations

\[
\| f^{m_0} \|_p - \| V^{m_0} \|_p < \eta \| f^{m_0} \|_p
\]

and

\[
\max_{0 \leq k \leq n} |g_{km_0} - g_{0m_0}|_r > \frac{\xi}{2} \| f^{m_0} \|_p^{p/r}.
\]

As \( F_m \) is finite, \( (T_1^{m_0}, \ldots, T_n^{m_0}) \) form the initial portion of a special norming sequence for which the hypothesis of the lemma fails. \( \square \)
We have reduced the proof of Theorem (1.2) to verifying that finitary versions of Estimates A and B hold for every special norming sequence. In order to show that this is true, we introduce a dilation of these operators similar to the one given in [A].

(5.7) Definitions. Let \((X, \mathcal{F}, \mu)\) be a measure space in which \(\mathcal{F}\) is a finite set. Let \(\{F_i\}_{i=1}^{d}\) be an enumeration of the atoms of \(\mathcal{F}\) of positive measure. Let the indices \(i\) and \(j\) range through the integers \(\{1, \ldots, d\}\). If \(T\) is a positive operator with a semi-invariant function \(u\), let \(u = \sum_i \alpha_i x_{F_i}\) and \(Tu = \sum_i \beta_i x_{F_i}\). We have \(\alpha_i > 0\) and \(\beta_i > 0\) for each \(i\). Let \(m_i = \mu(F_i)\) and let \(a_{ij} = \omega[\pi^{-1} F_i \cap (F_j \times [0, 1])]\), with \(\pi\) and \(\omega\) as given in §4. Observe that \(\sum_i a_{ij} = m_j\) for each \(j\), and that for each \(i\),

\[\sum_j a_{ij} = m_i\]

Let

\[b_{ij} = \left(\frac{\beta_i}{\alpha_i}\right)^p \frac{a_{ij}}{m_i}.

It is easy to verify that \(\sum_j b_{ij} = 1\). Observe also that \(a_{ij} = 0\) if and only if \(b_{ij} = 0\).

We are going to construct a set \(Z\) in the coordinate plane \(\mathbb{R}^2\) and an isometry of its \(L_p\) space. The construction is virtually identical to the one given in [A] and used in [AS2], except that some of the subrectangles may have measure zero. However, because of the last observation, this will cause no problems.

Let \(\{I_i\}_{i=1}^{d}\) be disjoint intervals on the \(x\)-axis of the coordinate plane, each of length \(m_i\). Let \(\{J_i\}_{i=1}^{d}\) be disjoint intervals on the \(y\)-axis, each of unit length. Let \(P_i = I_i \times J_i\) and \(Z = \bigcup_i P_i\). Let \(Z = (Z, \mathcal{B}, \lambda)\), where \(\mathcal{B}\) is the Borel \(\sigma\)-algebra of \(Z\) and \(\lambda\) is the restriction of Lebesgue measure on \(\mathbb{R}^2\) to \(Z\). Let \(L_p\) denote \(L_p(Z)\), and let \(\mathcal{P}\) be the partition \(\{P_i\}_{i=1}^{d}\) of \(Z\). Let \(E = E(\cdot|\mathcal{P})\) and let \(L^p_p = E L_p\).

Define a further partitioning of \(Z\) as follows. Each \(I_j\) is partitioned into \(d\) subintervals \(\{I_{ij}\}_{i=1}^{d}\), each of length \(a_{ij}\). Each \(J_i\) is partitioned into \(d\) subintervals \(\{J_{ij}\}_{j=1}^{d}\), each of length \(b_{ij}\). Let \(R_{ij} = I_i \times J_{ij}\), a horizontal strip of \(P_i\), and \(S_{ij} = I_{ij} \times J_j\), a vertical strip of \(P_j\).

Define a point transformation \(\tau: Z \to Z\) by mapping each \(R_{ij}\) of nonzero measure to the corresponding \(S_{ij}\), in such a way that the Radon-Nikodým derivative for the mapping of these rectangles is constant. Thus, \(\tau\) "squeezes" the width of \(R_{ij}\) from \(m_i\) to \(a_{ij}\) and "stretches" its height from \(b_{ij}\) to 1; this deformation determines the constant value of

\[\rho = \frac{d(\lambda \circ \tau^{-1})}{d\lambda}\]
on \(S_{ij}\).
$\lambda(R_{ij}) = 0$ if and only if $\lambda(S_{ij}) = 0$, because of the corresponding property of $a_{ij}$ and $b_{ij}$, and so $\tau$ is an automorphism of $Z$. An automorphism of $Z$ determined in this manner by any pair of sequences of $a_{ij}$'s and $b_{ij}$'s satisfying $\sum_i a_{ij} = m_j$, $\sum_j b_{ij} = 1$, and $a_{ij} = 0$ if and only if $b_{ij} = 0$, is called an admissible automorphism. Each admissible automorphism induces an admissible $L_p$ isometry $Q$ in the usual manner by $Qf = \rho^{1/p}(f \circ \tau^{-1})$.

(5.8) Theorem. The action of $EQ$ on $L_p$ is isomorphic to the action of the original operator $T$ on $L_p(X)$.

Proof. Let $i$ range through $\{1, \ldots, d\}$. Let $\Phi$ be given by

$$\sum_i c_i \chi_{F_i} \in L_p \mapsto \sum_i c_i \chi_{\nu_i} \in L_p(X).$$

This is an isometric isomorphism since $\lambda(P_i) = \mu(F_i) = m_i$.

Let $W = (W, \mathcal{F}, \omega)$, $\pi$, $\mathcal{F}$, and $\nu$ be as given in Theorem (4.5). According to that theorem, if we define $R : L_p(d\mu) \to L_p(d\nu)$ by

$$Rg = \frac{\nu}{\mu \circ \pi}(g \circ \pi),$$

then $Tg = E(Rg|\mathcal{F})$ for every $g \in L_p(d\mu)$. Since $u = \sum_i \alpha_i \chi_{F_i}$ and $Tu = \sum_i \beta_i \chi_{\nu_i}$, we have $Rg = (\beta_j/\alpha_i)c_i$ on each $\pi^{-1}F_i \cap (F_j \times [0, 1]) \subseteq F_j \times I$.

When $f \in L_p$, then $Qf = \rho^{1/p}_{ij}c_i$ on each $S_{ij} \subseteq P_j$, where $\rho_{ij}$ is the constant value of the Radon-Nikodym derivatives $\rho$ on the rectangle $S_{ij}$. Observe that

$$\rho_{ij} = \frac{\lambda(R_{ij})}{\lambda(S_{ij})} = \frac{m_i b_{ij}}{a_{ij}} = \left(\frac{\beta_j}{\alpha_i}\right)^{\rho}.$$

We also have $\nu[\pi^{-1}F_i \cap (F_j \times [0, 1])] = \lambda(S_{ij}) = a_{ij}$. This means that $Qf$ and $Rg$ are simple functions taking the same range of values over sets of identical measure. Therefore, $Tg = E(Rg|\mathcal{F}) = \Phi(EQf)$ as desired. \Box

The proof of the convergence of the alternating sequence is now reduced to an examination of the actions of admissible isometries of $Z$, intertwined with the conditional expectation operator with respect to $\mathcal{P}$.

(5.9) Definitions. Let $G$ be a subset $\mathbb{R}^2$. A subset $F$ of $G$ is called a vertical subset of $G$ if

$$F = (F' \times \mathbb{R}) \cap G$$

for some subset $F'$ of the $x$-axis. Similarly, if

$$H = (\mathbb{R} \times H') \cap G$$

for some subset $H'$ of the $y$-axis, then $H$ is called a horizontal subset of $G$.

We say that a function $f$ is constant on vertical lines if $f(x_1, y_1) = f(x_2, y_2)$ whenever $x_1 = x_2$. We say that $f$ is constant on horizontal lines if $f(x_1, y_1) = f(x_2, y_2)$ whenever $y_1 = y_2$.

The following is a summary of Lemmas (4.5) through (4.12) from [AS2].
(5.10) Lemma. Let $\tau$ be an admissible automorphism, and let $Q$ be the induced $L_p$ isometry.

(a) Suppose $\mathcal{F}$ is a finite partition of $\mathbb{Z}$ in which each atom is a vertical subset of some $P_i$. Let $f$ be an $L_p$ function which is constant on vertical lines. Then

$$QE(f|\mathcal{F}) = E(Qf|\mathcal{P} \vee \tau\mathcal{F}).$$

(a') Suppose $\mathcal{H}$ is a finite partition of $\mathbb{Z}$ in which each atom is a horizontal subset of some $P_i$. Let $f$ be an $L_p$ function which is constant on horizontal lines. Then

$$Q^{-1}E(f|\mathcal{H}) = E(Q^{-1}f|\mathcal{P} \vee \tau\mathcal{H}).$$

(b) If $f_1$ and $f_2$ are $L_p$ functions that are constant on vertical lines and $E f_1 = Ef_2$, then also $EQf_1 = EQf_2$.

(b') If $f_1$ and $f_2$ are $L_p$ functions that are constant on horizontal lines and $E f_1 = Ef_2$, then also $EQ^{-1}f_1 = EQ^{-1}f_2$.

(c) If $f$ is constant on vertical lines, then $Qf$ is constant on vertical lines.

(c') If $f$ is constant on horizontal lines, then $Q^{-1}f$ is constant on horizontal lines.

(5.11) Definitions. Let $n$ be a fixed integer, $n \geq 1$, and let $k$ range through \{0, 1, \ldots, n\}. If $1 \leq k \leq n$, let $\tau_k$ be an admissible isometry of $\mathbb{Z}$, let $Q_k$ be the $L_p$ isometry induced by $\tau_k$, and let $R_k$ be the $L_r$ isometry induced by $\tau_k^{-1}$. Let $Q_0$ and $R_0$ be the identities on $L_p$ and $L_r$ respectively. Let

$$T_k = EQ_kE, \quad V_k = T_k \cdots T_0, \quad W_k = Q_k \cdots Q_0,$$
$$S_k = ER_kE, \quad U_k = S_0 \cdots S_k, \quad D_k = R_0 \cdots R_k.$$

Observe that $S_k = (T_k^*)$, by Lemma (3.20).

Let $f$ be a fixed but arbitrary function in $L_p$. Let $g_k = U_k \psi(V_k f)$ and $\phi_k = W_k^{-1}EW_kEf$. Observe that $g_0 = \psi \phi_0 = \psi Ef$.

(5.12) Lemma. For any $f \in L_p$, $V_k f = EW_kEf$.

Proof. This is Lemma (4.14) of [AS2]. When $k = 0$ this is immediate from the definitions. The inductive step is given by Lemma (5.10)(b) and (c). \qed

(5.13) Lemma. For any $g \in L_r$, $U_k g = ED_k Eg$.

Proof. We will show that

$$S_i \cdots S_j g = ER_i \cdots R_j Eg$$

for every pair $i, j$ with $0 \leq i \leq j \leq n$. This will prove the lemma, since the desired identity is (5.14) with $i = 0$ and $j = k$. The proof is by induction on $j - i$. When $i = j$, (5.14) is simply the definition of $S_i g$. 
Now suppose (5.14) holds for some pair $i+1, j+1$ with $0 \leq i < j < n$. We have
\[ ER_{i+1} \cdots R_{j+1} E g = ES_{i+1} \cdots S_{j+1} g , \]
by the inductive hypothesis and the idempotence of $E$, the outermost operator in $S_{i+1}$. $R_{i+1} \cdots R_{j+1} E g$ is constant on horizontal lines, by repeated application of Lemma (5.10)(c'). Thus, by Lemma (5.10)(b'), we have
\[ ER_{i+1} \cdots R_{j+1} E g = ER_{i+1} S_{i+1} \cdots S_{j+1} = S_{i} \cdots S_{j+1} g . \]
This completes the induction. \( \square \)

(5.15) Lemma. $g_k = E \psi(\phi_k)$. 
Proof. 
\[ g_k = U_k \psi(V_k f) = ED_k E \psi(EW_k E f) = ED_k \psi(EW_k E f) = E \psi[(R_0)^p \cdots (R_k)^p EW_k E f] . \]
The second line follows from the two previous lemmas. The third line follows because $\psi$ maps $\mathcal{P}$-measurable functions to $\mathcal{P}$-measurable functions. For the fourth line, we use $(R_i)^p$ to denote the $L_p$ isometry induced by $\tau_i^{-1}$. Thus, this line follows by an application of Lemma (2.19). By Lemma (2.17), that isometry is $Q_i^{-1}$. Thus
\[ g_k = E \psi(W_k^{-1} RW_k E f) = E \psi(\phi_k) , \]
as desired. \( \square \)

(5.16) Lemma. There exists a monotone sequence $\mathcal{G}_n \subseteq \mathcal{G}_{n-1} \subseteq \cdots \subseteq \mathcal{G}_0$ of finite $\sigma$-algebras such that
\[ \phi_k = W_n^{-1} E(W_n E f|\mathcal{G}_k) . \]
Proof. This is Lemma (4.16) of [AS2]. We may take $\mathcal{G}_n = \mathcal{P}$. Lemma (5.10)(a) provides the induction step needed to show that we may take
\[ \mathcal{G}_{n-k} = \mathcal{P} \lor \tau_n \mathcal{P} \lor \cdots \lor \tau_n \cdots \tau_{n-k+1} \mathcal{P} \]
when $1 \leq k \leq n$. \( \square \)

(5.17) Definition. Let $u_k = E(W_n E f|\mathcal{G}_k)$, where the $\mathcal{G}_k$'s are as in the previous lemma. Observe that $\phi_k = W_n^{-1} u_k$.

(5.18) Theorem. The sequence $\langle u_0, \ldots, u_n \rangle$ is an $L_p$ martingale. Furthermore,
\[ \left\| \max_{0 \leq k \leq n} |u_k| \right\|_p \leq q \| u_0 \|_p \]
and
\[ \left\| \max_{0 \leq k \leq n} |u_k - u_n| \right\|_p \leq q \| u_0 - u_n \|_p . \]
Proof. 
\[ u_k = E(W_n E f|\mathcal{G}_k) = E(E(W_n E f|\mathcal{G}_0)|\mathcal{G}_k) = E(u_0|\mathcal{G}_k) , \]
since $\mathcal{F}_k \subseteq \mathcal{F}_0$ for every $k$, $0 \leq k \leq n$. As well,
\[ u_k - u_n = E(u_0 | \mathcal{F}_k) - E(u_n | \mathcal{F}_k) = E(u_0 - u_n | \mathcal{F}_k). \]
In the first case, this follows from the above computation. In the second case, $u_n = E(u_n | \mathcal{F}_k)$ because $u_n$ is already constant on the atoms of $\mathcal{F}_k$.

The lemma now follows by an application of the martingale convergence theorem for $L_p$. □

(5.19) **Theorem.** $\| \max_{0 \leq k \leq n} |g_k| \|_r \leq (q \|f\|_p)^{p/r}$.

**Proof.** Since $\phi_k = W_n^{-1} u_k$ and $W_n^{-1}$ is a positive isometry, we have $|\phi_k| = W_n^{-1} |u_k|$ and $\max_{0 \leq k \leq n} |\phi_k| = W_n^{-1}(\max_{0 \leq k \leq n} |u_k|)$ and so
\[ (5.20) \quad \left\| \max_{0 \leq k \leq n} |\phi_k| \right\|_p = \left\| \max_{0 \leq k \leq n} |u_k| \right\|_p \leq q \|u_0\|_p \leq q \|f\|_p. \]

The inequalities follow by an application of Theorem (5.18) and the fact that $\|u_0\|_p = \|E f\|_p$.

Since $g_k = E \psi(\phi_k)$, we have
\[ \max_{0 \leq k \leq n} |g_k| \leq E \left( \max_{0 \leq k \leq n} |\psi(\phi_k)| \right) = E \psi \left( \max_{0 \leq k \leq n} |\phi_k| \right), \]
where Lemma (2.24) was used for the inequality. Thus
\[ \left\| \max_{0 \leq k \leq n} |g_k| \right\|_r \leq \left\| \psi \left( \max_{0 \leq k \leq n} |\phi_k| \right) \right\|_r = \left\| \max_{0 \leq k \leq n} |\phi_k| \right\|_p^{p/r} \leq (q \|f\|_p)^{p/r}. \] □

(5.21) **Theorem.** For any $\xi > 0$ there is an $\eta > 0$ depending only on $\xi$, $p$, and $r$ such that
\[ \left\| \max_{0 \leq k \leq n} |g_k - g_0| \right\|_r < \xi \|f\|_p^{p/r} \]
whenever $\|f\|_p - \|V_n f\|_p < \eta \|E f\|_p$.

**Proof.** Since $u_n = E(u_0 | \mathcal{F}_n)$, we may apply Lemma (2.25) to choose an $\eta > 0$, depending only on $\delta$ (which will be specified later) and $p$ so that
\[ \|u_0\|_p - \|u_n\|_p < \eta \|u_0\|_p \]
implies
\[ \|u_0 - u_n\|_p < \frac{\delta}{2q} \|u_0\|_p. \]

We have already observed that $\|u_0\|_p = \|E f\|_p$. As well, we note that $\|u_n\|_p = \|V_n f\|_p$. Thus, if $\|f\|_p - \|V_n f\|_p < \eta \|E f\|_p$, we have
\[ \left\| \max_{0 \leq k \leq n} |u_k - u_0| \right\|_p \leq 2 \left\| \max_{0 \leq k \leq n} |u_k - u_n| \right\|_p \leq 2q \|u_0 - u_n\|_p < \xi \|u_0\|_p, \]
where the second inequality follows from Theorem (5.18).
As in the proof of the previous theorem, we deduce

\[ \max_{0 \leq k \leq n} |\phi_k - \phi_0| = \left( \max_{0 \leq k \leq n} |u_k - u_0| \right)^{-1}, \]

and so \( \| \max_{0 \leq k \leq n} |\phi_k - \phi_0| \|_p \leq \delta \| Ef \|_p \).

Since the inequality \( \| \max |\phi_k| \|_p \leq q \| Ef \|_p \) is simply a restatement of (5.20), we are in a position to apply Lemma (2.29). Choose \( \delta \) from that lemma corresponding to \( \xi \), \( q \) (which depends only on \( p \)), \( p \) and \( r \), and conclude that

\[ \left\| \max_{0 \leq k \leq n} |\psi(\phi_k) - \psi(\phi_0)| \right\|_r < \xi \| Ef \|_p^{p/r}, \]

whenever \( \| f \|_p - \| E_n f \|_p < \eta \| Ef \|_p \).

Now apply Lemma (2.24):

\[
\left\| \max_{0 \leq k \leq n} |g_k - g_0| \right\|_r \leq \left\| E \left( \max_{0 \leq k \leq n} |\psi(\phi_k) - \psi(\phi_0)| \right) \right\|_r < \xi \| Ef \|_p^{p/r}.
\]

This completes the proof of this theorem, and hence of Theorem (1.2). \( \Box \)

**References**


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