ON SUBORDINATED HOLOMORPHIC SEMIGROUPS

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ABSTRACT. If \([e^{-tA}]\) is a uniformly bounded \(C_0\) semigroup on a complex Banach space \(X\), then \(-A^\alpha\), \(0 < \alpha < 1\), generates a holomorphic semigroup on \(X\), and \([e^{-tA^\alpha}]\) is subordinated to \([e^{-tA}]\) through the Lévy stable density function. This was proved by Yosida in 1960, by suitably deforming the contour in an inverse Laplace transform representation. Using other methods, we exhibit a large class of probability measures such that the subordinated semigroups are always holomorphic, and obtain a necessary condition on the measure's Laplace transform for that to be the case. We then construct probability measures that do not have this property.

1. Introduction

Let \(X\) be a complex Banach space, and let \(C_0(X)\) be the class of uniformly bounded \(C_0\) semigroups \([T(t)]\), \(t \geq 0\), on \(X\). For fixed \(\alpha\), \(0 < \alpha < 1\), let \([p^\alpha_u(t)]\) be the family of functions implicitly defined as follows in Laplace transform space:

\[
\mathcal{L}\{p^\alpha_u(t)\} = \int_0^\infty p^\alpha_u(t)e^{-uz}du = e^{-tz^\alpha}, \quad \text{Re} z > 0.
\]

The principal branch of \(z^\alpha\) is understood in (1). For each fixed \(t > 0\), \(p^\alpha_u(t)\) is a Lévy 'stable' probability density function on \(u \geq 0\). Given \([T(u)] \in C_0(X)\), one may use (1) to construct a new semigroup \([U(t)] \in C_0(X)\), by means of

\[
U(0) = I, \quad U(t)x = \int_0^\infty p^\alpha_u(t)T(u)xdu, \quad t > 0, \ x \in X.
\]

We express this symbolically by \(U(t) = (p^\alpha(t), T)\), where, for fixed \(t\), \(p^\alpha(t)\) is the probability distribution with density \(p^\alpha_u(t)\). We write \(T(t) = e^{-tA}\), where \(-A\) is the infinitesimal generator of \([T(t)]\). Whenever multivalued functions \(\psi(z)\) appear, the particular branch where \(\text{Re} \psi(z) > 0\) for \(\text{Re} z > 0\), is understood.

The above is an example of a subordinated semigroup: \([U(t)]\) is said to be subordinated to \([T(t)]\) through the directing process \([p^\alpha(t)]\). See e.g. Feller.
The concept originated with Bochner, [3, 4], who used (1) and (2) to construct $A^\alpha$, $0 < \alpha < 1$. Subsequently, Phillips, [10], Nelson, [7], and Balakrishnan, [1], considered arbitrary \textit{infinitely divisible} probability distributions on $u \geq 0$, and developed a functional calculus for semigroup generators. Alternative methods of constructing fractional powers of operators, independent of subordination, were later devised by several authors, spawning a large literature; see Pazy, [9, p. 257]. Returning to (2), Yosida, [13–15], drew attention to the fact that in that case the semigroup $[U(t)]$ is \textit{holomorphic}, and that (1) and (2) together provide a method of constructing a large subclass of holomorphic semigroups within the class $C_0$. However, no other examples of families $[p(t)]$ leading to subordinated holomorphic semigroups seem generally known.

In this paper, we exhibit a rich variety of semigroups $[p(t)]$ of probability measures, such that $[U(t)] = [(p(t), T)]$ is holomorphic whenever $[T(t)] \in C_0(X)$, and we obtain a necessary condition on $\mathcal{L}\{p(t)\}$ in order that this be the case. We also construct families $[p(t)]$ that do not have this property.

2. Semigroups of probability measures

This section summarizes known results; see Phillips, [10], Hille and Phillips, [6, pp. 660–663], and Feller, [5]. Let $B(X)$ be the Banach algebra of bounded linear operators on $X$. Let $S$ be the Banach algebra of complex Borel measures $\mu$ on $X^+ \equiv \{u \geq 0\}$, with convolution as multiplication, and normed by the total variation. If $V$ is a Borel set in $X^+$, $\mu(V)$ denotes the value of $\mu$ on $V$, while $\int_V g(u)\mu(du)$ is the integral with respect to $\mu$ of the Borel measurable function $g$. Let $L$ be the Banach space of Borel measurable functions $f$ on $X^+$ such that

\begin{equation}
\|f\|_L = \int_{X^+} |f(u)| du < \infty.
\end{equation}

For each $\mu \in S$, define $Z_\mu \in B(L)$ by

\begin{equation}
Z_\mu f = (\mu * f)(\tau) \equiv \int_{X^+} f(\tau - u)\mu(du), \quad f \in L, \tau \geq 0.
\end{equation}

Then, the map $\mu \mapsto Z_\mu$ is an isometric isomorphism of $S$ into $B(L)$. Let $[T(t)] \in C_0(X)$. For each $\mu \in S$, define

\begin{equation}
\langle \mu, T \rangle = \int_{X^+} T(u)\mu(du).
\end{equation}

Then, $\mu \mapsto \langle \mu, T \rangle$, is a continuous homomorphism of $S$ into $B(X)$. In particular, $\langle \mu * \nu, T \rangle = \langle \mu, T \rangle \langle \nu, T \rangle$.

For each $x \geq 0$, let $\delta_x$ denote the Dirac measure at $x$, i.e., $\delta_x(V) = 1$ if $x \in V$, $\delta_x(V) = 0$ if $x \notin V$. Let $P$ be the set of all algebraic semigroups $[p(t)]$, $t \geq 0$, of probability measures on $X^+$. Thus, for fixed $t$, $p(t) \in S$, $p(t) \geq 0$, $\|p(t)\|_S = 1$, $p(t) * p(s) = p(t + s)$, $s, t \geq 0$, and $p(0) = \delta_0$. If $[p(t)] \in P$, then $[Z_p(t)] \equiv [Z_{p(t)}]$ forms an algebraic contraction semigroup.
on \( L \), and for \([T(u)] \in C_0(X), [U(t)] = [(p(t), T)]\) is a uniformly bounded algebraic semigroup on \( X \). \([U(t)]\) is subordinated to \([T(t)]\).

**Definition 1.** \( \mathcal{F} \) is the set of all \( [p(t)] \in P \) such that, given an arbitrary complex Banach space \( X \), \([p(t), T]\) \( \in C_0(X) \) whenever \([T(t)] \in C_0(X)\).

**Theorem 1.** Let \([p(t)] \in P \). The following statements are equivalent:

1. \( [p(t)] \in \mathcal{F} \).
2. \( [Zp(t)] \in C_0(L) \).
3. For every \( x > 0 \), \( p(t)(V_x) \to 1 \) as \( t \downarrow 0 \), where \( V_x = \{0 < u < x\} \).

For fixed \( t \geq 0 \), define the Laplace transform of \( p(t) \in S \) by

\[
\mathcal{L}\{p(t)\} = \int_{\mathbb{R}^+} e^{-uz} p(t)(du), \quad \text{Re} \ z > 0.
\]

**Theorem 2.** The following statements are equivalent:

1. \( [p(t)] \in \mathcal{F} \).
2. \( \mathcal{L}\{p(t)\} = e^{-t\psi(z)} \), \( t \geq 0 \), where \( \psi(z) \) is holomorphic for \( \text{Re} \ z > 0 \) and continuous for \( \text{Re} \ z \geq 0 \), with \( \text{Re} \ \psi(z) \geq 0 \). Moreover, \( \psi(0) = 0 \), and \( \psi'(x) \) is completely monotone for \( x > 0 \).

When \([p(t)] \in \mathcal{F}\), the function \( \psi(z) \) is called the exponent of \([p(t)]\). An equivalent characterization of \( \psi(z) \) is the following: There exists a positive measure \( \rho \) on \( \mathbb{R}^+ \), finite or infinite, such that \( \int_{u > 1} u^{-1} \rho(du) < \infty \), and

\[
\psi(z) = \int_{\mathbb{R}^+} (1 - e^{-uz}) u^{-1} \rho(du), \quad \text{Re} \ z > 0.
\]

A few objects \( \in \mathcal{F} \) are known explicitly as functions of \( u \) for all \( t \geq 0 \). In the following examples, \( p_u(t) \) denotes the density of the probability distribution \( p(t) \) on \( \mathbb{R}^+ \).

**Degenerate.**

\[
p(t) = \delta_1, \quad \mathcal{L}\{p(t)\} = e^{-t}z, \quad t > 0.
\]

**Inverse Gaussian.** This is the special case \( \alpha = 1/2 \) in (1).

\[
p_u(t) = \frac{te^{-t^2/4u}}{\sqrt{4\pi u^3}}, \quad \mathcal{L}\{p(t)\} = e^{-t\sqrt{z}}, \quad t > 0.
\]

**Gamma.** With fixed \( b > 0 \),

\[
p_u(t) = \frac{b^u t^{-1} e^{-bu}}{\Gamma(t)}, \quad \mathcal{L}\{p(t)\} = b^t(z + b)^{-t}, \quad t > 0.
\]

**Negative binomial.** This is a discrete family consisting of a weighted sum of Dirac measures. With fixed \( 0 < b < 1 \) and \( a = 1 - b \),

\[
p(t) = b^t \sum_{j=0}^{\infty} \binom{t}{j} (-a)^j \delta_j, \quad \mathcal{L}\{p(t)\} = b^t(1 - ae^{-z})^{-t}, \quad t > 0.
\]
Poisson. This is also a discrete family. With fixed \( c > 0 \)

\[
(12) \quad p(t) = e^{-ct} \sum_{j=0}^{\infty} \frac{(ct)^j}{j!} \delta_j, \quad \mathcal{L}\{p(t)\} = e^{ct(e^{-z}-1)}, \quad t > 0.
\]

Compound Poisson. Let \( q \) be an arbitrary probability measure on \( \mathbb{R}^+ \), and let \( Q(z) = \mathcal{L}\{q\} \). With \( \{q\} = \delta_0 \) and fixed \( c > 0 \),

\[
(13) \quad p(t) = e^{-ct} \sum_{j=0}^{\infty} \frac{(ct)^j}{j!} \{q\}^j, \quad \mathcal{L}\{p(t)\} = e^{ctQ(z)-1}, \quad t > 0.
\]

This construction includes many explicitly known semigroups \( e^S \) as special cases. Thus, (12) corresponds to the choice \( q = \delta_1 \). Similarly, (11) is a special case of (13) with \( c = -\log b > 0 \), and

\[
(14) \quad cQ(z) = -\log(1 - ae^{-z}) = \sum_{j=1}^{\infty} \frac{a^j e^{-jz}}{j}, \quad \text{Re} z \geq 0,
\]

so that \( c q = \sum_{j=1}^{\infty} a^j \delta_j / j \). As another example, let \( q \) have the density \( q_u = be^{-bu} \), \( b > 0 \). Then

\[
(15) \quad \{q_u\}^n = \frac{b^n u^{n-1} e^{-bu}}{\Gamma(n)}, \quad n \geq 1,
\]

and \( p(t) \) can be expressed in terms of the modified Bessel function \( I_1 \). With \( c = 1 \), \( p(t) = e^{-t} \delta_0 + r(t) \), where \( r(t) \) has the density

\[
(16) \quad r_u(t) = e^{-t} (bt/u)^{1/2} e^{-bu} I_1(2\sqrt{btu}), \quad t > 0,
\]

and

\[
(17) \quad \mathcal{L}\{p(t)\} = e^{-t} e^{bt(z+b)^{-1}}, \quad t > 0.
\]

3. Holomorphic semigroups

We consider bounded holomorphic semigroups \([S(t)]\) on \( X \), for which \( t \) can assume complex values in a sector

\[
(18) \quad \Sigma_\omega = \{t \in \mathbb{C}: \text{Re} t > 0, |\text{Arg}(t)| < \omega\}, \quad 0 < \omega \leq \pi/2,
\]

with \( \omega \) fixed. The family \([S(t)]\) is assumed to satisfy the following:

(a) \( S(t) \) is a holomorphic function of \( t \in \Sigma_\omega \).

(b) \( S(t_1 + t_2) = S(t_1) S(t_2) \), \( t_1, t_2 \in \Sigma_\omega \).

(c) If \( 0 < \epsilon < \omega \), then \( \|S(t)\|_X \leq M_\epsilon < \infty \), for \( t \in \Sigma_{\omega-\epsilon} \).

(d) \( S(0) = I \), and, within any sector \( \Sigma_{\omega-\epsilon} \) with \( 0 < \epsilon < \omega \), \( S(t) \) is strongly continuous at \( t = 0 \).

The following result, due to Yosida, [12], tells us when a given semigroup \([U(t)] \in C_0(X)\), defined on \( t \geq 0 \), can be extended to a bounded holomorphic semigroup \([S(t)]\) in some sector \( \Sigma_\omega \). Note that (20) below together with \( \|U(t)\|_X \leq M < \infty \), imply

\[
(19) \quad \sup_{t > 0} \{t\|e^{-\beta t} U(t)\|_X\} \leq C_\beta < \infty,
\]

for any \( \beta > 0 \).
Theorem 3. Let \([U(t)], \ t \geq 0, \in C_0(X)\) with infinitesimal generator \(-A\). Let \(U(t)X \subset D(A)\) for all \(t > 0\), and let
\[
(20) \limsup_{t \to 0} \{t\|AU(t)\|_X\} < \infty.
\]
Then, for any \(\beta > 0\), \([e^{-\beta t}U(t)]\) can be extended to a bounded holomorphic semigroup \([S(0)]\) in some sector \(\Sigma_\omega\).

4. Subordination and the class \(\mathcal{H}\)

Definition 2. For any complex Banach space \(X\), \(H(X) \subset C_0(X)\) is the class of semigroups on \(X\) satisfying the hypotheses of Theorem 3; \(G(X) \subset H(X)\) is the class of semigroups with bounded generators; \(\mathcal{H}\) [resp. \(\mathcal{G}\)] is the set of all \([p(t)] \in \mathcal{F}\) such that for every \(X\), \([\langle p(t), T \rangle] \in H(X)\) [resp. \(G(X)\)] whenever \([T(t)] \in C_0(X)\).

We have \(\mathcal{G} \subset \mathcal{H} \subset \mathcal{F}\). The degenerate family (8) is evidently \(\notin \mathcal{H}\), while the inverse Gaussian (9), and all other one-sided Lévy families (1), belong to \(\mathcal{H}\) as shown by Yosida [13].

Theorem 4. Let \([p(t)] \in \mathcal{F}\). The following conditions are equivalent:
(a) \([p(t)] \in \mathcal{H}\).
(b) \([Z_p(t)] \in H(L)\).
(c) \(p(t)\) is continuously differentiable \(\in S\) for \(t > 0\), with \(\|p'(t)\|_S = O(t^{-1})\) as \(t \to 0\).

Moreover, \([p(t)] \in \mathcal{H}\) only if \(\psi(z)\) maps \(\text{Re } z > 0\) into a truncated sector of opening \(< \pi\), and there exist constants \(K > 0\), and \(\gamma\), \(0 < \gamma < 1\), such that
\[
|\psi(z)| \leq K|z|^\gamma, \quad |z| \geq 1, \quad \text{Re } z > 0.
\]

Proof. (a) \(\Rightarrow\) (b). Let \([p(t)] \in \mathcal{H}\). Using (4) and (5), \(Z_p(t) = \langle p(t), T \rangle\), where \([T(u)]\) is the semigroup of right translations on \(L\). Hence, \([Z_p(t)] \in H(L)\).
(b) \(\Rightarrow\) (c). Since \([Z_p(t)]\) satisfies the hypotheses of Theorem 3 on \(L\), the \(B(L)\) limit as \(h \to 0\) of \(h^{-1}\{Z_p(t + h) - Z_p(t)\}\) exists for each fixed \(t > 0\). By the isometric isomorphism \(p(t) \mapsto Z_p(t), \ h^{-1}\{p(t + h) - p(t)\}\) has a corresponding \(S\) limit \(p'(t)\), and \(\|p'(t)\|_S = \|Z_p'(t)\|_L, \ t > 0\). Therefore, from (20)
\[
(22) \limsup_{t \to 0} \{t\|p'(t)\|_S\} < \infty.
\]
If \(p'(t) \in S\) for each \(t > 0\), the same is true of \(p'(t/2)*p'(t/2)\). By considering \(\mathcal{L}\{p(t)\} = e^{-t\psi(z)}\), it follows that \(p'(t/2)*p'(t/2) = p''(t)\), so that \(\|p''(t)\|_S \leq \|p'(t/2)\|_S^2\). In particular, \(\|p'(t)\|_S\) is a bounded continuous function of \(t\) on any interval \(0 < t_0 < t \leq t_1 < \infty\).

(c) \(\Rightarrow\) (a) Given any \([T(u)] \in C_0(X)\), differentiation with respect to \(t\) under the integral sign is justified in \(\langle p(t), T \rangle\). Hence, \(U'(t) = \langle p'(t), T \rangle, \ t > 0\), and
\[
(23) \|U'(t)\|_X \leq \text{const. } \|p'(t)\|_S, \quad t > 0.
\]
From (23) and (22), it follows that \( \{ t \|U'(t)\|_{\mathcal{X}} \} \) remains bounded as \( t \downarrow 0 \), so that \( \{ p(t) \} \in \mathcal{H} \). This proves the first part of Theorem 4.

The second part is proved in two steps. First, a function-theoretic argument is used to obtain (21) for \( z \) on the positive real axis. Next, the representation (7) is used to extend the estimate to the right half-plane. Fix any \( \beta > 0 \). Since \( [Z_p(t)] \in H(L) \), \( e^{-\beta t} [Z_p(t)] \) can be continued analytically in \( t \), in a sector \( \Sigma_t \equiv \{ \Re t > 0, |\Arg(t)| \leq \omega/2 < \pi/2 \} \), with \( e^{-\beta t} \|Z_p(t)\|_L \) bounded in \( \Sigma_t \). In fact, with \( C_\beta \) the constant in (19)

\[
\| (e^{-\beta t} Z_p(t))^{(n)} \|_L \leq \| (e^{-\beta (t/n)} Z_p(t/n))^{(n)} \|_L \leq (nt^{-1} C_\beta)^n, \quad t > 0, n \geq 1.
\]

Fix \( \omega \) with \( 0 < \omega < 2 \tan^{-1}\{1/(eC_\beta)\} \). Then, for \( \Re t > 0 \), \( |\Arg(t)| \leq (\omega/2) \), the Taylor series

\[
e^{-\beta t} Z_p(t) = e^{-\beta \Re t} Z_p(\Re t) + \sum_{n=1}^{\infty} (n!)^{-1} (t - \Re t)^n (e^{-\beta \Re t} Z_p(\Re t))^{(n)},
\]

converges uniformly in \( B(L) \). Using the isometric isomorphism of \( S \) into \( B(L) \), it follows that \( p(t) \) is holomorphic \( \in S \) for \( t \in \Sigma_t \), and

\[
\| e^{-\beta t} p(t) \|_S \leq (1 - eC_\beta (\Re t)^{-1} |t - \Re t|)^{-1} \leq K_\beta < \infty, \quad t \in \Sigma_t.
\]

Taking the Laplace transform of \( p(t) \), we get

\[
| e^{-\beta(z + \psi(z))} | \leq \| e^{-\beta t} p(t) \|_S \leq K_\beta, \quad \Re z \geq 0, t \in \Sigma_t.
\]

Let \( \xi(z) = \beta + \psi(z) \). It follows from (27) that \( \xi \) maps the half-plane \( \Pi \equiv \Re z > 0 \), into the sector \( \{ \Sigma_z \equiv |\Arg(z)| \leq (\pi - \omega)/2 \} \). From Theorem 2, \( \psi(z) \) is holomorphic in \( \Pi \) with \( \psi(1) \geq 0 \). Hence

\[
f(z) \equiv z^{\omega/\pi} \xi(z)/\xi(1),
\]

maps \( \Pi \) conformally into itself with \( f(1) = 1 \). Put

\[
z = \frac{1+w}{1-w}, \quad h(w) = f\left(\frac{1+w}{1-w}\right), \quad g(w) = h(w) - 1 \frac{h(w)}{h(w)+1}.
\]

Then, \( h(w) \) maps the unit disc into \( \Pi \) with \( h(0) = 1 \), and \( g(w) \) maps the unit disc into itself with \( g(0) = 0 \). From the Schwarz Lemma applied to \( g(w) \), we get

\[
|f(z)| = |h(w)| \leq \frac{1+|w|}{1-|w|} = \frac{|z+1|+|z-1|}{|z+1|-|z-1|}.
\]

Hence, for real \( x \geq 1 \), \( 0 < f(x) \leq x \). Therefore, from (28)

\[
0 \leq \psi(x) \leq Ax^\gamma, \quad x \geq 1; \quad A = \psi(1) + \beta, \quad \gamma = (\pi - \omega)/\pi.
\]

We now use the representation (7) to obtain a similar estimate valid in the half-plane \( \Re z \geq 0 \). The following elementary estimates will be needed:

\[
1 - e^{-z} \leq \Min\{|z|, 2\}, \quad \Re z \geq 0;
\]

\[
|1 - e^{-z}| \leq \Min\{|z|, 2\}, \quad \Re z \geq 0;
\]

\[
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(33) \[ 1 - e^{-x} \geq \sigma, \quad x \geq 1, \quad 1 - e^{-x} \geq \sigma x, \quad 0 \leq x \leq 1, \]

where \( \sigma = 1 - e^{-1} \). From (7) and (31), we have for \( x \geq 1 \),

\[ A x^\gamma \geq \left( \int_0^{1/x} + \int_{1/x}^{\infty} \right) (1 - e^{-ux}) u^{-1} \rho(du) \]

(34)

\[ \geq \sigma x \int_0^{1/x} \rho(du) + \sigma \int_{1/x}^{\infty} u^{-1} \rho(du). \]

Therefore, with \( \varepsilon = 1/x \leq 1 \),

\[ \int_0^\varepsilon \rho(du) \leq \sigma^{-1} A e^{1-\gamma}, \quad \int_\varepsilon^{\infty} u^{-1} \rho(du) \leq \sigma^{-1} A e^{-\gamma}. \]

If \( \text{Re} z \geq 0 \), we obtain using (32)

\[ |\psi(z)| \leq |z| \int_0^\varepsilon \rho(du) + 2 \int_\varepsilon^{\infty} u^{-1} \rho(du) \]

\[ \leq \sigma^{-1} A (|z| e^{1-\gamma} + 2 e^{-\gamma}), \]

if \( \varepsilon \leq 1 \), on using (35). Setting \( \varepsilon = 1/|z| \), \( |z| \geq 1 \), we get

\[ |\psi(z)| \leq 3\sigma^{-1} A |z|^\gamma, \quad |z| \geq 1, \quad \text{Re} z \geq 0. \]

This concludes the proof of Theorem 4.

Remark. The restriction \( |z| \geq 1 \) in (21) is natural: if \( \psi(z) = z^{1/3} \) for example, the estimate \( |\psi(z)| \leq K |z|^{1/2} \) is not valid as \( z \to 0 \).

Theorem 5. Let \([p(t)] \in \mathcal{F} \). The following statements are equivalent:

(a) \([p(t)] \in \mathcal{F} \).
(b) \([Z_p(t)] \in G(L) \).
(c) \( p(t) \) is continuously differentiable \( \in S \) for \( t > 0 \), with \( \|p'(t)\|_S = O(1) \) as \( t \downarrow 0 \).
(d) \( \psi(z) \) is bounded on \( \text{Re} z \geq 0 \).
(e) \( \psi(x) \) is bounded on \( x \geq 0 \).
(f) \([p(t)] \) is a Compound Poisson family.

Proof. (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c). The argument is the same as that in the first part of Theorem 4, using \( \|Z_p'(t)\|_L = O(1) \) as \( t \downarrow 0 \).

(c) \( \Rightarrow \) (d) \( \Rightarrow \) (e). For sufficiently small \( t > 0 \), we have, on differentiating under the integral sign in \( e^{-t\psi(z)} = \mathcal{L}\{p(t)\} \),

\[ |\psi(z)| e^{-t\psi(z)} \leq \|p'(t)\|_S < K < \infty, \quad \text{Re} z \geq 0. \]

Hence, \( |\psi(z)| < K < \infty, \text{Re} z \geq 0 \).

(e) \( \Rightarrow \) (f). Let \( 0 \leq \psi(x) < K \) on \( x \geq 0 \). Since \([p(t)] \in \mathcal{F} \), we know from Theorem 2 that \( \psi(0) = 0 \) and \( \psi'(x) \) is completely monotone for \( x > 0 \).

Define

(39) \[ Q(z) = 1 - \psi(z)/K, \quad \text{Re} z \geq 0. \]
Then, \( Q(0) = 1 \), and \( Q(x) \) is completely monotone for \( x > 0 \). It follows from Bernstein’s theorem, (Feller, [5, p. 439]), that \( Q(z) \) is the Laplace transform of some probability measure \( q \) on \( \mathbb{R}^+ \). Since \( \psi(z) = K(1 - Q(z)) \), \( K > 0 \), \( p(t) \) has the form (13).

(f) \( \Rightarrow \) (a). Let \( p(t) \) have the form (13) and let \( U(t) = \langle p(t), T \rangle \) for given \( [T(t)] \in C_0(X) \). Using the continuous homomorphism \( q \mapsto \langle q, T \rangle \) of \( S \) into \( B(X) \), we get

\[
U(t) = e^{-ct} \sum_{n=0}^{\infty} \frac{(ct)^n}{n!} \langle q, T \rangle^n, \quad t > 0.
\]

Thus, \([U(t)]\) has the bounded operator \( c\{\langle q, T \rangle - I\} \) as its infinitesimal generator, and \([p(t)] \in \mathcal{G} \). This concludes the proof of Theorem 5.

Theorem 6. If \([p(t)], [q(t)] \in \mathcal{F} [\text{resp. } \mathcal{H}, \mathcal{G}]\), then \([p(t) * q(t)] \in \mathcal{F} [\text{resp. } \mathcal{H}, \mathcal{G}]\).

Proof. That \( \mathcal{F} \) is closed under convolution is immediate from Theorem 2. Using \((p(t) * q(t))' = p'(t) * q(t) + p(t) * q'(t)\), together with statement (c) in Theorem 4 [resp. Theorem 5], it follows that \( \mathcal{H} [\text{resp. } \mathcal{G}] \) is closed under convolution.

5. Applications

Example 1. Gamma families \( \mathcal{H} \).

With fixed \( b > 0 \), let

\[
p_u(t) = \frac{b^t u^{t-1} e^{-bu}}{\Gamma(t)}, \quad t > 0.
\]

Then

\[
(\partial / \partial t)p_u(t) = \left\{ \log b + \log u - \frac{\Gamma'(t)}{\Gamma(t)} \right\} p_u(t), \quad t > 0.
\]

For \( 0 < u < 1 \), write

\[
(\log u)p_u(t) = \{2b^{t/2} \Gamma(1 + t/2)u^{t/2}(\log u)p_u(t/2)\}{\Gamma(1 + t)}^{-1},
\]

and for \( u \geq 1 \), write

\[
(\log u)p_u(t) = \{2^t (e^{-bu/2} \log u)(b/2)^t u^{t-1} e^{-bu/2}\}{\Gamma(t)}^{-1}.
\]

Let

\[
K_1 = \max_{0 \leq v \leq 1} \{v|\log v|\}, \quad K_2 = \sup_{u \geq 1} \{e^{-bu/2} \log u\}.
\]

From (42)-(45), we have

\[
\|p'(t)\|_S \leq 2^t K_2 + |\log b| + \frac{|\Gamma'(t)|}{\Gamma(t)} + \frac{4K_1 b^{t/2} \Gamma(1 + t/2)}{t\Gamma(1 + t)}, \quad t > 0.
\]
The only singularity in $\Gamma'(t)/\Gamma(t)$, $t \geq 0$, is a simple pole at $t = 0$; see Olver, [8, p. 39]. Also, $p''(t) = p'(t/2)^2 + p'(t/2)$. Thus, $p(t)$ is continuously differentiable in $S$ for $t > 0$, and $\{\|p'(t)\|_S\}$ remains bounded as $t \downarrow 0$. By Theorem 4, $[p(t)] \in H$.

**Example 2 (Corollary).** If $[T(t)] = [e^{-tA}] \in C_0(X)$, then $-\log(A + I)$, where

$$\{\log(A + I)\} x = \int_1^\infty s^{-1}(A + sI)^{-1} A x \, ds, \quad x \in D(A),$$

is the infinitesimal generator of $[S(t)] = [(A + I)^{-t}] \in H(X)$.

Let $[p(t)] \in S$ have the exponent $\psi(z)$, and let $\rho$ be the measure on $\mathbb{R}^+$ in (7). Let $U(t) = (p(t), T)$. A formula for the generator of $[U(t)]$ is known, which generalizes Theorem 2 and the representation (7); see Phillips, [10], Nelson, [7], and Feller, [5, p. 458]. We have

$$[U(t)] = [e^{-t\psi(A)}],$$

$$\psi(A) x = \int_{\mathbb{R}^+} u^{-1}(I - e^{-uA}) x \rho(du), \quad x \in D(A).$$

In fact, $\psi(A)$ is the closure of its restriction to $D(A)$. Choosing $[p(t)]$ to be the Gamma family (41) with $b = 1$, we have $\psi(z) = \log(1 + z)$, $\rho(du) = e^{-u} du$, and

$$\{\log(A + I)\} x = \int_{\mathbb{R}^+} (I - e^{-uA}) x \left\{ \int_1^\infty e^{-us} ds \right\} du$$

$$= \int_1^\infty s^{-1}(A + sI)^{-1} A x \, ds, \quad x \in D(A).$$

The result follows on viewing $[S(t)]$ as being subordinated to $[T(t)]$ through the Gamma family.

The Hausdorff-Young theorem on Fourier transforms may be combined with Theorem 4 to obtain another proof of the fact that the one-sided Lévy stable families $\in H$.

**Example 3.** Fix $\alpha$ with $0 < \alpha < 1$, and let $[p^\alpha(t)]$ have the exponent $\psi(z) = z^\alpha$. Then, $[p^\alpha(t)] \in H$.

From Theorem 2, $[p^\alpha(t)] \in S$ for $0 < \alpha < 1$. We verify statement (c) of Theorem 4 for $(d/dt)p^\alpha(t)$. Let $b = \cos(\alpha \pi/2) > 0$, and let $C$ be a generic positive constant. Let $q_{t,\alpha}(u) \equiv (\partial/\partial t)p^\alpha(t)$, $u \geq 0$. For real $y$, $Q_{t,\alpha}(y) \equiv -(iy)^{\alpha}e^{-t(y)^{\alpha}}$ and $(\partial/\partial y)Q_{t,\alpha}(y)$ are, respectively, the Fourier transforms of the densities $q_{t,\alpha}(u)$ and $g_{t,\alpha}(u) \equiv i u q_{t,\alpha}(u)$. Moreover

$$|Q_{t,\alpha}(y)| = |y|^\alpha e^{-bt|y|^\alpha},$$

$$|(\partial/\partial y)Q_{t,\alpha}(y)| \leq \alpha |y|^{\alpha-1}(1 + t|y|^\alpha)e^{-bt|y|^\alpha}. $$
Let $1 < r < \min\{2, (1 - \alpha)^{-1}\}$, let $s = r/(r - 1)$, and let $\| \cdot \|_r$ denote the $L^r(-\infty, \infty)$ norm. The change of variables $v = t^{1/\alpha}$ shows that

$$t\| Q_{t, \alpha} \|_r \leq C t^{-1/\alpha}, \quad t\| (\partial/\partial y) Q_{t, \alpha} \|_r \leq C t^{1/\alpha}. \quad (52)$$

Using the Hausdorff-Young inequality, (Rudin, [11, p. 247]), we obtain

$$t\| Q_{t, \alpha} \|_s \leq C t^{-1/\alpha}, \quad t\| Q_{t, \alpha} \|_s \leq C t^{1/\alpha}. \quad (53)$$

Next, for any $v > 0$, Hölder’s inequality gives

$$\int_0^v |q_{t, \alpha}(u)| \, du \leq \|q_{t, \alpha}\|_r \leq t^{1/\alpha}, \quad \int_0^v u t^{1/\alpha} \, du \leq g_{t, \alpha}(u) \leq C t^{1/\alpha}. \quad (54)$$

Therefore, on choosing $v = t^{1/\alpha}$, it follows from (53), (54), and (55), that

$$q_{t, \alpha}(u) \in L^1(\mathbb{R}^+) \text{ for each } t > 0, \quad \|q_{t, \alpha}\|_1 \equiv \|(d/dt)p^{\alpha}(t)\|_s = O(t^{-1}) \quad \text{as} \quad t \downarrow 0. \quad \text{Hence, } [p^{\alpha}(t)] \in \mathcal{H}. \quad (55)$$

Evidently, a large number of objects $e \in \mathcal{H}$ can be created by convolutions. Further objects $e \in \mathcal{H}$ may be generated by means of the following construction:

**Example 4. Subordination of convolution semigroups.**

Let $[p_1(t)] \in \mathcal{H}, \ [p_2(t)] \in \mathcal{F}$, and consider the convolution semigroup $[Z_{p_2}(t)] \in C_0(L)$. Let $U(t) = \langle p_1(t), Z_{p_2} \rangle$. Then, $[U(t)] \in H(L)$ is the convolution semigroup $[Z_{p_3}(t)]$, where

$$p_3(t) = \int_{\mathbb{R}^+} p_2(u) p_1(t)(du), \quad t > 0. \quad (56)$$

Using Laplace transforms, it is easily seen that if $\psi_j(z)$ is the exponent $[p_j(t)]$, $j = 1, 2$, then $[p_3(t)]$ has the exponent $\psi_3(z) = \psi_1(\psi_2(z))$. Since $\psi_3(x)$ vanishes at 0 and has a completely monotone derivative on $x > 0$, it follows from Theorem 2 that $[p_3(t)] \in \mathcal{F}$. Statement (b) of Theorem 4 shows that $[p_3(t)] \in \mathcal{H}$.

We now construct two distinct classes of objects $e \in \mathcal{F} \setminus \mathcal{H}$.

**Example 5.** If $[p(t)] \in \mathcal{F}$, then $[q(t)] = [p(t) * \delta_1] \in \mathcal{F} \setminus \mathcal{H}$.

If $[p(t)]$ has the exponent $\psi(z)$, $[q(t)]$ has the exponent $z + \psi(z)$. From (21) in Theorem 4, $[q(t)] \notin \mathcal{H}$.

**Example 6.** Let $\{\beta_n\}_{n=0}^\infty$ and $\{a_n\}_{n=0}^\infty$ be any two sequences satisfying

$$0 < \beta_n < 1; \quad 0 < a_n; \quad \lim_{n \to \infty} \beta_n = 1; \quad \sum_{n=0}^\infty a_n < \infty; \quad (57)$$

and define

$$\psi(z) = \sum_{n=0}^\infty a_n z^{\beta_n}, \quad \text{Re } z > 0. \quad (58)$$
Then, $\psi(z)$ is the exponent of some $[p(t)] \in \mathcal{F} \setminus \mathcal{H}$. The same is true for $\psi(\varphi(z))$, whenever $\varphi(z)$ is a function of the form (7) that does not satisfy (21).

The infinite series of holomorphic functions (58) converges uniformly on compact subsets of the half-plane $\text{Re} z > 0$, to a holomorphic $\psi(z)$ with $\psi(0) = 0$. In particular, termwise differentiation is permissible. It follows that $\psi(x)$ has a completely monotone derivative on $x > 0$. By Theorem 2, $[p(t)] \in \mathcal{F}$. Since each $a_n > 0$, $\psi(z)$ cannot satisfy (21) on $x > 0$, and $[\rho(t)] \notin \mathcal{H}$. Similarly, $\psi(\varphi(z))$ is the exponent of some $[r(t)] \in \mathcal{F}$, since it vanishes at zero, and has a completely monotone derivative for $x > 0$. That $[r(t)] \notin \mathcal{H}$ may be seen by examining the series:

\[ \psi(\varphi(x)) = \sum_{n=0}^{\infty} a_n \varphi(x)^{\beta_n}, \quad x > 0, \]

\[ > a_m \varphi(x)^{\beta_m}, \quad m > 0. \]

Fix any $\gamma$ with $0 < \gamma < 1$, fix $m > 0$ such that $\gamma < \beta_m < 1$, and let $\alpha_m = (\gamma/\beta_m) < 1$. Then, $x^{-\gamma} \psi(\varphi(x)) > a_m \{x^{-\alpha_m} \varphi(x)\}^{\beta_m}$, and cannot remain bounded as $x \to \infty$.

REFERENCES

2. _, Fractional powers of closed operators and the semigroups generated by them, Pacific J. Math. 10 (1960), 419–437.


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