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RADON-NIKODYM PROPERTIES ASSOCIATED WITH SUBSETS OF COUNTABLE DISCRETE ABELIAN GROUPS

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ABSTRACT. With any subset of a countable discrete abelian we associate with it three Banach space properties. These properties are Radon-Nikodym type properties. The relationship between these properties is investigated. The results are applied to give new characterizations of Riesz subsets and Rosenthal subsets of countable discrete abelian groups.

1. INTRODUCTION

The aim of this paper is to study three families of Banach space properties, each of which includes the Radon-Nikodym property, the analytic Radon-Nikodym property and the property of not containing a subspace which is isomorphic to c_0. The properties we consider are Radon-Nikodym type properties associated with subsets of countable discrete abelian groups. This line of study was initiated by Edgar [E]. Edgar used his results to give a new characterization of Riesz subsets of countable discrete abelian groups. In this paper we will introduce two more families of Radon-Nikodym type properties. We will consider the three types of properties and discuss their interplay. If Λ is a subset of a countable discrete abelian group, then the properties associated with Λ will be called type I-Λ-Radon-Nikodym property, type II-Λ-Radon-Nikodym property and type III-Λ-Radon-Nikodym property.

In §2, we will give the appropriate definitions and preliminaries.

§3 will be used to study type I-Λ-Radon-Nikodym property. This property will be used to give new characterizations of Riesz subsets and Rosenthal subsets of countable discrete abelian groups. These characterizations improve the results of Edgar [E] and Lust-Piquard [L].

In §4, we give some characterizations of type II-Λ-Radon-Nikodym property. It is unknown if the type I-Λ-Radon-Nikodym property and the type II-Λ-Radon-Nikodym property are equivalent, even when Λ is a Riesz subset. We will give a necessary and sufficient condition for these two properties to be equivalent when Λ is a Riesz subset of the circle $\mathbb{T}$.

Finally, in §5, we prove two results related to the type III-Λ-Radon-Nikodym property. The first result states that type III-Λ-Radon-Nikodym property and

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type II-Λ-Radon-Nikodym property are equivalent when Λ is a Riesz subset. The second result shows that type III-Λ-Radon-Nikodym property and the Radon-Nikodym property are equivalent if and only if Λ is not a Riesz subset.

2. Preliminaries and definitions

Throughout this note, G will denote a compact abelian metrizable group, $\mathscr{B}(G)$ is the σ-algebra of Borel subsets of G, and λ is normalized Haar measure on G. We will denote the dual group of G by Γ. Then Γ is a countable discrete abelian group. We note that all countable discrete abelian groups can be seen to arise in this manner [Ru].

Let X be a Banach space and let $1 \leq p \leq \infty$. For an X-valued measure, μ, on $\mathscr{B}(G)$ we define

$$\mathbb{E}(\mu|\pi) = \sum_{E \in \pi} \frac{\mu(E)}{\lambda(E)} \chi_E ,$$

where π is a finite measurable partition of G, along with the convention $\frac{0}{0} = 0$. The space $V^p(G; X)$ consists of all the X-valued measures, μ, on $\mathscr{B}(G)$ with $\|\mu\|_p < \infty$, where

$$\|\mu\|_p = \sup_{\pi} \|\mathbb{E}(\mu|\pi)\|_{L^p(G;X)}$$

and where the supremum is taken over all finite measurable partitions of G.

If $\mu \in V^p(G; X)$ and $\gamma \in \Gamma$ then the Fourier coefficient, $\hat{\mu}(\gamma)$, is defined by

$$\hat{\mu}(\gamma) = \int_G \overline{\gamma(x)}\, d\mu(x) .$$

If $f \in L^p(G; X)$ and $\gamma \in \Gamma$ then $\hat{f}(\gamma)$ is defined by

$$\hat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}\, d\lambda(x) .$$

If $\Lambda \subseteq \Gamma$ we define

$$L^p_\Lambda(G; X) = \{f \in L^p(G, X) : \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}$$

and

$$V^p_\Lambda(G; X) = \{\mu \in V^p(G; X) : \hat{\mu}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\} .$$

Definition 1. A Banach space X is said to have type I-Λ-Radon-Nikodym property (type I-Λ-RNP) if and only if $V^\infty_\Lambda(G; X) = L^\infty_\Lambda(G; X)$.

Remarks. (1) Type I-Λ-RNP was introduced by Edgar [E] under the name of the Λ-Radon-Nikodym property.

(2) If $G = \mathbb{T}$, the circle group, then $\Gamma = \mathbb{Z}$. Then we get that type I-$\mathbb{Z}$-RNP is equivalent to the usual Radon-Nikodym property and type I-$\mathbb{N}$-RNP is equivalent to the analytic Radon-Nikodym property (see [E]).

Before we get to the next type of Radon-Nikodym property we should note that if $\mu \in V^p_\Lambda(G; X)$, where $1 < p \leq \infty$, then μ is absolutely continuous with respect to λ. If $p = 1$, this may no longer be true. Thus we define

$$V^1_{\Lambda,\mathrm{ac}}(G; X) = \{\mu \in V^1_\Lambda(G; X) : \mu \text{ is absolutely continuous with respect to } \lambda\} .$$

Definition 2. A Banach space X is said to have type II-Λ-Radon-Nikodym property (type II-Λ-RNP) if $V^1_{\Lambda,\mathrm{ac}}(G;X) = L^1_\Lambda(G;X)$.

Remarks. (1) It is clear from the definition that type II-Λ-RNP implies type I-Λ-RNP.

(2) If $G = \mathbb{T}$ then $\Gamma = \mathbb{Z}$. Then type II-$\mathbb{Z}$-RNP is equivalent to the Radon-Nikodym property and type II-$\mathbb{N}$-RNP is equivalent to the analytic Radon-Nikodym property (see [E]).

(3) If $\Lambda \subseteq \Gamma$, then Λ is called a Riesz subset of Γ if every $\mu \in V^1_\Lambda(G)$ $(= V^1_\Lambda(G;\mathbb{C}))$ has a Radon-Nikodym derivative with respect to λ. That is, Λ is a Riesz subset of Γ if and only if $V^1_\Lambda(G) = L^1_\Lambda(G)$. It is easy to show that if Λ is a Riesz subset of Γ and X is a Banach space then $V^1_\Lambda(G;X) = V^1_{\Lambda,\mathrm{ac}}(G;X)$. Consequently, if Λ is a Riesz subset of Γ then a Banach space X has type II-Λ-RNP if and only if $V^1_\Lambda(G;X) = L^1_\Lambda(G;X)$.

The last of the Radon-Nikodym type properties we wish to consider is modeled on the main results of chapter VI of the Diestel and Uhl monograph [DU]. Before we get to the definition let us introduce some notation.

If $\Lambda \subseteq \Gamma$ then $\Lambda' = \{\gamma \in \Gamma : \bar{\gamma} \notin \Lambda\}$. We denote by $C(G)$, the space of continuous functions on G, with the supremum norm. The subspace $C_{\Lambda'}(G)$ of $C(G)$ consists of continuous functions on G whose Fourier coefficients vanish off of Λ'. Let $T\colon X \to Y$ be a bounded linear operator from the Banach space X to the Banach space Y. Then T is said to be absolutely summing if T maps weakly unconditionally Cauchy series in X into absolutely convergent series in Y. T is said to be nuclear if there exists sequences $\{x_n^*\}_{n=1}^\infty$ in X^* and $\{y_n\}_{n=1}^\infty$ in Y such that $\sum_{n=1}^\infty \|x_n^*\|\,\|y_n\| < \infty$ and such that $T(x) = \sum_{n=1}^\infty x_n^*(x)y_n$ for all $x \in X$.

Definition 3. A Banach space X is said to have type III-Λ-Radon-Nikodym property (type III-Λ-RNP) if every absolutely summing operator $T\colon C(G) \to X$, which satisfies $T|_{C_{\Lambda'}(G)} \equiv 0$, is nuclear.

Remarks. (1) Type III-Λ-RNP may seem to be slightly contrived, so a few words about it are in order. Let us recall the following result from [DU];

Theorem 1. *Let X be a Banach space. A bounded linear operator $T : C(G) \to X$ is absolutely summing if and only if its representing measure is of bounded variation. T is nuclear if and only if its representing measure is of bounded variation and has a Radon-Nikodym derivative with respect to its variation.*

Now let us apply Theorem 1, in the setting of type III-Λ-RNP. If $T\colon C(G) \to X$ is absolutely summing then its representing measure, $\mu\colon \mathscr{B}(G) \to X$, is of bounded variation. If T also satisfies $T|_{C_{\Lambda'}(G)} \equiv 0$ then $\hat{\mu}(\gamma) = 0$ for all $\gamma \notin \Lambda$. In particular, $\mu \in V^1_\Lambda(G;X)$. If such an operator, T, is also nuclear then μ has a Radon-Nikodym derivative with respect to $|\mu|$, the variation of μ. Thus a Banach space X has type III-Λ-RNP if and only if every $\mu \in V^1_\Lambda(G;X)$ has a Radon-Nikodym derivative with respect to $|\mu|$.

(2) If $\mu \in V^1_{\Lambda,\mathrm{ac}}(G; X)$ and μ has a Radon-Nikodym derivative with respect to $|\mu|$, then μ has a Radon-Nikodym derivative with respect to λ [B, p. 16]. Consequently type III-Λ-RNP implies type II-Λ-RNP.

3. Type I-Λ-Radon-Nikodym property

We wish to begin this section with a result of Edgar [E]. For this we need the notion of a "good approximate identity".

A sequence $\{i_n\}_{n=1}^{\infty}$ of measurable functions $i_n: G \to \mathbb{R}$ is called a good approximate identity on G if

(a) $i_n \geq 0$ for all $n \in \mathbb{N}$,
(b) $\int_G i_n(x)\, d\lambda(x) = 1$ for all $n \in \mathbb{N}$,
(c) $\sum_{\gamma\in\Gamma} \hat{i}_n(\gamma) < \infty$, and
(d) $\lim_{n\to\infty} \int_U i_n(x)\, d\lambda(x) = 1$ for all neighborhoods U of 1 in G.

We note that if G is a compact abelian metrizable group then there exists a good approximate identity on G [Ru, p. 23].

Theorem 2 (Edgar). *Let G be a compact abelian metrizable group, let $\Lambda \subseteq \Gamma$ and let $\{i_n\}_{n=1}^{\infty}$ be a good approximate identity on G. For a Banach space X the following conditions are equivalent;*

(i) *X has type* I-Λ-*RNP*;

(ii) *if $\{a_\gamma\}_{\gamma\in\Lambda} \subset X$ and $f_n = \sum_{\gamma\in\Lambda} \hat{i}_n(\gamma) a_\gamma \gamma$ is bounded in $L^\infty_\Lambda(G; X)$, then there exists a function $f \in L^\infty_\Lambda(G; X)$ with $\hat{f}(\gamma) = a_\gamma$ for all $\gamma \in \Lambda$;*

(iii) *if $\{a_\gamma\}_{\gamma\in\Lambda} \subset X$ and the sequence $\{f_n\}_{n=1}^{\infty}$, as in* (ii), *is bounded in $L^\infty_\Lambda(G; X)$, then $\{f_n\}_{n=1}^{\infty}$ converges in $L^1(G; X)$-norm;*

(iv) *if $T: L^1(G)/L^1_{\Lambda'}(G) \to X$ is a bounded linear operator, where $\Lambda' = \{\gamma \in \Gamma : \overline{\gamma} \notin \Lambda\}$, and $Q: L^1(G) \to L^1(G)/L^1_{\Lambda'}(G)$ is the natural quotient map, then TQ is a representable operator* (*see* [DU] *for unexplained terms*).

Edgar used Theorem 2 to get the following:

Proposition 3 (Edgar). *Let G be a compact abelian metrizable group and let $\Lambda \subseteq \Gamma$. Then Λ is a Riesz subset of Γ if and only if $L^1[0, 1]$ has type* I-Λ-*RNP.*

Remarks. (1) Since G is a metrizable group, $L^1(G)$ is separable. Consequently, the operators considered in Theorem 2(iv) have separable range and so type I-Λ-RNP is a separably determined property.

(2) In [BR] Bourgain and Rosenthal show that if X is a separable Banach space and Y is a Banach space with the Radon-Nikodym property and if X semi-embeds in Y then X has the Radon-Nikodym property. (A bounded linear operator $T: X \to Y$ is a semi-embedding if $T(B_X)$ is closed in Y.) Again using Theorem 2(iv) and the Bourgain-Rosenthal technique of proof one can easily show that if X is a separable Banach space which semi-embeds in a Banach space with type I-Λ-RNP then X has type I-Λ-RNP.

(3) In [GR] Ghoussoub and Rosenthal consider the question of what type of Banach spaces semi-embed in $L^1[0, 1]$. By Proposition 3, $L^1[0, 1]$ has type I-Λ-RNP if Λ is a Riesz subset of Γ. Combining these results with remarks (1) and (2) we get that if Λ is a Riesz subset of Γ then Banach lattices not containing a copy of c_0 and Banach spaces with both finite cotype and G-L l.u.st. have type I-Λ-RNP (see [GR]).

(4) If Λ is an infinite subset of Γ then c_0 fails to have type I-Λ-RNP (use Theorem 2(ii)). Consequently, by Theorem 2(iv), $L^1(G)/L^1_{\Lambda'}(G)$ fails to have type I-Λ-RNP whenever Λ is infinite. Combining this with Remark 3, we get that if Λ is an infinite Riesz subset of Γ then $L^1(G)/L^1_{\Lambda'}(G)$ either fails to have finite cotype or fails to have G-L l.u.st.

In [D2] it is shown that if Λ is a Sidon subset of Γ then every Banach space not containing a copy of c_0 has type I-Λ-RNP (Λ is a Sidon set if $C_\Lambda(G)$ is isomorphic to $l^1(\Lambda)$). Clearly, for infinite subsets Λ, this is the weakest property we can get. On the other hand, the Radon-Nikodym property is the strongest property possible, while the analytic Radon-Nikodym property can be viewed as an intermediate property. The obvious question now is: How many other distinct properties exist? While we do not know the answer to this question, our efforts to solve it have led to new characterizations of Riesz subsets and Rosenthal subsets. These characterizations are slight improvements of results of Lust-Piquard [L] and Proposition 3 [E].

Proposition 4. *Let Λ be a subset of Γ. Then Λ is a Riesz set if and only if $L^1_\Lambda(G)$ has type* I-Λ-*RNP.*

Proof. If Λ is a Riesz set then $L^1_\Lambda(G)$ has a Radon-Nikodym property [L] and so has type I-Λ-RNP.

Conversely, suppose $L^1_\Lambda(G)$ has type I-Λ-RNP and let $\mu \in V^1_\Lambda(G)$. Define $T: L^1(G) \to L^1_\Lambda(G)$ by $T(f) = \mu * f$ for all $f \in L^1(G)$. Clearly, T is bounded and linear and $T(\gamma) = \hat{\mu}(\gamma)\gamma$ for all $\gamma \in \Gamma$. In particular, $T(\gamma) = 0$ for all $\gamma \notin \Lambda$. In the notation of Theorem 2(iv), $T(\gamma) = 0$ for all $\gamma \in \overline{\Lambda}'$. Therefore $T|_{L^1_{\overline{\Lambda}'}} \equiv 0$ and so there exists a bounded linear operator $\widetilde{T}: L^1(G)/L^1_{\overline{\Lambda}'}(G) \to X$ such that $T = \widetilde{T}Q$ where $Q: L^1(G) \to L^1(G)/L^1_{\overline{\Lambda}'}(G)$ is the natural quotient map. Since $L^1_\Lambda(G)$ has type I-Λ-RNP it has type I-$\overline{\Lambda}$-RNP. Therefore, by Theorem 2(iv), T is a representable operator. An application of Costé's Theorem [DU, p. 91] says that μ has a Radon-Nikodym derivative with respect to λ. Therefore Λ is a Riesz set. □

Definition. Let G be a compact abelian metrizable group and let Λ be a subset of Γ. Then Λ is called a Rosenthal set if $L^\infty_\Lambda(G) = C_\Lambda(G)$.

Proposition 5. *Let Λ be a subset of Γ. Then Λ is a Rosenthal set if and only if $C_\Lambda(G)$ has type* I-Λ-*RNP.*

Proof. If Λ is a Rosenthal set then $C_\Lambda(G)$ has the Radon-Nikodym property [L] and so has type I-Λ-RNP.

Conversely, suppose $C_\Lambda(G)$ has type I-Λ-RNP and let $f \in L^\infty_\Lambda(G)$. Define $T: L^1(G) \to C_\Lambda(G)$ by $T(g) = f * g$ for all $g \in L^1(G)$. Clearly, T is a bounded linear operator and $T(\gamma) = \hat{f}(\gamma)\gamma$ for all $\gamma \in \Gamma$. In particular, $T(\gamma) = 0$ for all $\gamma \notin \Lambda$. As in Proposition 4, this implies that $T|_{L^1_{\overline{\Lambda}'}(G)} \equiv 0$ and so we can find a bounded linear operator $\widetilde{T}: L^1(G)/L^1_{\overline{\Lambda}'}(G) \to C_\Lambda(G)$ such that $T = \widetilde{T}Q$ where $Q: L^1(G) \to L^1(G)/L^1_{\overline{\Lambda}'}(G)$ is the natural quotient map. Since $C_\Lambda(G)$ has type I-Λ-RNP it has type I-$\overline{\Lambda}$-RNP and thus an application of Theorem 2(iv) shows that T is a representable operator. However, Lust-Piquard [L] proved that if such a T is representable then $f \in C_\Lambda(G)$. Therefore $L^\infty_\Lambda(G) = C_\Lambda(G)$ and so Λ is a Rosenthal set. □

4. Type II-Λ-Radon-Nikodym property

In this section we give some characterizations of type II-Λ-RNP similar to Theorem 2.

Theorem 6. *Let G be a compact abelian metrizable group, let $\{i_n\}_{n=1}^\infty$ be a good approximate identity on G, let Λ be a subset of Γ and let X be a Banach space.*

(a) *X has type* II-Λ-*RNP*;

(b) *if $\{a_\gamma\}_{\gamma\in\Lambda} \subset X$ and $f_n = \sum_{\gamma\in\Lambda} \hat{i}_n(\gamma)a_\gamma\gamma$ is bounded in $L^1_\Lambda(G; X)$, then there exists a function $f \in L^1_\Lambda(G; X)$ with $\hat{f}(\gamma) = a_\gamma$ for all $\gamma \in \Lambda$;*

(c) *if $\{a_\gamma\}_{\gamma\in\Lambda} \subset X$ and the sequence $\{f_n\}_{n=1}^\infty$, as in* (b), *is bounded in $L^1_\Lambda(G; X)$, then $\{f_n\}_{n=1}^\infty$ converges in $L^1(G; X)$-norm.*

Then (b) $\Leftrightarrow$ (c) $\Rightarrow$ (a) *and* (a) $\Rightarrow$ (b) *if and only if Λ is a Riesz set.*

Proof. (b) $\Leftrightarrow$ (c) can be proved the same way as (ii) $\Leftrightarrow$ (iii) in Theorem 2 (see [E]). (c) $\Rightarrow$ (a): Let $\mu \in V^1_{\Lambda,\mathrm{ac}}(G; X)$ and define $a_\gamma = \hat{\mu}(\gamma)$ for all $\gamma \in \Lambda$. Form $\{f_n\}_{n=1}^\infty$ by $f_n = \sum_{\gamma\in\Lambda} \hat{i}_n(\gamma)a_\gamma\gamma$.

Then

$$(i_n * \mu)(x) = \int_G i_n(xy^{-1})\,d\mu(y) = \sum_{\gamma\in\Lambda} \hat{i}_n(\gamma)a_\gamma\gamma(x) = f_n(x)$$

for all $x \in G$. Therefore

$$\|f_n\|_1 = \|i_n * \mu\|_1 \le \|i_n\|_1\|\mu\|_1 = \|\mu\|_1 .$$

Hence $\{f_n\}_{n=1}^\infty$ is bounded in $L^1_\Lambda(G; X)$ and so $\{f_n\}_{n=1}^\infty$ converges to an element f, in $L^1(G; X)$-norm, by (c). It is easy to see that $f \in L^1_\Lambda(G; X)$ and that f is the Radon-Nikodym derivative of μ with respect to λ, so (a) holds.

We now assume that Λ is a Riesz set and that (a) holds. Let $\{a_\gamma\}_{\gamma\in\Lambda} \subset X$ with the corresponding $\{f_n\}_{n=1}^\infty$ being bounded in $L^1_\Lambda(G; X)$. Define a sequence of measures $\mu_n: \mathscr{B}(G) \to X$ by $\mu_n(E) = \int_E f_n(x)d\lambda(x)$ for all $E \in \mathscr{B}(G)$. Then $\|\mu_n\|_1 = \|f_n\|_1$ for each $n \in \mathbb{N}$ so $\sup_n \|\mu_n\|_1 = \sup_n \|f_n\|_1 < \infty$. Therefore there exists a subnet $\{\mu_{n_\alpha}\}$ of $\{\mu_n\}$ which converges weak* to a

measure $\mu: \mathscr{B}(G) \to X^{**}$. Clearly, $\|\mu\|_1 < \infty$. Also $\hat{\mu}(\gamma) = 0$ if $\gamma \notin \Lambda$ and $\hat{\mu}(\gamma) = a_\gamma$ for all $\gamma \in \Lambda$ and in particular $\hat{\mu}(\gamma) \in X$ for all $\gamma \in \Gamma$. From this it is easily shown that μ is, in fact, an X-valued measure. By combining these results we see that $\mu \in V^1_\Lambda(G; X)$. However, since Λ is a Riesz set $\mu \in V^1_{\Lambda,\mathrm{ac}}(G; X)$. Since (a) holds, μ has a Radon-Nikodym derivative with respect to λ. That is, there is a function $f \in L^1(G, X)$ such that $\mu(E) = \int_E f(x)\, d\lambda(x)$ for all $E \in \mathscr{B}(G)$. A simple calculation shows that $\hat{\mu}(\gamma) = \hat{f}(\gamma)$ for all $\gamma \in \Gamma$ and thus $f \in L^1_\Lambda(G; X)$ and satisfies $\hat{f}(\gamma) = a_\gamma$ for all $\gamma \in \Lambda$ so (b) holds.

Finally, we assume that Λ is not a Riesz set. Then there is a $\mu_1 \in V^1_\Lambda(G)$ with $\mu_1 \notin L^1_\Lambda(G)$. Let $x \in X$, $x \neq 0$ and define $\mu: \mathscr{B}(G) \to X$ by $\mu(E) = x\mu_1(E)$. Let $a_\gamma = \hat{\mu}(\gamma)$ for all $\gamma \in \Lambda$. Then $\{a_\gamma\}_{\gamma\in\Lambda} \subset X$ and the sequence $\{f_n\}_{n=1}^\infty$, $f_n = \sum_{\gamma\in\Lambda} \hat{i}_n(\gamma) a_\gamma \gamma$, is bounded in $L^1_\Lambda(G; X)$. If there exists an $f \in L^1_\Lambda(G; X)$ such that $\hat{f}(\gamma) = a_\gamma$ for all $\gamma \in \Lambda$ then $\mu = f$, which is impossible. Therefore (b) does not hold. □

In [E], Edgar asked the following question: If Λ is a Riesz set are the type I-Λ-RNP and the type II-Λ-RNP equivalent properties?

In [D2], the following sufficient condition is given:

Proposition 7. *Let G be a compact abelian metrizable group and let Λ be a Riesz subset of Γ. Suppose that a Banach space X has type* I-Λ-*RNP. Then X has type* II-Λ-*RNP if $L^1(G; X)$ has type* I-Λ-*RNP.*

In the remainder of this section we will show that the converse of Proposition 7 is also true when G is the circle group, $\mathbb{T}$. The main ingredient in obtaining this result is our choice of the good approximate identity. Since Theorem 6(a) is independent of the choice of the good approximate identity, we can use the Poisson kernel. More specifically, if $n \in \mathbb{N}$, we let $r_n = 1 - 1/n$ and $i_n = P_{r_n}$ where

$$P_{r_n}(e^{it}) = \sum_{m=-\infty}^{\infty} r_n^{|m|} e^{imt}.$$

Proposition 8. *Let Λ be a Riesz subset of $\mathbb{Z}$ and let X be a Banach space. Then X has type* II-Λ-*RNP if and only if $L^1(\mathbb{T}; X)$ has type* I-Λ-*RNP.*

Proof. If $L^1(\mathbb{T}, X)$ has type I-Λ-RNP then X has type II-Λ-RNP by Proposition 7.

Conversely, suppose X has type II-Λ-RNP. Let $\{a_m\}_{m\in\Lambda} \subset L^1(\mathbb{T}; X)$ and define

$$f_n(t) = \sum_{m\in\Lambda} \hat{i}_n(m) a_m e^{imt}.$$

Suppose $\{f_n\}_{n=1}^\infty$ is bounded in $L^1(\mathbb{T}; L^1(\mathbb{T}; X))$. since $i_n = P_{r_n}$, we have $\hat{i}_n(m) = r_n^{|m|}$. Therefore $f_n(t) = \sum_{m\in\Lambda} r_n^{|m|} a_m e^{imt}$. Note that

$$P_{r_n/r_{n+1}} * f_{n+1} = f_n \quad \text{and} \quad \|P_{r_n/r_{n+1}}\|_1 = 1.$$

Therefore

$$\|f_n\|_{L^1(\mathbb{T};L^1(\mathbb{T};X))} \leq \|f_{n+1}\|_{L^1(\mathbb{T};L^1(\mathbb{T},X))}$$

and so we have

$$\lim_{n\to\infty} \|f_n\|_{L^1(\mathbb{T};L^1(\mathbb{T};X))} = \sup_n \|f_n\|_{L^1(\mathbb{T};L^1(\mathbb{T};X))} < \infty .$$

By the same reasoning if we define $f_n(\cdot)(\theta)$ by

$$f_n(t)(\theta) = \sum_{m\in\Lambda} r_n^{|m|} a_m(e^{i\theta}) e^{imt}$$

then

$$\lim_{n\to\infty} \|f_n(\cdot)(\theta)\|_{L^1(\mathbb{T};X)} = \sup_n \|f_n(\cdot)(\theta)\|_{L^1(\mathbb{T};X)}$$

where $\theta \in [0, 2\pi)$.

$$\begin{aligned}
\infty > \sup_n \|f_n\|_{L^1(\mathbb{T};L^1(\mathbb{T};X))} &= \lim_{n\to\infty} \|f_n\|_{L^1(\mathbb{T};L^1(\mathbb{T};X))} \\
&= \lim_{n\to\infty} \int_0^{2\pi}\int_0^{2\pi} \|f_n(t)(\theta)\|_X \frac{d\theta}{2\pi}\frac{dt}{2\pi} \\
&= \lim_{n\to\infty} \int_0^{2\pi}\int_0^{2\pi} \|f_n(t)(\theta)\|_X \frac{dt}{2\pi}\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \left(\lim_{n\to\infty} \int_0^{2\pi} \|f_n(t)(\theta)\|_X \frac{dt}{2\pi} \right) \frac{d\theta}{2\pi}
\end{aligned}$$

(by the monotone convergence theorem)

$$\begin{aligned}
&= \int_0^{2\pi} \left(\sup_n \int_0^{2\pi} \|f_n(t)(\theta)\|_X \frac{dt}{2\pi} \right) \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \left(\sup_n \|f_n(\cdot)(\theta)\|_{L^1(\mathbb{T};X)} \right) \frac{d\theta}{2\pi} .
\end{aligned}$$

Therefore, $\sup_n \|f_n(\cdot)(\theta)\|_{L^1(\mathbb{T};X)} < \infty$ for almost all $\theta \in [0, 2\pi)$. Consequently, since X has type II-Λ-RNP, we get that for almost all $\theta \in [0, 2\pi)$ the sequence $\{f_n(\cdot)(\theta)\}_{n=1}^{\infty}$ converges in $L^1(\mathbb{T}; X)$-norm (Theorem 6(c)). Also,

$$\begin{aligned}
\|f_n - f_k\|_{L^1(\mathbb{T};L^1(\mathbb{T};X))} &= \int_0^{2\pi}\int_0^{2\pi} \|f_n(t)(\theta) - f_k(t)(\theta)\|_X \frac{d\theta}{2\pi}\frac{dt}{2\pi} \\
&= \int_0^{2\pi}\int_0^{2\pi} \|f_n(t)(\theta) - f_k(t)(\theta)\|_X \frac{dt}{2\pi}\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \left(\|f_n(\cdot)(\theta) - f_k(\cdot)(\theta)\|_{L^1(\mathbb{T};X)} \right) \frac{d\theta}{2\pi} .
\end{aligned}$$

But $\|f_n(\cdot)(\theta) - f_k(\cdot)(\theta)\|_{L^1(\mathbb{T};X)} \to 0$ as $n, k \to \infty$ for almost all $\theta \in [0, 2\pi)$ and

$$\|f_n(\cdot)(\theta) - f_k(\cdot)(\theta)\|_{L^1(\mathbb{T};X)} \leq 2 \sup_n \|f_n(\cdot)(\theta)\|_{L^1(\mathbb{T};X)} .$$

Hence, by the dominated convergence theorem $\|f_n - f_k\|_{L^1(\mathbb{T}; L^1(\mathbb{T}; X))} \to 0$ as $n, k \to \infty$ and so $\{f_n\}_{n=1}^{\infty}$ is convergent in $L^1(\mathbb{T}; L^1(\mathbb{T}; X))$-norm.

By Theorem 6(c), $L^1(\mathbb{T}; X)$ has type II-Λ-RNP and so it has type I-Λ-RNP. □

Remark. The proof of Proposition 8 is very similar to the proof of Theorem 2 of [D1]. In fact, the proof can be modified to show;

Proposition 9. *Let Λ be a subset of $\mathbb{Z}$, let X be a Banach space and $1 < p < \infty$. Then $V_\Lambda^p(\mathbb{T}; X) = L_\Lambda^p(\mathbb{T}; X)$ if and only if $L^p(\mathbb{T}; X)$ has type* I-Λ-*RNP.*

5. Type III-Λ-Radon-Nikodym property

We begin this section by proving a result similar to Proposition 3. The proof we present here was shown to us by Joe Diestel.

Proposition 10 (Diestel). *If G be a compact abelian metrizable group and let Λ be a subset of Γ. Then Λ is a Riesz set if and only if $L^1[0, 1]$ has type* III-Λ-*RNP.*

Proof. If $L^1[0, 1]$ has type III-Λ-RNP then it has type I-Λ-RNP and therefore Λ is a Riesz set by Proposition 3.

Conversely, suppose Λ is a Riesz set and let $T\colon C(G) \to L^1[0, 1]$ be an absolutely summing operator such that $T|_{C_{\Lambda'}(G)} \equiv 0$. Then there exists a bounded linear operator $\widetilde{T}$ so that the following diagram commutes:

$$\begin{tikzcd} C(G) \arrow[rr, "T"] \arrow[dr, "q"'] & & L^1[0, 1] \\ & C(G)/C_{\Lambda'}(G) \arrow[ur, "\widetilde{T}"'] & \end{tikzcd}$$

where q is the natural quotient map. Since T is absolutely summing on $C(G)$ it is Pietsch integral [DU, p. 169] and hence it is integral [DU, p. 235]. Since the range of T is contained in $L^1[0, 1]$, $T(B_{C(G)})$ is lattice bounded by a result of Grothendieck [DU, p. 258]. But

$$2T(B_{C(G)}) = 2\widetilde{T}(q(B_{C(G)})) \supseteq \widetilde{T}(B_{(C(G)/C_{\Lambda'}(G))}).$$

So $\widetilde{T}$ is integral by the converse of Grothendieck's result [DU, p. 258]. Note that $\widetilde{T}\colon C(G)/C_{\Lambda'}(G) \to L^1[0, 1]$ and also $(C(G)/C_{\Lambda'}(G))^* \cong L_\Lambda^1(G)$, since Λ is a Riesz set. By [L], $L_\Lambda^1(G)$ has the Radon-Nikodym property. Also, one can easily show that $L_\Lambda^1(G)$ has the approximation property. Therefore $\widetilde{T}$ is a nuclear operator [DU, p. 248] and so T is also nuclear. Thus $L^1[0, 1]$ has type III-Λ-RNP. □

Our next result shows that the type II-Λ-RNP and the type III-Λ-RNP coincide whenever Λ is a Riesz set.

Theorem 11. *Let G be a compact abelian metrizable group and let Λ be a Riesz subset of Γ. Then a Banach space X has* II-Λ-*RNP if and only if it has type* III-Λ-*RNP.*

Proof. If X has type III-Λ-RNP then it has type II-Λ-RNP from §2.

Conversely, suppose X has type II-Λ-RNP and let $T\colon C(G) \to X$ be absolutely summing and such that $T|_{C_{\Lambda'}(G)} \equiv 0$. Let μ be the representing measure for T. That is, $T(f) = \int_G f d\mu$ for all $f \in C(G)$. Since T is absolutely summing, μ is of bounded variation by Theorem 1. Also $\hat{\mu}(\gamma) = T(\overline{\gamma}) = 0$ for all $\gamma \notin \Lambda$ since $T|_{C_{\Lambda'}(G)} \equiv 0$. Therefore $\mu \in V^1_\Lambda(G; X)$ and since Λ is a Riesz set $\mu \in V^1_{\Lambda,\mathrm{ac}}(G; X)$. Now X has type II-Λ-RNP so there exists a function $g \in L^1_\Lambda(G; X)$ such that $\mu(E) = \int_E g\, d\lambda$ for all $E \in \mathscr{B}(G)$.

Let $\varepsilon > 0$. By [DU, p. 172] there exists sequences $\{x_n\}_{n=1}^\infty$ in X and $\{E_n\}_{n=1}^\infty$ in $\mathscr{B}(G)$ such that

$$g = \sum_{n=1}^\infty x_n \chi_{E_n}, \quad \lambda\text{-almost everywhere},$$

and

$$\int_G \|g\|\, d\lambda \le \sum_{n=1}^\infty \|x_n\| \lambda(E_n) \le \int_G \|g\|\, d\lambda + \varepsilon.$$

For each $f \in C(G)$,

$$\begin{aligned} T(f) &= \int_G f\, d\mu = \int_G f g\, d\lambda \\ &= \int_G f \cdot \left(\sum_{n=1}^\infty x_n \chi_{E_n} \right) d\lambda = \sum_{n=1}^\infty \left(\int_{E_n} f\, d\lambda \right) x_n. \end{aligned}$$

Define $T_n \in (C(G))^*$ by $T_n(f) = \int_{E_n} f\, d\lambda$ for all $f \in C(G)$. Note that $\|T_n\| = \lambda(E_n)$. Thus we have $\{T_n\}_{n=1}^\infty$ in $(C(G))^*$, $\{x_n\}_{n=1}^\infty$ in X with

$$T(f) = \sum_{n=1}^\infty T_n(f) x_n \quad \text{for all } f \in C(G),$$

and

$$\sum_{n=1}^\infty \|T_n\|\, \|x_n\| = \sum_{n=1}^\infty \lambda(E_n) \|x_n\| \le \int_G \|g\|\, d\lambda + \varepsilon < \infty.$$

That is, T is a nuclear operator and so the proof is complete. □

Remark. Diestel's result, Proposition 10, can also be derived from Theorem 11, Proposition 7 and Proposition 3. If Λ is a Riesz set then $L^1(G \times G)$ has type I-Λ-RNP by Proposition 3. However $L^1(G \times G)$ is isomorphic to $L^1(G; L^1(G))$ and so $L^1(G)$ has type II-Λ-RNP by Proposition 7. Now, Theorem 11 says that $L^1(G)$ has type III-Λ-RNP and so $L^1[0, 1]$ has type III-Λ-RNP.

Corollary 12. *Let G be a compact abelian metrizable group and let Λ be a Sidon subset of Γ. If X is a Banach space not containing a copy of c_0 then every absolutely summing operator $T\colon C(G) \to X$ with $T|_{C_{\Lambda'}(G)} \equiv 0$ is nuclear.*

Proof. If Λ is a Sidon subset of Γ and X does not contain a copy of c_0 then X has type II-Λ-RNP [D2]. Sidon sets are Riesz sets so apply Theorem 11 to complete the proof. □

Corollary 13. *Let G be a compact abelian metrizable group and let Λ be a Sidon subset of Γ. Let X be a Banach space not containing c_0 and let $T\colon C_{\Lambda'}(G) \to X$ be a Pietsch integral operator. If T_1 and T_2 are two Pietsch integral extensions of T to $C(G)$ then $T_1 - T_2$ is nuclear.*

Remark. An analogous result to Corollary 13 was proved in [BD] for the subset $\mathbb{N}$ of $\mathbb{Z}$. Also in [BD], Theorem 11 was proved for the special case of the subset $\mathbb{N}$ of $\mathbb{Z}$ using a slightly different proof that makes use of the special properties of analytic measures.

Our final result characterizes type III-Λ-RNP whenever Λ is a non-Riesz set.

Theorem 14. *Let G be a compact abelian metrizable group and let Λ be a non-Riesz subset of Γ. If X is a Banach space with type* III-Λ-*RNP then X has the Radon-Nikodym property.*

Proof. To show that X has the Radon-Nikodym property it suffices to show that every Pietsch integral operator from $C(G)$ to X is nuclear.

If Λ is a non-Riesz set then $(C(G)/C_{\Lambda'}(G))^*$ is isomorphic to $V^1_\Lambda(G)$ and $V^1_\Lambda(G)$ is nonseparable. Let $q\colon C(G) \to C(G)/C_{\Lambda'}(G)$ be the natural quotient map. Then $q^*((C(G)/C_{\Lambda'}(G))^*)$ is nonseparable since it is isomorphic to $V^1_\Lambda(G)$. By a result of Rosenthal [R, p. 362], there is a subspace Z of $C(G)$ with Z isometrically isomorphic to $C(G)$ and such that the restriction map $q|_Z\colon Z \to q(Z)$ is an isomorphism. Let $S\colon C(G) \to X$ be a Pietsch integral operator and consider the following diagram:

$$\begin{tikzcd}
C(G) \arrow[rr, "S"] \arrow[d, shift left, "j^{-1}"] \arrow[dr, phantom, "\widetilde{S}"] & & X \\
q(Z) \arrow[u, shift left, "j"] \arrow[r, hook, "i"] & C(G)/C_{\Lambda'}(G) \arrow[ur, phantom, "\widetilde{T}"] & C(G) \arrow[l, "q"'] \arrow[u, "T"']
\end{tikzcd}$$

The map j is an isomorphism. This is possible since $C(G)$ is isomorphic to Z and Z is isomorphic to $q(Z)$. $\widetilde{S} = Sj$ and thus $\widetilde{S}$ is Pietsch integral since S is Pietsch integral. The map i is the inclusion map and $\widetilde{T}$ is a Pietsch integral extension of $\widetilde{S}$. The map T is defined as $T = \widetilde{T}q$.

Now T is Pietsch integral and so is absolutely summing. Also $T|_{C_{\Lambda'}(G)} \equiv 0$. Therefore, since X has type III-Λ-RNP, T is a nuclear operator. Thus $T|_Z\colon Z \to X$ is also a nuclear operator. That is, $\widetilde{T}q|_Z$ is a nuclear operator.

Note that

$$\widetilde{T}|_{q(Z)} = (\widetilde{T}q|_Z) \circ (q|_Z)^{-1} : q(Z) \to X$$

and so it is also nuclear. However $\widetilde{T}|_{q(Z)} = \widetilde{S}$. Finally, $S = \widetilde{S} \circ j^{-1}$, thus making S a nuclear operator. This completes the proof. □

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References

[BD] A. Bèlanger and P. N. Dowling, *Two remarks on absolutely summing operators*, Math. Nachr. **136** (1988), 229–232.

[BR] J. Bourgain and H. P. Rosenthal, *Applications of the theory of semi-embeddings to Banach space theory*, J. Funct. Anal. **52** (1983), 149–188.

[B] R. Bourgin, *Geometric aspects of convex sets with the Radon-Nikodym property*, Lecture Notes in Math., vol. 933, Springer-Verlag, Berlin, 1983.

[DU] J. Diestel and J. J. Uhl, *Vector measures*, Math. Surveys, No. 15, Amer. Math. Soc., Providence, R. I., 1977.

[D1] P. N. Dowling, *The analytic Radon-Nikodym property in Lebesgue-Bochner function spaces*, Proc. Amer. Math. Soc. **99** (1987), 119–122.

[D2] ——, *Duality in some vector-valued function spaces*, preprint.

[E] G. A. Edgar, *Banach spaces with the analytic Radon-Nikodym property and compact abelian groups* (to appear).

[GR] N. Ghoussoub and H. P. Rosenthal, *Martingales, G_δ-embeddings and quotients of L^1*, Math. Ann. **264** (1983), 321–332.

[L] F. Lust-Piquard, *Ensembles de Rosenthal et ensembles de Riesz*, C.R. Acad. Sci. Paris **282** (1976), A833–835.

[Ro] H. P. Rosenthal, *On factors of $C[0, 1]$ with nonseparable dual*, Israel J. Math. **13** (1972), 361–378; Correction: *ibid.* **21** (1975), 93–94.

[Ru] W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Appl. Math., No. 12, Interscience, New York, 1962.

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