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A MEASURE OF SMOOTHNESS RELATED TO THE LAPLACIAN

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ABSTRACT. A K-functional on $f \in C(R^d)$ given by

$$\widetilde{K}(f, t^2) = \inf(\|f - g\| + t^2\|\Delta g\|;\ g \in C^2(R^d))$$

will be shown to be equivalent to the modulus of smoothness

$$\tilde{w}(f, t) = \sup_{0<h\le t}\left\|2df(x) - \sum_{i=1}^{d}[f(x + he_i) + f(x - he_i)]\right\| .$$

The situation for other Banach spaces of functions on R^d will also be resolved.

1. INTRODUCTION

The Laplacian of $f(x)$, $x \in R^d$ given by

$$\Delta f(x) = \frac{\partial^2}{\partial x_1^2} f(x) + \cdots + \frac{\partial^2}{\partial x_d^2} f(x) \tag{1.1}$$

was related recently [3] to the discrete Laplacian given by

$$\widetilde{\Delta}_h f(x) = 2df(x) - \sum_{i=1}^{d}(f(x + he_i) + f(x - he_i)) \tag{1.2}$$

where e_i is a fixed given orthonormal base of R^d. It was shown in [3] that for $C(R^d)$, $L_p(R^d)$ where $1 < p < \infty$ or $L_1(R^d)$ the inequality

$$\|h^{-2}\widetilde{\Delta}_h f\| \le M \tag{1.3}$$

(where M is independent of h) is equivalent to the statement: Δf exists in the weak (Sobolev) sense as an element of $L_\infty(R^d)$, $L_p(R^d)$, $1 < p < \infty$, or $\mathscr{M}(R^d)$ (the space of finite measures) respectively and $\|\Delta f\| \le M_1$.

For a space B of functions on R^d on which translation is a continuous isometry, we may define the natural K-functional

$$\widetilde{K}(f, t^2)_B = \inf_{g\in S}\{\|f - g\|_B + t^2\|\Delta g\|_B\} \tag{1.4}$$

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where $g \in S$ means that $\frac{\partial}{\partial x_i} g$ and $(\frac{\partial}{\partial x_i})^2 g$ exist as strong derivatives in B. If the space B is continuously imbedded in $\mathscr{D}'$, the space of Schwartz distributions, we define $\widetilde{K}_*(f, t^2)_B$ by replacing S in (1.4) by S^*, where S^* is the collection of all g, such that $\Delta g \in B$ and Δg is taken in the weak (distributional) sense. It turns out that $\widetilde{K}(f, t^2)_B$ is equivalent to $\widetilde{K}_*(f, t^2)_B$. The main result of the paper is the equivalence of the K-functional $\widetilde{K}$ given by (1.4) with the modulus of smoothness given by

$$(1.5) \qquad \tilde{\omega}(f, t)_B = \sup_{0<h\le t} \left\| 2\,df(x) - \sum_{i=1}^{d} [f(x+he_i) + f(x-he_i)] \right\|_B$$

for some given orthonormal base $\{e_i\}_{i=1}^d$ of R^d. That is, it will be shown that

$$(1.6) \qquad \tilde{\omega}(f, t)_B \sim \widetilde{K}(f, t^2)_B,$$

which means

$$C^{-1}\tilde{\omega}(f, t)_B \le \widetilde{K}(f, t^2)_B \le C\tilde{\omega}(f, t).$$

Proving (1.6), we see that (1.5) is independent of the orthonormal base $\{e_i\}_{i=1}^d$.

For $L_p(R^d)$ the K-functional

$$(1.7) \qquad K_2(f, t^2)_{L_p(R^d)} = \inf_{g\in s} \left(\|f-g\|_{L_p(R^d)} + t^2 \sum_{i,j} \left\| \frac{\partial^2}{\partial x_i \partial x_j} g \right\|_{L_p(R^d)} \right)$$

where g has distributional derivatives and $(\partial^2/\partial x_i \partial x_j)g \in L_p(R^d)$ was shown to be equivalent to

$$(1.8) \qquad \omega^2(f, t)_{L_p(R^d)} = \sup_{0<h\le t} \sup_{\substack{e\in R^d \\ |e|=1}} \|f(x+he) - 2f(x) + f(x-he)\|_{L_p(R^d)}$$

(see [1, pp. 337–339]). In fact the equivalence between (1.7) and (1.8) is valid with the same proof if $L_p(R^d)$ is replaced by a Banach space of functions on R^d for which translation is a continuous isometry and S is replaced by the class of functions for which $(\partial^2/\partial x_i \partial x_j)g$ is a strong derivative in B or when $B \subset \mathscr{D}'$ by the class of functions g for which the distributional derivative $(\partial^2/\partial x_i \partial x_j)g$ is in B.

We should note that $\widetilde{K}(f, t^2)_{C(R^d)}$ and $K_2(f, t^2)_{C(R^d)}$ are not equivalent, and also that $\widetilde{K}(f, t^2)_{L_1(R^d)}$ and $K_2(f, t^2)_{L_1(R^d)}$ are not equivalent. It is virtually obvious that $\widetilde{K}(f, t^2)_{L_p(R^d)}$ and $K_2(f, t^2)_{L_p(R^d)}$ are equivalent for $1 < p < \infty$. A further relation between $\widetilde{K}(f, t^2)_B$ and $K_2(f, t^2)_B$, and therefore between $\tilde{\omega}(f, t)_B$ and $\omega^2(f, t)_B$, is that the Besov spaces generated by them are equivalent. This means that the K-functionals $\widetilde{K}(f, t^2)_B$ and $K_2(f, t^2)_B$ differ only at optimal or near optimal rate.

As characterization of a given K-functional by the behaviour of the function is considered quite a basic problem in interpolation of spaces, I hope the present somewhat combinatorial result will be of interest.

2. Some inequalities

Throughout this section we will assume that $\{e_i\}_{i=1}^d$ is a given fixed orthonormal base of R^d. We define the operator Δ_v^2 for $v \in R^d$

$$(2.1)\qquad \Delta_v^2 f(x) \equiv f(x+v) - 2f(x) + f(x-v), \quad \Delta_v^{2l+2} f(x) \equiv \Delta_v^2(\Delta_v^{2l} f(x)).$$

A Banach space B of functions or distributions on R^d is called translation invariant if

$$(2.2)\qquad \|f(\cdot + v)\|_B = \|f(\cdot)\|_B \quad \text{for all } v \in R^d.$$

We can now state and prove the following result which will be helpful later.

Theorem 2.1. *For a Banach space of functions or distributions on* R^d *which satisfies* (2.2), $\Delta_{he_{i_j}}^{2s_j}$ *given by* (2.1) *and* $\widetilde{\Delta}_h$ *given by* (1.2) *we have*

$$(2.3)\qquad \left\| \sum \Delta_{he_{i_1}}^{2s_1} \cdots \Delta_{he_{i_m}}^{2s_m} f(x) \right\|_B \le C(m, s) \sum_{k=1}^{s_1+\cdots+s_m=s} \|\widetilde{\Delta}_{kh} f\|_B$$

where the sum on the left of (2.3) *is taken on all possibilities* $e_{i_1} \cdots e_{i_m}$ *such that* $e_{i_s} \ne e_{i_n}$ *for* $r \ne s$ *and where* $e_{i_j} \in \{e_i\}_{i=1}^d$.

Remark. Note that for $\widetilde{\Delta}_h f$ to be defined by (1.2) it is sufficient that B is a Banach space of distribution on R^d. Observe that in the sum (2.3) two different orders of $e_{i_1} \cdots e_{i_r}$ are two entries in the sum.

Proof. We first prove our theorem for $m = 1$. We can easily obtain the identity

$$(2.4)\qquad \sum_{i=1}^d \Delta_{he_i}^{2s} f = \sum_{l=0}^{s-1} \binom{2s}{l} (-1)^{l+1} \widetilde{\Delta}_{(s-l)h} f.$$

This will imply (2.3) for $m = 1$. We can now write the identity

$$(2.5)\qquad \left(\sum_{i=1}^d \Delta_{he_i}^{2s_l}\right) \left(\sum_{\substack{e_{i_r} \ne e_{i_t} \\ r \ne t}} \Delta_{he_{i_1}}^{2s_1} \cdots \Delta_{he_{i_{l-1}}}^{2s_{l-1}} \right) f$$
$$= \sum_{\substack{i_r \ne i_t \\ r \ne t}} \Delta_{he_{i_1}}^{2s_1} \cdots \Delta_{he_{i_l}}^{2s_l} f + \sum_{j=1}^{l-1} \sum_{\substack{i_r \ne i_t \\ r \ne t}} \Delta_{he_{i_1}}^{2s_1} \cdots \Delta_{he_{i_j}}^{2s_j + 2s_l} \cdots \Delta_{he_{i_{l-1}}}^{2s_{l-1}} f.$$

Assuming (2.3) for $m < l$ and utilising (2.2), we have

$$\left\| \left(\sum_{i=1}^{d} \Delta_{he_i}^{2s_l} \right) \sum_{\substack{i_r \neq i_t \\ r \neq t}} \Delta_{he_1}^{2s_1} \cdots \Delta_{he_{i_{l-1}}}^{2s_{l-1}} f \right\| \leq C(l-1, s_1 + \cdots + s_{l-1}) 4^{s_l} d \sum_{k=1}^{s_1 + \cdots + s_{l-1}} \|\widetilde{\Delta}_{kh} f\|$$

and

$$\left\| \sum_{\substack{i_r \neq i_t \\ r \neq t}} \Delta_{he_{i_1}}^{2s_1} \cdots \Delta_{he_{i_j}}^{2s_j + 2s_l} \cdots \Delta_{he_{i_{l-1}}}^{2s_l - 1} f \right\| \leq C(l-1, s_1 + \cdots + s_l) \sum_{k=1}^{s_1 + \cdots + s_l} \|\widetilde{\Delta}_{kh} f\|,$$

which, together with (2.5), implies (2.3) for $m = l$. □

We can now prove

Theorem 2.2. *For a Banach space B of functions or distributions on R^d satisfying* (2.2) *and $f \in B$ we have*

$$\|\Delta_{he_i}^{2d} f\|_B \leq C(d) \sum_{k=1}^{d} \|\widetilde{\Delta}_{kh} f\|_B \tag{2.6}$$

for $i = 1, \ldots, n$.

Proof. We may assume with no loss of generality that $i = 1$. We now write

$$\begin{aligned} -\Delta_{he_1}^2 (\widetilde{\Delta}_h) &= \Delta_{he_1}^2 \left(\sum_{i=1}^{d} \Delta_{he_i}^2 \right) = \Delta_{he_1}^4 + \Delta_{he_1}^2 \sum_{i=2}^{d} \Delta_{he_i}^2 \\ &= \Delta_{he_1}^4 + \sum_{i \neq j} \Delta_{he_i}^2 \Delta_{he_j}^2 - \sum_{\substack{i \neq j \\ i, j \neq 1}} \Delta_{he_i}^2 \Delta_{he_j}^2 . \end{aligned}$$

Using

$$\|\Delta_{he_1}^2 \widetilde{\Delta}_h f\| \leq 4 \|\widetilde{\Delta}_h f\|$$

and

$$\left\| \sum_{i \neq j} \Delta_{he_i}^2 \Delta_{he_j}^2 f \right\| \leq C(2)(\|\widetilde{\Delta}_{2h} f\| + \|\widetilde{\Delta}_h f\|),$$

we have

$$\left\| \Delta_{he_1}^4 f - \sum_{\substack{i \neq j \\ i, j \neq 1}} \Delta_{he_i}^2 \Delta_{he_j}^2 f \right\| \leq C_1(2) \sum_{l=1}^{2} \|\widetilde{\Delta}_{lh} f\| .$$

As for $d = 2$, the set $i \neq j$ and $i, j \neq 1$ is empty, (2.6) is proved for $d = 2$.

We continue and show by induction that

$$\left\| \Delta_{he_1}^{2k} f + (-1)^{k-1} \sum_{\substack{i_r \neq i_t \\ r \neq t \\ i_r \neq 1}} \Delta_{he_{i_1}}^2 \cdots \Delta_{he_{i_k}}^2 f \right\| \leq C_1(k) \sum_{l=1}^{k} \|\widetilde{\Delta}_{lh} f\| . \tag{2.7}$$

One can operate on the expression inside the norm on the left of (2.7) to obtain

$$\begin{aligned}
&\Delta_{he_1}^2 (\Delta_{he_1}^{2k}) f + (-1)^{k-1} \Delta_{he_1}^2 \sum_{\substack{i_r \neq i_t \\ r \neq t \\ i_r \neq 1}} \Delta_{he_{i_1}}^2 \cdots \Delta_{he_{i_k}}^2 f \\
&\qquad = \Delta_{he_1}^{2(k+1)} f + (-1)^k \sum_{\substack{i_r \neq i_t \\ r \neq t \\ i_r \neq 1}} \Delta_{he_{i_1}}^2 \cdots \Delta_{he_{i_{k+1}}}^2 f + (-1)^{k-1} \sum_{\substack{i_r \neq i_t \\ r \neq t}} \Delta_{he_{i_1}}^2 \cdots \Delta_{he_{i_{k+1}}}^2 f .
\end{aligned}$$

As (2.3) implies

$$\left\| \sum_{\substack{i_r \neq i_t \\ r \neq t}} \Delta_{he_{i_1}}^2 \cdots \Delta_{he_{i_{k+1}}}^2 f \right\| \leq C \left(\sum_{l=1}^{k+1} \|\widetilde{\Delta}_{lh} f\| \right) ,$$

we have (2.7) for $k+1$. For $k = d$ the sum

$$\sum_{\substack{i_r \neq i_t \\ r \neq t \\ i_r \neq 1}} \Delta_{he_{i_1}}^2 \cdots \Delta_{he_{i_k}}^2$$

is a sum on an empty set, and our theorem is proved. □

We will now obtain an estimate which, while as stated will be a multivariate result, is in fact a result about functions of one variable.

A Banach space of functions on R^d is called strongly continuous if

$$\|f(\cdot + he) - f(\cdot)\|_B = o(1) , \qquad h \to 0 \text{ for all unit vectors } e \text{ in } R^d . \tag{2.8}$$

Theorem 2.3. *Suppose B is a Banach space of functions on R^d for which translation is a strongly continuous isometry, that is (2.2) and (2.8) are satisfied. Then for $0 < u < t$ we can find $\alpha_i(y)$ $(y \in R)$ fixed even polynomials of degree $2i$ such that*

$$\left\| \sum_{i=1}^{k-1} \alpha_i \left(\frac{u}{t} \right) \Delta_{te}^{2i} f - \Delta_{ue}^2 f \right\|_B \leq M \sup_{0 < h \leq t} \|\Delta_{he}^{2k} f\|_B . \tag{2.9}$$

Proof. We define

$$F(s) = \langle f(\cdot + se) , g(\cdot) \rangle \quad \text{for } g \in B^* , \; s \in R ,$$

where B^* is the dual space to B. Obviously, $F(s) \in C(R)$. We first show that (2.9) for F, that is,

$$\left\| \sum_{i=1}^{k-1} \alpha_i \left(\frac{u}{t}\right) \Delta_t^{2i} F - \Delta_u^2 F \right\|_{C(R)} \le M \sup_{0<h\le t} \|\Delta_h^{2k} F\|_{C(R)} \tag{2.10}$$

implies (2.9) in general. Of course, when we deal with $F(s)$ which is a function on $C(R)$, we may write Δ_t^2 rather than Δ_{te}^2 with e the unit vector in $R = R^1$. We choose g such that $\|g\|_{B^*} = 1$ to obtain

$$\begin{aligned} \left| \sum_{i=1}^{k-1} \alpha_i \left(\frac{u}{t}\right) \Delta_t^{2i} F(0) - \Delta_u^2 F(0) \right| &\le \left\| \sum_{i-1}^{k-1} \alpha_i \left(\frac{u}{t}\right) \Delta_t^{2i} F - \Delta_u^2 F \right\|_{C(R)} \\ &\le M \sup_{0<h\le t} \|\Delta_h^{2k} F\|_{C(R)} \le M \sup_{0<h\le t} \|\Delta_{he}^{2k} f\|_B . \end{aligned}$$

We can now choose a g (of those that satisfy $\|g\|_{B^*} = 1$) that depends on u, t, f and $\varepsilon > 0$ such that

$$\begin{aligned} \left\| \sum_{i=1}^{k-1} \alpha_i \left(\frac{u}{t}\right) \Delta_{te}^{2i} f - \Delta_{he}^2 f \right\|_B - \varepsilon &\le \left| \sum_{i=1}^{k-1} \alpha_i \left(\frac{u}{t}\right) \Delta_t^{2i} F(0) - \Delta_u^2 F(0) \right| \\ &\le M \sup_{0<h\le t} \|\Delta_{he}^{2k} f\|_B , \end{aligned}$$

and as ε is arbitrary, we complete the proof of (2.9) given (2.10). In fact, we established that the present result is essentially about functions on R.

We now construct α_i such that the operator

$$O_{t,u,k} \equiv \sum_{i=1}^{k-1} \alpha_i \left(\frac{u}{t}\right) \Delta_t^{2i} - \Delta_u^2 \tag{2.11}$$

annihilates polynomials of degree $2k-1$. It is easy to see, and probably helpful in understanding the direction of the proof, that $\alpha_1(u/t) = u^2/t^2$ and that $(u^2/t^2)\Delta_t^2 - \Delta_u^2$ annihilates polynomials of degree 3. The construction follows by induction. We assume that we have constructed for $i < l$ even polynomials of degree $2i$, $\alpha_i(y)$ so that $O_{t,u,l}$ annilhilates polynomials of degree $2l-1$. As Δ_t^{2l} also annihilates polynomials of degree $2l-1$, we construct $\alpha_l(y)$ so that at $x = 0$ we have

$$\alpha_l \left(\frac{u}{t}\right) \Delta_t^{2l} x^{2l} = \alpha_l \left(\frac{u}{t}\right) (2l)! t^{2l} = -O_{t,u,l} x^{2l} .$$

As $O_{t,u,l}$ annihilates polynomials of degrees $2l-1$, it is sufficient to calculate $O_{t,u,l}(x-a)^{2l}$ at a or $O_{t,u,l}x^{2l}$ at 0 (which is equal), and therefore,

$$-O_{t,u,l} x^{2l} = 2u^{2l} - 2\sum_{i=1}^{l-1} \alpha_i \left(\frac{u}{t}\right) \left\{ \sum_{r=0}^{i} \binom{2i}{r} (-1)^r (i-r)^{2l} \right\} t^{2l} .$$

As $\alpha_i(\frac{u}{t})$ is $\frac{-1}{(2l)!}O_{t,u,l}x^{2l}$, it is even polynomial in $(\frac{u}{t})$ of degree $2l$, and in fact we can effectively estimate its coefficients. We have now shown that $O_{t,u,l+1}$ annihilates polynomials of degree $2l$. As $O_{t,u,l+1}(x-a)^{2l+1}$ at a is equal to zero for all a, and as polynomials can be expanded in $(x-a)^j$, we have constructed α_i as desired.

We now recall (see [1, p. 339, Theorem 4.12] for instance) that for any function $F \in C(R)$ we can find a function G_t such that $G_t^{(2k-1)} \in \text{A.C.}_{\text{loc}}$, $G_t^{(2k)} \in L_\infty$,

$$\|F - G_t\|_{C(R)} \leq A \sup_{0<h\leq t} \|\Delta_h^{2k} F\|_{C(R)}$$

and

$$t^{2k}\|G_t^{(k)}\| \leq \sup_{0<h\leq t} \|\Delta_h^{2k} F\|_{C(R)}.$$

As the operator $O_{t,u,k}$ is linear, we can estimate it separately on $F - G_t$ and G_t. The estimate of $O_{t,u,k}(F-G_t)$ is trivial since $O_{t,u,k}$ is bounded. The estimate of $O_{t,u,k}G_t$ follows from Taylor's formula, the annihilation of $(x-a)^j$ for $j < 2k$ and easy computation on the remainder. □

In fact, we can prove an analogue of Theorem 2.3 for a Banach space of distribution (generalized functions) on R^d, that is a space $B \subset \mathscr{D}'$ (where $\mathscr{D}'$ is the Schwartz space of distributions) such that B is continuously imbedded in $\mathscr{D}'$.

Theorem 2.4. *Suppose B is a Banach space of distributions which is strongly, weakly or weakly* continuous and on which translations are isometries. Then for the $\alpha_i(y)$ $(y \in R)$ of Theorem 2.3, (2.9) is valid.*

Proof. We recall that B is weakly continuous if

(2.12) $\langle f(\cdot + v) - f(\cdot), g(\cdot)\rangle = o(1)$, $\quad |v| \to 0$ for all $f \in B$ and $g \in B^*$

and weakly* continuous if $B = X^*$ and

(2.12)$'$ $\langle f(\cdot + v) - f(\cdot), g(\cdot)\rangle = o(1)$, $\quad |v| \to 0$ for all $f \in B$ and $g \in X$.

We now define $F(s)$ as in Theorem 2.3 for B which is strongly or weakly continuous and by

$$F(s) = \langle f(\cdot + se), g(\cdot)\rangle \quad \text{for } g \in X,\ f \in B \text{ where } B = X^*$$

for B which is weakly* continuous. We now follow the first part of the proof of Theorem 2.3 to show that (2.10) is sufficient for our result. The assertion (2.10) was proved while proving Theorem 2.3. □

3. The upper estimate

We recall the definition $\widetilde{K}(f, t^2)_B$ and $\widetilde{K}_*(f, t^2)_B$. The K-functional $\widetilde{K}(f, t^2)_B$ is given by

$$\widetilde{K}(f, t^2)_B = \inf_{g\in S}(\|f - g\|_B + t^2\|\Delta g\|_B)$$

where S is the class of elements of B for which the strong first two derivatives exist and are in B. The K-functional $\widetilde{K}_*(f, t^2)_B$ is defined for $B \subset \mathscr{D}'$ (B is continuously imbedded in the Schwartz space of distributions $\mathscr{D}'$) with S^* replacing S, and S^* is the class of elements of $\mathscr{D}'$ for which the weak Laplacian belongs to B. We note that it is possible that $\widetilde{K}(f, t^2) = O(t^2)$ and at the same time $(\partial/\partial x_i)^2 f$ is not in B or does not exist even locally.

Theorem 3.1. *Suppose $f \in B$, translations on B are isometries and the expressions $\widetilde{K}(f, t^2)_B$ and $\tilde{\omega}(f, t)_B$ are given by* (1.4) *and* (1.5) *respectively. Then we have for a Banach space B over R^d for which translations are strongly continuous*

$$\widetilde{K}(f, t^2)_B \leq C\tilde{\omega}(f, t)_B. \tag{3.1}$$

If $B \subset \mathscr{D}'$ and translations on B are strongly, weakly or weakly continuous, then*

$$\widetilde{K}_*(f, t^2)_B \leq C\tilde{\omega}(f, t)_B. \tag{3.1$'$}$$

Proof. Using the definitions of $\widetilde{K}(f, t^2)$ and $\widetilde{K}_*(f, t^2)$,

$$\widetilde{K}(f, (at)^2) \leq a^2\widetilde{K}(f, t^2), \text{ and } \widetilde{K}_*(f, (at)^2) \leq a^2\widetilde{K}_*(f, t^2) \quad \text{for } a > 1. \tag{3.2}$$

As d is a fixed finite integer, it is sufficient to prove

$$\widetilde{K}(f, t^2) \leq C\tilde{\omega}(f, td^2) \quad \text{and} \quad \widetilde{K}_*(f, t^2) \leq C\tilde{\omega}(f, td^2). \tag{3.3}$$

To prove (3.3) we construct the function

$$g_t(x) = t^{-2d}\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2} f(x + (u_1 + u_2)e_1 + \cdots + (u_{2d-1} + u_{2d})e_d)du_1\cdots du_{2d} \tag{3.4}$$

and show that

$$\|g_t - f\| \leq C\tilde{\omega}(f, td^2) \tag{3.5}$$

and

$$\|\Delta g_t\| \leq Ct^{-2}\tilde{\omega}(f, td^2). \tag{3.6}$$

In (3.6) Δg_t is taken as a sum of strong derivatives for the estimate of $\widetilde{K}(f, t^2)_B$ and a weak distributional derivative for the estimate of $\widetilde{K}_*(f, t^2)_B$.

We prove (3.5) first. For $1 \leq l \leq d$ we denote

$$\begin{aligned} I(f, l) &\equiv I(f, l, e_{i_1}, \dots, e_{i_d}) \\ &\equiv t^{-2d+2l-2}\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2} f\left(x + \sum_{j=l}^{d}(u_{2j-1} + u_{2j})e_{i_j}\right) du_{2l-1}\cdots du_{2d}, \\ I(f, d+1) &\equiv f(x). \end{aligned} \tag{3.7}$$

We may now write

$$g_t(x) - f(x) = \frac{1}{d!}\sum\sum_{l=1}^{d}[I(f, l) - I(f, l+1)]$$

where the first sum indicates summation on all permutations of $e_1 \cdots e_d$. Therefore, it is sufficient to estimate

$$\sum(I(f, l) - I(f, l+1))$$

for a fixed l, $1 \le l \le d$, where the sum is taken on all permutations of $e_1 \cdots e_d$. We can write

$$\begin{aligned}
&I(f, l) - I(f, l+1) \\
&\quad = t^{-2}\int_0^{t/2}\int_0^{t/2} \Delta^2_{(u_{2l-1}+u_{2l})e_{i_l}} I(f, l+1)\, du_{2l-1}\, du_{2l} \\
&\quad = t^{-2}\int_0^{t/2}\int_0^{t/2} \Delta^2_{(u_{2l-1}+u_{2l})e_{i_l}} f(x)\, du_{2l-1}\, du_{2l} \\
&\qquad + \sum_{k=l+1}^{d} t^{-2}\int_0^{t/2}\int_0^{t/2} \Delta^2_{(u_{2l-1}+u_{2l})e_{i_l}} (I(f, k) - I(f, k+1))\, du_{2l-1}\, du_{2l} \\
&\quad = \sum_{\{l_i\}_{i=1}^m \subset \{j\}_l^d,\, l_1 = l} t^{-2m} \\
&\qquad \times \int_0^{t/2} \cdots \int_0^{t/2} \left\{\prod_{k=1}^{m} \Delta^2_{(u_{2k-1}+u_{2k})e_{l_k}}\right\} f(x)\, du_1 \cdots du_{2m}.
\end{aligned}$$

To summarize,

$$\begin{aligned}
g_t(x) - f(x) &= \frac{1}{d} t^{-2}\int_0^{t/2}\int_0^{t/2} \tilde{\Delta}_{(u_1+u_2)} f(x)\, du_1\, du_2 \\
&\quad + \frac{1}{d!}\sum\nolimits^{*} \sum_{m=2}^{d} t^{-2m} \int_0^{t/2} \cdots \int_0^{t/2} \\
&\quad \times \left\{\sum\nolimits^{**} \Delta^2_{(u_1+u_2)e_{j_1}} \cdots \Delta^2_{(u_{2m-1}+u_{2m})e_{j_m}} f(x)\right\} du_1 \cdots du_{2m} \\
&\equiv I + J,
\end{aligned}$$

where $\sum^*$ is the sum on all permutations of $\{e_i\}$, and $\sum^{**}$ is the sum on all possible collections $e_{j_1} \cdots e_{j_m}$ such that $e_{j_r} \in \{e_j^*\}$, (where $\{e_j^*\}$ is the given permutation) and $e_{j_r} \ne e_{j_s}$ for $r \ne s$. The estimate of J is obvious. For the estimate of I we will need the results in Theorems 2.2 and 2.3.

To estimate I it is clear that we have only to estimate terms of the type

$$\sum \Delta^2_{(u_1+u_2)e_{j_1}} \cdots \Delta^2_{(u_{2m-1}+u_{2m})e_{j_m}} f(x) \tag{3.8}$$

where the sum is taken on all sequences of m different vectors $e_{j_1} \cdots e_{j_m}$ such that $e_{j_l} \in \{e_i\}_{i=1}^d$ and u_i satisfy $u_i \ge 0$ and $u_{2j-1} + u_{2j} \le t$. We can write the expression in (3.8) as

$$\sum \Delta^2_{v_1 e_{j_1}} \cdots \Delta^2_{v_m e_{j_m}} f(x)$$

with $0 < v_i \le t$. Following Theorems 2.3 and 2.4, we have

$$\begin{aligned} &\sum \Delta^2_{v_1 e_{j_1}} \cdots \Delta^2_{v_m e_{j_m}} f(x) \\ &\quad = (-1)^m \sum \prod_{l=1}^{m} \left(\sum_{i=1}^{d-1} \alpha_i \left(\frac{v_l}{t}\right) \Delta^{2i}_{te_{j_l}} - \Delta^2_{v_l e_{j_l}} \right) f(x) \\ &\qquad + (-1)^{m+1} \sum \prod_{l=1}^{m} \sum_{i=1}^{d-1} \alpha_i \left(\frac{v_l}{t}\right) \Delta^{2i}_{te_{j_l}} f(x) + R \\ &\quad \equiv S + L + R \end{aligned}$$

where R is a finite sum of translations of

$$\sum_{i=1}^{d-1} \alpha_i \left(\frac{v_l}{t}\right) \Delta^{2i}_{te_{j_l}} - \Delta^2_{v_l e_{j_l}} f(x)$$

multiplied by finite products of $\alpha_i(v_j/t)$, and $\alpha_i(v_j/t)$ are those given in Theorems 2.3 and 2.4, and therefore, bounded for $0 \le v_j \le t$. We now use Theorems 2.3, 2.4 and 2.2 to obtain

$$\begin{aligned} \left\| \left\{ \sum_{i=1}^{d_1} \alpha_i \left(\frac{v_l}{t}\right) \Delta^{2i}_{te_{j_l}} - \Delta^2_{v_l e_{j_l}} \right\} f \right\| &\le M \sup_{0<h\le t} \|\Delta^{2d}_{he_{j_l}} f\| \\ &\le M_1 \sup_{0<h\le t} \sum_{k=1}^{d} \|\tilde{\Delta}_{kh} f\| \le M_2 \tilde{\omega}(f, td). \end{aligned} \tag{3.9}$$

The inequality (3.9) yields the appropriate estimate for S and R.

To estimate L we have to estimate terms like

$$\sum \Delta^{2i_l}_{te_{j_1}} \cdots \Delta^{2i_m}_{te_{j_m}} f(x)$$

where the sum is taken to all subcollections of different $e_{j_1} \cdots e_{j_m}$, and therefore, using Theorem 2.1,

$$\begin{aligned} \left\| \sum \Delta^{2i_1}_{te_{j_1}} \cdots \Delta^{2i_m}_{te_{j_m}} f \right\| &\le M_2 \sum_{k=1}^{i_1+\cdots+i_m} \|\tilde{\Delta}_{kt} f\| \\ &\le M_3 \tilde{\omega}(f, (d-1)mt) \\ &\le M_3 \tilde{\omega}(f, d^2 t). \end{aligned}$$

To prove the second crucial inequality, i.e. the estimate (3.6), we write
(3.10)

$$\Delta g_t = t^{-2}\sum_{i=1}^{d}\Delta_{t_{e_i}} t^{-2d+2} \times \left\{\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2} f\left(x+\sum_{l\neq i}(u_{2l-1}+u_{2l})e_l\right) du_1\cdots du_{2i-2} \times du_{2i+1}\cdots du_{2d}\right\}.$$

The expression (3.10) for Δg_t follows as a strong derivative of g_t in case B is strongly continuous and as a weak (distributional) derivative for $B\subset\mathscr{D}'$.

We now follow earlier considerations to obtain

$$\Delta g_t = t^{-2}\sum_{i=1}^{d}\Delta_{te_i}^2 f(x)+\frac{1}{d!}t^{-2}\sum_{i=1}^{d}\Delta_{te_i}^2 \times\left\{\sum_{e_{j_l}\neq e_i}^{*} t^{-2d+2}\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2} f\left(x+\sum_{r=1}^{d-1}(u_{2r-1}+u_{2r})e_{j_r}\right) \times du_1\cdots du_{2d-2}-f(x)\right\}$$

where the sum $\sum^*$ is taken on all permutations of e_j with $j\neq i$. The estimate (3.6) now follows the estimate used for (3.5) with the change that here we have to estimate

$$t^{-2}\sum\Delta_{te_{j_1}}^2\Delta_{(u_1+u_2)e_{j_2}}^2\cdots\Delta_{(u_{2m-3}+u_{2m-2})e_{j_m}}^2$$

for $m>1$ and the sum is on all $j_1\cdots j_m$ such that $e_{j_k}\in\{e_i\}_{i=1}^d$ and $e_{j_r}\neq e_{j_s}$ for $r\neq s$ rather than (3.8). The fact that the first term is already $\Delta_{te_{j_1}}^2$ does not make much difference to the proof which follows that of (3.8) but leads to somewhat smaller bounds. □

4. The lower estimate

We now estimate $\tilde{\omega}$ by $\widetilde{K}$ or $\widetilde{K}_*$ to complete the proof of the equivalence of $\tilde{\omega}$ with these K-functionals. We first prove the result for $B=C(R^d)$.

Theorem 4.1. *For* $f\in C(R^d)$

$$\tilde{\omega}(f,t)_{C(R^d)}\leq C\widetilde{K}(f,t^2)_{C(R^d)}. \tag{4.1}$$

Remark. For the sake of Theorem 4.1 the infimum in the definition of $\widetilde{K}$ (see (1.4)) is on all g such that Δg exists in the strong sense and is in $C(R^d)$.

However. as we will see later, existence can be taken in the weak (Sobolev) sense or weak* sense and in $L_\infty(R^d)$ with no effect on (4.1).

Proof. From the definition of $\widetilde{K}(f, t^2)_{C(R^d)}$ it follows that a function $g_t \in C(R^d)$ exists satisfying

$$\|f - g_t\|_B \le 2\widetilde{K}(f, t^2)_B \tag{4.2}$$

and

$$t^2\|\Delta g_t\|_B \le 2\widetilde{K}(f, t^2)_B \tag{4.3}$$

for $B = C(R^d)$ where Δg_t is the strong derivative in $C(R^d)$. Obviously,

$$\tilde{\omega}(f, t)_B \le \tilde{\omega}(f - g_t, t)_B + \tilde{\omega}(g_t, t)_B \tag{4.4}$$

for $B = C(R^d)$ as well as for other B. For any g_t satisfying (4.2) we have

$$\omega(f - g_t, t)_B \le 4d\|f - g_t\|_B \le 8d\widetilde{K}(f, t^2)_B. \tag{4.5}$$

To estimate $\omega(g_t, t)_{C(R^d)}$ we use the result of [3, p. 112] and write

$$\begin{aligned}\tilde{\omega}(g_t, t)_{C(R^d)} &\le \sup_{0<h\le t} \|\widetilde{\Delta}_h g_t\|_{C(R^d)} \\ &\le t^2 \sup_{0\le h\le t} \|h^{-2}\widetilde{\Delta}_h g_t\|_{C(R^d)} \le Mt^2\|\Delta g_t\|_{C(R^d)},\end{aligned} \tag{4.6}$$

which concludes the proof of our theorem. □

We can now deduce the lower estimate, for other spaces B, of $\widetilde{K}(f, t^2)_B$ as well as of $\widetilde{K}_*(f, t^2)_B$.

Theorem 4.2. *Suppose $f \in B$, translation on B are isometries and the expressions $\widetilde{K}(f, t^2)_B$ and $\tilde{\omega}(f, t)_B$ are given by* (1.4) *and* (1.5) *respectively. Then for a Banach space for which translations are strongly continuous*

$$\tilde{\omega}(f, t)_B \le A\widetilde{K}(f, t^2)_B; \tag{4.8}$$

and for a Banach space B, $B \subset \mathscr{D}'$, for which translations are weakly continuous and $\mathscr{D}$ dense in B^* or weakly continuous and $\mathscr{D}$ dense in X $(X^* = B)$*

$$\tilde{\omega}(f, t)_B \le A\widetilde{K}_*(f, t^2)_B. \tag{4.9}$$

Proof. We follow the proof of Theorem 4.1. From the definitions of $\widetilde{K}$ and $\widetilde{K}_*$ we have a function g_t satisfying

$$\|f - g_t\|_B \le 2\widetilde{K}(f, t^2)_B \quad \text{or} \quad \|f - g_t\|_B \le 2\widetilde{K}_*(f, t^2)_B$$

and

$$t^2\|\Delta g_t\| \le 2\widetilde{K}(f, t^2)_B \quad \text{or} \quad \|\Delta g_t\|_B \le 2\widetilde{K}_*(f, t^2)_B.$$

This implies (4.5) for the first step in the proof of (4.8) or with $\widetilde{K}_*(f, t^2)_B$ replacing $\widetilde{K}(f, t^2)_B$ for the corresponding step in the proof of (4.9). To continue with the proof of (4.8) we write

$$G_t(x) \equiv \langle g_t(\cdot + x), \varphi(\cdot)\rangle, \qquad \varphi \in B^*, \ \|\varphi\|_{B^*} \le 1,$$

and therefore, ΔG_t exists in the sense of strong derivatives in $C(R^d)$ and it satisfies

$$\|\Delta G_t\|_{C(R^d)} \le \|\Delta g_t\|_B .$$

This implies for $0 < h \le t$

$$\|\widetilde{\Delta}_h G_t\|_{C(R^d)} \le Mt^2 \|\Delta G_t\|_{C(R^d)} \le Mt^2 \|\Delta g_t\|_B$$

with M that is independent of t or φ. Therefore, for $0 < h \le t$

$$\|\widetilde{\Delta}_h g_t\| \le Mt^2 \|\Delta g_t\|_B \le 2Mt^2 \widetilde{K}(f, t^2)_B ,$$

which completes the proof of (4.8).

To continue with the proof of (4.9) we define again

$$G_t(x) \equiv \langle g_t(\cdot + x), \varphi(\cdot)\rangle$$

with $\varphi \in B^*$ when translations are weakly continuous and $\mathscr{D}$ is dense in B^* and with $\varphi \in X$ when translations are weakly* continuous on $B = X^*$ and $\mathscr{D}$ is dense in X. We now complete the proof of (4.9) following the steps used earlier for the proof of (4.8). □

5. Remarks

Remark 5.1. We obtain the K-functional $\widetilde{K}^{**}(f, t^2)_B$ by replacing S of $\widetilde{K}(f, t^2)_B$ by S^{**} where S^{**} is the collection of all the functions g for which Δg exists in the weak or weak* sense and B is such that translations are weakly or weakly* continuous respectively. This K-functional is also equivalent to $\tilde{\omega}(f, t)_B$.

Remark 5.2. If for $L_p(R^d)$, $1 \le p \le \infty$, we define S^{***} by

$$S^{***} = \left\{ g \in L_p(R^d);\ \frac{\partial}{\partial u} g(x + he_i) \in \text{A.C.}_{\text{loc}} \text{ and } \Delta g \in L_p(R^d) \right\} ,$$

the corresponding K-functional, $K_{***}(f, t^2)_p$, is equivalent to $\tilde{\omega}(f, t)_p$.

Remark 5.3. From the above results and those in earlier sections we see that for some spaces the K-functionals $\widetilde{K}(f, t^2)$, $\widetilde{K}_*(f, t^2)$, $\widetilde{K}_{**}(f, t^2)$ and $\widetilde{K}_{***}(f, t^2)$ are defined and are equivalent. In fact in $L_p(R^d)$ and in other situations they are equal.

Remark 5.4. The K-functionals $\widetilde{K}(f, t^2)_B$ and $\widetilde{K}_*(f, t^2)_B$ are independent of the orthogonal system $\{e_i\}$, and therefore, so is the corresponding $\tilde{\omega}(f, t)_B$.

Remark 5.5. For $1 \le p \le \infty$ one usually investigates the K-functional

$$K_2(f, t^2)_p = \inf_{g \in A} \left(\|f - g\| + t^2 \sum_{i,j} \left\| \frac{\partial^2}{\partial x_i \partial x_j} g \right\|_p \right)$$

where A is the class for which $\partial^2 g/\partial x_i \partial x_j \in L_p$ (and the derivatives are taken in the Sobolev sense). For $1 < p < \infty$, $\widetilde{K}(f, t^2)_p$ and $K_2(f, t^2)_p$ are equivalent as

$$\left\| \frac{\partial^2}{\partial x_i \partial x_j} g \right\|_p \le C(p) \|\Delta g\|_p \quad (\text{for } 1 < p < \infty).$$

Remark 5.6. For $p = \infty$ and $p = 1$, $K_2(f, t^2)_p$ is no longer equivalent to $\widetilde{K}(f, t^2)_p$; otherwise the inequality

$$\left\| \frac{\partial^2 f}{\partial x \partial y} \right\|_p \le C \|\Delta f\|_p,$$

which is valid only for $1 < p < \infty$, would be valid for $p = 1$ and $p = \infty$ as well, and this is not true. For $p = \infty$ an example can be given that shows that $K_2(f, t^2)_\infty$ is not equivalent to $\widetilde{K}(f, t^2)_\infty$ (see for instance [2]).

Remark 5.7. In the next section we will compare Besov space induced by $\widetilde{K}(f, t^2)_B$ and $K_2(f, t^2)_B$ and find them equivalent, and hence only classes of functions with optimal or near optimal behaviour are different.

6. Besov type spaces

For a given K-functional the related Besov space is given by its norm

$$\|f\|_{B_q^\alpha} = \left\{ \int_0^1 (t^{-\alpha} K(f, t))^q \frac{dt}{t} \right\}^{1/q}, \qquad 1 \le q < \infty. \tag{6.1}$$

$$\|f\|_{B_\infty^\alpha} = \sup_t t^{-\alpha} K(f, t).$$

Generally, an additional index p is given to indicate that the K functional is between the space L_p and a subspace of L_p. However, we allow here a general class of spaces B rather than L_p to be the underlying space for our K-functional.

We define the spaces $\widetilde{B}_q^\alpha$ and $\overline{B}_q^\alpha$ as the Besov spaces introduced by (6.1) with the K-functionals $\widetilde{K}(f, t^2)_B$ and $K_2(f, t^2)_B$ respectively.

We can now state and prove the following equivalence theorem.

Theorem 6.1. *For $1 \le q \le \infty$ and $0 < \alpha < 2$ the spaces $\widetilde{B}_q^\alpha$ and $\overline{B}_q^\alpha$ are equivalent.*

Proof. For $\alpha < 2$, $\overline{B}_q^\alpha$ is clearly equivalent to B_q^α given by

$$\|f\|_{B_q^\alpha} = \begin{cases} \left\{ \int_0^1 (t^{-\alpha} \omega^{-2d}(f, t)_B)^q \frac{dt}{t} \right\}^{1/q}, & 1 \le q \le \infty, \\ \sup_t t^{-\alpha} \omega^{2d}(f, t)_B, & q = \infty, \end{cases} \tag{6.2}$$

where

$$\omega^{2d}(f, t)_B = \sup_{\substack{0<h\le t \\ e \in S}} \|\Delta_{he}^{2d} f\|_B$$

as it is valid also for any integer r replacing $2d$ in (6.2).

Using Theorem 2.2, it follows that

$$\omega^{2d}(f, t)_B \leq C\widetilde{K}(f, d^2 t) \tag{6.3}$$

since the direction e_i given in that theorem is immaterial because the Laplacian can be expressed in different coordinates. Therefore,

$$\|f\|_{B^\alpha_q} \leq C\|f\|_{\overline{B}^\alpha_q} \leq C_1\|f\|_{\overline{B}^\alpha_q} \leq C_2\|f\|_{B^\alpha_q}. \quad \square \tag{6.4}$$

7. On an a priori estimate

A priori estimates are discussed in many articles (see for instance [4]) and we would like to note that the present result makes a small contribution in this direction.

Theorem 7.1. *Suppose B is a Banach space for which translations are isometries on R^d, and $B \subset \mathscr{D}'$ (where $\mathscr{D}'$ is the space of distributions) translations are strongly continuous on B and $\mathscr{D}$ is dense in the dual to B or in the predual of B. Then Δf exists in $\mathscr{D}'$ and $\Delta f \in \mathrm{Lip}(\alpha, B)$, $0 < \alpha < 1$, implies $f \in \mathrm{Lip}^*(2 + \alpha, B)$.*

Remark 7.2. There are other analogues of Theorem 7.1, but I believe the present form will be of use. The condition $0 < \alpha < 1$ in Theorem 7.1 can be replaced by any $\beta > 0$ such that $[\beta] \neq \beta$. Recall that $g \in \mathrm{Lip}^*(\beta, B)$ means

$$\|\Delta_h^r g\| \leq Mh^\beta \quad \text{for } r > \beta. \tag{7.1}$$

Proof. We can write

$$\|\Delta\Delta_{he} f\|_B = \|\Delta_{he}\Delta f\|_B \leq Ch^\alpha.$$

Using [3], we have

$$\|\eta^{-2}\widetilde{\Delta}_\eta g\| \leq M\|\Delta g\|,$$

and therefore,

$$\|\eta^{-2}\widetilde{\Delta}_\eta \Delta_{he} f\| \leq \|\Delta_{he}\Delta f\|.$$

Using Theorem 2.2, we have

$$\begin{aligned}
\|\Delta_{he}^{2d}\Delta_{he} f\| &\leq C\sum_{j=1}^{d}\|\widetilde{\Delta}_{jh}\Delta_{he} f\| \\
&\leq C_1 h^2 \sum_{j=1}^{d^2}\|(jh)^{-2}\widetilde{\Delta}_{jh}\Delta_{he} f\| \\
&\leq C_2 h^2 \sup_\eta \|\eta^{-2}\widetilde{\Delta}_\eta \Delta_{he} f\| \\
&\leq C_2 h^{2+\alpha},
\end{aligned}$$

which concludes the proof of our theorem. $\square$

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