AN EQUIVARIANT TORUS THEOREM FOR INVOLUTIONS

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Abstract. A complete classification is given for equivariant surgery on incompressible tori with respect to involutions with possible 1- or 2-dimensional fixed sets.

1. Introduction

In [8] an equivariant torus theorem was proved for involutions which have at most isolated fixed points. The main result of this paper is an equivariant torus theorem (Theorem 4.5) for involutions with possible 1- or 2-dimensional fixed sets. If additional restrictions are imposed, various equivariant surgery theorems have been proved. If the manifold is closed, orientable, and irreducible with infinite first homology, a theorem was given in [12].

The proof of the equivariant torus theorem proceeds by isotoping a given incompressible torus to a new torus such that the new torus, its image under the involution, and the fixed point set of the involution are “almost pairwise transversal.” The weaker “almost pairwise transversality” condition is used since, in general, pairwise transversality cannot be ensured (see Remark 3.3). The curves of intersection between the new torus and its image under the involution are of certain types. The curves are changed or removed by various surgeries, thus obtaining an equivariant torus or one of the exceptional cases as listed in the theorem.

An application of this theorem is the classification of involutions with 1- or 2-dimensional fixed sets on orientable torus bundles or unions of twisted I-bundles on Klein bottles. Surgery on an equivariant torus reduces these spaces to spaces on which the involutions are known. Details of this application are given in [4].

2. Preliminaries

Throughout we use the piecewise linear category. A piecewise linear homeomorphism will be called an isomorphism.

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Let $M$ be a connected compact 3-manifold. An involution $i$ is an isomorphism with $i \neq \text{id}$ and $i^2 = \text{id}$.

Let $\text{Fix}$ denote the fixed set $\text{Fix} = \text{Fix}(i) = \{x : i(x) = x\}$. Let $i$ be an involution on a manifold $M$ and $i'$ an involution on a manifold $M'$. $i$ and $i'$ are conjugate if there is an isomorphism $h : M \to M'$ with $i' = h \circ i \circ h^{-1}$. Call $h$ a conjugation between $i$ and $i'$.

**Lemma 2.1.** Given a simplicial subdivision $K$ of $M$ and an involution $i$ of $M$ there is a subdivision $L$ of $K$ with $i$ simplicial with respect to $L$.

**Proposition 2.2.** Let $i$ be an involution on a manifold $M$. Let $L$ be a subdivision of $M$ with $i : L \to L$ simplicial, and let $L'$ be the first barycentric subdivision of $L$. Then $\text{Fix} = \text{Fix}(i)$ is a subcomplex of $L'$. $i$ is the union of disjoint 0-, 1- and 2-dimensional proper submanifolds. Write $\text{Fix}^0$, $\text{Fix}^1$, and $\text{Fix}^2$ respectively for the unions of the 0-, 1- and 2-dimensional components of $\text{Fix}$.

If $v \in \text{Fix}^0 \cup \text{Fix}^2$ then $i$ is locally orientation reversing at $v$. If $v \in \text{Fix}^1$ then $i$ is locally orientation preserving at $v$. In particular, if $M$ is orientable then $i$ is orientation reversing if $\text{Fix}^0 \cup \text{Fix}^2 \neq \emptyset$ and $i$ is orientation preserving if $\text{Fix}^1 \neq \emptyset$.

**Proof.** Use the following:

1. Let $\Delta$ be a standard $m$-simplex (with standard subdivision) invariant under $i$. Then $\text{Fix} \cap \Delta$ is a subcomplex of the first barycentric subdivision of $\Delta$.

2. If $\text{Fix}$ contains a 3-simplex then $i = \text{id}$.

3. If $v \in \text{Fix}$ is a vertex of $\text{int}(L)$ consider the link $Lk$ of $v$.

4. If $Lk \cap \text{Fix}$ contains a 1-cell then $Lk \cap \text{Fix}$ is one 1-sphere. So $v \in \text{Fix}^2$.

5. If $Lk \cap \text{Fix}$ consists of $m \geq 0$ vertices then

$$\chi(Lk / i) - m = \frac{1}{2}(\chi(Lk) - m).$$

Since $Lk / i$ is a surface and $Lk$ is a 2-sphere, it follows that $m = 2$ and hence $v \in \text{Fix}^1$, or $m = 0$ and hence $v \in \text{Fix}^0$. □

### 3. Equivariant transversality

In order to be able to perform surgeries on a surface $F_0$ in a 3-manifold $M$ we would like to perform an ambient isotopy on $F_0$ such that the isotopic surface $F$ has the property that $F$, $iF$, and $\text{Fix}$ are pairwise transversal. This can be done if the manifold is orientable. If $\text{Fix}^2 \neq \emptyset$ and $M$ is nonorientable, however, pairwise transversality is not possible in general. This necessitates using a somewhat weaker form of transversality.

**Lemma 3.1.** Let $F$ be a proper surface in a 3-manifold with $F$, $iF$, and $\text{Fix}$ pairwise transversal. Then the components of $F \cap iF$ are 1-spheres and proper 1-cells. If $C$ is a component of $F \cap iF$ with $C \cap \text{Fix}^2 \neq \emptyset$ then $C \subseteq \text{Fix}^2$. 

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Proof. The first statement follows by transversality of $F$ and $iF$. The second statement follows by considering the star of a point in $C \cap \text{Fix}^2$. □

For the 3-cell $B^3 = \{(x, y, z) : |x| \leq 1, |y| \leq 1, |z| \leq 1\}$ in $\mathbb{R}^3$ let $i : B^3 \to B^3$ be the map $i(x, y, z) = (-x, y, z)$. Then $\text{Fix}(i)$ is the intersection of $B^3$ with the $yz$ plane. Let $S$ be the 1-sphere obtained as the join of $\{(1, 1, 1), (-1, -1, 1)\}$ with $\{(-1, 1, -1), (1, -1, -1)\}$ and let $D$ be the cone from $(0, 0, 0)$ on $S$. $D$ is a saddle shaped region (see Figure 1). (We could alternately take $D$ defined by $\{z = xy/\sqrt{x^2 + y^2}\} \cup \{(0, 0, 0)\}$.) Notice that $D \cap iD$ is the part of the $x$ and $y$ axis in $B^3$ while $D \cap \text{Fix}(i)$ is part of the $y$ axis. $D$ and $\text{Fix}(i)$ are transversal and $iD$ and $\text{Fix}(i)$ are transversal, but $D$ and $iD$ are not transversal at $(0, 0, 0)$. There is a subdivision making these spaces simplicial with all the vertices on $\partial B^3 \cup (0, 0, 0)$.

Definition 3.2. Let $F$ be a proper surface in a 3-manifold $M$ and $i$ an involution on $M$ with fixed set $\text{Fix}$. Call a point $v$ a saddle point if $v \in F \cap \text{Fix}^2$ and if $(T_7, T_7, \text{Fix}) \cap \text{star}(v)$ is isomorphic to $(D, iD, \text{Fix}(i))$.

Remark 3.3. Saddle points exist since $i$ is an involution with fixed set $\text{Fix}(i)$. Although $\partial D$, $\partial iD$, $\partial \text{Fix}(i)$ are pairwise transversal there is no 2-cell $E$ with $\partial E = \partial D$ and $E$, $iE$, $\text{Fix}(i)$ pairwise transversal. Otherwise, since $\partial E \cap \partial iE - \text{Fix}(i) = (\pm 1, 0, 0)$ there is a 1-cell $I$ of $E \cap iE$ with $(1, 0, 0) \in \partial I$ and this 1-cell must meet $\text{Fix}$, contradicting the previous lemma.

Let $d$ denote the identification $(x, 1, z) \sim (x, -1, -z)$ for all $x$ and $z$. Then $D/d$ is an annulus in a solid Klein bottle $B^3/d$ and no isotopy of $D/d$ moves it to an annulus with $F$, $iF$, and $\text{Fix}(i)/d$ pairwise transversal.

Definition 3.4. Let $F$ be a proper surface in a 3-manifold and $i$ an involution on $M$ with fixed set $\text{Fix}$. Then $F$, $iF$, and $\text{Fix}$ are almost pairwise transversal.
if:
(1) $F$, $iF$, and $\text{Fix}$ are pairwise transversal except at a finite number of saddle points, and
(2) the only components of $F \cap \text{Fix}$ containing saddle points are 1-spheres and each such 1-sphere contains at most one saddle point.

Let $\mathcal{G}$ be the closure of $(F \cap iF) - \text{Fix}^2$. $\mathcal{G}$ consists of disjoint 1-spheres and proper 1-cells: in a neighborhood of a saddle point, $F \cap iF - \text{Fix}^2$ corresponds to $[-1,0) \times 0 \times 0 \cup (0,1] \times 0 \times 0$ in the $B^3$ model for saddle points.

Let $E$ be a component of $\mathcal{G}$ that contains a saddle point $v$. Then $E$ has a fixed point and is invariant under $i$. Therefore, either $E$ is a 1-cell with no fixed points other than $v$ or $E$ is a 1-sphere with exactly two fixed points $v$ and $w$. By transversality $w$ is in $\text{Fix}^1$ or $\text{Fix}^2$. In the latter case $w$ is a saddle point. Together with Lemma 3.1 this shows:

**Proposition 3.5.** Let $F$, $iF$, and $\text{Fix}$ be almost pairwise transversal. Then the components of $F \cap iF$ are of one of the following forms:

1. Components with no saddle points (standard components):
   a. proper 1-cell $I$ with $I \cap \text{Fix} = \emptyset$ or $I \subseteq \text{Fix}^2$,
   b. proper 1-cell $I$ with $I \cap \text{Fix} = I \cap \text{Fix}^1 = v$, $v$ a point,
   c. 1-sphere $S$ with $S \cap \text{Fix} = \emptyset$ or $S \subseteq \text{Fix}^2$,
   d. 1-sphere $S$ with $S \cap \text{Fix} = S \cap \text{Fix}^1 = v_1 \cup v_2$, where $v_1$ and $v_2$ are points.

2. Components with saddle points:
   Type I component: $S_1 \cup I$ with $S_1 \cap I = \text{Fix} \cap I = w$, $S_1 \subseteq \text{Fix}^2$, and $w$ is the only saddle point on $S_1 \cup I$.
   Type II component: $S_1 \cup S$ with $S_1 \cap S = w$, $S_1 \subseteq \text{Fix}^2$, $S \cap \text{Fix} = v \cup w$, $v \in \text{Fix}$, and $w$ is the only saddle point on $S_1 \cup S$.
   Type III component: $S_1 \cup S_2 \cup S$ with $S_1 \cap S_2 = \emptyset$, $S_1 \cap S = w_i$, $S_i \subseteq \text{Fix}^2$, $S \cap \text{Fix} = w_1 \cup w_2$, and $w_1$ and $w_2$ are the only saddle points on $S_1 \cup S_2 \cup S$.

Here $S$, $S_1$, and $S_2$ are 1-spheres, the $I$ are 1-cells and $w_i$ are points, where $i = 1, 2$.

Note that case (2) does not arise if the manifold is orientable because a regular neighborhood $N$ of $S_1$ is always a solid Klein bottle. For if $N$ is instead a solid torus instead, then $\text{Fix} \cap N$, $F \cap N$, and $iF \cap N$ are all annuli or all Möbius bands. Consider annulus $A$, a component of $\partial N - \text{Fix}$. $A \cap F$ and $A \cap iF$ are two 1-spheres intersecting transversally in $A$. One sees these 1-spheres have nonzero intersection number, a contradiction.

**Corollary 3.6.** If $F$, $iF$, and $\text{Fix}$ are almost pairwise transversal, then they are pairwise transversal if one of the following holds:

a. $M$ is orientable.
b. $F$ is a 2-cell.
(c) $F$ is an annulus with $\partial F \cap i\partial F = \emptyset$. 
Proof. In case (a) regular neighborhoods of 1-spheres are solid tori.

In cases (b) and (c) Type II or III components are excluded since the 1-sphere $S$ is nonseparating. In case (c) Type I components are excluded a priori, while in case (b) the 1-sphere $S_1$ separates so a proper 1-cell $C$ cannot intersect $S_1$ transversally at one point.

In a surface $F$ a proper 1-cell $I$ bounds a disc $D$ if $I = \partial D - \partial F$.

**Corollary 3.7.** Let $F$, $iF$, and Fix be almost pairwise transversal and $C$ a proper 1-cell or 1-sphere component of $F \cap iF$, that is, let $C$ be a standard component. Then any disc in $F$ or $iF$ bounded by $C$ contains only standard components.

**Proof.** The proof is similar to the proof of case (b) in the previous corollary.

**Equivariant Transversality Theorem 3.8.** Let $\tau$ be an involution on a 3-manifold $M$ with $\text{Fix} = \text{Fix}(\tau)$ and let $F_0$ be a proper surface in $M$. Then there is an ambient $\varepsilon$-isotopy on $M$ taking $F_0$ to a proper surface $F$ such that $F$, $iF$, and $\text{Fix}$ are almost pairwise transversal. In $\partial M$, if $\partial F$, $\partial iF$, and $\text{Fix}$ are pairwise transversal then the isotopy may be taken to be the identity on $\partial M - N$, where $N$ is a given neighborhood of $\partial \text{Fix}^2 \cap \partial F$.

**Proof.** Let $F = F_0$ be a proper surface. By Proposition 2.2 and Lemma 2.1 subdivide $M$ so that $\tau$ is simplicial with respect to the subdivision, $\text{Fix}$ is a subcomplex of the subdivision, and $\text{Fix}$ is a disjoint union of 0-, 1- and 2-dimensional components $\text{Fix}^0$, $\text{Fix}^1$, and $\text{Fix}^2$. All isotopies performed in the construction will be done in the star neighborhoods of certain simplexes. By taking a sufficiently fine subdivision $\varepsilon$-isotopies are obtained.

**Step 1. Adjust $F$ near $\text{Fix}^2$.** By isotopies similar to those in the third step below we can assume $F$ and $\text{Fix}$ are transversal, the isotopy not moving $\partial F$ and $\partial \text{Fix}$ are nontransversal. In particular, $F \cap \text{Fix}^0 = \emptyset$. Then $F \cap \text{Fix}^2$ consists of disjoint 1-spheres and 1-cell components proper in $M$.

Let $S$ be a 1-sphere component of $F \cap \text{Fix}^2$. Let $N'$ be a regular neighborhood of $S$ with $N' \cap F$ and $N' \cap \text{Fix}$ transversal and each an annulus or Möbius band. $S$ has a regular neighborhood $N$ contained in $\text{int}(N')$ invariant under $\tau$ with no vertices on $\text{int}(N) - S$ such that $N \cap \text{Fix}$ is a regular neighborhood of $S$ and $\text{Fix} \cap \partial N$ has a regular neighborhood $Q$ in $\partial N$ which is invariant under $\tau$ and has no vertices except on $\text{Fix} \cup \partial Q$.

**Case 1.** $F \cap N$ and $\text{Fix} \cap N$ are annuli. Then $N$ is a solid torus, $\partial Q$ has four components, and $N - \text{Fix}$ consists of two components $N_1$ and $N_2$ which are interchanged by $\tau$. Let $J_i$ and $J_j$ be components of $\partial Q$ with $J_i \subseteq N_i$ and $\tau J_1 \neq J_2$. Let $A_i$ be the annulus with $\partial A_i = J_i \cup S$ having no vertices except on $\partial A_i$. $F$ is isotopic to a surface $F'$ by an ambient isotopy which is the identity on $M - N'$ and such that $F' \cap N' \cap \text{Fix} \subseteq N$ and $F' \cap N = A_1 \cup A_2$. Since $\tau J_1 \neq J_2$ it follows $F' \cap N \cap \tau (F' \cap N) = S$ and $F' \cap N$, $\tau (F' \cap N)$, $\text{Fix} \cap N$ are pairwise transversal.
Case 2. \( F \cap N \) and \( \text{Fix} \cap N \) are Möbius bands. Then \( N \) is a solid torus, and \( \partial Q \) has two components that are interchanged by \( \iota \). If \( J \) is one of these, then \( J \) and \( S \) determine a Möbius band \( A \) with \( \partial A = J \). Proceed as in Case 1.

If \( M \) is orientable then Cases (3) and (4) do not arise. Only in these cases do saddle points arise.

Case 3. \( F \cap N \) is an annulus and \( \text{Fix} \cap N \) is a Möbius band. Then \( N \) is a solid Klein bottle. Let \( A \) be one of the two (open) Möbius band components of \( \partial N - \text{Fix} \). There are two 1-spheres \( J_1 \) and \( J_2 \) which represent generators of \( H_1(A) = \mathbb{Z} \) with \( J_1 \) and \( J_2 \) intersecting transversally and at only one point \( x \). \( J_1 \) and \( S \) bound an annulus \( A_k \) with \( A_1 \cap A_2 = S \cup I \), where \( I \) is a 1-cell with \( \partial I = x \cup y \), \( y \in S \). Proceed as in Case 1 using \( F' \cap N = A_1 \cup tA_2 \). Then \( y \) is a saddle point and \( F' \cap N \), \( t(F' \cap N) \), and \( \text{Fix} \cap N \) intersect pairwise transversally elsewhere in \( N \).

Case 4. \( F \cap N \) is a Möbius band and \( \text{Fix} \cap N \) is an annulus. This case is similar to Case 3. Here \( A = \partial N - Q \) is an invariant annulus under \( \iota \). Find a curve \( J \) that bounds a Möbius band \( B \) with \( B \cap iB = S \) by lifting (from annulus \( A/\iota \)) a curve \( J' \) which represents twice a generator and which is embedded in \( A/\iota \) except for one transversal self-intersection. Now proceed as before.

When \( S \) is a 1-cell component of \( F \cap \text{Fix}^2 \), use an isotopy similar to the one of Case 1 above. This isotopy may change \( \partial F \) in \( N \cap \partial M \).

Step 2. Adjust \( F \) near \( \text{Fix}^1 \). By Step 1, \( F \cap \text{Fix}^1 \) consists of a number of vertices in \( \text{int}(M) \). If \( v \in F \cap \text{Fix}^1 \), let \( N' \) be a regular neighborhood of \( v \) and let \( N \) be the star neighborhood of \( v \). Take the subdivision so that \( N \) is in the interior of \( N' \), \( F \cap N \) is a proper 2-cell in \( N \), and \( \text{Fix} \cap N \) is a proper 1-cell. Since \( F \) is transversal, \( F \cap \partial N \) is a generator of \( H_1(N - \text{Fix}) \). Let \( J' \) be a curve in the annulus \( (\partial N - \text{Fix})/\iota \) representing twice a generator of this annulus. Take \( J' \) embedded except for one transversal self-intersection. \( J' \) lifts to two 1-spheres \( J \) and \( tJ \), which on coning to \( v \) give 2-cells \( D \) and \( iD \). \( D \), \( iD \), and \( \text{Fix} \) are pairwise transversal in \( \text{int} N \). Proceed as in Case 1 of Step 1.

We obtain a surface \( F \) and a neighborhood \( N \) of \( \text{Fix} \) such that \( F \) has the required transversality properties in \( N \). The following construction adjusts \( F \) only on star neighborhoods of simplexes of \( F - N \) where \( F \) and \( iF \) are not already pairwise transversal. By subdividing sufficiently we may assume without loss that \( \text{Fix} = \emptyset \). For convenience assume also \( \partial F = \emptyset \).

Let \( K \) be a subdivision of \( M \) with \( \iota \) simplicial and \( F \) a subcomplex of \( K \). Let \( \Delta \) be an \( m \)-simplex of \( F \) in \( K \) with \( m = 0, 1, \) or 2. Define \( \text{St}(\Delta) \), the reduced star of \( \Delta \) in \( K \), to be all 3-simplexes \( \sigma \) of \( K \) with \( \Delta \subseteq \sigma \) together with their faces. Let \( \text{St}_\iota(\Delta) \), the reduced star of \( \Delta \) in \( F \), be all 2-simplexes \( \sigma \) of \( K \) with \( \Delta \subseteq \sigma \subseteq F \) together with their faces. Let \( \rho: M \to M/\iota \) be the projection.
Step 3. There is a subdivision of $M$ and a proper surface $F'$ $\epsilon$-isotopic to $F$ such that for every simplex $\Delta$ of $F'$ either $\partial^{-1} p(\Delta) \cap F' = \Delta$ or $\Delta$ is a 0- or 1-simplex with $\text{int}(\text{St}_{F'}(\Delta))$ and $\text{int}(\text{St}_{iF'}(\Delta))$ transversal.

Call a simplex exceptional if it fails to satisfy these conditions and is of the highest possible dimension $m = 0, 1, \text{ or } 2$. Induct on the number of such simplexes. If there are no exceptional simplexes the theorem is established.

Suppose there is an exceptional simplex $\Delta$. Adjoin all the vertices (and their translates under $i$) of form $\frac{m+3}{m+3} b + \frac{1}{m+3} v$, where $b$ is the barycenter of $\Delta$ and $v$ is a vertex of $\text{St}(\Delta) - \Delta$. This determines a refinement $K'$ of $K$ with the same number of exceptional simplexes; no $m$-simplexes are subdivided for $m = 1, 2$, while for $m = 0$ transversality already holds away from vertices of $K$. Consider the reduced stars in $K'$. $\partial \text{St}_{F'}(\Delta)$ is a 1-sphere that decomposes $\partial \text{St}_{F'}^{'}(\Delta)$ into two components $D_+$ and $D_-$. There is an ambient isotopy taking $F$ to $F_1 = (F - \text{St}_{F'}(\Delta)) \cup D_+$ which is the identity except on $\text{St}_{F'}(\Delta)$. $F_1$ has fewer exceptional simplexes. When $m \neq 2$ this follows since $D_+ \cup D_-$ intersects the interior of any 2-simplex of $\text{St}(\Delta)$ transversally. $\Box$

Regular neighborhoods of the standard components of $F \cap iF$ can be taken in a special form.

**Proposition 3.9.** Let $F$, $iF$, and $\text{Fix}$ be almost pairwise transversal and $S$ be a 1-sphere component of $F \cap iF$. Suppose, in addition, that the regular neighborhood of $S$ in $F$ and $iF$ is an annulus. Then there exists a regular neighborhood $V \subseteq \text{int}(M)$ of $S$, called a standard neighborhood of $S$, with the following properties:

1. $V \cap F$ and $V \cap iF$ are annuli. Since these intersect transversally, $V$ is a solid torus.
2. $\text{Fix} \cap V$ intersect transversally, $\text{Fix} \cap F \cap V \subseteq S$, and the closure of each component of $(\text{Fix} \cap V) - S$ meets $S$ and $\partial V$. In particular, $\text{Fix} \cap V = \emptyset$.
3. $\text{Fix} \cap V$ is an annulus, two proper 1-cells, or empty.
4. If $iS = S$ then $iV = V$.
5. If $iS \neq S$ then $iV \cap V = \emptyset$ and the above properties hold simultaneously for $iV$.

Property (3) can be arranged since if $\text{Fix} \cap S \neq \emptyset$ then $iS = S$. $i$ is an involution on a 1-sphere so either $i = \text{id}$ or $i$ has exactly two fixed points.

The four 1-spheres $(F \cup iF) \cap \partial V$ decompose $\partial V$ into four (closed) annuli $\alpha_1$, $\alpha_2$, $\beta_1$, and $\beta_2$, with $\alpha_1 \cap \alpha_2 = \emptyset$ and $\beta_1 \cap \beta_2 = \emptyset$. Call these annuli the standard annuli corresponding to the standard neighborhood of $V$. Suppose $iS = S$. Relabeling, if necessary, we may assume $i(\alpha_1 \cap \beta_1) = (\alpha_1 \cap \beta_2)$. It follows that $i\alpha_1 = \alpha_1$. Then $i\beta_1 = \beta_2$ and $i\alpha_2 = \alpha_2$. When $\text{Fix} \cap V \neq \emptyset$ we obtain $\text{Fix} \cap \alpha_1 \neq \emptyset$, $\text{Fix} \cap \alpha_2 \neq \emptyset$, $\text{Fix} \cap \beta_1 = \emptyset = \text{Fix} \cap \beta_2$, and each component of $\text{Fix} \cap V$ meets both $\alpha_1$ and $\alpha_2$.

**Proposition 3.10.** Let $S$ be a 1-cell component of $F \cap iF$, where $F$, $iF$, $\text{Fix}$ are pairwise transversal (near $S$). Then there exists a regular neighborhood $V$ of
$S$ with $V \cap \partial M$ a regular neighborhood of $\partial S$, called a standard neighborhood of $S$, with the following properties:

1. $V \cap F$ and $V \cap iF$ are 2-cells with $\partial M \cap V \cap F$ and $\partial M \cap V \cap iF$ each two 1-cells. Necessarily $V$ is a 3-cell.

2. (2), (4) and (5) as for 1-sphere standard neighborhoods.

3. $\text{Fix} \cap V$ is a disc, one proper 1-cell, or empty.

The four 1-cells $(F \cup iF) \cap \partial V - \partial M$ subdivide $\partial V - \partial M$ into four discs $\alpha_1$, $\alpha_2$, $\beta_1$, and $\beta_2$ with $\alpha_1 \cap \alpha_2 = \emptyset$, $\beta_1 \cap \beta_2 = \emptyset$, and the properties as in the previous situation. Call these discs the standard discs corresponding to $V$.

**Remark 3.11.** In the following theorem certain 1-sphere components $S$ of $F \cap iF$ have standard neighborhoods because $S$ bounds discs in $F$ and $iF$. In the disc theorem $F$ is orientable so again there are standard neighborhoods. In the torus theorem the construction will be made so as to keep $S$ in this form always.

**Theorem 3.12.** Let $M$ be an irreducible 3-manifold with involution $i$ and $F_0$ be an incompressible proper surface. Then there is an ambient isotopy of $M$ which is an $i$-isotopy on $\partial M$ taking $F_0$ to a proper surface $F$ such that $F$, $iF$, and $\text{Fix}$ are almost pairwise transversal and no 1-spheres in $F \cap iF$ bound 2-cells in $F$. On $\partial M$, if $\partial F$, $i\partial F$, and $\partial \text{Fix}$ are pairwise transversal then the isotopy may be taken to be the identity on $\partial M - N$, where $N$ is a given neighborhood of $\partial \text{Fix}^2 \cap \partial F$.

**Proof.** By the preceding transversality theorem there is an $F$ with all the above properties except possibly 1-spheres in $F \cap iF$ bound 2-cells in $F$. By Proposition 3.5 those 2-cells contain no saddle components. Let $S$ be a 1-sphere of $F \cap iF$ bounding an innermost 2-cell in $iF$, that is, there is a 2-cell $D \subseteq iF$ with $D \cap F = \partial D = S$. Since $F$ is compressible, $S$ bounds a 2-cell $B$ in $F$. If $iS = S$ then we may assume $iB = D$.

Let $V$ be a standard neighborhood of $S$. Such a neighborhood exists since $S$ bounds a disc in $F$ and $iF$. Let $\alpha$ be the standard annulus in $\partial V$ meeting $D$ but not $B$. Then $i\alpha \cap \alpha = \emptyset$. There is a bicollar $D \times [-1, 1]$ of $D = D \times 0$ with

$$\partial D \times [-1, 1] = D \times [-1, 1] \cap F = S \times [-1, 1]$$

and with $D \times 1 \cap \alpha \neq \emptyset$. Since $D$ is innermost it follows that for a sufficiently thin collar $(D \times 1) \cap (D \times 1) = \emptyset$ and $F \cap (D \times 1) = \emptyset$. Consider $F' = (F - (B \cup S \times [-1, 1])) \cup D \times 1$. Then $F' \cap iF' \subseteq (F \cap iF) - S$ and $F'$, $iF'$, and $\text{Fix}$ are almost pairwise transversal. Since $M$ is irreducible and $D \cup B$ is a 2-sphere, $F'$ and $F$ are ambient isotopic by an isotopy being the identity on $\partial M$. By induction, all 1-spheres bounding 2-cells can be removed. □

A 2-cell $B$ in a 3-manifold is essential if it is proper and $\partial B$ does not bound a 2-cell in $\partial M$. In an irreducible 3-manifold a nonseparating proper 2-cell is essential. The following theorem is well known (see [1, 2, 6, or 10] for instance).
Disc Theorem 3.13. Let $M$ be an irreducible 3-manifold with involution $\iota$. Suppose $M$ has an essential 2-cell $B_0$. Then there is an essential 2-cell $B \subseteq M$ such that $B$ and $\Fix$ are transversal and either $B \cap \iota B = \emptyset$ or $\iota B = B$. In the former case $B \cap \Fix = \emptyset$ and in the latter case $B \cap \Fix$ is a proper 1-cell of $B$ or one point in the interior of $B$. If $\partial B_0 \cap \iota \partial B_0 = \emptyset$ then one can take $\partial B = \partial B_0$ and $B$ and $B_0$ are ambient isotopic by an isotopy that is the identity on $\partial M$.

Proof. By Theorem 3.12 and Corollary 3.6 there is an essential 2-cell $B$ with $B$, $\iota B$, and $\Fix$ pairwise transversal, $B$ and $B_0$ ambient isotopic, and $B \cap \iota B$ is either empty or consists of proper 1-cells only. Assume $B \cap \iota B \neq \emptyset$ (in particular, then $\partial B_0 \cap \iota \partial B_0 \neq \emptyset$). By induction it suffices to show how to obtain a new 2-cell $B_1$ with fewer 1-cells in $B_1 \cap \iota B_1$.

Let $D$ be an outermost disc of $B$: $D \subseteq B$ with $D \cap \iota B = \partial D \cap \iota B = \emptyset$ a proper 1-cell of $B$ and $\partial D - I \subseteq \partial B$. If $I = \iota I$ define $D' = \iota B - D$. If $I \neq \iota I$ define $D'$ to be the closure of the component of $\iota B - I$ that does not contain $\iota I$ (see Figure 2). Let $V$ be a standard neighborhood of $I$ and let $\alpha_1$, $\alpha_2$, and $\beta$ be standard discs of $V$ with $\alpha_1 \cap \alpha_2 = \emptyset$, $\alpha_1 \cap \beta \cap D \neq \emptyset$, and $\beta \cap D' \neq \emptyset$. Consider

$$B_1 = (D \cup \beta \cup D') - \text{int}(V) \quad \text{and} \quad B_2 = D \cup (\iota B - D').$$

Then $B_1 \cap \iota B_1 \subseteq (B \cap \iota B) - I$. If $B_1$ is essential we are done by induction or we arrive at the case $B_1 \cap \iota B_1 = \emptyset$. If $B_1$ is not essential then $\partial B_1$ bounds a 2-cell $E$ of $\partial M$. Since $M$ is irreducible, the 2-sphere $B_1 \cup E$ bounds a 3-cell. This 3-cell does not meet $I$, otherwise $\iota B$ would not be essential. Using the 3-cell construct an ambient isotopy taking $B_2$ to $\iota B$.

So we may assume $B_2$ is essential. If $I = \iota I$ we have $\iota B_2 = B_2$ and note that $\Fix \cap B_2 \subseteq \Fix \cap \iota I$, which is necessarily a point of $I$ or all of $I$. If $I \cap \iota I = \emptyset$ consider a sufficiently thin bicollar $D \times [-1, 1]$ of $D = D \times 0$ such that $D \times [-1, 1] \cap I$ is a bicollar of $I$ in $\iota B$ and $D \times 1$ meets $\alpha_1$. Then $B_2' = (D \times 1 \cup \iota B) - (I \times [-1, 1] \cup D')$ is essential since it is isotopic to $B_2$ and $B_2 \cap \iota B_2' \subseteq B \cap \iota B - I$. $\Box$

**Figure 2**
4. Equivariant torus theorem

A proper annulus $A$ in a 3-manifold $M$ is trivial if $A$ decomposes $M$ into a solid torus $V = D^2 \times S^1$ and a submanifold $M_0$ such that:

$$M = M_0 \cup V, \quad M_0 \cap V = \partial M_0 \cap \partial V = A,$$
and there exists a nonseparating proper 2-cell $B \subseteq V$ with $B \cap A = \partial B \cap A$ a nonseparating 1-cell in $A$.

Otherwise call $A$ nontrivial. Call $V$ a solid torus that trivializes $A$.

Note that if $A$ does not separate $M$ or if $\partial A$ is in different boundary components of $M$ then $A$ is nontrivial.

Call a nontrivial incompressible proper annulus an essential annulus. Call an incompressible proper Möbius band an essential Möbius band.

Let $F$ be a surface and $S$ a component of $F \cap iF$. In some surgeries performed later we will wish to replace $F$ by $F' = F \times 1$, where $F \times [-1, 1]$ is a bicollar of $F$. To insure that $F'$, $iF'$, and $\text{Fix}$ are transversal at least in standard neighborhoods (see Propositions 3.9 and 3.10), the following lemma is used.

**Lemma 4.1.** Let $V$ be a solid torus and $i: V \to V$ an involution. Let $A_0$ and $A_1$ be annuli in $\partial V$ with $\partial A_0 = S_0 \cup S_1$, $\partial A_1 = S_1 \cup S_2$, $A_0 \cap A_1 = S_1$, $iS_0 = S_0$, $(A_0 \cup A_1) \cap i(A_0 \cup A_1) = S_0$, and $\text{Fix} \cap (A_0 \cup A_1) \subseteq S_0$. Then there is a proper annulus $A \subseteq V$ such that $A$, $iA$, and $\text{Fix}$ intersect transversally with $A \cap iA = S$ a 1-sphere and $\partial A$ having one component in $\text{int}(iA_0)$ and the other component is $S_2$.

A similar statement holds if $V$ is a 3-cell, $A_0$, $A_1$ are 2-cells, and the $S_i$ are 1-cells (see Figure 3).

**Proof.** By transversality of $\text{Fix}$, by taking a sufficiently small regular neighborhood $N$ of $A_0 \cup A_1 \cup iA_0 \cup iA_1$ we may assume one of the following holds:

1. $\text{Fix} \cap N = \emptyset$, 

![Figure 3](https://www.ams.org/journal-terms-of-use)
(2) Fix \( \cap N \) consists of two disjoint 1-cells \( I_i \) with exactly one point of \( \partial I_i \) in \( S_0 \) and the other in \( \text{int}(V) \cap \partial N \), or (3) Fix \( \cap N \) is an annulus such that one boundary component is \( S_0 \) and the other is in \( \text{int}(V) \cap \partial N \).

Further, there is a regular neighborhood \( N' \) of \( S_0 \subseteq N \) such that \( iN' = N' \), \( A'_0 = N' \cap A_0 \) is an annulus, \( A'_0 \cup iA'_0 = N' \cap \partial V \), and properties (1)-(3) hold with respect to \( N' \).

Let \( B \) be the annulus which is the closure of \( \partial N' - A'_0 \cup iA'_0 \). Then there is a 1-sphere \( S \) in \( \text{int}(B) \) with \( iS = S \) and \( iA' \cap S = \emptyset \). There is an annulus \( A'' \) in \( V - N' \) such that \( \partial A'' = A'' \cap V - N' = S_2 \cup S \) and \( A'' \cap iA'' = S \). Let \( A' \) be the component of \( B - S \) that meets \( iA_0 \). Then \( A = A' \cup A'' \) is the desired annulus. \( \Box \)

A solid Klein bottle is a twisted \( I \)-bundle over an annulus. The annulus is essential but it does not separate the boundary.

**Lemma 4.2.** Let \( U \) be a solid torus or a solid Klein bottle. If \( U \) is a solid torus then \( U \) has no essential annuli. If \( U \) is a solid Klein bottle then \( U \) has no essential annuli that separate \( \partial U \).

Moreover, suppose \( A' \) is an annulus contained in \( \partial U \) such that a nonseparating proper disc \( D \) of \( U \) intersects \( A' \) in exactly one nonseparating 1-cell of \( A' \). If \( A \) is an incompressible proper annulus disjoint from \( A' \) then the solid torus which trivializes \( A \) may be taken to be disjoint from \( A' \).

**Proof.** Suppose \( A \) is an essential annulus. Then let \( D \) be any proper nonseparating 2-cell of \( U \). (When \( A' \) is given, take \( D \) as in the statement.) Make \( A \) and \( D \) transversal. Since \( A \) is incompressible, adjust \( D \) so that \( A \cap D \) consists of 1-cells only.

If \( A \cap D = \emptyset \) then \( A \) is contained in a 3-cell obtained by removing a sufficiently small regular neighborhood of \( D \) from \( U \). This contradicts incompressibility.

If \( A \cap D \neq \emptyset \) let \( B \) be an outermost 2-cell of \( D \) (and disjoint from \( A' \) if \( A' \) is given): so \( B \cap A = \partial B \cap A = I \) is a 1-cell and \( B \cap \partial U = \overline{\partial B - I} \). If \( I \) bounds a 2-cell in \( A \), then by an isotopy moving \( B \), obtain a disc \( D' \) with fewer 1-cells in \( A \cap D' \). Assume now that \( I \) does not bound a 2-cell in \( A \). Then \( I \) does not separate \( A \). Let \( V \) be the closure of the component of \( U - A \) that meets \( \text{int}(B) \). \( \partial A \) decomposes \( \partial U \) into two annuli or possibly, in the case where \( U \) is a solid Klein bottle, into an annulus and two Möbius bands. However, in the latter case \( \partial B \cap \partial U \) must meet the annulus. It follows that \( \partial V \cap \partial U \) is an annulus and \( V \) is a solid torus with the properties making \( A \) trivial. \( \Box \)

In order to prove the torus theorem we will need to know the involutions up to conjugacy on a solid torus.

Let \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \). Define maps on \( S^1 \) by \( \kappa(z) = \overline{z} \) and \( \alpha(z) = -z \). Let \( D^2 = \{ z \in \mathbb{C} : |z| \leq 1 \} \). Define maps on \( D^2 \) by \( \hat{\kappa}(z) = \overline{z} \) and
\( \hat{\alpha}(z) = -z \). Also define \( \hat{\rho} \) on \( D^2 \times S^1 \) by \( \hat{\rho}(z, w) = (zw, w) \). For later use, define \( \tau \) on \( I = [-1, 1] \) by \( \tau(t) = -t \).

Define the following involutions on the solid torus \( V = D^2 \times S^1 = \{(z, w) : |z| \leq 1, |w| = 1, z, w \in \mathbb{C}\} : \)

\[ j_A = \hat{k} \times \text{id} \text{ having as fixed set the annulus } \mathbb{R}e \times S^1. \]
\[ j_M = \hat{\rho} \circ (\hat{k} \times \text{id}) \text{ having as fixed set the Möbius band } \{(s \cdot e^{\pi i t}, e^{2\pi i t}) : 0 \leq s \leq 1, -1 \leq t \leq 1\}. \]
\[ j_{2D} = \text{id} \times \kappa \text{ having as fixed set two 2-cells } D^2 \times \pm 1. \]
\[ j_{DP} = \hat{\rho} \circ (\text{id} \times \kappa) \text{ having as fixed set a 2-cell and a point } D^2 \times 1 \cup 0 \times -1. \]
\[ j_S = \hat{\alpha} \times \text{id} \text{ having as fixed set one 1-sphere } 0 \times S^1. \]
\[ j_{2C} = \hat{k} \times \kappa \text{ having as fixed set two 1-cells } \mathbb{R}e \times \pm 1. \]
\[ j_{2P} = \hat{\alpha} \times \kappa \text{ having as fixed set two points } 0 \times \pm 1. \]
\[ j_N = \hat{k} \times \alpha, \text{ fixed point free and orientation reversing.} \]
\[ j_O = \text{id} \times \alpha, \text{ fixed point free and orientation preserving.} \]

The next theorem follows easily by applying the Disc Theorem 3.13 and using the fact (see [13]) that there are only three involutions up to conjugacy on a 3-cell (for details see [4]).

**Theorem 4.3.** If \( \iota \) and \( \iota' \) are involutions on \( V = D^2 \times S^1 \) with nonempty isomorphic fixed point sets or if \( \iota \) and \( \iota' \) are fixed point free and of the same orientation type, then \( \iota \) and \( \iota' \) are conjugate. An involution on \( V \) is conjugate to one of the nine involutions listed above.

**Lemma 4.4.** Let \( M \) be an irreducible, \( P^2 \)-irreducible 3-manifold containing an incompressible torus. Let \( F \) be a 1-sided Klein bottle in the interior of \( M \) and \( W \) a regular neighborhood of \( F \) in \( M \) with \( \partial W \) a torus. Then \( \partial W \) is an incompressible torus.

**Proof.** Otherwise \( M = W \cup U \), where \( M - W = U \), \( \partial U = U \cap (\overline{U - W}) \), and \( U \) is a solid torus. Necessarily \( W \) is an orientable twisted I-bundle over \( T \) and \( M \) is orientable. The inclusion of \( U \) in \( M \) determines an index two subgroup of \( \pi_1(M) \). Consider \( p: \tilde{M} \to M \), the 2-sheeted covering corresponding to that subgroup. Then \( p^{-1}(W) = T \times [-1, 1] \), where \( T \) is a torus with \( p(T \times 0) = F \), and \( p^{-1}(U) = V_1 \cup V_2 \), where \( V_1 \) and \( V_2 \) are two disjoint solid tori. \( \tilde{M} \) is a lens space. But \( M \) and hence \( \tilde{M} \) contains a 2-sided incompressible torus. \( \square \)

**Equivariant Torus Theorem 4.5.** Let \( M \) be an irreducible, \( P^2 \)-irreducible 3-manifold with involution \( \iota \). Suppose \( M \) contains an incompressible torus. Then one of the following holds:

(I) There is a 2-sided incompressible torus or Klein bottle \( T \) in \( \text{int}(M) \) transversal to \( \text{Fix} \) with \( T \cap \iota T = \emptyset \) or \( \iota T = T \).

(II) \( M = V_{-1} \cup V_1 \cup U_{-1} \cup U_1 \), where \( V_i \) and \( U_i \) are solid tori and \( \iota V_i = V_i \) and \( \iota U_{-1} = U_1 \). There are annuli \( A_i \), \( i = \pm 1 \), with

\[ A_1 \cap A_{-1} = A_i \cap \iota A_i = \partial A_i = \partial \iota A_i = V_i \cap V_{-1} = U_1 \cap U_{-1} \]
and \( V_i \cap U_i = A_i, \ V_i \cap U_{-i} = iA_i, \ \partial V_i = A_i \cup iA_i, \ \partial U_i = A_i \cup iA_{-i} \), (see Figure 4). \( A_1 \cup A_{-1} \) is a 2-sided incompressible torus transversal to \( \text{Fix} \). \( i|V_i \) is orientation preserving.

(III) \( M = V_1 \cup V_2 \cup V \), where \( V_1, V_2 \), and \( V \) are solid tori each invariant under \( \iota \) such that \( \iota \) is orientation preserving when restricted to any of \( V_1, V_2, \) and \( V \). There is a 1-sided Klein bottle \( T \) with \( T \cap \iota T = S \subseteq \text{int}(V) \) a generator of \( \pi_1(V) \). \( V_1 \cap V_2 = (T \cap \iota T) - \text{int}(V) \) are two annuli. \( T, \ iT, \) and \( \text{Fix} \) are pairwise transversal and \( \text{Fix} \cap \partial V_2 = \emptyset \) and \( \text{Fix} \cap S = \emptyset \). \( V \) is a standard neighborhood of \( S \), (see Figure 4).

(IV) \( M = W \cup V \), where \( W \) is a twisted I-bundle over a torus \( T \subseteq W \) and \( V \) is a solid torus with \( \partial W = \partial V = W \cap V \) and \( \iota W = W, \ iT = T, \) and \( iV = V \). \( \text{Fix} \) is transversal to \( \partial W \) and \( T \) except for a possible 1-sphere component \( S \) of \( \text{Fix}^1 \) contained in \( T \).

Proof. Let \( T_0 \) be an incompressible torus in \( \text{int}(M) \). By Theorem 3.12 assume \( T_0, \ iT_0 \), and \( \text{Fix} \) are almost pairwise transversal and that no 1-spheres in \( T_0 \cap iT_0 \) bound 2-cells in \( T_0 \).

As a first step we handle the cases where saddle components arise. Only Type III and Type II components are possible. In both cases since \( S \) and \( S_1 \) intersect transversally at one point, there can be only one component in \( T_0 \cap iT_0 \).

Suppose \( T_0 \cap iT_0 \) is a Type III component \( S \cup S_1 \cup S_2 \). Then \( S_1 \) and \( S_2 \) bound an annulus \( A \) in \( T_0 \) since \( S_1 \cap S_2 = \emptyset \) and both intersect \( S \) transversally once. Then \( T = A \cup iT \) is a torus with \( iT = T \) and \( T \) and \( \text{Fix} \) are transversal. \( T \) is 1-sided since a regular neighborhood of \( S_1 \) is a solid Klein bottle. Let \( N \) be a regular neighborhood of \( T \) invariant under \( N \). If \( \partial N \) is incompressible then it is a 2-sided torus satisfying (I). If \( \partial N \) is compressible we arrive at (IV).

Suppose \( T_0 \cap iT_0 \) is a Type II component \( S \cup S_1 \). First we construct a torus \( T' \) isotopic to \( T = T_0 \) with \( iT' = T' \). Let \( N(S) \) and \( N(S_1) \) be regular neighborhoods of \( S \) and \( S_1 \) respectively, both invariant under \( \iota \) such that \( N = N(S) \cup N(S_1) \) is a regular neighborhood of \( S \cup S_1 \) and such that \( T \cap N(S) \)
and \( T \cap N(S_1) \) are annuli, \( N(S_1) \cap \text{Fix} \) is a Möbius band, \( N(S) \cap \text{Fix}^2 \) is a proper 2-cell, and \( N(S) \cap \text{Fix}^1 \) is a proper 1-cell. Both \( N(S) \) and \( N(S_1) \) are solid Klein bottles. By transversality there are two disjoint open 2-cell components \( K_1 \) and \( K_2 \) of \( N(S) - (T \cup iT) \) that meet \( \text{Fix}^1 \) and there are two disjoint open 2-cell components \( L_1 \) and \( L_2 \) of \( N(S_1) - (T \cup iT) \) that do not meet \( \text{Fix}^2 \). By considering the effect of \( i \) near saddle points we see \( A = (K_1 \cup K_2 \cup L_1 \cup L_2) \cap \partial N \) is an annulus with \( \partial A = C \cup iC \), where \( C = \partial N \cap T \). The closure of \( A \cup (T - N) \cup i(T - N) \) is a 2-sphere which by the irreducibility of \( M \) bounds a 3-cell \( E \). \( E \) cannot contain the proper punctured torus \( T \cap N \) so \( E \cap \text{int}(N) = \emptyset \). Since \( \text{Fix}^1 \) is transversal to \( \partial E \) and \( i\partial E = \partial E \) it follows \( iE = E \). In particular, \( iE \) is conjugate to \( j_1 \), the standard involution of a 3-cell with fixed set one 1-cell. \( A \) is invariant and contains \( \text{Fix}^1 \cap \partial E \).

Hence one shows there is a proper 2-cell \( D \) with \( \partial D \) a generator of \( H_1(A) \) such that \( \text{Fix}^1 \cap E \) is a proper 1-cell of \( D \) and \( iD = D \). Since \( \partial D = \partial D \), by taking \( N \) sufficiently small we can construct a proper punctured torus \( P \) in \( N \) with \( \partial P = \partial D \) and \( iP = P \) (namely isotope \( T \cap N \)). Consider the torus \( T' = P \cup D \). \( \text{Fix}^2 \) intersects \( T' \) transversally at \( S_1 \). The component of \( \text{Fix} \) meeting \( N \) is contained in \( T' \). \( T' \) is 1-sided. Let \( W \) be a regular neighborhood of \( T' \) invariant under \( i \). If \( \partial W \) is incompressible then it is a 2-sided torus satisfying (I). If \( \partial W \) is compressible we arrive at (IV).

We may now assume \( T_0 \cap iT_0 \) has no saddle components. \( T_0, iT_0 \), and \( \text{Fix} \) are pairwise transversal and \( T_0 \cap iT_0 \) consists of disjoint 1-spheres bounding annuli in \( T_0 \) and \( iT_0 \). We successively construct incompressible tori or Klein bottles \( T \) with fewer 1-spheres in \( T \cap iT \), but always keep \( T \cap iT \) consisting of 1-spheres bounding annuli in \( T \) and \( iT \) and always retain the property that each component of \( T \cap iT \) has a standard neighborhood, (see Propositions 3.9 and 3.10). It also follows then that any 1-sided Klein bottle arising from such a construction has a regular neighborhood \( W \) with \( \partial W \) a torus. So Lemma 4.4 is applicable.

Note: Suppose \( T \) satisfies all the conditions of (I) except that \( T \) is 1-sided instead of 2-sided. Let \( W \) be a regular neighborhood of \( T \). We can take \( W \) so that \( \partial W \) and \( \text{Fix} \) are transversal and \( iW = W \) or \( W \cap iT = \emptyset \). \( \partial W \) is 2-sided. If \( \partial W \) is incompressible, \( \partial W \) satisfies (I). Assume now that \( \partial W \) is compressible. \( T \) cannot be a Klein bottle since the previous paragraph shows that \( \partial W \) is a torus and thus Lemma 4.4 would show \( \partial W \) to be incompressible. So \( T \) is a torus. Now \( V = M - W \) is a solid torus. If \( iT = T \) we have (IV). If \( iT \cap T = \emptyset \) then the solid torus \( V \) contains an embedded 1-sided torus \( iT \), a contradiction.

We give constructions that reduce the number of components of \( T \cap iT \), producing a new torus or Klein bottle. Repeat these constructions with the new torus or Klein bottle produced until eventually one satisfying condition (I), (II), (III), or (IV) is constructed. There are four cases depending on the number of 1-spheres of \( T \cap iT \) and the compressibility of certain surfaces.
Assume $T \cap iT$ consists of at least two 1-spheres. Let $A \subseteq iT$ be an innermost annulus: $A \cap T = \partial A$. $\partial A$ decomposes $T$ into two annuli $A'$ and $A''$ with $T = A' \cup A''$ and $\partial A = \partial A' = \partial A'' = A' \cap A''$. $T' = A' \cup A$ and $T'' = A'' \cup A$ are tori or Klein bottles (see Figure 5).

**Case (1).** $T'$ is incompressible.

**Case (1.1):** $i\partial A = \partial A$ and $iA = A'$. Then $iT' = T'$. One sees Fix is transversal to $T'$ by considering the standard neighborhoods of $\partial A$. We arrive at (I) or (IV).

**Case (1.2):** Either $i\partial A = \partial A$ and $iA = A''$ or $i\partial A \cap \partial A$ is a single 1-sphere $S$ and $iA \subseteq A''$. In the latter case $iS = S$. Let $V_1$ and $V_2$ be distinct standard neighborhoods of $\partial A$ and let $\gamma_1$ and $\gamma_2$ be the two distinct standard annuli that meet both $A$ and $A'$. Let $T_1 = (A' \cup A \cup \gamma_1 \cup \gamma_2) - \text{int}(V_1 \cup V_2)$. Then $T_1 \cap iT_1 \subseteq (T \cap iT) - \partial A$ and $T_1 \cap iT_1 \neq T \cap iT$ because $(\gamma_1 \cup \gamma_2) \cap i(\gamma_1 \cup \gamma_2) = \emptyset$. $T'$ and $T_1$ are ambient isotopic so $T_1$ is incompressible. Fix $\cap (\gamma_1 \cup \gamma_2) = \emptyset$ and $A$ is innermost so $T_1$, $iT_1$, and Fix are pairwise transversal. Proceed with $T_1$.

**Case (1.3):** Either $i\partial A \cap \partial A$ is a single 1-sphere $S$ and $iA \subseteq A'$ or $i\partial A \cap \partial A = \emptyset$. Let $\partial A = S \cup S'$. Let $V$ be a standard neighborhood of $S$ and let $\gamma$ be the standard annulus that meets both $A$ and $A'$. Let $A \times [0, \varepsilon]$ be a sufficiently thin collar of $A = A \times 0$ in $M$ such that

$$S' \times [0, \varepsilon] \subseteq A', \quad S \times [0, \varepsilon] \subseteq T, \quad (A \cap \partial V) \times [0, \varepsilon] = (A \times [0, \varepsilon]) \cap \partial V.$$ 

The collar exists since $V$ is a solid torus. In the first case, $iS = S$ and $i\gamma = \gamma$. By Lemma 4.1 if $(A \times \varepsilon) \cap \gamma \neq \emptyset$ we may assume $(A \times \varepsilon) \cap V$ and $i((A \times \varepsilon) \cap V)$ intersect transversally in a 1-sphere $S_1$ and that both are transversal to Fix. In all other cases set $S_1 = S$. Define

$$T_1 = (A \times \varepsilon) \cup A' - ((S' \cup S) \times [0, \varepsilon]) \cup ((S \times [0, \varepsilon]) - A').$$

Then $T_1 \cap iT_1 \subseteq ((T \cap iT) - \partial A) \cup S_1$. $T_1$ is incompressible since it is ambient isotopic to $T'$. $T_1$, $iT_1$, and Fix are pairwise transversal. Proceed with $T_1$.

By Case (1) we may now assume $T'$ and $T''$ are compressible.
Case (2). For every annulus $A \subseteq iT$ with $A \cap T = \partial A$, both corresponding surfaces $T'$ and $T''$ are compressible and $T \cap iT$ contains more than two 1-spheres. Then let $A_1$ and $A_2$ in $iT$ be annuli with $A_i \cap T = \partial A_i$ and with $\partial A_i = S_0 \cup S_1$, where $S_0$, $S_1$, and $S_2$ are 1-spheres with $S_1 \neq S_2$. Let $A$, $A_1'$, and $A_2'$ be the three annuli of $T$ that these 1-spheres decompose $T$ into: $\partial A = S_1 \cup S_2$ and $\partial A_i = S_0 \cup S_i$ for $i = 1, 2$ (see Figure 6).

Define $T_1 = A \cup A_1 \cup A_2$. $T_1$ is incompressible. Otherwise $T_1$ bounds a solid torus or a Klein bottle $U$. Say $A' \subseteq U$. $A'_1$ is trivial in $U$ by Lemma 4.2. If $A' \cup A_1$ bounds the trivializing torus then the incompressible $T = A'_1 \cup A'_2 \cup A$ is ambient isotopic to $A \cup A'_2 \cup A$ which was compressible by hypothesis. If $A'_1 \cup A_2 \cup A$ bounds the trivializing torus then, since $A'_1$ and $A_2$ meet on $S_0$, $A_2$ must also be trivial in $A'_1 \cup A_2 \cup A$. So $T$ is ambient isotopic to $A'_2 \cup A_2$ which was assumed compressible.

We have five cases:

Case (2.1): $i(S_1 \cup S_2) = S_1 \cup S_2$ and $iS_0 \subseteq A$. Then $i(A_1 \cup A_2) = A$.

Case (2.2): $i(S_1 \cup S_2) = S_1 \cup S_2$ and $iS_0 \subseteq A'$. Then $i(A_1 \cup A_2) = A'_1 \cup A'_2$ and $iS_0 = S_0$.

Case (2.3): $iS_1 = S_1$ and $iS_0 = S_0$. Then $iA_1 = A'_1$ and $iS_2 \subseteq A'_2$.

Case (2.4): $i(S_1 \cup S_2) \cap (S_1 \cup S_2) = \emptyset$.

Case (2.5): $i(S_1 \cup S_2) \cap (S_1 \cup S_2)$ is one 1-sphere.

These cases cover all possibilities. In each case we find a $T_i$ with fewer 1-spheres.

Case (2.5) follows from the other cases. After relabeling assume $S_1$ is the 1-sphere in the intersection. Then $iS_1 = S_1$. By Case (2.3) we assume $iS_0 \neq S_0$. Let $A_3$ be the innermost annulus adjacent to $A_1$: $A_3 \subseteq iT_1$ with $A_3 \cap T_1 = \partial A_3 = S_1 \cup S_3$, where $S_3 \neq S_0$. By Case (2.3) again we may assume $iS_3 \neq S_3$. By Cases (2.1) and (2.2) we may assume $iS_0 \neq S_3$. So we have $i(S_0 \cup S_3) \cap (S_0 \cup S_3) = \emptyset$ and Case (2.4) gives the reduction.

In Case (2.1) use $T_1 = iT_1$. Fix is transversal to $T_1$ since the standard annulus meeting $A_1$ and $A$ is invariant.

In Case (2.2) and Case (2.3): For $i = 1, 2$ let $V_i$ be the standard neighborhoods of $S_i$ with $\gamma_i$ the standard annuli that meet both $A$ and $A_i$. In all cases $(\gamma_1 \cup \gamma_2) \cap (\gamma_1 \cup \gamma_2) = \emptyset$. Define $T_2$ to be the incompressible surface ambient
isotopic to $T_1$ given by

$$T_2 = T_1 \cup \gamma_1 \cup \gamma_2 \cap \text{int}(V_1 \cup V_2).$$

Then $T_2$, $iT_2$, and $\text{Fix}$ are pairwise transversal and

$$T_2 \cap iT_2 \subseteq T_1 \cap iT_1 - (S_1 \cup S_2).$$

In Case (2.4): First assume $iS_0 \neq S_1$. By symmetry assume $iS_0 \neq S_1$. Let $(A_1 \cup A_2) \times [0, \varepsilon]$ be a sufficiently thin collar of $A_1 \cup A_2 = (A_1 \cup A_2) \times 0$ in $M$ such that $(S_1 \cup S_2) \cup [0, \varepsilon] \subseteq T$ and $S_0 \times [0, \varepsilon] \subseteq A'$. Define

$$T_2 = (A_1 \cup A_2) \times \varepsilon \cup A - ((S_1 \cup S_2) \times [0, \varepsilon]) \cup ((S_1 \cup S_2) \times [0, \varepsilon] - A).$$

Then $T_2 \cap iT_2 \subseteq i(T \cap iT) - S_0$. Also $T_2$ is ambient isotopic to incompressible $T_1 \cap iT_2$, and $\text{Fix}$ are pairwise transversal.

If $iS_0 = S_0$, proceed as above but replace the condition $S_0 \times [0, \varepsilon] \subseteq A'$ by $S_1 \times [0, \varepsilon] \subseteq A$. Use Lemma 4.1 on a standard neighborhood of $S_0$ to adjust the collar so that $(A_1 \cup A_2) \times \varepsilon$ and $i(A_1 \cup A_2) \times \varepsilon$ intersect transversally in one 1-sphere $S_3$. Then $T_2 \cap iT_2 \subseteq (iT \cap iT) - (S_0 \cup S_1) \cup S_3$.

Case (3). For each annulus $A \subseteq iT$ with $A \cap T = \partial A$, both corresponding surfaces $T'$ and $T''$ are not incompressible and $T \cap iT$ is exactly two 1-spheres. Set $iT = A_{-1} \cup A_1$ with $A_{-1} \cap A_1 = \partial A_1 = \partial A_{-1} = T \cap iT = S_1 \cup S_2$. Then $T = iA_{-1} \cup iT_1$. There are solid tori or Klein bottles $U_i$ and $V_i$ for $i = \pm 1$, pairwise disjoint on their interiors with $\partial V_i = A_i \cup iT_i$ and $\partial U_i = A_i \cup iT_{-i}$. None of $U_i$ or $V_i$ are solid Klein bottles. Otherwise, if, say, $V_1$ is a solid Klein bottle, then since $S_1 \cup S_2$ decomposes into two annuli it follows that $S_1$ bounds a disc in $V_1$. This contradicts the incompressibility of $T$. By considering the standard annuli of a standard neighborhood of $S_1$ we see $iV_i = V_i$ and $ iT_i = U_{-i}$. That $A_1 \cup A_{-1}$ is a torus and not a Klein bottle follows from the fact that $V_1$ and $U_{-1}$ are solid tori.

If both $iV_1$ and $iV_{-1}$ are orientation preserving then we arrive at (II). So assume that $iV_1$, say, is orientation reversing. We will show $iV_1$ is conjugate to $j_M$ and arrive at (I). By Theorem 4.3, $iV_1$ is conjugate to $j_A$, $j_{2D}$, $j_N$, $j_M$, or $j_{DP}$, the standard involutions on a solid torus. $j_{2D}$ and $j_{DP}$ are not possible since $S_1$ or $S_2$ would bound a disc contradicting the incompressibility of $T$.

If $iV_1$ is conjugate to $j_M$ then, say, $S_1 \subseteq \text{Fix}$ and $S_2 \cap \text{Fix} = \emptyset$. Then $iV_{-1}$ has a 2-dimensional fixed set component that has only one boundary component. It follows that $iV_{-1}$ is also conjugate to $j_M$. So $\text{Fix}$ contains a Klein bottle $K$. There is a regular neighborhood $W$ of $K$ with $iW = \partial W$ and $W \cap \text{Fix} = \emptyset$. Since $V_i$ are solid tori and $K \cap V_i$ is a Möbius band, $\partial W$ is a torus. By Lemma 4.4, $iW$ is incompressible. We arrive at (I).

If $iV_1$ is conjugate to $j_A$, then $[S_1]$ represents a generator of $H_1(V_i)$ and hence there is an ambient isotopy taking $iT$ to $\partial U_{-1}$ (move $A_1$ to $iA_1$). This contradicts that $iT$ is incompressible.
Finally suppose \( t|V_1 \) is conjugate to the involution \( j_N = \kappa \times \alpha \) on \( D^2 \times S^1 \).

If \( tS_1 = S_1 \) then \( S'_1 = 1 \times S^1 \) and \( S'_2 = -1 \times S^1 \) determine annuli \( A' \) and \( tA' \) of \( \partial D^2 \times S^1 \). It is possible to construct a conjugation \( \partial V_1 \to \partial D^2 \times S^1 \) taking \( A_1 \) to \( A' \). This conjugation extends to a conjugation \( V_1 \to D^2 \times S^1 \). But \([S'_1]\) is a generator of \( H_1(D^2 \times S^1)\) and we get a contradiction as for the \( j_A \) case above. If \( tS'_1 = S'_2 \) then use \( S'_1 = \partial D^2 \times 1 \) and \( S'_2 = \partial D^2 \times -1 \) and proceed as above but this time obtaining a contradiction as for \( j_2D \) above.

Case (4). \( T \cap iT \) is a single 1-sphere \( S \). Then \( iS = S \). Let \( V \) be a standard neighborhood of \( S \) and let \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) be the standard annuli with \( \alpha_1 \cap \alpha_2 = \emptyset, \beta_1 \cap \beta_2 = \emptyset, \tau\alpha_1 = \alpha_1, \tau\alpha_2 = \alpha_2, \) and \( \tau\beta_1 = \beta_2 \). Define

\[
T_1 = (T \cup iT \cup \alpha_1 \cup \alpha_2) - \text{int}(V)
\]

and

\[
T_2 = (T \cup iT \cup \beta_1 \cup \beta_2) - \text{int}(V).
\]

If \( T \) is 2-sided then \( T_1 \) is 2-sided. Also \( iT_1 = T_1 \). Since \( T \) is 2-sided it follows that a sufficiently thin collar \( T \times [0, \varepsilon] \) of \( T = T \times 0 \) can intersect only one of \( \text{int}(\alpha_1) \) and \( \text{int}(\alpha_2) \). Hence \( T_1 \) cannot separate and therefore \( T_1 \) is incompressible. We arrive at (I).

From now on assume \( T \) is 1-sided. \( T_1 \) and \( T_2 \) are tori. This follows since \( V \) is a solid torus and either both of the annuli \( T - \text{int}(V) \) and \( iT - \text{int}(V) \) are "twisted" relative to \( V \) (if \( T \) is a Klein bottle) or neither is (if \( T \) is a torus).

If either of \( T_1 \) or \( T_2 \) is incompressible we arrive at (I). Assume then that \( T_1 \) and \( T_2 \) are compressible. Then \( T_1 \) bounds a solid torus \( V_1 \). If \( S \subseteq V_1 \) then \( V_1 \) contains a 1-sided torus or Klein bottle, a contradiction. So \( M = V \cup V_1 \cup V_2 \) with \( \text{int}(V), \text{int}(V_1), \) and \( \text{int}(V_2) \) pairwise disjoint.

By the choice of \( \alpha_1 \) and \( \alpha_2 \), \( i \) interchanges the components of \( \partial \alpha_i \). Therefore \( i|\alpha_i \) is conjugate to one of \( \text{id} \times \tau, \kappa \times \tau, \) or \( \alpha \times \tau \), the standard involutions of \( S^1 \times I \), where \( I = [-1, 1] \) and \( \tau(t) = -t \). Let \( S_i \) be a 1-sphere of \( \alpha_i \) that is the image of \( S^1 \times 0 \) under some conjugation. Note that \( S_i \) does not bound a disc \( D \) in \( V_1 \), otherwise \( T \) would be compressible.

If there is an annulus \( A \subseteq V_1 \) with \( \partial A = S_1 \cup S_2 \) and \( tA = A \) then we arrive at (IV) as follows. Torus \( V_1 \) is separated by \( A \). Since \( i \) interchanges the components of \( \partial \alpha_1 \), \( i \) interchanges the components of \( A - V_1 \). \( A \) is trivial in \( V_1 \) so it follows that \( V_1 \) can be given a trivial \( I \)-bundle structure over \( A \). There is an annulus \( B \subseteq V \) with \( \partial B = S_1 \cup S_2 \) and \( tB = B \). \( V \) is an \( I \)-bundle over \( B \). Consider \( T_3 = A \cup B \). It follows that \( V \cup V_1 \) is an \( I \)-bundle over \( T_3 \) with \( \partial(V \cup V_1) = T_2 \) a torus. Moreover \( T_3 \) does not separate \( T \) so \( T_3 \) is 1-sided. Since \( T_2 \) is compressible, \( T_3 \) is not a Klein bottle in view of Lemma 4.4. Thus \( T_3 \) is a torus and we arrive at (IV).

Assume now that such an annulus \( A \) does not exist. Since \( V \) is a standard neighborhood, \( \text{Fix} \cap \alpha_1 = \emptyset \) if and only if \( \text{Fix} \cap \alpha_2 = \emptyset \). Therefore \( i|\alpha_1 \) and \( i|\alpha_2 \) are conjugate.
Case (4.1): $i|\alpha_1$ is conjugate to $id \times \tau$. Then $i|V_1$ has a 2-dimensional fixed set that meets $\partial V_1$ in two fixed 1-spheres. It follows that $S_1$ bounds a disc or $S_1 \cup S_2$ bounds an annulus $A$ fixed by $i|V_1$. By the above comments, we arrive at (IV).

Case (4.2): $i|\alpha_1$ is conjugate to $\alpha \times \tau$. Then $i|V_1$ is orientation reversing. $i|\partial V_1$ is conjugate to $\alpha \times \kappa$ on $S^1 \times S^1$ by a conjugation taking $S_1$ to $S^1 \times (-1)^t$. Then $i|V_1$ is conjugate to $\hat{\alpha} \times \kappa$ or $\alpha \times \hat{\kappa}$ by a conjugation extending the one given on the boundaries. In the first case $S_1$ bounds a disc and in the second case $S_1 \cup S_2$ bounds an annulus with $iA = A$. Again by the above comments we arrive at (IV).

Case (4.3): $i|\alpha_1$ is conjugate to $\kappa \times \tau$. Then $i|V_1$ and $i|V$ are orientation preserving. Now $i|V_2$ is orientation reversing if and only if $T$ is a torus. To see this let $S_1 = \alpha_1 \cap \beta_1$ and without loss say $S_1 \subseteq T$. Orient $S_1$. $S_1$ and $iS_1$ bound two annuli $A_1$ and $A_2$ of $\partial V_2$ with $iA_1 = A_2$. Consider the ways of inducing an orientation on $iS_1$. The orientation induced by $A_1$ and the orientation induced by $\alpha_1$ are the same if and only if $T$ is a torus. Since $i|\alpha_1$ is orientation reversing the orientation induced by $\alpha_1$ and the orientation induced by $i$ are opposite. So $i$ and $A_1$ induce opposite orientations on $iS_1$ if and only if $T$ is a torus. Since $iA_1 = A_2$ the claim follows.

If $T$ is a torus then $i|V_2$ is orientation reversing, so $i|\partial V_2$ is conjugate to the involution $\alpha \times \kappa$ on $S^1 \times S^1$ by a conjugation taking $S_1$ to $i \times S^1$. As in Case (4.2) we arrive at (I) or (IV).

If $T$ is a Klein bottle then we arrive at (III). $i|V_2$ is fixed point free so $i|V_2$ is conjugate to $j_S$ or $j_O$ while $i|V_1$ is conjugate to $j_{2C}$. □

Corollary 4.6. Let $M$ be an irreducible, $P^2$-irreducible 3-manifold with involution $i$. Suppose $M$ contains an incompressible torus. Suppose $M$ is neither an orientable Seifert fiber space over the 2-sphere with four exceptional fibers nor a nonorientable Seifert fiber space over the projective plane with at most one exceptional fiber. Then there is a 2-sided incompressible torus or Klein bottle $T$ in $\text{int}(M)$ transversal to $\text{Fix}$ with $T \cap iT = \emptyset$ or $iT = T$.

Proof. Case (II) in the Equivariant Torus Theorem can only occur if $M$ is an orientable Seifert fiber space over the 2-sphere with four exceptional fibers, while Cases (III) and (IV) can only occur if $M$ is a nonorientable Seifert fiber space over the projective plane with at most one exceptional fiber. Since these possibilities have been excluded, only Case (I) of the Equivariant Torus Theorem remains. □

References


7. K. Kwun and J. Tollefson, PL involutions of $S^1 \times S^1 \times S^1$, Trans. Amer. Math. Soc. 203 (1975), 97–106.


