VANISHING OF $H^2_w(M, K(H))$ FOR CERTAIN FINITE VON NEUMANN ALGEBRAS

FLORIN RÂDULESCU

ABSTRACT. We prove the vanishing of the second Hochschild cohomology group $H^2_w(M, K(H))$, whenever $M \subset B(H)$ is a finite countably decomposable von Neumann algebra not containing a non $\Gamma$-factor or a factor without Cartan subalgebra as a direct summand. Here $H$ is a Hilbert space, and $K(H)$ the compact operators.

1. Introduction

The cohomology of operator algebras introduced by B. E. Johnson, R. V. Kadison, and J. R. Ringrose in a series of three papers is a useful tool for obtaining new invariants for operator algebras or to prove stability results by the vanishing of their cohomology groups (see [14]).

If $M$ is a von Neumann algebra and a Banach bimodule over $M$, and if $n$ is a positive integer, then the $n$th cohomology group of $M$ is denoted by $H^n_c(M, X)$. If $X$ is also a normal dual bimodule, then $H^n_w(M, X)$ is the $n$th weakly continuous cohomology group and it is proved in [14] that $H^n_c(M, X)$ is isomorphic to $H^n_w(M, X)$ and vanishes whenever $M$ is approximately finite dimensional. Actually, the results of Alain Connes show that the vanishing of $H^n(M, X)$ for every normal dual Banach bimodule over $M$ is equivalent to the injectivity of $M$ (see [2]).

If $M \subseteq B(H)$ (the space of all linear bounded operators on a Hilbert space $H$), then $B(H)$ itself is a normal dual Banach bimodule over $M$; the most interesting examples of dual normal bimodules are $B(H)$ and $M$.

By the work of E. Christensen (see [3]) it is known that $H^1_c(M, B(H)) = 0$ in most cases (see also [5] for results concerning the higher cohomology groups). It was very well known that $H^k_c(M, M)$ for nonnegative $M$ vanishes for $k = 1$, but for $k = 2$ nothing was known (excepting an example of B. E. Johnson [9]) until recently. It was proved by E. Christensen and A. Sinclair [6] that it vanishes if $M$ has property $\Gamma$.

When $X$ is no longer a normal dual bimodule, the proof of vanishing of $H^n_c(M, X)$ can be difficult even for injective $M$ (see [9]).

Received by the editors July 10, 1989. Presented to the International Conference on Operator Algebras and Ergodic Theory, Craiova, Romania, September '89.


©1991 American Mathematical Society
0002-9947/91 $1.00 + $.25 per page

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
An interesting example of nondual bimodule over $M \subseteq B(H)$ is $K(H)$ (the compact linear operators on $H$). Computation of the cohomology groups with coefficients into $K(H)$ may be interesting for its connections with the Brown-Douglas-Fillmore theory of extensions of $C^*$-algebras.

Johnson and Parrott proved in the early 70's [10] that $H^1_c(M, K(H))$ vanishes whenever $M$ is abelian. They deduced that the same holds true if $M$ does not contain type II$_1$ factors without Cartan subalgebras as direct summands. Sorin Popa proved the striking results that $H^1_c(M, K(H)) = 0$ for arbitrary type II$_1$ factors $M$, so that this group vanishes for any von Neumann algebra $M$ [12] (see also [13] for a related result).

Note also that in the case of the first cohomology no mention of topology is necessary since the cochains are this time derivations and, by [14], any derivation from $M$ into $B(H)$ is automatically ultraweakly continuous.

In particular, $H^1_w(M, K(H))$ is isomorphic to $H^1_c(M, K(H))$ and hence also null.

In this paper we are concerned with the vanishing of the second cohomology group $H^2_w(M, K(H))$ for finite von Neumann algebras of countable type not containing certain type II$_1$ factors as direct summands. (The subscript $w$ means that we consider only cochains which are ultraweakly continuous with respect to the restriction of the $\sigma(B(H), B(H)_w)$-topology on $K(H)$.)

Our main result (Theorem 13) is $H^2_w(M, K(H))$ vanishes whenever $M \subseteq B(H)$ is a finite von Neumann algebra of countable type not containing type II$_1$ factors without Cartan subalgebras, or type II$_1$ factors without property $\Gamma$ as direct summands.

(In fact, instead of property $\Gamma$ we use a more general property whose definition is given at the end of §2.) This partially answers a question raised by S. Popa.

As a technical tool we prove (Proposition 3) that any ultraweakly continuous linear map defined on finite von Neumann algebras of countable type with values into $K(H)$ is also continuous from the unit ball of $M$ with the strong operator topology into $K(H)$ with the norm topology on $K(H)$ (and an analogous result for bilinear mappings). This fact was known only for derivations by [12], but the arguments there, combined with the fact (Corollary 2) that any linear map $\xi : l^\infty \rightarrow K(H)$ which is $\sigma(l^\infty, l^1) - \sigma(K(H), B(H))$ continuous is also continuous with respect to the norm topology on $K(H)$, yield this more general result. As a by-product, which is also used in the proof of Theorem 13, we show that if $M \subseteq B(H)$ has a Cartan subalgebra $A$, then $M$ has certain properties which are very close to the vanishing of $H^2_w(M, K(H))$. In fact for every cochain $\eta$ (i.e., $\eta$ is a separately ultraweakly continuous bilinear map from $M \times M$ into $K(H)$ with $\Delta \eta = 0$) we prove that there exists an ultraweakly continuous linear map $\xi$ such that $\Delta \xi = \eta$ (where $\Delta$ is the Hochschild coboundary operator) and such that $\xi(u)$ belongs to $K(H)$ for every $u$ in the normalizer $N_M(A)$ of $A$. From this we easily infer vanishing of $H^2_w(M, K(H))$ in the case of diffuse center.
If we could prove that \( \zeta \) (the composition of \( \zeta \) with the canonical surjection onto the Calkin algebra \( Q(H) = B(H)/K(H) \)) is continuous from the unit ball of \( M \) with the strong operator topology into \( Q(H) \) with the norm topology, then it would follow that \( H^2_Z(M, K(H)) = 0 \). We are able to prove only a weaker continuity property for \( \zeta \) but which is fortunately sufficient when combined with the arguments from [12, Proof of Proposition] to prove the vanishing of the cohomology group when \( M \) is a type II\(_1\) factor with property \( \Gamma \).

Note also that \( \zeta \) is obtained by perturbing with a suitable element in \( Q(H) \) the linear map that is obtained from \( \eta \) by the averaging techniques of [14].

Finally the analysis of the discrete abelian case (which is also based on the continuity result) gives a procedure for reduction in the case of discrete center. In fact we prove (Corollary 8) that \( H^2_Z(M, K(H)) = 0 \) whenever \( M \) is finite of countable type and there are nonzero projections \( p_n \) in \( Z(M) \) with sum 1 and such that \( H^2_Z(M_{p_n}, K(p_n H)) = 0 \) (with a certain control of the norm) for every \( n \).

2. Definitions

Let \( M \subseteq B(H) \) be a von Neumann algebra, \( P(M) \) the set of projections in \( M \), \( U(M) \) the unitary group of \( M \), \( Z(M) \) the center of \( M \), and \( M^* \) the predual of \( M \). By \( \pi_H: B(H) \rightarrow Q(H) = B(H)/K(H) \) we denote the usual projection of \( B(H) \) into the Calkin algebra. If \( A \subset M \) is a von Neumann subalgebra, then the normalizer of \( A \) in \( M \) is the multiplicative group \( N_M(A) = \{ u \in U(M) | u^* Au = A \} \) and the groupoid normalizer is the set

\[
GN_M(A) = \{ u e | u \in N_M(A), \ e \in P(A) \}.
\]

If \( S \subseteq M \) is a subset of \( M \), then \( S'' \) is the von Neumann subalgebra of \( M \) generated by \( S \). For a normal form \( \phi \) on \( M \) and \( x \) in \( M \) we denote \( \| x \|_\phi = \phi(x^* x)^{1/2} \). A von Neumann subalgebra \( B \) of \( M \) is diffuse if it has no minimal projections and discrete if it is generated by its minimal projections.

Recall that a maximal abelian diffuse subalgebra \( A \) of \( M \) is a Cartan subalgebra if \( (N_M(A))'' = M \).

Moreover \( M \) is of countable type if each orthogonal family of nonzero projections in \( M \) is at most countable.

Let \( X = B(H) \) or \( K(H) \) with the canonical structure of Banach bimodules over \( M \), and let \( n \) be a positive integer. Let \( C^n_w(M, X) \) be the space of \( n \)-linear maps from \( M \) into \( X \) which are separately \( \sigma(M, M^n) - \sigma(B(H), B(H)^n) \) continuous. (When \( X = K(H) \) we take the restriction of the \( \sigma(B(H), B(H)^n) \)-topology on \( X \).) By an obvious application of the Banach-Steinhaus principle we obtain that any element \( \eta \) in \( C^n_w(M, X) \) is also globally norm-bounded so that the quantity

\[
\| \eta \| = \sup \{ \| \eta(x_1, \ldots, x_n) \| | x_i \in M, \| x_i \| \leq 1, \ i = 1, 2, \ldots, n \}
\]

is finite.
Let $\Delta : C^n_w(M, X) \to C^{n+1}_w(M, X)$ be the usual Hochschild coboundary operator defined by

$$(\Delta \eta)(x_1, \ldots, x_{n+1}) = x_1 \eta(x_2, \ldots, x_{n+1}) + \sum_{k=1}^n (-1)^k \eta(x_1, \ldots, x_k x_{k+1}, \ldots, x_{n+1}) + (-1)^{n+1} \eta(x_1, \ldots, x_n) x_{n+1}$$

for $\eta$ in $C^n_w(M, X)$ and $x_1, \ldots, x_{n+1}$ in $M$.

Recall that the corresponding space of cochains is

$Z^n_w(M, X) = \{\eta \in C^n_w(M, X) | \Delta \eta = 0\}$

and for $n \geq 1$ the space of coboundaries is $B^n_w(M, X) = \{\Delta \xi | \xi$ is an element in $C^{n-1}_w(M, X)\}$. The Hochschild cohomology groups $H^n_w(M, X)$ are then defined as the vector space quotient $Z^n_w(M, X)/B^n_w(M, X)$. Note that $C^0_w(M, X)$ is simply identified with $X$, and $Z^1_w(M, X)$ is the space of derivations from $M$ into $X$ (i.e., the space of linear mappings $\delta : M \to X$ that satisfy $\delta(xy) = x\delta(y) + \delta(x)y$ for $x, y$ in $M$; by [14] any derivation from $M$ into $B(H)$ is automatically $\sigma(M, M_e) - \sigma(B(H), H(H))$ continuous). Also, the vanishing of $H^1_w(M, X)$ means that any derivation $\delta$ of $M$ into $X$ is inner, i.e., there is $T$ in $X$ such that $\delta(x) = \text{Ad} T(x) = [T, x] = Tx - xT$ for $x$ in $M$.

The vanishing of $H^2_w(M, X)$ means that for every bilinear map $\eta$ which is separately $\sigma(M, M_e) - \sigma(B(H), B(H))$ continuous and $\Delta \eta = 0$ there is a $\sigma(M, M_e) - \sigma(B(H), B(H))$ continuous linear map $\xi : M \to X$ such that $\eta$ measures the obstruction of $\xi$ to be a derivation, i.e., $\eta(x, y) = x\xi(y) - \xi(xy) + \xi(x)y$ for all $x, y$ in $M$.

Let $e$ be a nonzero projection in $M$, $M_e$ the reduced von Neumann algebra, and $\rho_e : M_e \to Q(eH) = B(eH)/K(eH)$ the composition of the canonical inclusion of $M_e$ into $B(eH)$ with the canonical map into the Calkin algebra $Q(eH)$. A linear mapping $\xi : M_e \to Q(eH)$ is a $\rho_e$ derivation if $\xi(xy) = \rho_e(x)\xi(y) + \xi(x)\rho_e(y)$ for all $x, y$ in $M_e$.

For any $C^*$-algebras $A, B$ and for a completely bounded linear map $\xi : A \to B$ we denote $\|\xi\|_{cb} = \sup\{\|\xi_n\| | n \in \mathbb{N}\}$, where $\xi_n : A \otimes M_n(\mathbb{C}) \to B \otimes M_n(\mathbb{C})$ is the canonical extension of $\xi$.

Our main technical definition is the following.

**Definition.** A type II$_1$ factor $M \subseteq B(H)$ with faithful normalized trace $\tau$ will be said to have property $S$ with respect to $H$ if there is a strictly positive number $C$ such that for any nonzero projection $e$ in $M$ with $\tau(e) \in \mathbb{Q}$ and for any $\rho_e$ derivation $\xi : M_e \to Q(eH)$ we have

$$\|\xi\|_{cb} \leq C\|\xi\|.$$
Note that by the work of Christensen [4], factors with property \( \Gamma \) always have property \( S \) (independent on the Hilbert space \( H \)) and that in this case \( C = 119 \).

3. Proof of the continuity result

Recall that \( l^1 \) is the Banach space of absolutely summable scalar sequences \( \{\lambda_n\}_{n \in \mathbb{N}} \) with \( \|\lambda_n\|_1 = \sum |\lambda_n| < \infty \), and that its dual is canonically identified with the Banach space \( l^\infty \) of all bounded scalar sequences with the uniform norm. Moreover by a well-known result of Schur (see [7]), every weakly convergent sequence in \( l^1 \) is also norm convergent. Using a vector-valued version of this result we prove that any continuous linear mapping from \( l^\infty \) with the \( \sigma \) topology into the space of compact operators \( K(H) \) on an arbitrary Hilbert space \( H \) with the \( \sigma(K(H), B(H)_*) \)-topology is also continuous with respect to the uniform norm on \( K(H) \).

Finally, using this and the arguments from the Appendix of [12] we prove that weakly continuous linear mappings from a finite von Neumann algebra \( M \) of countable type into \( K(H) \) automatically have the special continuity properties we mentioned in the Introduction.

Lemma 1. Let \( \{X_n\}_{n \in \mathbb{N}} \) be Banach spaces, and let \( \mathcal{F}_n \subseteq X_n^* \) be linear subspaces such that \( \|x\| = \sup\{\|\varphi(x)\| : \varphi \in \mathcal{F}_n, \|\varphi\| \leq 1\} \) for every \( x \) in \( X_n \) and every \( n \in \mathbb{N} \). Suppose further that \( F_n : l^\infty \rightarrow X_n \) are linear and weak*-\( \sigma(X_n, \mathcal{F}_n) \) continuous and that \( \|F_n(a)\| \rightarrow 0 \) for every \( a \) in \( l^1 \). Then \( \|F_n\| \rightarrow 0 \).

Proof. Suppose on the contrary that there exist \( \{a_n\}_{n \in \mathbb{N}} \) in \( l^\infty \) and \( c > 0 \) such that
\[
\|F_n(a_n)\| > 2c, \quad \|a_n\| \leq 1.
\]

By the property of \( \mathcal{F}_n \) it follows that there exists \( \varphi_n \) in \( \varphi_n \) such that
\[
|\varphi_n(F_n(a_n))| \geq c, \quad \|\varphi_n\| \leq 1.
\]

Then \( \psi_n = \varphi_n \circ F_n \) are weak* continuous functionals on \( l^\infty \) and hence may be identified with an element in \( l^1 \) for each \( n \) in \( \mathbb{N} \). Moreover for every \( a \) in \( l^\infty \) we have
\[
|\psi_n(a)| \leq \|F_n(a)\| \rightarrow 0
\]
so that \( \{\psi_n\}_n \) is weakly convergent to zero, viewed as a sequence in \( l^1 \). But then by the Schur lemma, \( \|\psi_n\|_1 \rightarrow 0 \) which obviously contradicts (1).

The next corollary is the basic ingredient for the continuity result, but will also be useful in the proof of the discrete abelian case.

Corollary 2. Let
\[
\xi : l^\infty \rightarrow K(H)
\]
be a linear mapping which is weak*-\( \sigma(K(H), B(H)_*) \) continuous. Then for any sequence \( \{p_n\}_n \subseteq B(H) \) of projections increasing to \( 1 \), we have
\[
\sup\{\|\xi(a)(1 - p_n)\| : a \in l^\infty, \|a\| \leq 1\} \rightarrow 0.
\]
In particular, $\xi$ is also continuous from $l^\infty$ with respect to the weak* topology into $K(H)$ with the norm topology.

**Proof.** We apply the preceding lemma with $X_n = B(H)(1 - p_n)$ and $T_n = \{\varphi | x_n | \varphi \in \sigma(B(H), B(H)_*)\}$. Lemma 1.9 from [15] shows that $T_n$ has the required properties.

Moreover it is clear that $F_n: l^\infty \to X_n$ defined by $F_n(a) = \xi(a)(1 - p_n)$ for any $a$ in $l^\infty$ is weak*-continuous and $\|F_n(a)\| = \|\xi(a)(1 - p_n)\| \to 0$ since $\xi(a)$ is compact. But then $\|F_n\| \to 0$ and this is exactly the first part of the statement. To prove the second let $\{a_i\}_{i \in I}$ be any net in $l^\infty$ weak* convergent to 0, and let $\varepsilon > 0$ be arbitrary. Let $M = \sup\{\|a_i\| | i \in I\}$.

By what we have just proved there exist two finite-dimensional projections $p, q$ in $B(H)$ such that $||\xi(a) - p\xi(a)q|| \leq \varepsilon/2$ whenever $a$ is any element in $l^\infty$ of norm smaller than $M$. Since the linear mapping defined on $l^\infty$ by $a \mapsto p\xi(a)q$ has finite dimensional range and hence is weak*-norm continuous it follows that there exists $i_\varepsilon$ such that $||p\xi(a_i)q|| \leq \varepsilon/2$ for $i \geq i_\varepsilon$. The rest of the argument is standard.

We come now to the continuity result previously announced.

**Proposition 3.** Let $M$ be a finite von Neumann algebra with finite faithful normal trace $\tau$, and $\eta: M \times M \to K(H)$ a bilinear map which is separately $\sigma(M, M_*) - \sigma(K(H), B(H)_*)$ continuous. Then for every $\varepsilon > 0$ there is a $\delta$ such that $||\eta(x, y)|| < \varepsilon$ whenever $x, y \in M$, $||x||, ||y|| \leq 1$, $||x||_\tau, ||y||_\tau \leq \varepsilon$.

In particular, any $\sigma(M, M_*) - \sigma(K(H), B(H)_*)$ continuous linear map $\xi: M \to K(H)$ is also continuous from the unit ball of $M$ with the strong operator topology into $K(H)$ with the norm topology.

**Proof.** It is sufficient to show that $||\eta(x_n, y_n)|| \to 0$ whenever $\{x_n\}, \{y_n\}$ are sequences in the unit ball of $M$ with $||x_n||_\tau \to 0$, $||y_n||_\tau \to 0$. By the arguments in [12, Appendix] we may reduce ourselves to the case when each of the sequences $\{x_n\}, \{y_n\}$ consists of orthogonal projections. Let $A = \{x_n\}''$, $B = \{y_n\}''$ so that $A, B$ are isomorphic as von Neumann algebras (and hence as Banach spaces) with $l^\infty$. Let $F_n: A \to K(H)$ be defined by $F_n(a) = \eta(a, y_n)$, $a \in A$. By the preceding corollary we have that $\|F_n(a)\| \to 0$ and therefore $\|F_n\| \to 0$ by Lemma 1. Consequently $\|\eta(x_n, y_n)\| \to 0$ and this completes the proof.

As a corollary we obtain that whenever $M$ is a finite von Neumann algebra and $\{p_n\}$ is a partition of the unity in the center of $M$, then we may reduce the analysis of the vanishing of $H^2_w(M, K(H))$ for certain cochains to the analysis of each term $H^2_w(M_{p_n}, K(p_nH))$.

**Corollary 4.** Let $M$ be a finite von Neumann algebra of countable type, and $\{p_n\}$ an orthogonal sequence of nonzero projections in $Z(M)$ and $\eta$ in $Z^2_w(M, K(H))$ such that $\eta(x, y) = 0$ whenever $x$ or $y$ belong to the von Neumann algebra $Z = \{p_n\}''$. For each $n$, the restriction of $\eta$ to $M_{p_n}$ defines an
element \( \eta_n \) in \( Z^2_w(M_{p_n}, K(p_nH)) \). Moreover if there is a number \( C > 0 \) and \( \xi_n \) in \( C^1_w(M_{p_n}, K(p_nH)) \) such that \( \Delta \xi_n = \eta_n \), \( \|\xi_n\| \leq C\|\eta_n\| \) for every \( n \in \mathbb{N} \), then there is \( \xi \) in \( C^1_w(M, K(H)) \) such that \( \Delta \xi = \eta \), \( \|\xi\| \leq C\|\eta\| \).

Proof. Since \( \eta(x, y) = 0 \) whenever \( x \) or \( y \) belong to \( Z \) and \( \Delta \eta = 0 \) it follows that \( p_n \eta(x, y) = \eta(p_n x, y) = \eta(x, p_n y) = \eta(x, y)p_n \) for every \( x, y \) in \( M \), \( n \in \mathbb{N} \). This shows that \( \eta_n \) is well defined as an element in \( Z^2_w(M_{p_n}, K(p_nH)) \).

For \( x \) in \( M \) we let \( \xi(x) \) be the Hilbert space direct sum (after \( n \)) of the compact operators \( \xi_n(p_n x) \). Clearly \( \|\xi\| \leq C\|\eta\| \), \( \xi \in C^1_w(M, B(H)) \), and \( \Delta \xi = \eta \). To end the proof we have only to prove that \( \xi \) takes its values into the compact operators. To do this we have only to prove that \( \lim_{n \to \infty} \|\xi_n(x p_n)\| = 0 \) for every \( x \) in \( M \). Since \( \|\xi_n\| \leq C\|\eta_n\| \), it is also sufficient to prove that \( \|\eta_n\| \to 0 \).

4. The technical results

If \( M \subseteq B(H) \) is a finite von Neumann algebra of countable type, \( A \subseteq M \) is an injective von Neumann subalgebra, and \( \eta \) is an element in \( Z^2_w(M, K(H)) \), then the averaging technique from [14] combined with the results from [8] allows us to find a \( \xi \) in \( C^1_w(M, B(H)) \) such that \( (\Delta \xi - \eta)(x, y) = 0 \) whenever \( x \) or \( y \) are in \( A \).

If \( A \) is discrete and abelian, then the statements in the preceding section will show that we may further assume that \( \xi \) takes its values into \( K(H) \). This will complete the reduction procedure begun in Corollary 4 and will show in particular that \( H^2_w(A, K(H)) = 0 \) (for discrete abelian \( A \)).

When \( A \) is abelian and diffuse we shall show that we may still assume that \( \xi(x) \in K(H) \) for \( x \) in \( A \cap M \) or for \( x \) in \( N_M(A) \). Again this will imply that \( H^2_w(A, K(H)) = 0 \) and, using the fact that there is no nonzero compact operator which commutes with a diffuse abelian algebra, it will follow that \( (\Delta \xi - \eta)(x, y) = 0 \) if \( x \) or \( y \) belong to the von Neumann subalgebra of \( M \) generated by \( N_M(A) \).

Lemma 5. Let \( M \) be a finite von Neumann algebra and \( A \subseteq M \) an injective von Neumann subalgebra. Let \( m \) be a mean on \( l^\infty(U(A)) \) such that

\[
\int_{U(A)} V(x u^*, u) \, dm(u) = \int_{U(A)} V(u^*, u x) \, dm(u),
\]

for any \( x \) in \( A \) and for any separately weakly continuous bilinear form \( V \) on \( A \).
[8]. Let $\eta$ be any element in $Z^2_w(M, K(H))$. Then for $x$ in $M$ the formulas

\begin{align}
\xi_0(x) &= \int_{U(A)} u^* \eta(u, x) \, dm(u), \\
\xi_1(x) &= \int_{U(A)} (-\eta(xu^*, u) + \xi_0(xu^*)u + xu^*\xi_0(u)) \, dm(u)
\end{align}

define elements in $C^1_w(M, B(H))$, and $(\eta - \Delta \xi_1)(x, y) = 0$ whenever $x$ or $y$ are in $A$ and $||\xi_1|| \leq 3||\eta||$.

**Proof.** Since $\eta \in Z^2_w(M, B(H))$ we have

$$u^* \eta(u, x) = \eta(1, x) - \eta(u^*, ux) + \eta(u^*, u)x, \quad u \in U(A), \ x \in M.$$ 

To show that $\xi_0$ defines an element in $C^1_w(M, B(H))$ it is therefore sufficient to prove that the mapping $x \mapsto \int_{U(A)} \eta(u^*, xu) \, dm(u)$ defines an element in $C^1_w(M, B(H))$—this is essentially done in [8, proof of Lemma 2.1].

Let $\eta_1 = \eta - \Delta \xi_1$ and

$$\xi' = \int -\eta_1(xu^*, u) \, dm(u).$$

Similarly, $\xi' \in C^1_w(M, B(H))$. The computations made in [14] show that $(\eta - \Delta(\xi_0 + \xi'))(x, y) = 0$ whenever $x$ or $y$ is in $A$. But

$$\xi'(x) = -\int \eta(xu^*, u) \, dm(u) + \int \Delta \xi_0(xu^*, u) \, dm(u)$$

$$= -\int \eta(xu^*, u) \, dm(u) + \int \xi_0(xu^*)u \, dm(u)$$

$$+ \int xu^*\xi_0(u) \, dm(u) - \xi_0(x),$$

whence $\xi_0 + \xi' = \xi_1$ and hence $\xi_1 \in C^1_w(M, B(H))$, $(\eta - \Delta \xi_1)(x, y) = 0$ for $x$ or $y$ in $A$. The last inequality in the statement is obvious since $||\xi_0|| \leq ||\eta||$.

If $M \subseteq B(H)$ is finite dimensional, then $M$ itself is injective and the mean invariant coincides with Haar measure on the compact group $U(M)$. Hence if $\eta \in Z^2_w(M, K(H))$, then the integrals in the definition of $\xi_0$ (which coincides this time with $\xi_1$) are norm-convergent and therefore $\xi_0$ belongs to $C^1_w(M, K(H))$. Consequently, we obtain

**Corollary 6.** Let $M \subseteq B(H)$ be a finite-dimensional von Neumann algebra. Then for any $\eta$ in $Z^2_w(M, K(H))$ there is $\xi$ in $C^1_w(M, K(H))$ such that $\Delta \xi = \eta$ and $||\xi|| \leq ||\eta||$.

We come now to the analysis of the discrete case.

As we said before an obvious corollary of the next proposition will be that $H^2_w(A, K(H)) = 0$ for discrete abelian $A$ of countable type.
Proposition 7. Let $M \subseteq B(H)$ be a finite von Neumann algebra of countable type and $A \subseteq M$ a discrete abelian von Neumann subalgebra. For any $\eta$ in $Z_w^2(M, K(H))$ there exists $\xi_1$ in $C_w^1(M, K(H))$ such that $(\eta - \Delta \xi_1)(x, y) = 0$ whenever $x$ or $y$ is in $A$ and $\|\xi_1\| \leq 3\|\eta\|$. 

Proof. Fix $\eta$ in $Z_w^2(M, K(H))$. Let $\xi_1$, $\xi_2$ be defined by (3), (4) and $\xi'$ as in the proof of Lemma 5. We prove first that $\xi_0$ belongs to $C_w^1(M, K(H))$. Suppose on the contrary that $\xi_0(x)$ is noncompact for some $x$ in $M$. Then there exist a sequence of finite dimensional projections $p_n \in B(H)$ with $p_n \uparrow 1$, and a number $c > 0$ such that $\|\xi_0(x)(1 - p_n)\| > c$ for every $n$ in $\mathbb{N}$. Since $\xi_0(x)$ is a weak limit of convex combinations of elements from the set $\{u^*\eta(u, x)|u \in U(A)\}$, it follows by the weak inferior semicontinuity of the uniform norm on $B(H)$ that there exists $u_n$ in $U(A)$ such that 

$$\|\eta(u_n, x)(1 - p_n)\| = \|u_n^*\eta(u_n, x)(1 - p_n)\| > c, \quad \text{for every } n \in \mathbb{N}.$$ 

Since $x$ is fixed this contradicts Corollary 2. Hence $\xi_0$ takes values into $K(H)$ and consequently $\eta_1 = \eta - \Delta \xi_0 \in Z_w^2(M, K(H))$. Moreover (by [14]), $\eta_1(a, y) = 0$ for $a$ in $A$ and $y$ in $M$ and hence for any $x$ in $M$ and $u$ in $U(A)$:

$$0 = (\Delta \eta_1)(x, u^*, u) = x\eta_1^*(u^*, u) - \eta_1(xu^*, u) + \eta_1(x, 1) - \eta_1(x, u^*)u,$$

whence

$$\eta_1(xu^*, u) = \eta_1(x, 1) - \eta_1(x, u^*)u.$$

From the proof of Lemma 5 we know that $\xi_1 = \xi_0 + \xi'$, where

$$\xi'(x) = -\int_{U(A)} \eta_1(xu^*, u) du dm(u) = -\eta_1(x, 1) + \int_{U(A)} \eta_1(x, u^*)u du dm(u)$$

for $x$ in $M$. Since $\eta_1(x, 1) \in K(H)$ for $x$ in $M$, we prove similarly as for $\xi_0$ that $\xi$ takes values into $K(H)$. This ends the proof.

By the previous proposition and using Corollary 4, we obviously obtain the argument for the reduction process in the case of discrete center:

Corollary 8. Let $M \subseteq B(H)$ be a finite von Neumann algebra of countable type, and $\{p_n\} \subseteq Z(M)$ a sequence of orthogonal nonzero projections with $\sum p_n = 1$. If there is a positive $C$ such that for every $n$ and for every $\eta_n$ in $Z_w^2(M_{p_n}, K(p_n H))$ there exists $\xi_n$ in $C_w^1(M_{p_n}, K(p_n H))$ with $\Delta \xi_n = \eta_n$ and $\|\xi_n\| \leq C\|\eta_n\|$, then for every $\eta$ in $Z_w^2(M, K(H))$ there exists $\xi$ in $C_w^1(M, K(H))$ with $\Delta \xi = \eta$, $\|\xi\| \leq (4C + 3)\|\eta\|$. (In particular, under this hypothesis $H_w^2(M, K(H)) = 0$.)

Proof. Let $Z = \{p_n\}''$ and let $\eta$ be any element in $Z_w^2(M, K(H))$. By Proposition 7 we may find $\xi_1$ in $C_w^1(M, K(H))$ with $\|\xi_1\| \leq 3\|\eta\|$ and $\eta'(x, y) = (\eta - \Delta \xi_1)(x, y) = 0$ whenever $x$ or $y$ are in $Z$. Note that $\|\eta'\| \leq 4\|\eta\|$. The hypotheses of Corollary 4 are thus fulfilled for $\eta'$ and therefore there is $\xi_2$ in
$C_w^1(M, K(H))$ such that $\eta' = \Delta \xi_2$, $\|\xi_2\| \leq C\|\eta'\| \leq 4C\|\eta\|$. Hence $\eta - \Delta \xi_1 = \Delta \xi_2$, $\eta = \Delta(\xi_1 + \xi_2)$, and $\xi_1 + \xi_2 \in Z_w^2(M, K(H))$, $\|\xi_1 + \xi_2\| \leq (4C + 3)\|\eta\|$. We turn now to the case when the injective von Neumann subalgebra $A$ of $M$ is diffuse. We first make a remark which shows that in the case of Cartan diffuse subalgebras the $\xi_1$ defined by Lemma 5 has in fact the property that $\Delta \xi_1 = \eta$ on $M$.

Remark 9. Let $M \subseteq B(H)$ be an arbitrary von Neumann subalgebra, $B \subseteq M$ a diffuse (i.e., without minimal projections) von Neumann subalgebra. If $\eta$ belongs to $Z^2_w(M, K(H))$ and $\xi$ is in $C^1_w(M, B(H))$ such that $(\eta - \Delta \xi)(x, y) = 0$ whenever $x$ or $y$ is in $B$ and $\xi(u)$ is compact for $u$ in $N_M(B)$, then $(\eta - \Delta \xi)(x, y) = 0$ also when $x$ and $y$ belong to the von Neumann subalgebra of $M$ generated by $N_M(B)$.

Proof. Denote by $N$ the von Neumann subalgebra of $M$ generated by $N_M(B)$ and $\eta_1 = \eta - \Delta \xi$. Since $\eta_1(x, y) = 0$ for $x$ or $y$ in $C$ and $\Delta \eta_1 = 0$ we have

$$b\eta_1(x, y) = \eta_1(bx, y), \quad \eta_1(xb, y) = \eta_1(x, by), \quad \eta_1(x, yb) = \eta_1(x, y)_b$$

for all $x, y$ in $M$ and $b$ in $B$.

Therefore for any $u, v$ in $N_M(B)$ and $b$ in $B$ we obtain

$$bu^*\eta_1(u, v)v^* = u^*(ubu^*)\eta_1(u, v)v^* = u^*\eta_1((ubu^*)u, v)v^* = u^*\eta_1(u, bv)v^* = u^*\eta_1(u, v)(bv)v^* = u^*\eta_1(u, v)v^*b.$$ 

Hence $u^*\eta_1(u, v)v^*$ belongs to the commutant of $B$ in $B(H)$ and by hypothesis it is a compact operator.

Since $B$ is diffuse, it follows that $u^*\eta_1(u, v)v^* = 0$ and consequently $\eta_1(u, v) = 0$ for every $u, v$ in $N_M(B)$. By the separately weak continuity of $\eta_1$ it follows that $\eta_1(x, y) = 0$ if $x$ or $y$ are in $N$. But $\eta_1 = \eta - \Delta \xi$ and this completes the proof.

We are now able to prove our main technical result.

Proposition 10. Let $M \subseteq B(H)$ be a finite von Neumann algebra of countable type, and $A \subseteq M$ a diffuse abelian subalgebra. Then for any $\eta$ in $Z^2_w(M, K(H))$ there is a $\xi$ in $C^1_w(M, B(H))$ such that $(\eta - \Delta \xi)(x, y) = 0$ whenever $x$ or $y$ are in $A$, or where $x$ and $y$ belong to the von Neumann subalgebra of $M$ generated by $N_M(A)$. Moreover $\|\xi\| \leq 10\|\eta\|$ and $\xi(x)$ belongs to $K(H)$ if $x$ is any element in $A' \cap M$ or in $N_M(A)$.

Finally if $\hat{\xi}$ is the composition of $\xi$ with the canonical surjection from $B(H)$ onto $B(H)/K(H)$ and $\tau$ is any normal finite faithful trace on $M$, then for any $\epsilon > 0$ there is a $\delta > 0$ such that $\|\hat{\xi}(ef)\| \leq \epsilon$ whenever $x \in M$, $e, f \in P(A)$, and $\|e\|_\tau, \|f\|_\tau \leq \delta$, $\|x\| \leq 1$.

Proof. The proof of the proposition is divided into six steps. For any fixed $\eta$ in $Z^2_w(M, K(H))$ we let $\xi_0, \xi_1$ be defined by (3) and (4). We prove first
that $\xi_0, \xi_1$ have similar continuity properties (to those required for $\bar{\xi}$) and we construct an element $\hat{T}$ in $Q(H)$ such that if $T$ in $B(H)$ is any lift of $\hat{T}$, then $\xi = \xi_1 + \text{Ad } T$ has the stated properties.

**Step I.** For any $\epsilon > 0$ there is a $\delta > 0$ such that $\|\epsilon\xi_1(xf)\| \leq \epsilon/2$ whenever $x \in M, \epsilon, f \in P(A)$, and $\|x\| \leq 1, \tau(\epsilon) < \delta, \tau(f) < \delta$.

Indeed, by Proposition 3, there is a $\delta > 0$ such that $\|\eta(x, y)\| \leq \epsilon/6$ for every $x, y$ in the unit ball of $M$ with $\|x\|_r, \|y\|_r \leq \delta$. Then by the invariance of $m$

\[
e\xi_0(xf) = \int_{U(A)} u^* \eta(ue, xf) \, dm(u),
\]

whence

\[
\|\epsilon\xi_0(xf)\| \leq \int_{U(A)} u^* \eta(ue, xf) \, dm(u) \leq \epsilon/6.
\]

Similarly,

\[
\|\epsilon\xi_1(xf)\| \leq \int_{U(A)} \|\eta(xfu^*, uf)\| \, dm(u) + \int_{U(A)} \|\epsilon\xi_0(xfu^*)\| \, dm(u)
\]

\[+ \int_{U(A)} \|\epsilon xf u^* \xi_0(uf)\| \, dm(u) \leq \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{2}.
\]

**Step II:** Construction of a canonical element $\hat{T}$ in $Q(H)$. Let $\{e_i^n | i = 1, 2, \ldots, k_n, n \in \mathbb{N}\}$ be a family of projections in $A$ with the following properties:

(a) $\sum_{i=1}^{k_n} e_i^n = 1$ ($e_i^n$ are orthogonal for any fixed $n \in \mathbb{N}$);

(b) for each $i, n$ there is a set $A_i^n$ such that $e_i^n = \sum_{j \in A_i^n} e_j^{n+1}$;

(c) $\{e_i^n\}'' = A$;

(d) $\sup\{\|e_i^n\|_r | i = 1, 2, \ldots, k_n\} \leq \delta_n$,

where $\delta_n$ is the number given by Step I corresponding to $\epsilon_n = 3\|\eta\|/2^{n+1}$. Let $\bar{\xi}_1 = \pi_n \circ \xi_1$ and observe that the restriction of $\bar{\xi}_1$ to $A$ is a derivation into $Q(H)$ (where we identify the elements from $M$ with their image into $Q(H)$).

Let $\hat{T}_n \in Q(H)$ be defined by $\hat{T}_n = \sum_i e_i^n \bar{\xi}_1(e_i^n)$, $n \in \mathbb{N}$. $\{\hat{T}_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $Q(H)$ since

\[
\|\hat{T}_{n+1} - \hat{T}_n\| = \left\|\sum_j e_j^{n+1} \bar{\xi}_1(e_j^{n+1}) - \sum_i e_i^n \bar{\xi}_1(e_i^n)\right\|
\]

\[= \left\|\sum_i e_i^n \left(\sum_{j \in A_i^n} e_j^{n+1} (\bar{\xi}_1(e_j^{n+1}) - e_j^{n+1} \bar{\xi}_1(e_j^n))\right)\right\|
\]

\[= \left\|\sum_i e_i^n \left(\sum_{j \in A_i^n} e_j^{n+1} (\bar{\xi}_1(e_i^n e_j^{n+1}) - e_j^{n+1} \bar{\xi}_1(e_i^n))\right)\right\|
\]

\[= \left\|\sum_i e_i^n \left(\sum_{j \in A_i^n} e_j^{n+1} \bar{\xi}_1(e_j^{n+1})\right)\right\|
\]

(continued)
Here $\lambda_i^n$ is the normalized Lebesgue measure on the cartesian product of $a_i^n$ copies of the torus $T = \{z \in \mathbb{C} | |z| = 1\}$, $a_i^n$ is the cardinality of $A_i^n$. For the last inequality we used property (d) and Step I. Since obviously $\|\hat{T}_n\| \leq \|\xi_1\| \leq 3\|\eta\|$, it follows that the sequence \{\hat{T}_n\} converges to an element $\hat{T}$ in $Q(H)$ which has the property that $\|\hat{T}\| \leq 6\|\eta\|$.

Let $T$ be any element in $Q(H)$ such that $\pi_H(T) = T$ and $\|T\| \leq 7\|\eta\|$. We define $\hat{\xi} = \xi_1 + \text{Ad } T$, $\hat{\xi} = \pi_H \circ \xi$ and we obviously have $\|\xi\| \leq 10\|\eta\|$ and $(\Delta\xi - \eta)(x, y) = 0$ for $x$ or $y$ in $A$. We will show now that $\hat{\xi}$ is the element we are looking for.

**Step III:**

$$\hat{\xi}_1(x) + [\hat{T}_n, x] = \sum_{i, v} e_i^n \hat{\xi}_1(e_i^n xe_i^n) e_i^n.$$

Indeed,

$$\hat{\xi}_1(x) + [\hat{T}_n, x] = \hat{\xi}_1(x) + \hat{T}_n x - x \hat{T}_n$$

$$= \hat{\xi}(x) - \sum_i e_i^n \hat{\xi}_1(e_i^n) + \sum_r e_r^n \hat{\xi}_1(e_r^n)x$$

$$= \hat{\xi}_1(x) - \sum_i \hat{\xi}_1(x e_i^n) + \sum_i \hat{\xi}(x e_i^n)e_i^n + \sum_r e_r^n \hat{\xi}_1(e_r^n)x$$

$$= \sum_{i, r} e_r^n (e_r^n \hat{\xi}_1(x e_i^n) + \hat{\xi}_1(e_r^n)x e_i^n) e_i^n$$

$$= \sum_{i, r} e_r^n \hat{\xi}_1(e_r^n x e_i^n) e_i^n.$$

**Step IV.** If $x$ belongs to $A' \cap M$, then $\hat{\xi}(x) = 0$ (or, what is the same, $\xi(x)$ belongs to $K(H)$).
This follows from

\[ \|\xi(x)\| = \lim_{n \to \infty} \|\xi_1(x) + [\hat{T}_n, x]\| \]

\[ = \lim_{n \to \infty} \left\| \sum_{i} \xi_{i}^{n} (e_{i}^{n} x e_{i}^{n}) e_{i}^{n} \right\| \]

\[ = \lim_{n \to \infty} \left\| \sum_{i} e_{i}^{n} \xi_{i}^{n} (e_{i}^{n} x e_{i}^{n}) e_{i}^{n} \right\| \]

\[ = \lim \sup_{n} \|e_{i}^{n} \xi_{i}^{n} (e_{i}^{n} x e_{i}^{n}) e_{i}^{n}\| = 0, \]

where the last equality follows from Step I and assumption (d).

**Step V.** For every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( \|\xi(xf)\| \leq \varepsilon \) when \( x \) belongs to the unit ball of \( M \) and \( e, f \) are projections in \( P(A) \) with \( \tau(e), \tau(f) < \delta \).

To prove this let \( \varepsilon > 0 \) be arbitrary, let \( \delta > 0 \) be the corresponding number given by Step I, and assume \( x, e, f \) are fixed with the properties in the statement of Step V.

Let \( n_{0} \) be a natural number such that \( \|\hat{T}_n - \hat{T}\| \leq \varepsilon/4 \) for every \( n \geq n_{0} \) and, therefore, \( \||[\hat{T}_n, x] - [\hat{T}, x]\| \leq \varepsilon/2 \) for any \( x \) in \( M \) with \( \|x\| \leq 1 \).

Letting \( n \) big enough, we may assume that there exist two sets of indices \( A, B \) with cardinalities \( a, b \) and such that

\[ e \leq \sum_{i \in A} e_{i}^{n}, \quad f \leq \sum_{j \in B} e_{j}^{n}, \quad \left\| \sum_{i \in A} e_{i}^{n}\right\|_{\tau} < \delta, \quad \left\| \sum_{j \in B} f_{j}^{n}\right\|_{\tau} < \delta. \]

By Step III we obtain

\[ \|\xi_1(xf) + [\hat{T}_n, xf]\| \]

\[ = \left\| \sum_{i \in A} \sum_{j \in B} e_{i}^{n} \xi_{i}^{n} (e_{i}^{n} x e_{i}^{n}) e_{j}^{n} \right\| \]

\[ \leq \int_{T_{a}} \int_{T_{b}} \left( \sum_{i \in A} z_{i} e_{i}^{n}\right)^{*} \xi_1 \left( \left( \sum_{i \in A} z_{i} e_{i}^{n}\right) x \left( \sum_{j \in B} w_{j} e_{j}^{n}\right) \right) \]

\[ \cdot \left( \sum_{j \in B} w_{j} e_{j}^{n}\right) d\lambda_{A}(z) d\lambda_{B}(w) \]

\[ \leq \int_{T_{a}} \int_{T_{b}} \left| e_{i}^{n} \xi_{i}^{n} \left( \sum_{i \in A} z_{i} e_{i}^{n}\right) x \left( \sum_{j \in B} w_{j} e_{j}^{n}\right) f \right| d\lambda_{A}(z) d\lambda_{B}(w) \leq \frac{\varepsilon}{2}, \]

where \( \lambda_{A}, \lambda_{B} \) are the normalized Lebesgue measures on the cartesian products \( T_{a} \) and \( T_{b} \), and where for the last inequality we used Step I.

Therefore \( \|\xi_1(xf) + [\hat{T}_n, x]\| \leq \varepsilon/2 + \|[\hat{T} - \hat{T}_n, xf]\| \leq \varepsilon \).
Step VI: $\hat{\xi}(u) = 0$ (or, what is the same, $\xi(u) \in K(H)$ if $u$ belongs to $N_M(A)$).

Indeed, since $(\Delta \xi - \eta)(x, y) = 0$ whenever $x$ or $y$ are in $A$, the restriction of $\xi$ to $A$ takes values into $K(H)$ and since $\eta$ belongs to $Z^2_w(M, K(H))$ it follows that $a\hat{\xi}(x) = \hat{\xi}(ax)$, $\hat{\xi}(xa) = \hat{\xi}(x)a$ for $x$ in $M$ and $a$ in $A$.

Consequently, for any $u$ in $N_M(A)$, $u^*\hat{\xi}(u)$ commutes in $Q(H)$ with $A$ (since $a(u^*\hat{\xi}(u)) = u(ua^*)\hat{\xi}(u) = u^*\hat{\xi}((ua^*)u) = u^*\hat{\xi}(ua) = u^*\hat{\xi}(u)a$ for $a$ in $A$).

We deduce therefore that for any natural $n$

$$\|u^*\xi(u)\| = \left\| \sum_i e_i^n(u^*\xi(u))e_i^n \right\| = \sup_i \|e_i^n(u^*\xi(u))e_i^n\|$$

and the last term tends to zero when $n$ tends to infinity. Hence $u^*\hat{\xi}(u) = 0$ and $\hat{\xi}(u) = 0$ for every $u$ in $N_M(A)$.

The proof of the proposition is now accomplished by Remark 9.

As an obvious corollary we obtain that $H^2_w(M, K(H)) = 0$ whenever $M$ has diffuse center. Precisely, we have

**Corollary 11.** Let $M \subseteq B(H)$ be a finite von Neumann algebra of countable type with diffuse center $Z(M)$. Then for every $\eta$ in $Z^2_w(M, K(H))$ there is a $\xi$ in $C^1_w(M, K(H))$ such that $\Delta \xi = \eta$ and $\|\xi\| \leq 10\|\eta\|$.

**Proof.** Applying the preceding proposition with $A = Z(M)$ it follows that there is $\xi$ in $C^1_w(M, B(H))$ such that $\|\xi\| \leq 10\|\eta\|$, $(\Delta \xi - \eta)(x, y) = 0$ if $x$ and $y$ are in $N_M(A)^\prime\prime$, and $\xi(u)$ belongs to $K(H)$ if $u$ is in $N_M(A)$.

Since $M$ is this time the linear span of $N_M(A)$, it follows that $\xi$ has compact values (by linearity) and $\Delta \xi = \eta$.

5. Proof of the main result

Using Proposition 10 and the arguments from [4, proof of Theorem 3.5] we prove that $H^2_w(M, K(H)) = 0$ for a type II$_1$ factor $M \subseteq B(H)$ with property $S$ with respect to $H$ and with a Cartan subalgebra $A$ (see §2 for the definition of property $S$).

The general result will then follow easy from this and from Corollaries 6 and 11.

**Proposition 12.** Let $M \subseteq B(H)$ be a type II$_1$ factor with property $S$ with respect to $H$ and with a Cartan subalgebra $A$. Then for every $\eta$ in $Z^2_w(M, K(H))$ there exists $\xi$ in $C^1_w(M, K(H))$ with $\Delta \xi = \eta$ and $\|\xi\| \leq 10\|\eta\|$.

**Proof.** Let $\xi$ be given by Proposition 10, relative to the abelian diffuse von Neumann subalgebra $A$. (We have only to show that $\xi$ has compact values or, what is the same, that $\hat{\xi}$ is identically zero, where $\hat{\xi}$ is the composition of $\xi$ with the projection from $B(H)$ onto the Calkin algebra.)
Since \( N_N(A)'' = M \) it follows that \( \Delta_\xi = \eta \) and hence that \( \hat{\xi} : M \rightarrow Q(H) \) is a derivation (since \( \eta \) has compact values). Moreover \( \hat{\xi} \) vanishes on \( GN_N(A) \).

It remains only to prove that \( \hat{\xi} \) vanishes everywhere.

Suppose on the contrary that this is not the case. Similar to the arguments in [4] it follows that there is an orthogonal sequence of nonzero projections \( \{f_n\}_n \subseteq A \) and \( x_n \) in \( f_nMf_n \) such that \( \|x_n\| \leq 1 \) and \( \|\hat{\xi}(x_n)\| > 1 \). But this obviously contradicts the continuity properties of \( \hat{\xi} \) stated in Proposition 10.

For the sake of completeness we repeat here the construction of the \( x_n \)'s and \( f_n \)'s. Let \( \{e_{ij}^n\}, i, j = 1, 2, \ldots, 2^k_n \), be a matrix unit for each \( n \in \mathbb{N} \) with the following properties [11]:

\[
\begin{align*}
(\alpha) & \quad e_{ii}^n \text{ belongs to } A, \sum_i e_{ii}^n = 1 \text{ for each } n; \\
(\beta) & \quad e_{ij}^n \text{ belongs to } N_M(A) \text{ for each } n, i, j; \\
(\gamma) & \quad e_{rs}^n \text{ is the sum of some } e_{ij}^n \text{ for every } pn.
\end{align*}
\]

By (\( \beta \)) and Proposition 10, we have that \( \hat{\xi}(e_{ij}^n) \) is zero for all \( n, i, j \). Let \( f_n = e_{i_1i_2}^n \) be any sequence of nonzero orthogonal projections in \( A \). Moreover \( M \) is canonically identified with \( M_{2^k_n}(\mathbb{C}) \otimes M_{f_n} \) (by means of \( e_{ij}^n \)); with this identification \( \hat{\xi} \) becomes \( I_{M_{2^k_n}(\mathbb{C})} \otimes \hat{\xi}^n \), where \( \hat{\xi}^n \) is the restriction of \( \hat{\xi} \) to \( M_{f_n} = f_nMf_n \). Since \( \hat{\xi} \neq 0 \), we may suppose that \( \|\hat{\xi}^n\| > C \), where \( C \) is the constant arising from the property \( S \) of \( M \). Since (by assumption \( S \)) \( \hat{\xi}^n \) is completely bounded with \( \|(\hat{\xi}^n)\|_{cb} \leq C\|\hat{\xi}^n\| \), it follows that there are \( x_n \) in \( f_nMf_n \) with \( \|x_n\| \leq 1 \) and \( \|\hat{\xi}(f_nx_nf_n)\| > 1 \). This completes the proof.

We are now able to prove our main result.

**Theorem 13.** Let \( M \subseteq B(H) \) be a finite von Neumann algebra of countable type and suppose that there is no nonzero central projection \( p \) in \( Z(M) \) such that \( M_p \) is a type \( II_1 \) factor without property \( S \) with respect to \( pH \) or a type \( II_1 \) factor without Cartan subalgebra.

Then \( H^2_{w}(M, K(H)) = 0 \) and moreover for every \( \eta \) in \( Z^2_{w}(M, K(H)) \) there exists \( \xi \) in \( C^1_{w}(M, K(H)) \) such that \( \Delta_\xi = \eta, \|\xi\| \leq 43\|\eta\|. \)

**Proof.** There exists a sequence \( \{p_n\}_{n \in \mathbb{N}} \subseteq Z(M) \) of nonzero orthogonal projections with sum 1 such that \( Mp_n \) has diffuse center and \( \{Mp_n\}_{n \geq 2} \) is a factor. By hypothesis it follows that \( Mp_n \) for \( n \geq 2 \) is a type \( II_1 \) factor with property \( S \) relative to \( p_nH \) and with a Cartan subalgebra, or it follows that \( Mp_n \) is a finite-dimensional factor.

By Corollaries 6 and 11 and by Proposition 12 we deduce that for every \( n \) in \( \mathbb{N} \) and for every \( \eta_n \) in \( Z^2_{w}(M_{p_n}, K(p_nH)) \) there exists \( \xi_n \) in \( C^1_{w}(M_{p_n}, K(p_nH)) \) such that \( \Delta_\xi_n = \eta_n \) and \( \|\xi_n\| \leq 10\|\eta_n\| \). Hence by Corollary 8 we obtain that for every \( \eta \) in \( Z^2_{w}(M, K(H)) \) there exists \( \xi \) in \( C^1_{w}(M, K(H)) \) such that \( \Delta_\xi = \eta \) and \( \|\xi\| \leq 43\|\eta\|. \)
Corollary 14. If $M$ is a type II$_1$ factor with property $\Gamma$ and with Cartan subalgebra, or if $M$ is finite, of countable type with diffuse center, then $H^2_w(M, K(H)) = 0$.

REFERENCES


DEPARTMENT OF MATHEMATICS, THE NATIONAL INSTITUTE FOR SCIENTIFIC AND TECHNICAL CREATION, BD. PĂCII 220, 79622 BUCHAREST, ROMANIA

Current address: Department of Mathematics, University of California, 405 Hilgard Avenue, Los Angeles, California 90024