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k-COBORDISM FOR LINKS IN S^3

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ABSTRACT. We give an explicit finite set of (based) links which generates, under connected sum, the k-cobordism classes of links. We show that the union of these generating sets, $2 \leq k < \infty$, is *not* a generating set for ω-cobordism classes or even ∞-cobordism classes.

For 2-component links in S^3 we define $(2, k)$-corbordism and show that the concordance invariants β^i, $i \in \mathbb{Z}^+$, previously defined by the author, are invariants under $(2, i+1)$-cobordism. Moreover we show that the $(2, k)$-cobordism classes of links (with linking number 0) is a free abelian group of rank $k-1$, detected precisely by $\beta^1 \times \cdots \times \beta^{k-1}$. We write down a basis. The union of these bases $(2 \leq k < \infty)$ is *not* a generating set for $(2, \infty)$ or $(2, \omega)$-cobordism classes. However, we can show that $\prod_{i=1}^{\infty} \beta^i(\ )$ is an isomorphism from the *group* of $(2, \infty)$-cobordism classes to the subgroup $\mathscr{R} \subset \prod_{i=1}^{\infty} \mathbb{Z}$ of linearly recurrent sequences, so a basis exists by work of T. Jin.

1. COBORDISM RELATIONS FOR LINKS

It is a problem of some interest to characterize the set $C(m)$ of equivalence classes of links $(\coprod_{i=1}^{m} S^1 \hookrightarrow S^3)$ under the equivalence relation of link concordance. We are interested in the case $m > 1$ and in this case $C(m)$ does not appear to have a natural group structure. Moreover, we consider only "linking properties" as opposed to "knotting" of the individual components of the link. The invariants we consider are independent of the concordance class of the component knots. Our links are oriented, ordered and tame.

Weaker equivalence relations have recently been considered by N. Sato, K. Orr, and the author [Sa, O2, C2, C3]. Let us immediately generalize these in order to state our main results.

Definition 1.1. A link $L = \{L_1, \ldots, L_m\}$ in S^3 is $(b_1, \ldots, b_m)$-cobordant to $L' = \{L_1, \ldots, L_m\}$, where b_i are ordinal numbers, if there are disjointly embedded, compact, connected, oriented surfaces V_i in $S^3 \times [0, 1]$ such that

(i) $V_i \cap (S^3 \times \{0, 1\}) = \partial V_i = \partial V_i^0 \amalg \partial V_i^1$, where $\{\partial V_i^0\} = L$ and $\{\partial V_i^1\} = -L'$;

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(ii) there are tubular neighborhoods $(V_i \times D^2, \phi_i)$ of the V_i which extend the "longitudinal" ones on ∂V_i, and such that the image of the homomorphism:

$$\pi_1(V_i) \xrightarrow{\phi_i} \pi_1(V_i \times \partial D^2) \longrightarrow \pi_1\left(S^3 \times [0, 1]) - \bigcup_{i=1}^{m} \text{image}\, \phi_i\right)$$

lies in the subgroup generated by the b_ith term of the (transfinite) lower-central-series and the image of $\pi_1(\partial V_i)$.

Remarks. Condition (ii) is equivalent to requiring that, for some system $\{\gamma_{ij}\}$ of embedded closed curves on V_i which represents a symplectic basis for $H_1(V_i/\partial V_i)$, the image of each γ_{ij} lies in the b_ith term of the lower-central-series.

If all $b_i = \alpha$, then this was called α-cobordism in [C2] and α-concordance in [O2]. If $m = 2$ and each $b_i = 2$ this was called β-equivalence in [Sa]. We say that a link $L = \{L_1, L_2\}$ is (k_1, k_2)-cobordant to L' $(k_i \in \mathbb{Z}^+ \cup \{\infty\})$ if L is (n_1, n_2)-cobordant to L' for each positive integer n_i less than $k_i + 1$ $(\infty + 1 = \infty)$. Two links are ∞-*cobordant* if they are (∞, ∞)-cobordant.

Let $C(m; (b_1, \dots, b_m))$, where b_i are ordinals or ∞, denote the set of equivalence classes of m-component links in S^3 under the relation of $(b_1, \dots, b_m)$-cobordism. A superscript zero, such as $C^0(m)$ will be used when we restrict to links of linking number zero.

Recall that the author has defined functions $\beta^i \colon C^0(2) \to \mathbb{Z}$, $i \in \mathbb{Z}^+$, which are *additive* under any connected sum and vanish on boundary links [C2]. These were shown to be invariants of what was there called "weak cobordism," which was a "2-cobordism" on the first component and a true concordance on the second component. Since $\pm\beta^i$ was shown to be a lifting of Milnor's invariant $\overline{\mu}(11\overbrace{22222\cdots2}^{2i})$ [C3, §6; St2] and both of these are known to be related to $(i+1)$-cobordism [C3, §9; O3], the following is the desired result.

Theorem 2.1. *On the class of* 2*-component links in* S^3 *with zero linking number, the invariant* β^i *of* [C2, §5.2] *is an invariant of* $(2, i+1)$*-cobordism.*

We remark that Milnor's $\overline{\mu}$-invariants, Massey's products and Orr's invariants are "higher-order" invariants, requiring the vanishing of lower-order invariants to be well-defined. Thus the β^i are the only of these which could hope to yield a direct classification result.

Our first major result is

Theorem 2.9. $C^0(2; (2, k))$ *is a free abelian group* (*under arbitrary connected sum*) *of rank* $(k-1)$ *and an isomorphism is given by* $\beta^1 \times \beta^2 \times \cdots \times \beta^{k-1}$ *where* $\beta^i(L)$ *is the* i*th derived invariant of* [C2]. *An explicit basis is drawn in Figure* 2.10.

We remark that the corresponding group for higher-dimensional links is trivial by work of Orr [O1] and the author [C4]. The techniques of 2.9 together with a result of T. Jin also show

Theorem 2.16. *$C^0(2; (2, \infty))$ is an abelian group (under arbitrary connected sum) and the map $B: C^0(2; (2, \infty)) \to \prod_{i=1}^{\infty} \mathbb{Z}$ given by $\prod_{i=1}^{\infty} \beta^i$ is an isomorphism onto the subgroup $\mathscr{R}$ of linearly-recurrent sequences.*

Unfortunately, the union of the bases of 2.9–2.10 (as k varies) is *not* a generating set for $C^0(2; (2, \infty))$. In fact, this union is precisely the set of equivalence classes which have a representative whose second component is unknotted! This is the content of 2.11. However, 2.16 and the aforementioned result of Jin [J] are enough to pinpoint (less explicitly) a basis.

Recall that in [C3, §7, §10] we defined, for positive integers m, w, a *finite* set $\mathbb{B}(m, w)$ of based, Brunnian, m-component links in S^3. These were obtained by beginning with a Hopf link or a Whitehead link, "Bing-doubling" many times and finally banding certain components together in the simplest possible manner. It was shown in 9.26 and 9.28 of that paper that an element of $\mathbb{B}(m, w)$ is $[\frac{w-1}{2}]$-cobordant to the trivial link where $[\,]$ denotes "greatest integer less than." These links were created to be "atomic" for Milnor's $\overline{\mu}$-invariants of length w just as monomials are "atomic" for power series. Two results were announced in that paper (10.7, 10.8) for which detailed proofs were not provided. We will herein give these details. The basings of the elements of $\mathbb{B}(m, w)$, as discussed in [C3], needed only to avoid hitting a certain finite collection of surfaces. Here, however, we must fix some basing once and for all so that we have sufficient information to define connected-sum. Also, since there are an infinite number of basings which satisfy the meager requirements of [C3], it is nonsense to say that $\mathbb{B}(m, w)$ is a *finite* generating set under connected-sum unless we base the elements of $\mathbb{B}$ and use the same basing throughout. In addition, to define *iterated* connected-sums we must define basings for connected-sums. Suffice it to say that this can be done in such a way that $(A\#B)\#C = A\#(B\#C)$ which is the only property we shall need. Obviously all of these preliminaries would be greatly clarified by the language of "disk links" as espoused in [HL]. If S is a set of based links, let $\langle S \rangle$ denote the set of iterated connected-sums of elements of S. Let $\mathbb{B}_m$ denote $\langle \bigcup_{w=2}^{\infty} \mathbb{B}(m, w) \rangle$.

Theorem 3.3. *Suppose L is an m-component link in S^3 whose $\overline{\mu}$-invariants of weight less than k vanish $(k \geq 2)$ and such that $\overline{\mu}_L(I) = a_I$ for each sequence I of weight k. Then there is some $L' \in \langle \mathbb{B}(m, k) \rangle$, whose $\overline{\mu}$-invariants of weight less than k vanish, and for which $\overline{\mu}_{L'}(I) = a_I$ for each I of weight k.*

Realization Theorem 3.1 (see 10.8 of [C3]). *Given positive integers m, k there is a positive integer d such that for any m-component link L in S^3, there is an $L' \in \langle \bigcup_{w=2}^{d} \mathbb{B}(m, w) \rangle$ which is k-cobordant to L.*

Corollary 3.2. *The inclusion $\langle \bigcup_{w=2}^{d} \mathbb{B}(m, w) \rangle \to C(m; (k, \dots, k))$ is onto. For any m and any k, the inclusion $\mathbb{B}_m \to C(m; (k, \dots, k))$ is onto, that is, $\bigcup_{w=2}^{\infty} \mathbb{B}(m, w)$ is a generating set for k-cobordism.*

Remark. An announced result of K. Igusa and K. Orr (that the vanishing of the $\overline{\mu}$-invariants of weight less than or equal to $2k$ implies k-null-cobordism), implies that the integer $d(m, k)$ may be taken to be $2k$, as we shall see in the proof of 3.1.

The point of proving 3.1 here is that we will show

Theorem 2.11. *The set $\mathbb{B}_m$ is not a set of generators for ∞-cobordism $(C(m; \infty, \dots, \infty))$. Moreover there are 2-component links L which are not ∞-cobordant to any element of $\mathbb{B}_2$ even if one adds local knots; so for such L there is no element L' of $\mathbb{B}_2$ such that $L\#L'$ is ∞-cobordant to a boundary link even after introducing local knots.*

Unfortunately, here there is no known analogue of the β^i invariants or of Jin's theorem, so we fail to give a nice generating set for $C(m; \infty)$. There are two missing ingredients. Firstly, we need more k-cobordism invariants which are additive (like β^i). Secondly, given these invariants, we would appear to need a "finiteness" result analogous to our result that $\sum_{i=1}^{\infty} \beta^i x^i$ is the power series of a *rational* function. Ongoing work of N. Habegger and X. S. Lin on disk links seems to hold the best hope for understanding the structure of these sets of cobordism classes, in particular because it seems Quixotian to hope that we will continue to see nice group structure as we see herein.

2. $(2, n)$-COBORDISM AND THE β^i INVARIANTS

In this section we demonstrate that β^i of [C2] is an invariant of $(2, i+1)$-cobordism and use this to completely characterize $(2, n)$-cobordism and $(2, \infty)$-cobordism.

We recall the construction of [C2]. If $L = \{K_1, K_2\}$ has linking number zero, then there are oriented Seifert surfaces $\{S_1, S_2\}$ in $E(L) \equiv S^3 - N(L)$ which meet transversely in a circle (or are disjoint). This circle acquires an orientation and $D(L)$ is the oriented link $\{S_1 \cap S_2, K_2\}$. By iterating, $D^n(L)$ is defined for $n \geq 0$. The Sato-Levine invariant $\beta^1(L)$ is the self-linking number of $S_1 \cap S_2$ as a loop on S_1. $\beta^n(L)$ is defined to be $\beta^{n-1}(D(L))$. Another sequence of invariants is defined by $\overline{\beta}^i(\{K_1, K_2\}) = \beta^i(\{K_2, K_1\})$. Therefore to prove something about the β^i it is prudent to prove something about $D(L)$ and proceed by induction. Hence the backbone of this paper is the following satisfying geometric result.

Theorem 2.1. *If L is a 2-component link in S^3 with 0 linking number, and L is $(2, n+1)$-cobordant to L' $(n \geq 1)$, then $D(L)$ is $(2, n)$-cobordant to $D(L')$.*

Corollary 2.2. *The invariant $\beta^i(-)$ is an invariant of $(2, i+1)$-cobordism. The two sequences of invariants $\{\beta^i(\ )|\ i \in \mathbb{Z}^+\}$ and $\{\overline{\beta}^i\}$ are invariants of ∞-cobordism, and hence of ω-cobordism.*

Proof of 2.2. Theorem 4.1 of [Sa] establishes the first statement for $i = 1$. Now assume the result for $i \leq k$. Suppose L is $(2, k+2)$-cobordant to L'.

Then $D(L)$ is $(2, k+1)$-cobordant to $D(L')$ by 2.1, so $\beta^{k+1}(L) \equiv \beta^k(D(L)) = \beta^k(D(L')) = \beta^{k+1}(L')$. $\square$

Proof of Theorem 2.1. Suppose we have $L = \{K_1, K_2\}$, $L' = \{K_1', K_2'\}$ with connected Seifert surfaces $\{S_1, S_2\}$, $\{S_1', S_2'\}$, as defined in [C2, p. 293], lying in $S^3 \times \{1\}$, $S^3 \times \{0\}$ respectively. By hypothesis there exist disjoint cobordisms C_1 (from K_1 to K_1') and C_2 (from K_2 to K_2') in $S^3 \times I$. Moreover, if $\{\gamma^i \mid i = 1, \dots, 2g\}$ is a standard symplectic curve system on C_2 (identified with the section of the circle bundle over C_2 as in Definition 1.1), then $\gamma^i \in (\pi_1(E(C)))_{n+1}$ where $C = C_1 \cup C_2$. We want to equate this algebraic data to topological data.

Definition 2.3. Suppose γ is an embedded circle in a 4-manifold X (possibly $\gamma \subset \partial X$). A *length* 1 *half-grope attached to* γ is an immersed surface G_1 (compact, oriented, connected) whose boundary is γ (properly embedded near ∂X if $\gamma \in \partial X$). Suppose that $\{\gamma_{11}, \dots, \gamma_{1r}\}$ is one-half of a symplectic curve system for G_1. After a small isotopy of G_1 we may assume these are embedded. A *length* 2 *half-grope* attached to γ is a collection of immersed surfaces $\{G_1, G_{21}, \dots, G_{2r_1}\}$ where G_{2i} is a length 1 half-grope attached to γ_{1i}, $1 \le i \le r_1$. A *length* n *half-grope* attached to γ is a collection $\{G_1; \mathscr{G}_1, \dots, \mathscr{G}_r\}$ where $\mathscr{G}_i$ is length $n-1$ half-grope attached to γ_i, $1 \le i \le r$. The surface G_1 is called a 1st-*stage surface.* Any $\mathscr{G}_i$ equals $\{G_{2i}; \mathscr{G}_1, \dots, \mathscr{G}_{2g}'\}$ ($g = \text{genus}\, G_{2i}$) and G_{2i} is called a 2nd-*stage surface*, etc. We omit a proof of the following.

Proposition 2.4. *The loop γ in X (as above) lies in $(\pi_1(X))_{n+1}$ $(n \ge 1)$ if and only if there exists a length n half-grope attached to γ.*

We may assume that the entire collection of immersed surfaces constituting the length n half-gropes $X_1, \dots, X_{2g}$ (attached to $\gamma^1, \dots, \gamma^{2g}$ in $E(C)$) is in general position. Thus they have only point intersections and self-intersections and these lie in the interiors of the surfaces.

We need only produce connected 3-manifolds M_i, $i = 1, 2$, in $E(C)$ such that $\partial M_i = S_i \cup C_i \cup (-S_i')$ and such that $M_1 \cap M_2 = C_{12}$ does not intersect the length $(n-1)$ half-gropes X_i' attached to γ^i ($X_i' = X_i - n$th-stage surfaces). For then $\gamma^i \in (\pi_1(E(C_2 \cup C_{12})))_n$. But $\{C_{12}, C_2\}$ is a cobordism from $D(L)$ to $D(L')$, and this is a $(2, n)$-cobordism. Note that loops on C_{12} are loops in M_2 so they will always lie in $[\pi_1(E(C_2)), \pi_1(E(C_2))]$ because they will have zero linking number with ∂M_2. Moreover, if the section of the S^1-bundle over C_{12} is chosen to be the "unlinked" one (as in [L, Lemma 2.6]), then the images of these loops will lie in $(\pi_1(E(C_2 \cup C_{12})))_2$ as well.

Now let Y be the submanifold of $E(C)$ consisting of all of the *boundaries* of the surfaces in the $\{X_i\}$. (Note that the interiors of the nth-stage surfaces are disjoint from Y.) Let Z be the set of (images of) singular points (intersections and self-intersections) of the grope surfaces. Note that the "linking homomorphism" (induced by inclusion) $H_1(C_2 \cup Y \cup Z) \to H_1(E(C_1)) \cong \mathbf{Z}$ is trivial (remember $n \geq 1$ so L is $(2, 2)$-cobordant to L'). Thus there is a compact, connected 3-manifold M_1 in $E(C_2 \cup Y \cup Z)$ whose boundary is $C_1 \cup S_1 \cup (-S_1')$. The spirit of this is already seen in 3.2 of [C2] and 2.1 of [Sa] but we reprove it.

Below $w = C_1 \cup S_1 \cup -S_1'$ and $F = C_2$.

Proposition 2.5. *Suppose W is an oriented, connected, closed 2-dimensional submanifold of $S^3 \times [0, 1]$ and X is a disjoint union of circles and a connected, compact oriented surface F, embedded in $(S^3 \times [0, 1]) - W$. Suppose that $X \cap (S^3 \times \{0, 1\})$ is ∂F and that ∂F consists of two circles, one in $S^3 \times \{0\}$ and one in $S^3 \times \{1\}$. Let $N = N(X)$ be a regular neighborhood of X in the complement of W. Then $[W] \in H_2(S^3 \times I - N)$ is zero if the "linking homomorphism" $H_1(X) \to H_1((S^3 \times I) - W) \cong \mathbf{Z}$, induced by inclusion, is trivial.*

Proof. Let $\mathring{N} = \text{interior}(N)$. From the diagram below, where ϕ is an excision isomorphism, we see

$$\begin{tikzcd}
 & H_3(N, \partial N) \arrow[r, "\partial'"] \arrow[d, "\phi", "\cong"'] & H_2(\partial N) \arrow[r] \arrow[d, "i"] & 0 \\
 {} \arrow[r] & H_3(S^3 \times I, S^3 \times I - \mathring{N}) \arrow[r, "\partial"] & H_2(S^3 \times I - \mathring{N}) \arrow[r] & 0
\end{tikzcd}$$

that i is onto. Choose β so that $i(\beta) = [W]$. ∂N consists of $S^1 \times S^2$ components and a component homeomorphic to $F \times S^1$. Thus $H_2(\partial N)$ splits naturally as $\mathbf{Z} \times H_2^+(\partial N)$ where the generator of $\mathbf{Z}$ is represented by the torus $\partial_0 F \times S^1$ in $(S^3 \times \{0\}) - \partial_0 F$. Since the latter has trivial H_2, we may choose β above to lie in $H_2^+(\partial N)$. Similarly, $H_1(F \times S^1) \cong \mathbf{Z} \times H_1(F)$ where a generator of the $\mathbf{Z}$ is $* \times S^1$, a "meridian" of F. Moreover it is a standard result that this decomposition may be chosen so that under $F \equiv F \times * \hookrightarrow F \times S^1 \hookrightarrow \partial N$ that the image of $H_1(F)$ does not link F ($H_1((S^3 \times I) - F)) \cong \mathbf{Z}$ generated by the meridian). This is proved in [Sa] for example. Thus $H_1(\partial N)$ splits as $\mathbf{Z} \oplus H_1^+(\partial N) \cong \mathbf{Z} \oplus H_1(X)$. Now, by inspection, the intersection pairing $H_2^+(\partial N) \xrightarrow{I} \text{Hom}(H_1^+(\partial N), \mathbf{Z})$ is nondegenerate, i.e., $\beta = 0$ if $I(\beta) = 0$.

Let $N(W)$ be a regular neighborhood of W in the complement of N. The following diagram commutes, where I' is intersection pairing.

$$\begin{tikzcd}
H_1(\partial N) \times \mathbf{Z} \cong H_1(\partial N \cup \partial N(W)) & \otimes & H_2^+(\partial N) \arrow[dr, "I"] \arrow[d, "i"] & \\
H_2((S^3 \times I) - \mathring{N} - \mathring{N}(W), \partial) \arrow[u, "\partial"] & \otimes & H_2((S^3 \times I) - \mathring{N} - \mathring{N}(W)) \arrow[r, "I'"'] & \mathbf{Z}
\end{tikzcd}$$

Since W has a trivial normal bundle, we may consider

$$[W] \in H_2((S^3 \times I) - \mathring{N} - \mathring{N}(W))$$

is $i(\beta)$. By hypothesis, each element $\alpha \in H_1^+(\partial N)$ is zero in $H_1((S^3 \times I) - \mathring{N}(W))$ and by our choice of splitting, each such α is trivial in $H_1((S^3 \times I) - N)$. Thus each such α is $\partial(\delta)$. But, since W has a trivial normal bundle, and since δ may be represented by a surface F whose boundary lies in ∂N, $\delta \cdot [W] = 0$. Hence $\alpha \cdot \beta = 0$, implying that $\beta = 0$. □

We now return to the proof of 2.1. Suppose S is *one* of the surfaces of *one* of the length $(n-1)$ half-gropes (so S is *not* an nth-stage surface). Then $S - \mathring{N}(Y)$ is (the image of) a planar surface P and M_1 intersects S in a collection of disjoint simple closed curves. Moreover $M_1 \cap P \cap Z = \varnothing$. Let $Z' = Z \cap P$. Choosing tubular neighborhoods in P for the components of $Z' \cup \partial P$, we may speak sensibly of the circle of radius r about $z \in Z'$ (r small) and the "parallel" of a component of ∂P at distance r from ∂P. It suffices to be able to modify M_1, without changing its intersections with other surfaces, so that $M_1 \cap S$ is a union of circles of radius $\frac{1}{N}$ (N odd) about points of Z' and parallels of ∂P at distance $\frac{1}{N}$ (N odd). For then we could progressively similarly modify M_1 near the other k-stage surfaces ($k < n$). Then, since $H_1(C_1 \cup Y \cup Z) \to H_1(E(C_2))$ is zero, there is a 3-manifold M_2 in $E(C_1 \cup Y \cup Z)$ whose boundary is $(S_2 \cup C_2 \cup -S_2')$. Modify M_2 near each S as above so that $M_2 \cap S$ is a union off circles of radius $\frac{1}{N}$ (N even). Thus for each X_i', $M_1 \cap M_2 \cap X_i' = \varnothing$ as desired.

The modifications to M ($M = M_i$, $i = 1$ or 2) for a fixed S may be accomplished via compositions of 3 moves.

Move A. If γ_0 is a component of $M \cap P$ and γ_0 lies on the boundary of an annulus A where $(A - \{\gamma_0\}) \cap M \cap Z = \varnothing$ then, by the isotopy extension theorem, M may be isotoped, in a 4-dimensional neighborhood of A, to M' where $M' \cap P = (M \cap P) - \{\gamma_0\} \cup \{\gamma_1\}$ where γ_1 is the other boundary component of A.

Move B. If γ_0 is a component of $M \cap P$ which bounds a disk Δ in $\operatorname{int}(P - M - Z)$, then M may be "ambiently Dehn-surgered" so that $M' \cap P = (M \cap P) - \{\gamma - 0\}$. Here we mean that $M' = (M - (\partial\Delta \times D^2)) \cup (\Delta \times \partial D^2)$.

Move C. If γ_0 is a component of $M \cap P$ and α is an arc in P connecting 2 points of γ_0, such that $\mathring{\alpha} \cap M \cap Z = \varnothing$, then we may define the oriented band sum γ_1 of γ_0 with itself along α; then M may be ambiently surgered to M' where $M' \cap P = (M \cap P) - \{\gamma_0\} \cup \{\gamma_1\}$. Here $M' = (M - (\partial\alpha \times D^3)) \cup (\alpha \times \partial D^3)$.

It is easy to see that these moves suffice. Choose a component of $M \cap P$ which is innermost. If Move B applies, use it. Otherwise use Moves A and C to modify this component in the desired fashion. Then repeat this process, ignoring the components already in proper position. This completes the proof of 2.1. □

Theorem 2.6. *If $\beta^i(L) = \beta^i(L')$ for $i = 1, \dots, k$, then L is $(2, k+1)$-cobordant to L'. Similarly, if $\overline{\beta}^i(L) = \overline{\beta}^i(L')$ for $i = 1, \dots, k$, then L is $(k+1, 2)$-cobordant to L'.*

Proof. The theorem follows easily by induction from the following and from Sato's result for $k = 1$.

Lemma 2.7 (see 6.5 of [C2]). *If $D(L)$ is $(2, k)$-cobordant to $D(L')$ and $\beta^1(L) = \beta^1(L')$, then L is $(2, k+1)$-cobordant to L'.*

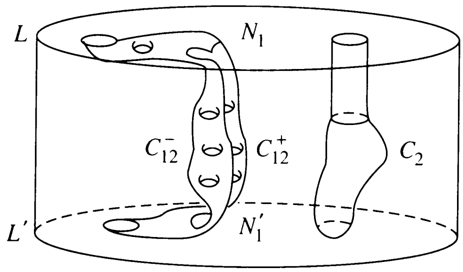

FIGURE 2.8

Proof of Lemma 2.7. Suppose $L = \{K_1, K_2\}$, $L' = \{K_1', K_2'\}$ with Seifert surfaces $\{S_1, S_2\}$, $\{S_1', S_2'\}$ lying in $S^3 \times \{1\}$ and $S^3 \times \{0\}$ respectively. Refer to Figure 2.8. There exists a $(2, k)$-cobordism $\{C_{12}, C_2\}$ from DL to DL' as shown. The normal vector field on ∂C_{12} induced by S_1 and S_1' can be extended to all of C_{12} if and only if $\beta^1(L) = \beta^1(L')$. Moreover, this vector field may be altered away from ∂C_{12} so that the linking map

$$H_1(C_{12}^+) \to H_1(S^3 \times [0, 1] - C_{12})$$

is zero. Using this "unlinked" vector field, push off oppositely-oriented copies C_{12}^+ and C_{12}^- of C_{12}. Let N_1 (respectively N_1') denote the complement in S_1 (respectively S_1') of an open tubular neighborhood of $C_{12} \cap S_1$ in S_1 (respectively $C_{12} \cap S_1'$ in S_1'). Then, setting $C_1 = N_1 \cup C_{12}^+ \cup C_{12}^- \cup N_1'$, as in Figure 2.8, we see that L is cobordant to L' via $\{C_1, C_2\}$. Since we chose an unlinked vector field it is easy to see that the homomorphism $\pi_1(C_1) \to H_1(E(C_1))$ is zero. Since $\{C_{12}, C_2\}$ was a $(2, k)$-cobordism, it is also easy to see that $\pi_1(C_1) \to H_1(E(C_2))$ is zero. Thus $\{C_1, C_2\}$ is a $(2, d)$-cobordism for some d. Let γ be a loop on the boundary of a tubular neighborhood of $\mathring{C}_2$ which lies in $\pi_1(E(C_{12} \cup C_2))_k$ $(k > 1)$. It suffices to show that $[\gamma]$ lies in $\pi_1(E(C_1 \cup C_2))_{k+1}$. The former implies that γ bounds a $(k-1)$-stage half-grope in $E(C_{12} \cup C_2)$ so γ bounds a $(k-1)$-stage half-grope G in $E(C_1 \cup C_2)$.

Let S be a last-stage surface of G. It suffices to show

$$[\partial S] \in (\pi_1(E(C_1 \cup C_2)))_3.$$

For this it suffices to show that there is a symplectic curve system (α_i, β_i) on S such that each $[\alpha_i] \in (\pi_1(E(C_1 \cup C_2)))_2$. But note that *any* loop in $E(C_{12})$ has zero linking number with $C_{12}^+ \cup C_{12}^-$ since these are "oppositely oriented." Thus it suffices to choose a system so that each $[\alpha_i] \in (\pi_1(E(C_2)))_2$. But this is easy since the linking homomorphism $H_1(S) \to H_1(E(C_2))$ has kernel of rank at least $2(\text{genus}\, S) - 1$. □

Theorem 2.9. *The equivalence classes of* 2*-component links in* S^3 (*with* 0*-linking number*) *modulo* $(2, k)$*-cobordism* $C^0(2; (2, k))$ *in our notation*) *is a free abelian group* (*under connected sum*) *of rank* $k-1$. *The map*

$$\beta^1(\ ) \times \beta^2(\ ) \times \cdots \times \beta^{k-1}(\ ) \colon C^0(2; (2, k)) \to \mathbf{Z}^{k-1}$$

is an isomorphism

Proof. By 2.1, the above function B is well-defined. By 5.6 of [C2] it is additive with respect to *any* band sum, in particular with respect to connected sum. The map B is one-to-one by 2.6; it is onto by 6.8 of [C2]. Finally, if $[L_1]$ and $[L_2]$ are in C, define $[L_1] \oplus [L_2]$ to be $[L_1 \# L_2]$ for any connected sum. If L_1' and L_2' are other representatives, $[L_1' \# L_2'] = [L_1 \# L_2]$ because B is additive and 1-1. By similar reasoning, C is associative and commutative. Another way to look at this is to define $[L_1] \oplus [L_2] = B^{-1}(B([L_1]) + B([L_2]))$. □

The reader might find it satisfying to see a basis for the above group. A basis is $\{W_i \mid i = 1, \ldots, k-1\}$ where the W_i are shown in Figure 2.10. Note that each component of W_i is unknotted. Here the second component is the one retained in the derivation process, the first component is replaced each time. In this example, for the obvious Seifert surfaces, $D^{2i}(W_k)$ is a trivial link! The set $\{W_i \mid i \le k-1\}$ is a subset of $\mathbb{B}(2, w = 2k) \subset \mathbb{B}_2$.

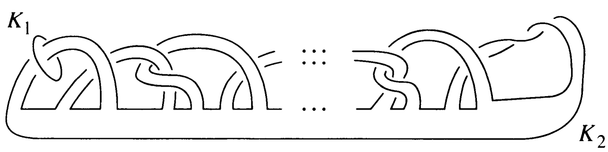

FIGURE 2.10

Theorem 2.11. *The set* $\mathbb{B}$ *is not a generating set for classical links modulo* ∞*-cobordism* $(C(m; \infty, \ldots, \infty))$. *Moreover, there are* 2*-component links* L *which are not* $(2, \infty)$*-cobordant to any element of* $\mathbb{B}_2$ *even if one adds local knots*; *so there is no element* L' *of* $\mathbb{B}_2$ *such that* $L \# L'$ *is* ∞*-cobordant to a boundary link, even after introducing local knots. The image of* $\mathbb{B}_2 \to C(2; (2, \infty))$ *is precisely those classes which have a representative whose second component is unknotted. The image is also those classes whose* β^i *are finitely nonzero.*

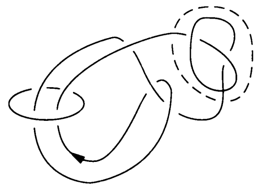

FIGURE 2.12

Proof of 2.11. Consider the innocent-looking link L shown in Figure 2.12, one of whose components is trivial and the second of which is a ribbon knot (the square knot). In [C2, 4.6, 4.7, p. 300], it is calculated that $\beta^i(L) = -1$ for all $i \geq 1$. The local knot shown inside the dotted circle is irrelevant to the calculations. By contrast:

Lemma 2.13. *For any element* $L' \in \mathbb{B}_2$, *there is an integer* N *such that* $\beta^i(L') = 0$ *if* $i > N$.

Proof of 2.13. Theorem 7.1(c) of [C2] implies that, if the second component of a link (K_1, K_2) is unknotted, then only a finite number of the β^i are nonzero. Since an element of $\mathbb{B}_2$ is, by definition, a (finite) connected sum of Brunnian links [C2, 7.2] and we have formed our connected sums carefully so that the result is still Brunnian; the result follows. □

If the link L of 2.12 were $(2, \infty)$-cobordant to $L' \in \mathbb{B}_2$, then for some large N, $\beta^N(L) = \beta^N(L') = 0$ by 2.2, a contradiction.

Now suppose $L \in C(2; (2, \infty))$ has its second component unknotted.

Again using 7.1(c) of [C2] we may choose N so that $\beta^i(L) = 0$ for $i > N$. Choose $L' \in \mathbb{B}_2$ such that $\beta^i(L) = \beta^i(L')$ for $i \leq N$, which can be done by 2.9. More precisely, using the generators W_i of Figure 2.11 we can ensure that $\beta^i(L') = 0$ for $i > N$. Thus $\beta^i(L) = \beta^i(L')$ for each i. By 2.9 L is $(2, i)$-cobordant to L' for each i and hence is $(2, \infty)$-cobordant to L'. This shows L is in the image of $\mathbb{B}$.

The above paragraph also proves

Lemma 2.14. *A 2-component link is* $(2, \infty)$*-cobordant to one whose second component is unknotted iff its* β^i *are finitely nonzero.*

This completes the proof of 2.11. □

Thus the set $\mathbb{B}$ is not rich enough to capture $C(2; (2, \infty))$. Its image is $\{[L] \mid \sum_{i=1}^{\infty} \beta^i(L)x^i$ is a finite polynomial$\}$. In [C2] this polynomial was observed to be related to Kojima's η-function by a change of variables and hence was the "power series of a rational function." Taek Jin showed that a sequence $\{n_i\}$ is $\{\beta^i(L)\}$ if and only if it is a "linearly recurrent sequence" (essentially if and only if the above power series is the power series of a rational function). In the process, he proved the existence of a set $\mathscr{T}$ containing one link for each $\{n_i\}$ (where $\{\beta^i(L)\} = \{n_i\}$) [J]. We can use this set to characterize $C(2; (2, \infty))$.

Consider the diagram 2.15 where $C(2;(2,k))$ is abbreviated $C(2,k)$.

$$\begin{tikzcd}
C(2,\infty) \arrow[dr, "i"] \arrow[ddd, "B"'] & & & & & & \\
& \varprojlim C(2,k) \arrow[r] \arrow[d, "\cong\ B"] & \cdots \arrow[r] & C(2,k) \arrow[r, two heads] \arrow[d, "\cong\ B_k"] & C(2,k-1) \arrow[r, two heads] \arrow[d, "\cong\ B_{k-1}"] & \cdots \arrow[r, two heads] & C(2,2) \arrow[d, "\cong\ B_2"] \\
& \varprojlim \mathbb{Z}^k \arrow[r] & \cdots \arrow[r] & \mathbb{Z}^{k-1} \arrow[r, two heads, "\pi_{k-1}"] & \mathbb{Z}^{k-2} \arrow[r, two heads] & \cdots \arrow[r, two heads] & \mathbb{Z} \\
\mathscr{R} \arrow[r, hook] & \prod_{i=1}^{\infty} & & & & &
\end{tikzcd}$$

Diagram 2.15

Here also $\mathscr{R}$ is the subgroup of $\varprojlim \mathbb{Z}^k$ (naturally identified with $\prod_{i=1}^{\infty} \mathbb{Z}$) consisting of linearly-recurrent sequences. Thus all the objects are naturally groups except $C(2,\infty)$.

Theorem 2.16. *$C(2;(2,\infty))$ is an abelian group under connected sum and the map $B\colon C(2,\infty) \to \prod_{i=1}^{\infty} \mathbb{Z}$ given by $\prod_{i=1}^{\infty} \beta^i$ is an isomorphism onto the subgroup $\mathscr{R}$ of linearly recurrent sequences. Thus $C(2,\infty)$ is an infinitely-generated torsion-free abelian group. An element of $C(2,\infty)$ is determined precisely by its Kojima-Yamasaki η-function* [C2, §7; KY].

Proof of 2.16. B is well-defined by 2.2 *onto* $\mathscr{R}$ by Jin's work [J], and additive an any connected sum by 5.6 of [C2]. Suppose $B([L]) = B([L'])$; then for any k, $B_k([L]) = B_k([L'])$ so L is $(2,k)$-cobordant to L'. Thus $[L]=[L']$ and B is one-to-one. Now define $[L]\oplus[L'] = [L\#L']$, which is well-defined by additivity and injectivity of B. The collection of β^i determines and is determined by the η-function [C2, §7].

3. A finite generating set for the k-cobordism classes of links in S^3

In this section we will show that there is a specific finite subset of $\mathbb{B}_m$ with the property that any element of $C(m;(k,k,\ldots,k))$ has a representative which is a finite connected sum of elements of the subset. If a certain conjecture of the author and K. Orr is true, then there is an algorithm to decide if such a connected sum represents the trivial element. In the process we provide details for the proofs of Theorems 10.7 and 10.8 of [C3].

Realization Theorem 3.1 (see 10.8 of [C3]). *Given positive integers m, k, there is an integer d such that for any m-component link L, there is, in the k-cobordism class of L, a connected sum of elements of $\bigcup_{w=2}^{d} \mathbb{B}(m,w)$ (see* §1 *for a definition of $\mathbb{B}(m,w)$).*

Corollary 3.2. *For any k, $\mathbb{B}_m$ is a generating set (under connected sum) for k-cobordism.*

Before proving 3.1, we need to refine Theorem 7.2 of [C3]. This states that if c is $\overline{\mu}(I)$, $I = i_1 i_2 \cdots i_k$, for *some* m-component link L whose $\overline{\mu}$-invariants of weight less than k vanish, then c is $\overline{\mu}(I)$ for some L' in $\langle\mathbb{B}(m, k)\rangle$. For 3.1 we need to realize all the weight k $\overline{\mu}$-invariants simultaneously. This is a bit more subtle.

Theorem 3.3. *Suppose L is an m-component link in S^3 whose $\overline{\mu}$-invariants of weight less than k vanish $(k \geq 2)$ and such that $\overline{\mu}_L(I) = a_I$ for each sequence I of weight k. Then there is some $L' \in \langle\mathbb{B}(m, k)\rangle$, whose $\overline{\mu}$-invariants of weight less than k vanish, and for which $\overline{\mu}_{L'}(I) = a_I$ for each I of weight k.*

Before proving 3.3, let us give the reader a feeling for the difficulty involved in deducing it from 7.2 of [C3]. The strategy of 7.2 is that, if a nice system of Seifert surfaces exists for L' which exhibits the Massey products associated to the $\overline{\mu}$-invariants, then each $\overline{\mu}(I)$ is equal to a certain linear combination of equivalence classes of "higher-order linking numbers." The latter are not always link invariants but they can be *independently* realized. The $\overline{\mu}$-invariants *are* link invariants, but satisfy complex relations. Suppose $\overline{\mu}(I_1) = l_1 + l_2$ and $\overline{\mu}(I_2) = l_2$. To realize $\overline{\mu}(I_1) = 1$ it certainly suffices to find a link (with a nice surface system) for which $l_1 = 1$ and $l_2 = 0$, and this was the strategy of 7.2. However, obviously to realize $\overline{\mu}(I_1) = 1$ and $\overline{\mu}(I_2) = 1$ simultaneously, one needs $l_1 = 0$ and $l_2 = 1$. How do we know in general that the system of linear equations $\{a_I = \overline{\mu}_L(I) = \sum a_i l_i \mid \text{all } I\}$ can be solved (for the l_i)? We must make crucial use of the fact that $\{a_I\}$ is a set of $\overline{\mu}$-invariants of *some* link.

Proof of 3.3. The hypotheses on L guarantee that, for any I of weight k, $\overline{\mu}_L(I)$ is a well-defined integer (we may assume, by the cyclic symmetry of $\overline{\mu}$-invariants, that $i_1 \neq i_k$) and is equal to the value of a Massey product

$$\langle x_{i_1}, \dots, x_{i_k}\rangle \in H^2(S^3 - L) \equiv \mathrm{Hom}(H_2(S^3 - L), \mathbb{Z})$$

evaluated on the boundary torus corresponding to the i_1st component. The hypotheses also ensure that each such product is uniquely defined. The calculation of a Massey product in $S^3 - L$ entails choices of specific cochains $a_{ij} \in C^1(S^3 - L)$ each of which is a solution of an equation $\delta a_{ij} = b_{ij}$ for $b_{ij} \in C^2(S^3 - L)$. Choose the necessary cochains for *all* Massey products of weight k in a compatible fashion, in the sense that, if b_{ij} occurs more than once, always choose the same solution a_{ij}. Now, by definition, the value of a Massey product $M_I = \langle x_{i_1}, \dots, x_{i_k}\rangle$ in $S^3 - L$ is a particular linear combination $\sum A_j^I z_j$ of cochains $z_j \in C^2(S^3 - L)$, $A_j^I \in \mathbb{Z}$. The index set depends only on I. As explained in §6 of [C3], these A_j^I are independent of L and of these choices b_{ij}. These z_j are the so-called "higher-order linking homomorphisms" which, in case a nice surface system exists, really are homomorphisms given by "linking number with" embedded 1-manifolds (see 6.3–6.5 of [C3]). It is there explained that the z_j are associated to iterated bracketings or commutators of the x_{ij}. An equivalence relation of 2.17–2.19 of [C3] is imposed to yield $\overline{\mu}(I) = \sum B_j^I[z_j]$. One crucial point is that not only are the B_j^I independent of L, but the index set, being over the equivalence classes (2.17) of formal brackets in the symbols x_i, is also independent of L. The other crucial point is that the z_j are fixed cochains independent of I because of the compatibility imposed above on the b_{ij}! Let c_j be the evaluation of $[z_j]$ (independent of representative by a generalization of 2.16 [C3]) on the (i_1) boundary component of $S^3 - L$. Thus, for any I, $a_I = \overline{\mu}_L(I) = \sum B_j^I c_j$ where the index set depends on I. Now if we construct a link L' with a nice system of surfaces then $\overline{\mu}_{L'}(I) = \sum B_j^I l_j$ for the same index set and B_j^I as above but where the l_j is a higher-order linking number as in [C3]. If we can choose L' so that $l_j = b_j$, then, for all I, $a_I = \overline{\mu}_{L'}(I)$. But, by [C3, 7.10, 7.11] the elements of $\mathbb{B}(m, k)$ have $l_j = 1$ for a single j. Since we have based our links in order to avoid the nice surface systems, any values of the weight k higher-order linking numbers may be achieved. This completes the proof of 3.3. □

Proof of 3.1. By 1.3 of [C4], given m, k there is a positive integer d such that, if all of the $\overline{\mu}$-invariants of weight no more than d vanish for a link, then that link is null-k-cobordant. Orr and Igusa have announced that d may be taken to be $2k$. Now fix L and k, hence m and d.

We shall show by induction on r, $1 \leq r \leq d$, that there is an $L'_r \in \langle \bigcup_{w=2}^{r} \mathbb{B}(m, r) \rangle$ such that, for $L\#L'_r$ (use any basing of L), $\overline{\mu}(I) = 0$ if $w(I) \leq r$. This is true for $r = 1$ by letting L'_1 be the trivial link. Assume it is true for r. By 3.3, there is $L''_{r+1} \in \langle \mathbb{B}(m, r+1) \rangle$ whose $\overline{\mu}$-invariants agree with those of $-(L\#L'_r)$ (a concordance inverse of $L\#L'_r$), up to weight $r+1$. Thus, for $(L\#L'_r)\#L''_{r+1}$, $\overline{\mu}(I) = 0$ if $w(I) \leq r+1$ [C3, 8.13; St1; O2]. Now it is not in general true that $(A\#_1 B)\#_2 C = A\#_1(B\#_2 C)$ because the bands defining $\#_2$ may wrap nastily about A. However, if we choose the basing of $(A\#_1 B)$ nicely, then it will work. This is important because we want to have fixed basings for our elements of $\mathbb{B}(m, w)$ and for $\langle \mathbb{B}(m, w) \rangle$. Thus, setting $L'_{r+1} = L'_r \# L''_{r+1}$, the claim is proved by induction on r.

By the result of the first paragraph there is an $L' \in \langle \bigcup_{w=2}^{d} \mathbb{B}(m, w) \rangle$ such that $L\#(-L')$ is null-k-cobordant. It is now easy to see that the null-bordism may be enlarged by adding m bands to a k-cobordism between L and L'. □

If the Cochran-Orr conjecture is true (as announced by Igusa-Orr and Xiao-Song Lin), then L is null-k-cobordant if and only if $\mu_L(I) = 0$ for $w(I) \leq 2k$. It follows easily that

Proposition 3.4. *There exists a concrete algorithm to decide whether or not a given element of* $\langle \bigcup_{w=2}^{2k} \mathbb{B}(m, w) \rangle$ *represents the trivial element of* $C(m; (k, k, \ldots, k))$.

This relies on algorithms to calculate $\overline{\mu}$-invariants given by Milnor [M2], and D. Stein [Ste2, St1].

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