LINEAR TOPOLOGICAL CLASSIFICATIONS
OF CERTAIN FUNCTION SPACES

VESKO M. VALOV

ABSTRACT. Some linear classification results for the spaces $C_p(X)$ and $C^*_p(X)$ are proved.

0. Introduction

If $X$ is a space then $C_p(X)$ denotes the set of all continuous real-valued functions on $X$ with the topology of pointwise convergence. We write $C^*_p(X)$ for the subspace of $C_p(X)$ consisting of all bounded functions. $R$ stands for the usual space of real numbers, $I$—for the unit segment $[0, 1]$ and $Q$ is the Hilbert cube $I^\omega$. If $n \geq 1$ then $\mu^n$ denotes the $n$-dimensional universal Menger compactum. Let $X$ be a separable metric space. A separable metric space $Y$ is called an $X$-manifold if $Y$ admits an open cover by sets homeomorphic to open subsets of $X$.

Results in [A1, A2 and Ps] show that the linear topological classification of the spaces $C_p(X)$ is very complicated. Below the linear topological classification results for the spaces $C_p(X)$ which I know are listed:

1. Let $X$ and $Y$ be non-zero-dimensional compact polyhedra. Then $C_p(X) \sim C_p(Y)$ if and only if dim $X = \text{dim } Y$ [Pv]. Here the symbol “$\sim$” stands for linear homeomorphism.

2. If $X$ is a locally compact subset of $R^n$ such that cl(Int($X$)) \cap (R^n - X) \neq \emptyset then $C_p(X) \sim C_p(R^n)$ [Dr1].

3. If $X$ is a 1-dimensional compact ANR with finite ramification points or a continuum $X$ is a one-to-one continuous image of $[0, \infty )$ then $C_p(X) \sim C_p(I)$ [KO].

For topological classification results of the spaces $C_p(X)$ see [BGM, BGMP, GH and M].

The aim of this paper is to prove the following results:

4. $C_p(X) \sim C_p(Q)$ if and only if $X$ is a compact metric space containing a copy of $Q$.

5. Let $X$ be a subset of $R^n$. Then $C_p(X) \sim C_p(I^n)$ iff $X$ is compact and dim $X = n$. 

Received by the editors February 8, 1989.

1980 Mathematics Subject Classification (1985 Revision). Primary 46E10, 54C35.

Key words and phrases. Linear homeomorphism, function space, manifold.
(6) \( C_p(X) \cong C_p(\mu^n) \) if and only if \( X \) is an \( n \)-dimensional compact metric space containing a copy of \( \mu^n \).

(7) \( C_p(X) \cong C_p(l_2) \) provided \( X \) is an \( l_2 \)-manifold (by \( l_2 \) is denoted the separable Hilbert space).

(8) Let \( X \) be one of the spaces \( Q, l^n \) or \( \mu^n \), and \( Y \) be a locally compact subset of an \( X \)-manifold. Then \( C_p(Y) \cong C_p(X)^\alpha \) if and only if \( Y \) contains a closed copy of the topological sum \( \sum X_i \) of infinitely many copies of \( X \).

Similar results are also proved for the spaces \( C_p^*(X) \).

I am indebted to A. Dranishnikov and the referee for many valuable comments.

1. Preliminaries

All spaces under discussion are Tychonoff and all mappings between topological spaces are continuous. By \( L_p(X) \) is denoted the dual linear space of \( C_p(X) \) with the weak (i.e. pointwise) topology. It is known that

\[
L_p(X) = \left\{ \sum_{i=1}^{k} a_i \delta_{x_i} : a_i \in \mathbb{R} - (0) \text{ and } x_i \in X \text{ for each } i \leq k \right\}.
\]

Here \( \delta_x \) is the Dirac measure at the point \( x \in X \). We denote

\[
P_\infty(X) = \left\{ \sum_{i=1}^{k} a_i \delta_{x_i} : a_i \in (0, 1) \text{ for each } i \text{ and } \sum_{i=1}^{k} a_i = 1 \right\}
\]

and \( \text{supp}(l) = (x_1, \ldots, x_k) \), where \( l = \sum_{i=1}^{k} a_i \delta_{x_i} \in L_p(X) \).

Let \( A \) be a closed subset of a space \( X \). Consider the following conditions:

(i) There is a continuous linear extension operator \( u : C_p(A) \to C_p(X) \) (recall that \( u : C_p(A) \to C_p(X) \) is an extension operator if \( u(f)|A = f \) for every \( f \in C_p(A) \));

(ii) There is a continuous linear extension operator \( u : C_p(A) \to C_p(X) \) and a positive constant \( c \) such that \( \|u(f)\| \leq c \cdot \|f\| \) for every \( f \in C_p^*(A) \). Here \( \|f\| \) is the supremum norm of \( f \);

(iii) There is a regular extension operator \( u : C_p(A) \to C_p(X) \) i.e. a continuous linear extension operator \( u \) with \( u(1_A) = 1_X \) and \( u(f) \geq 0 \) provided \( f \geq 0 \).

\( A \) is said to be \( l \)-embedded (resp., \( l^* \)-embedded) in \( X \) if the condition (i) (resp., the condition (ii)) holds. If (iii) is satisfied then \( A \) is called strongly \( l \)-embedded in \( X \). Dugundji [D] proved that every closed subset of a metric space \( X \) is strongly \( l \)-embedded in \( X \) (he did not state this explicitly in this form). It is known (see [AÇ, Dr1]) that \( A \) is \( l \)-embedded (resp., strongly \( l \)-embedded) in \( X \) if and only if there is a mapping \( r : X \to L_p(A) \) (resp., \( r : X \to P_\infty(A) \)) such that \( r(x) = \delta_x \) for every \( x \in A \). Such a mapping will be called an \( L_p \)-valued (resp., a \( P_\infty \)-valued) retraction. Every \( L_p \)-valued retraction \( r : X \to L_p(A) \) defines a continuous linear extension operator \( u_r : C_p(A) \to C_p(X) \) by setting
ur(f)(x) = r(x)(f). If the operator $u_r$ satisfies the condition (ii), $r$ is said to be a bounded $L_p$-valued retraction.

Let $u: C_p(A) \to C_p(X)$ be a continuous linear extension operator. Then the mapping $v(f, g) = u(f) + g$ is a linear homeomorphism from $C_p(A) \times C_p(X; A)$ onto $C_p(X)$, where

$$C_p(X; A) = \{ g \in C_p(X) : g|A = 0 \}.$$  

Analogously, if $A$ is $l^*$-embedded in $X$ then $C_p(A) \times C_p(X; A)$ is linearly homeomorphic to $C_p^*(X)$.

Let $\mathcal{F}$ be a family of bounded subsets of a space $X$ (i.e. $f|K$ is bounded for every $K \in \mathcal{F}$ and $f \in C_p(X)$) and $E$ be a linear topological subset of $C_p(X)$. Then we set

$$(\prod E)_{\mathcal{F}} = \{ (f_1, \ldots, f_n, \ldots) \in E^\omega : \lim_{n} \|f_n\|_K = 0 \text{ for every } K \in \mathcal{F} \}$$

and

$$(\prod E)^*_\mathcal{F} = \{ (f_1, \ldots, f_n, \ldots) \in (\prod E)_{\mathcal{F}} : \sup_n \|f_n\| < \infty \}.$$ 

$(\prod E)_{\mathcal{F}}$ and $(\prod E)^*_\mathcal{F}$ are considered as topological linear subspaces of $C_p(X)^{\omega}$. We write $(\prod E)_b$ and $(\prod E)_c$ (resp. $(\prod E)_*^c$ and $(\prod E)_*^0$) if $\mathcal{F}$ is the family of all bounded (resp., of all compact) subsets of $X$. In the above notations $\|f\|_K$ stands for the set $\sup\{|f(X)| : x \in K\}$. Let us note that if $X$ is pseudocompact and $E$ is a linear subset of $C_p(X)$, the space

$$(\prod E)^0_0 = \{ (f_1, \ldots, f_n, \ldots) \in E^\omega : \lim_n \|f_n\| = 0 \}$$

is considered in [GH].

We need also the following notion: a space $X$ is said to be a $k_R$-space [N] if every function $f: X \to R$ is continuous provided that $f|K$ is continuous for each compact subset $K$ of $X$.

2. Linear topological classifications of $C_p(X)$

2.1 Lemma. Let $A$ be a strongly $l$-embedded (resp., $l$-embedded or $l^*$-embedded) subset of a space $X$. Then $A \times Y$ is strongly $l$-embedded (resp., $l$-embedded or $l^*$-embedded) in $X \times Y$ for every space $Y$.

Proof. Suppose $A$ is strongly $l$-embedded in $X$. So, there exists a $P_\infty$-valued retraction $r_1: X \to P_\infty(A)$. Define a mapping $r: X \times Y \to P_\infty(A \times Y)$ by setting

$$r(x, y) = \sum_{i=1}^{k} a_i \delta_{(x_i, y)}, \quad \text{where } r_1(x) = \sum_{i=1}^{k} a_i \delta_{x_i}.$$ 

It is easily shown that $r$ is a $P_\infty$-valued retraction. Thus, $A \times Y$ is strongly $l$-embedded in $X \times Y$. One can also prove that $r$ is a (bounded) $L_p$-valued retraction provided $r_1$ is a (bounded) $L_p$-valued retraction. Hence, if $A$ is $l$ (resp., $l^*$)-embedded in $X$ then $A \times Y$ is $l$ (resp., $l^*$)-embedded in $X \times Y$. 


2.2 Lemma. Let $A$ be an $l^*$-embedded subset of a space $X$. Then $(\prod C_p(X))_b$ is linearly homeomorphic to $(\prod C_p(A))_b \times (\prod C_p(X; A))_b$.

Proof. Let $u: C_p(A) \to C_p(X)$ be a continuous linear extension operator such that $\|u(f)\| \leq c \cdot \|f\|$ for every $f \in C_p^*(A)$, where $c > 0$. Since $\|f\| = \infty$ provided $f \in C_p(A) - C_p^*(A)$, the inequality $\|u(f)\| \leq c \cdot \|f\|$ holds for every $f \in C_p(A)$. Then the mapping $r: X \to L_p(A)$, defined by $r(x)(f) = u(f)(x)$, is an $L_p$-valued retraction. Consider the linear homeomorphism $v$ from $C_p(A) \times C_p(X; A)$ onto $C_p(X)$, $v(f, g) = u(f) + g$. Suppose $(f_1, \ldots, f_n, \ldots) \in C_p(A)^\omega$ and $(g_1, \ldots, g_n, \ldots) \in C_p(X; A)^\omega$. Put

$$H(K) = \text{cl}_A \left( \bigcup \{\text{supp}(r(x)) : x \in K\} \right),$$

where $K$ is a subset of $X$. Obviously, $\|u(f_n)\|_K \leq c \cdot \|f_n\|_H(K)$ for every $n \in \mathbb{N}$. By a result of Arhangel'skii [A2], $H(K)$ is a bounded subset of $A$ provided $K$ is a bounded subset of $X$. Hence, $(f_1, \ldots, f_n, \ldots) \in (\prod C_p(A))_b$ if and only if $(u(f_1), \ldots, u(f_n), \ldots)$ belongs to $(\prod C_p(X))_b$. Consequently, $(v(f_1, g_1), \ldots, v(f_n, g_n), \ldots)$ belongs to $(\prod C_p(X))_b$ if $(g_1, \ldots, g_n, \ldots) \in (\prod C_p(X; A))_b$ and $(f_1, \ldots, f_n, \ldots) \in (\prod C_p(A))_b$. Suppose

$$v(f_1, g_1), \ldots, v(f_n, g_n), \ldots) \in \left( \prod C_p(X) \right)_b.$$

Then $(f_1, \ldots, f_n, \ldots) \in (\prod C_p(A))_b$ because $v(f_n, g_n)|A = f_n$ for every $n$. Therefore $(u(f_1), \ldots, u(f_n), \ldots) \in (\prod C_p(X))_b$. So we have $(g_1, \ldots, g_n, \ldots) \in (\prod C_p(X; A))_b$. Thus, $(v(f_1, g_1), \ldots, v(f_n, g_n), \ldots)$ belongs to $(\prod C_p(X))_b$ if $(g_1, \ldots, g_n, \ldots) \in (\prod C_p(X; A))_b$ and $(f_1, \ldots, f_n, \ldots) \in (\prod C_p(A))_b$. Hence, the formula $v_0((f_1, \ldots, f_n, \ldots), (g_1, \ldots, g_n, \ldots)) = (v(f_1, g_1), \ldots, v(f_n, g_n), \ldots)$ defines a linear mapping from $(\prod C_p(A))_b \times (\prod C_p(X; A))_b$ onto $(\prod C_p(X))_b$ which is a homeomorphism.

2.3 Lemma. Let $A$ be an $l^*$-embedded subset of a space $X$. If every closed and bounded subset of $A$ is compact then $(\prod C_p(X \times Y))_c \sim (\prod C_p(A \times Y))_c \times (\prod C_p(X \times A \times Y))_c$ for any space $Y$.

Proof. Let $u_1: C_p(A) \to C_p(X)$ be a continuous linear extension operator such that $\|u_1(f)\| \leq c \cdot \|f\|$ for every $f \in C_p^*(A)$, where $c > 0$, and $r_1: X \to L_p(A)$ be defined by $r_1(x)(f) = u_1(f)(x)$. Obviously, $r_1$ is an $L_p$-valued retraction. For a given space $Y$ the equality $r(x, y) = \sum_{i=1}^k a_i \delta_{(x_i, y_i)}$, where $r_1(x) = \sum_{i=1}^k a_i \delta_{x_i}$, defines an $L_p$-valued retraction from $X \times Y$ into $L_p(A \times Y)$. Next, set $u(f)(x, y) = r(x, y)(f)$ for every $(x, y) \in X \times Y$ and $f \in C_p(A \times Y)$. It is easily shown that $u: C_p(A \times Y) \to C_p(X \times Y)$ is a continuous linear extension operator.
Claim 1. \( \|u(f)\| \leq c \cdot \|f\| \) for every \( f \in C_p^*(A \times Y) \).

Fix a point \((x, y) \in X \times Y\) and an \( f \in C_p(A \times Y)\). It follows from the definition of \( u \) that

\[
 u(f)(x, y) = \sum_{i=1}^{k} a_i f(x_i, y), \quad \text{where } r_1(x) = \sum_{i=1}^{k} a_i \delta_{x_i}. 
\]

So, \( |u(f)(x, y)| \leq \sum_{i=1}^{k} |a_i| \cdot \|f\| \). Take a function \( g \in C_p^*(A) \) with \( \|g\| = 1 \) and \( g(x_i) = \text{sgn}(a_i) \) for each \( i = 1, \ldots, k \). Then

\[
 u_1(g)(x) = r_1(x)(g) = \sum_{i=1}^{k} |a_i|. 
\]

Since \( |u_1(g)| \leq c \cdot \|g\| \), we have \( \sum_{i=1}^{k} |a_i| \leq c \). Hence, \( |z_u(f)(x, y)| \leq c \cdot \|f\| \). Claim 1 is proved.

Claim 2. For every compact subset \( K \) of \( X \times Y \) the set

\[
 H(K) = \text{cl}_{A \times Y} \left( \bigcup \{\text{supp}(r(x, y)): (x, y) \in K\} \right),
\]

is also compact.

Let \( n_X: X \times Y \to X \) and \( n_Y: X \times Y \to Y \) be the natural projections. Then \( n_X(K) \) and \( n_Y(K) \) are compact subsets of \( X \) and \( Y \) respectively. By a result of Arhangel'skii [A2],

\[
 H_1(K) = \text{cl}_{A} \left( \bigcup \{\text{supp}(r_1(x)): x \in n_X(K)\} \right)
\]

is a bounded subset of \( A \). Thus, \( H_1(K) \) is compact. So \( H_1(K) \times n_Y(K) \) is a compact subset of \( A \times Y \). Since \( r(x, y) = (\text{supp}(r_1(x))) \times \{y\} \) for every point \( (x, y) \in X \times Y \), we have \( H(K) \subset H_1(K) \times n_Y(K) \). Hence, \( H(K) \) is compact as a closed subset of \( H_1(K) \times n_Y(K) \). Claim 2 is proved.

Now, the proof of Lemma 2.3 follows from the above two claims and the arguments used in the proof of Lemma 2.2.

2.4 Corollary. Let \( X \) be a product of metric spaces and \( A \) be an \( l^* \)-embedded subset of \( X \). Then \( (\prod C_p(X))_c \sim (\prod C_p(A))_c \times (\prod C_p(X \setminus A))_c \).

Proof. Since \( A \) is closed in \( X \), every closed bounded subset of \( A \) is compact. Thus, the proof follows from Lemma 2.3, where \( Y \) is the one-point space.

2.5 Lemma. Suppose \( X \) is a space such that both \( X \times I \) and \( X \times T \) are \( k_R \)-spaces, where \( T = \{0, 1/n: n \in \mathbb{N}\} \). Then \( C_p(X \times I) \) is linearly homeomorphic to \( (\prod C_p(X \times I))_c \).

Proof. Since, by Lemma 2.1, \( X \times T \) is strongly \( l \)-embedded in \( X \times I \) we have

\[
 C_p(X \times I) \sim C_p(X \times T) \times C_p(X \times I; X \times T).
\]

Let \( I_n = [1/n+1, 1/n] \) and \( E_n = C_p(X \times I_n; X \times \{1/n+1, 1/n\}) \) for every \( n \in \mathbb{N} \). Consider the set

\[
 \left( \prod E_n \right)_c = \left\{ (f_1, \ldots, f_n, \ldots) \in \prod E_n : \lim_n \|f_n\|_{K \times I_n} = 0 \right\}
\]

for every compact subset \( K \) of \( X \).

as a topological linear subset of \( \prod \{E_n : n \in \mathbb{N}\} \). Since \( X \times I \) is a \( k_R \)-space
we have \( C_p(X \times I; X \times T) \sim (\prod E_n)_c \). Identifying each \( E_n \) with the space \( E = C_p(X \times I; X \times \{0, 1\}) \) we get

\[
(2) \quad C_p(X \times I; X \times T) \sim \left( \prod E \right)_c.
\]

Analogously, \( C_p(X \times T) \sim C_p(X \times \{0\}) \times C_p(X \times T; X \times \{0\}) \) and

\[
C_p(X \times T; X \times \{0\}) \sim \left( \prod C_p(X) \right)_c.
\]

Thus,

\[
(3) \quad C_p(X \times T) \sim C_p(X \times \{0\}) \times \left( \prod C_p(X) \right)_c \sim \left( \prod C_p(X) \right)_c.
\]

By Lemma 2.3, the following holds

\[
(4) \quad \left( \prod C_p(X \times I) \right)_c \sim \left( \prod C_p(X \times \{0, 1\}) \right)_c \times \left( \prod E \right)_c.
\]

Obviously,

\[
(5) \quad \left( \prod C_p(X \times \{0, 1\}) \right)_c \sim \left( \prod C_p(X) \right)_c \times \left( \prod C_p(X) \right)_c \sim \left( \prod C_p(X) \right)_c.
\]

So we have

\[
C_p(X \times I) \sim C_p(X \times T) \times C_p(X \times I; X \times T) \quad \text{by (1)}
\]

\[
\sim \left( \prod C_p(X) \right)_c \times \left( \prod E \right)_c \quad \text{by (2) and (3)}
\]

\[
\sim \left( \prod C_p(X \times I) \right)_c \quad \text{by (4) and (5)}.
\]

2.6 Corollary. Let \( X \) be as in Lemma 2.5. Then \( C_p(X \times I) \) is homeomorphic to \( C_p(X \times I)^\omega \).

Proof. S. Gul'ko and T. Hmyleva [GH] proved that \( (\prod C_p(X))_0 \) is homeomorphic to \( C_p(X)^\omega_0 \times (\prod C_p(X))_0 \) for every pseudocompact space \( X \). Using the same arguments one can see that \( (\prod C_p(X))_c \) is homeomorphic to \( C_p(X)^\omega_0 \times (\prod C_p(X))_c \) for each \( X \). Now, the proof of Corollary 2.6 follows from Lemma 2.5.

2.7 Lemma. Suppose a space \( X \) contains an \( l \)-embedded copy \( F_1 \) of a space \( Y \) and \( Y \) contains an \( l^* \)-embedded copy \( F_2 \) of \( X \). Then \( C_p(X) \sim C_p(Y) \) provided one of the following conditions is fulfilled:

(i) \( C_p(Y) \sim (\prod C_p(Y))_b \):

(ii) \( C_p(Y) \sim (\prod C_p(Y))_c \sim (\prod C_p(F_2))_c \times (\prod C_p(Y; F_2))_c \).

Proof. We have \( C_p(X) \sim C_p(F_1) \times E_1 \) and \( C_p(Y) \sim C_p(F_2) \times E_2 \), where \( E_1 = C_p(X; F_1) \) and \( E_2 = C_p(Y; F_2) \). Thus, \( C_p(X) \sim C_p(Y) \times E_1 \). Suppose \( C_p(Y) \sim (\prod C_p(Y))_b \). By Lemma 2.2,

\[
\left( \prod C_p(Y) \right)_b \sim \left( \prod C_p(F_2) \right)_b \times \left( \prod E_2 \right)_b,
\]

so

\[
\left( \prod C_p(Y) \right)_b \sim \left( \prod C_p(X) \right)_b \times \left( \prod E_2 \right)_b.
\]
Therefore,
\[ C_p(Y) \sim \left( \prod C_p(Y) \right)_b \sim C_p(Y) \times \left( \prod C_p(Y) \right)_b. \]

Hence,
\[ C_p(X) \sim E_1 \times C_p(Y) \sim E_1 \times C_p(Y) \times (\prod C_p(X))_b \times (\prod E_2)_b \sim C_p(X) \times (\prod C_p(X))_b \times (\prod E_2)_b \sim (\prod C_p(X))_b \times (\prod E_2)_b \sim C_p(Y). \]

If condition (ii) is fulfilled we use the same arguments.

2.8 Theorem. (i) Let \( X \) be a subspace of \( R^n \). Then \( C_p(X) \sim C_p(I^n) \) if and only if \( X \) is compact and \( \dim X = n \);
(ii) \( C_p(X) \sim C_p(Q) \) if and only if \( X \) is a compact metric space containing a copy of \( Q \).

Proof. We prove only the first part of Theorem 2.8. The proof of (ii) is analogous to that of (i).

Suppose \( C_p(X) \sim C_p(I^n) \). Then by [A2 and A3] \( X \) is a compact metric space. Next, it follows from a result of Pavlovskii [Pv] that there is a nonempty open subset of \( I^n \) which can be embedded in \( X \). Thus, \( \dim X = n \).

Now, let \( X \) be a compact \( n \)-dimensional subset of \( R^n \). Then \( X \) contains a copy of \( I^n \). On the other hand \( X \) can be considered as a subset of \( I^n \). Hence, by Corollary 2.4, \( (\prod C_p(I^n))_c \sim (\prod C_p(X))_c \times (\prod C_p(I^n; X))_c \). Since \( C_p(I^n) \sim (\prod C_p(I^n))_c \) (see Lemma 2.5), we derive from Lemma 2.7(ii) that \( C_p(X) \sim C_p(I^n) \).

2.9 Theorem. Let \( \mu^n \) be the \( n \)-dimensional universal Menger compactum. Then \( C_p(X) \sim C_p(\mu^n) \) if and only if \( X \) is an \( n \)-dimensional compact metric space containing a copy of \( \mu^n \).

Proof. Let \( C_p(X) \sim C_p(\mu^n) \). Then, by results of Arhangel'skii [A2, A3] and Pestov [Ps], \( X \) is an \( n \)-dimensional compact metric space. It follows from [Pv] that there exists an open subset of \( \mu^n \) which can be embedded in \( X \). But each open subset of \( \mu^n \) contains a copy of \( \mu^n \) [Bt]. Thus, \( X \) contains a copy of \( \mu^n \).

Suppose \( X \) is an \( n \)-dimensional compact metric space containing a copy of \( \mu^n \). Since \( X \) can be embedded in \( \mu^n \), by Lemma 2.7(ii) and Corollary 2.4 it is enough to show that \( C_p(\mu^n) \sim (\prod C_p(\mu^n))_c \). For proving this fact we need the following result of Dranishnikov [Dr2]: There is a mapping \( f_n \) from \( \mu^n \) onto \( Q \) such that \( f_n^{-1}(P) \) is homeomorphic to \( \mu^n \) for every \( LC^{n-1} & C^{n-1} \)-compact subspace \( P \) of \( Q \). Now, consider \( Q \) as a product \( Q_1 \times I \), where \( Q_1 \) is a copy of \( Q \). Let \( T = \{0, 1/k ; k \in N \} \) and \( T^* = f_n^{-1}(Q_1 \times T) \). Then
\begin{align*}
C_p(\mu^n) \sim C_p(T^*) \times C_p(\mu^n; T^*)
\end{align*}

and
\begin{align*}
C_p(T^*) \sim C_p(f_n^{-1}(Q_1 \times \{0\})) \times C_p(T^*; f_n^{-1}(Q_1 \times \{0\})).
\end{align*}
Since each of the sets $f_n^{-1}(Q_1 \times \{1/k\}), \ k \in N,$ and $f_n^{-1}(Q_1 \times \{0\})$ is homeomorphic to $\mu^n$, we have

$$C_p(T^*; f_n^{-1}(Q_1 \times \{0\})) \sim \left( \prod C_p(\mu^n) \right)_c$$

and

$$C_p(f_n^{-1}(Q_1 \times \{0\})) \sim C_p(\mu^n).$$

Thus,

$$C_p(T^*) \sim C_p(\mu^n) \times \left( \prod C_p(\mu^n) \right)_c \sim \left( \prod C_p(\mu^n) \right)_c$$

(7)

$$\sim \left( \prod C_p(\mu^n) \right)_c \sim \left( \prod C_p(\mu^n) \right)_c \sim \left( \prod C_p(\mu^n) \right)_c \times C_p(T^*).$$

Finally,

$$C_p(\mu^n) \sim C_p(T^*) \times C_p(\mu^n; T^*) \quad \text{by (6)}$$

$$\sim \left( \prod C_p(\mu^n) \right)_c \times C_p(T^*) \times C_p(\mu^n; T^*) \quad \text{by (7)}$$

$$\sim \left( \prod C_p(\mu^n) \right)_c \times C_p(\mu^n) \sim \left( \prod C_p(\mu^n) \right)_c.$$

2.10 Theorem. Let $X$ be a metric space and $\tau$ be an infinite cardinal. Suppose $Y$ is an $I^*$-embedded subspace of the product $X^\tau$ and $Y$ contains an $I^*$-embedded copy of $X^\tau$. Then $C_p(Y) \sim C_p(X^\tau)$.

Proof. By Corollary 2.4 and Lemma 2.7(ii), it is enough to show that $C_p(X^\tau) \sim \left( \prod C_p(X^\tau) \right)_c$. Since $\tau$ is infinite we have $X^\tau = (X^\omega)^\tau$. So we can suppose that $X$ is not discrete. Thus, there exists a nontrivial converging sequence $\{x_n\}_{n \in N}$ in $X$ with $\lim x_n = x_0$. Let $T = \{x_0, x_n; n \in N\}$. By Lemma 2.1, $X^\tau \times T$ is $l$-embedded in $X^\tau \times X$. Therefore,

$$C_p(X^\tau) \sim C_p(X^\tau \times T) \times C_p(X^\tau \times X; X^\tau \times T).$$

But $C_p(X^\tau \times T) \sim C_p(X^\tau \times \{x_0\}) \times C_p(X^\tau \times T; X^\tau \times \{x_0\})$ because $X^\tau \times \{x_0\}$ is also $l$-embedded in $X^\tau \times T$. Since $X^\tau \times T$ is a $k_R$-space [N] we have $C_p(X^\tau \times T; X^\tau \times \{x_0\}) \sim \left( \prod C_p(X^\tau) \right)_c$. Hence,

$$C_p(X^\tau \times T) \sim C_p(X^\tau \times \{x_0\}) \times \left( \prod C_p(X^\tau) \right)_c \sim \left( \prod C_p(X^\tau) \right)_c$$

$$\sim \left( \prod C_p(X^\tau) \right)_c \times \left( \prod C_p(X^\tau) \right)_c \sim C_p(X^\tau \times T) \times \left( \prod C_p(X^\tau) \right)_c.$$

Then

$$C_p(X^\tau) \sim C_p(X^\tau \times T) \times C_p(X^\tau \times X; X^\tau \times T)$$

$$\sim \left( \prod C_p(X^\tau) \right)_c \times C_p(X^\tau \times T) \times C_p(X^\tau \times X; X^\tau \times T)$$

$$\sim \left( \prod C_p(X^\tau) \right)_c \times C_p(X^\tau) \sim \left( \prod C_p(X^\tau) \right)_c.$$
2.11 Corollary. Let $X$ be a separable metric space and $\tau > \omega$. Then $C_p(X^\tau) \sim C_p(Y)$ for every closed $G_\delta$-subset $Y$ of $X^\tau$.

Proof. Suppose $Y$ is a closed $G_\delta$-subset of $X^\tau$. It is well known (see for example [PP]) that modulo a permutation of the coordinates, $Y = Z \times X^{\tau - \omega}$, where $Z$ is a closed subset of $X^{\omega}$. Thus, by Lemma 2.1, $Y$ is $l^*$-embedded in $X^\tau$. On the other hand $\{z\} \times X^{\tau - \omega}$ is an $l^*$-embedded copy of $X^\tau$ in $Y$ for each $z \in Z$. Now, Theorem 2.10 completes the proof.

2.12 Corollary. Let $U$ be a functionally open subset of $R^\tau$, $\tau \geq \omega$. Then $C_p(U) \sim C_p(R^\tau)$.

Proof. Modulo a permutation of the coordinates, $U = V \times R^{\tau - \omega}$, where $V$ is open in $R^\omega$. Obviously, $U$ contains an $l^*$-embedded copy of $R^\tau$. Since there is an embedding of $V$ in $R^\omega$ as a closed subset, by Lemma 2.1, $U$ can be $l^*$-embedded in $R^\tau$. Thus, by Theorem 2.10, $C_p(U) \sim C_p(R^\tau)$.

Let $f$ be a mapping from a space $X$ onto a space $Y$. Recall that a continuous linear operator $u : C_p(X) \to C_p(Y)$ is said to be an averaging operator for $f$ if $u(h \circ f) = h$ for every $h \in C_p(Y)$. If $f$ admits a regular averaging operator $u : C_p(X) \to C_p(Y)$ we can define a mapping $r : Y \to P_{\omega}(X)$ by the formula $r(y)(g) = u(g)(y)$. The mapping $r$ has the following property [Dr1]: $\text{supp}(r(y))$ is contained in $f^{-1}(y)$ for each $y \in Y$. Conversely, if there is a mapping $r : Y \to P_{\omega}(X)$ such that $\text{supp}(r(y)) \subset f^{-1}(y)$ for every $y \in Y$, then the formula $u(g)(y) = r(y)(g)$ defines a regular averaging operator $u$ for $f$.

It is easily seen that if $u$ is a regular averaging operator for $f$ the mapping $v(g) = (u(g), g - u(g) \circ f)$ is a linear homeomorphism from $C_p(X)$ onto $C_p(Y) \times E$, where $E = \{g - u(g) \circ f; g \in C_p(X)\}$. Dranishnikov proved [Dr1, Theorem 9] that $C_p(R^n) \sim C_p(U)$ for every open subset $U$ of $R^n$. The same arguments are used in the proof of Proposition 2.13 below.

2.13 Proposition. Let $\{U_i; i \in N\}$ be an infinite locally finite functionally open cover of a space $X$. Suppose there is a space $Y$ with $C_p(\text{cl}_X(U_i)) \sim C_p(Y)$ for each $i \in N$. Then $C_p(X) \sim C_p(Y)^\omega$ provided $X$ contains an $l$-embedded copy of a topological sum $\sum_{i=1}^\infty F_i$ such that $C_p(F_i) \sim C_p(Y)$ for every $i \in N$.

Proof. For every $i \in N$ take an $f_i \in C_p(X)$ such that $f_i^{-1}(0) = X - U_i$ and $f_i \geq 0$. Without loss of generality we can suppose that $\sum_{i=1}^\infty f_i = 1$. Let $f \in C_p(\sum \text{cl}_X(U_i))$ such that $f|\text{cl}_X(U_i) = f_i|\text{cl}_X(U_i)$. Consider the natural mapping $p : \sum \text{cl}_X(U_i) \to X$ with all preimages finite. Let $r : X \to P_{\omega}(\sum \text{cl}_X(U_i))$ be defined by $r(x) = \sum \{f(y) \cdot \delta_y; y \in p^{-1}(x)\}$. It is easily seen that $r$ is continuous and $\text{supp}(r(x)) \subset p^{-1}(x)$ for every $x \in X$. Thus, there is a regular averaging operator $u : C_p(\sum \text{cl}_X(U_i)) \to C_p(X)$ for $p$. Hence, $C_p(\sum \text{cl}_X(U_i))$ is linearly homeomorphic to $C_p(X) \times E$, where $E$ is a linear subspace of $C_p(\sum \text{cl}_X(U_i))$. Since $\sum F_i$ is $l$-embedded in $X$ we have $C_p(X) \sim C_p(\sum F_i) \times C_p(X; \sum F_i)$. Observe that

$$C_p \left( \sum \text{cl}_X(U_i) \right) \sim \prod_{i=1}^\infty C_p(\text{cl}_X(U_i)) \sim C_p(Y)^\omega \sim C_p \left( \sum F_i \right).$$
Now, using the technique of Pelczynski [P] and Bessaga [B] we have
\[ C_p(X) \sim C_p \left( \sum F_i \right) \times C_p \left( X; \sum F_i \right) \sim C_p(Y)^\omega \times C_p \left( X; \sum F_i \right) \sim (C_p(Y)^\omega \times \cdots \times C_p(Y)^\omega \times \cdots ) \times C_p \left( X; \sum F_i \right) \sim (C_p(Y)^\omega \times \cdots \times C_p(Y)^\omega \times \cdots ) \times C_p(X) \sim (C_p(Y) \times E \times \cdots \times C_p(Y) \times E \times \cdots ) \times C_p(X) \sim C_p(Y)^\omega \times E^\omega \sim (C_p(X) \times E)^\omega \sim C_p \left( \sum \text{cl}_X(U_i) \right)^\omega \sim C_p(Y)^\omega. \]

2.14 Theorem. Let \( Y \) be a noncompact separable metric space and \( X \) be one of the spaces \( Q, I^n, \mu^n, l_2 \). Then \( C_p(Y) \sim C_p(X)^\omega \) provided \( Y \) is an \( X \)-manifold.

Proof. Let \( \{ U_i : i \in \mathbb{N} \} \) be an infinite locally finite open cover of \( Y \) such that each \( \text{cl}_Y(U_i) \) is regularly closed subset of \( X \). It is clear that a topological sum \( \sum F_i \) of infinitely many regularly closed subsets \( F_i \) of \( X \) is contained in \( Y \) as a closed subset. Since each of the sets \( \text{cl}_Y(U_i) \) and \( F_i, i \in \mathbb{N} \), contains a closed copy of \( X \), it follows from Theorem 2.8, Theorem 2.9 and Theorem 2.10 that \( C_p(\text{cl}_Y(U_i)) \sim C_p(F_i) \sim C_p(X) \) for every \( i \in \mathbb{N} \). Hence, by Proposition 2.13, \( C_p(Y) \sim C_p(X)^\omega \).

2.15 Theorem. Let \( U \) be a functionally open subset of \( I^\tau \) and \( \tau \) be an uncountable cardinal. Then \( C_p(U) \sim C_p(I^\tau)^\omega \).

Proof. There exists a projection \( p \) from \( I^\tau \) onto a countable face of \( I^\tau \) such that \( p^{-1}(p(U)) = U \) (see [PP]). Take a locally finite open cover \( \{ U_i : i \in \mathbb{N} \} \) of \( p(U) \) such that \( \text{cl}_r(p^{-1}(U_i)) \subset U \) for every \( i \in \mathbb{N} \). Since each \( \text{cl}_r(p^{-1}(U_i)) \) is a closed \( G_\delta \)-subset of \( I^\tau \), by Corollary 2.11, \( C_p(\text{cl}_r(p^{-1}(U_i))) \sim C_p(I^\tau) \).

Now, let \( \{ x_i : i \in \mathbb{N} \} \) be a closed discrete infinite subset of \( p(U) \). So, the topological sum \( \sum p^{-1}(x_i) \) is \( l \)-embedded in \( U \) (by Lemma 2.1) and obviously, each \( p^{-1}(x_i) \) is homeomorphic to \( I^\tau \). Thus, by Proposition 2.13, \( C_p(U) \sim C_p(I^\tau)^\omega \).

2.16 Theorem. Let \( X \) be one of the spaces \( Q, I^n, \mu^n, l_2 \), and \( Y \) be a locally compact subset of an \( X \)-manifold. Then \( C_p(Y) \sim C_p(X)^\omega \) if and only if \( Y \) contains a closed copy of the topological sum \( \sum X \) of infinitely many copies of \( X \).

Proof. The proof of the part "if" is based on a Dranishnikov's idea from [Drl, Theorem 9'], where it is shown that \( C_p(P) \sim C_p(R^n) \) for every locally compact subset \( P \) of \( R^n \) with \( \text{cl}_{R^n}(\text{Int}(P)) \cap (R^n - P) \neq \emptyset \).

Suppose \( Y \) is a locally compact subspace of an \( X \)-manifold \( Z \) and contains a closed copy of the topological sum \( \sum X \). Then \( C_p(Y) \sim C_p(\sum X) \times C_p(Y; \sum X) \). Next, take a locally finite open cover \( \{ V_i : i \in \mathbb{N} \} \) of \( Y \) such that each \( \text{cl}_Y(V_i) \) is compact. For every \( i \in \mathbb{N} \) there exists an open subset \( U_i \).
of $\mathbb{Z}$ such that $V_i = U_i \cap Y = U_i \cap \text{cl}_Y(V_i)$. Since every set $V_i$ is closed in $U_i$, $\sum V_i$ is closed in $\sum U_i$. Thus, $C_p(\sum U_i) \sim C_p(\sum V_i) \times C_p(\sum U_i; \sum V_i)$.

Let $\{f_i : i \in \mathbb{N}\}$ be a partition of unity subordinated to the cover $\{V_i : i \in \mathbb{N}\}$. Define a continuous mapping $r: Y \to P_\infty(\sum V_i)$ as in the proof of Proposition 2.13 and by the same arguments we get that $C_p(\sum V_i)$ is linearly homeomorphic to $C_p(Y) \times E$, where $E$ is a linear subspace of $C_p(\sum V_i)$. It follows from Theorem 2.14 that $C_p(\sum V_i) \sim C_p(X_i)$ for every $i \in \mathbb{N}$. Hence

$$C_p(X)^\omega \sim C_p\left(\sum U_i\right) \sim C_p\left(\sum V_i\right) \times C_p\left(\sum U_i; \sum V_i\right)$$

$$\sim C_p(Y) \times E \times C_p\left(\sum U_i; \sum V_i\right).$$

Now, using the scheme of Pelczynski and Bessaga we get $C_p(Y) \sim C_p(X)^\omega$.

Suppose there is a linear homeomorphism $\theta$ from $C_p(\sum X) = C_p(X)^\omega$ onto $C_p(Y)$. Let $K$ be the set $\{y \in Y; \text{ every neighborhood of } y \text{ in } Y \text{ contains a copy of } X\}$. We use the following property of $X$ (for $Q$ and $I^n$ this is obvious, and for $\mu^n$ see [Bt]):

(*) Every open subset of $X$ contains a copy of $X$.

Now we show that $K$ is nonempty. Indeed, by [Pv], $Y$ contains an open subset of $\sum X$. So, by (*), $Y$ contains a copy $F$ of $X$ and $F \subset K$. Obviously $K$ is closed in $Y$ and it follows also from (*) that $Y - K$ does not contain a copy of $X$. Next, assume $K$ is compact. Consider the set

$$L = \text{cl}\left(\bigcup\{\text{supp}(\theta^*(\delta_y)): y \in K\}\right),$$

where $\theta^*: L_p(Y) \to L_p(\sum X)$ is the dual homeomorphism of $\theta$. By a result of Arhangel’skii [A2], $L$ is a compact subset of $\sum X$. Therefore, there is a $k \in \mathbb{N}$ such that $L \subset \sum_{i=1}^k X_i$. Let $P = \sum_{i=1}^k X_i$, $f \in C_p(\sum X; P)$ and $y \in K$. We have $\theta^*(\delta_y)(f) = \delta_y(\theta(f)) = \theta(f)(y)$. But $\theta^*(\delta_y)(f) = 0$ because $\text{supp}(\theta^*(\delta_y)) \subset P$. Thus, $\theta(f)$ belongs to $C_p(Y; K)$ for every $f \in C_p(\sum X; P)$. Let $p$ be the linear projection from $C_p(\sum X) = C_p(P) \times C_p(\sum X; P)$ onto $C_p(\sum X; P)$. Then $\theta \circ p \circ \theta^{-1}: C_p(Y; K) \to \theta(C_p(\sum X; P))$ is a continuous linear retraction. This means that there is a closed linear subspace $E$ of $C_p(Y; K)$ such that $C_p(Y; K)$ is linearly homeomorphic to $C_p(\sum X; P) \times E$. Clearly, $C_p(Y; K) \sim C_p(Y/K; (K))$, where $(K)$ is the identification point of $K$ in the quotient space $Y/K$. Analogously, $C_p(\sum X; P) \sim C_p((\sum X)/P; (P))$. Since $C_p(Y/K) \sim R \times C_p(Y/K; (K))$ and

$$C_p\left(\left(\sum X\right)/P; (P)\right) \times R \sim C_p\left(\left(\sum X\right)/P\right),$$

we get that $C_p(Y/K) \sim C_p(\left(\sum X\right)/P) \times E$. Now, we need the following result of Dranishnikov [Dr1, Theorem 6]: Let $X_1$ and $X_2$ be compact metric spaces and $C_p(X_i)$ be linearly homeomorphic to a product $C_p(X_i) \times E_i$. Then $\dim X_2 \leq \dim X_1$. Actually, it is proved that $X_2$ is a union of countably many compact subsets which are embeddable in $X_1$. It follows from Dranishnikov’s arguments that the last statement remains valid if $X_1$ and $X_2$ are separable locally compact
metric spaces. Hence, there is a countable family \( \{F_i : i \in \mathbb{N}\} \) of compact subsets of \((\sum X)/P\) such that \((\sum X)/P = \bigcup\{F_i : i \in \mathbb{N}\}\) and each \(F_i\) can be embedded in \(Y/K\). Since \((\sum X)/P\) has the Baire property, there exists an \(i_0 \in \mathbb{N}\) with \(\text{Int}(F_{i_0}) \neq \emptyset\). Then the set \(\text{Int}(F_{i_0}) - \{(P)\}\) is both open in \(\sum X\) and embeddable in \(Y/K\). Thus, by (+), \(Y/K\) contains a copy of \(X\). So \(Y - K\) contains also a copy of \(X\). But we have already seen that this is not possible. Therefore \(K\) is not compact.

Take a countable infinite discrete family \(\{W_i : i \in \mathbb{N}\}\) in \(K\) consisting of open subsets of \(K\). Let \(W_i^*\) be an open subspace of \(Y\) with \(W_i^* \cap K = W_i\) for each \(i \in \mathbb{N}\). For every \(i \in \mathbb{N}\) there is a copy \(X_i\) of \(X\) such that \(X_i \subset W_i^*\). It follows from (+) that \(X_i \subset K\) because \(Y - K\) does not contain a copy of \(X\). Hence, \(X_i \subset W_i^*\) for every \(i \in \mathbb{N}\). So \(\{X_i : i \in \mathbb{N}\}\) is a discrete family in \(K\). Thus, \(\sum X_i\) is a closed subset of \(Y\).

2.17 Corollary. Let \(X\) be a locally compact (\(n\)-dimensional) separable metric space. Then \(C_p(X) \sim C_p(\mathbb{Q})^0\) (resp., \(C_p(X) \sim C_p(\mu^n)^0\)) if and only if \(X\) contains a closed copy of the topological sum \(\sum Q\) (resp., \(\sum \mu^n\)).

Proof. Since \(X\) can be embedded in \(Q\) (resp., in \(\mu^n\)), the proof follows from Theorem 2.16.

3. Linear topological classifications of \(C_p^*(X)\)

The proofs of the Lemmas 3.1-3.4 below are similar to the proofs of the corresponding lemmas from §2.

3.1 Lemma. Let \(A\) be an \(l^*\)-embedded subset of a space \(X\). Then \((\prod C_p^*(X))^*_b \sim (\prod C_p^*(A))^*_b \times (\prod C_p^*(X; A))^*_b\).

3.2 Lemma. Let \(A\) be an \(l^*\)-embedded subset of a space \(X\). If every closed bounded subset of \(A\) is compact then \((\prod C_p^*(X \times Y))^*_c \sim (\prod C_p^*(A \times Y))^*_c \times (\prod C_p^*(X \times Y))^*_c\) for any space \(Y\).

3.3 Corollary. Let \(A\) be an \(l^*\)-embedded subset of a product \(X\) of metric spaces. Then \((\prod C_p^*(X))^*_c \sim (\prod C_p^*(A))^*_c \times (\prod C_p^*(X; A))^*_c\).

3.4 Lemma. Suppose \(X\) is a space such that both \(X \times T\) and \(X \times I\) are \(k\)-spaces, where \(T = \{0, 1/n : n \in \mathbb{N}\}\). Then we have \(C_p^*(X \times I) \sim (\prod C_p^*(X \times I))^*_c\).

3.5 Corollary. Let \(X = \sum I^\tau\) be a topological sum of infinitely many copies of \(I^\tau\), \(\tau \geq 1\). Then \(C_p^*(X) \sim (\prod C_p^*(X))^*_c\).

3.6 Lemma. Suppose a space \(X\) contains an \(l^*\)-embedded copy \(F_1\) of a space \(Y\) and \(Y\) contains an \(l^*\)-embedded copy \(F_2\) of \(X\). Then:

\(\text{(i) } C_p^*(X) \sim (\prod C_p^*(X))^*_c \sim C_p^*(Y)\) if \(C_p^*(Y) \sim (\prod C_p^*(Y))^*_c\); \(\text{(ii) } C_p^*(X) \sim (\prod C_p^*(X))^*_c \sim C_p^*(Y)\) if \(C_p^*(Y) \sim (\prod C_p^*(Y))^*_c \sim (\prod C_p^*(F_2))^*_c\) \times (\prod C_p^*(Y \times F_2))^*_c\).

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Proof. Let \( C_p^*(Y) \sim (\prod C_p^*(Y))_c \). Using the same arguments as in the proof of Lemma 2.7(i), one can show that \( C_p^*(X) \sim C_p^*(Y) \). Next, by Lemma 3.1, we have

\[
\left( \prod C_p^*(X) \right)_b \sim \left( \prod C_p^*(F_1) \right)_b \times \left( \prod C_p^*(X; F_1) \right)_b
\]

and

\[
\left( \prod C_p^*(Y) \right)_b \sim \left( \prod C_p^*(F_2) \right)_b \times \left( \prod C_p^*(Y; F_2) \right)_b.
\]

Thus,

\[
\left( \prod C_p^*(X) \right)_b \sim \left( \prod C_p^*(F_1) \right)_b \times \left( \prod C_p^*(X; F_1) \right)_b
\]

Using the same arguments as in the proof of Theorem 2.10.

3.7 Corollary. Let \( \{X_i : i \in N\} \) be an infinite family of spaces such that each \( X_i \) is strongly \( l \)-embedded in a space \( Y \) and contains a strongly \( l \)-embedded copy \( Y_i \) of \( Y \). Then \( C_p^*(\sum Y_i) \sim (\prod C_p^*(\sum X_i))_c \sim C_p^*(\sum X_i) \) if \( C_p^*(\sum Y_i) \sim (\prod C_p^*(\sum Y_i))_c \).

Proof. Let for each \( i \) \( u_i : C_p(X_i) \rightarrow C_p(Y) \) be a regular extension operator. Then the mapping \( u : C_p(\sum X_i) \rightarrow C_p(\sum Y_i) \), defined by \( u(f) = \sum u_i(f|X_i) \), is also a regular extension operator. Thus, \( \sum X_i \) is \( l^* \)-embedded in \( \sum Y_i \). Analogously, \( \sum Y_i \) is \( l^* \)-embedded in \( \sum X_i \). Now the proof follows from Lemma 3.6(i).

3.8 Theorem. Let \( X \) be a metric space and \( \tau \) be an infinite cardinal. Suppose \( Y \) is an \( l^* \)-embedded subspace of the product \( X^\tau \) and \( Y \) contains an \( l^* \)-embedded copy \( X^\tau \) of \( X^\tau \). Then \( C_p^*(Y) \sim C_p^*(X^\tau) \sim (\prod C_p^*(X^\tau))_c \).

Proof. By Corollary 3.3 and Lemma 3.6(ii), it is enough to show that \( C_p^*(X^\tau) \sim (\prod C_p^*(X^\tau))_c \). The last can be proved using the same arguments as in the proof of Theorem 2.10.

3.9 Corollary. Let \( X \) be a separable metric space and \( \tau > \omega \). Then \( C_p^*(X^\tau) \sim C_p^*(Y) \) for every closed \( G_\delta \)-subset \( Y \) of \( X^\tau \).
3.10 **Corollary.** Let $U$ be a functionally open subset of $R^\tau$, $\tau \geq \omega$. Then $C_p^*(R^\tau) \sim C_p^*(U)$.

The proofs of Corollaries 3.9 and 3.10 are similar respectively to the proofs of Corollaries 2.11 and 2.12.

3.11 **Proposition.** Let $\sum \mu^n_i$ be a topological sum of infinitely many copies of the $n$-dimensional Menger compactum. Then $C_p^*(\sum \mu^n_i) \sim (\prod C_p^*(\sum \mu^n_i))_c$.

**Proof.** For each $i \in N$ take a mapping $f_n^i$ from $\mu^n_i$ onto a copy $Q_i$ of the Hilbert cube $Q$ such that $(f_n^i)^{-1}(P)$ is homeomorphic to $\mu^n$ for every $LC^{n-1} \& C^{n-1}$-compact subspace $P$ of $Q_i$ (see [Dr2]). Define $f_n: \sum \mu^n_i \rightarrow \sum Q_i$ by $f_n|\mu^n_i = f_n^i$. Consider $Q_i$ as a product $Q_i^1 \times I$, where $Q_i^1$ is a copy of $Q$.

Let $T_i = Q_i^1 \times \{0, 1/k: k \in N\}$ and $T = f_n^{-1}(\sum T_i)$. Then we have

$$C_p^*\left(\sum \mu^n_i\right) \sim C_p^*(T) \times C_p^*\left(\sum \mu^n_i ; T\right)$$

and

$$C_p^*(T) \sim C_p^*\left(f_n^{-1}\left(\sum (Q_i^1 \times \{0\})\right)\right) \times C_p^*\left(T ; f_n^{-1}\left(\sum (Q_i^1 \times \{0\})\right)\right).$$

Since each of the sets $f_n^{-1}(\sum (Q_i^1 \times \{0\}))$ and $f_n^{-1}(\sum (Q_i^1 \times \{1/k\}))$ for $k \in N$ is homeomorphic to $\sum \mu^n_i$, the following holds

$$C_p^*\left(f_n^{-1}\left(\sum (Q_i^1 \times \{0\})\right)\right) \sim C_p^*\left(\sum \mu^n_i\right)$$

and

$$C_p^*\left(T ; f_n^{-1}\left(\sum (Q_i^1 \times \{0\})\right)\right) \sim \left(\prod C_p^*\left(\sum \mu^n_i\right)\right).$$

Thus,

$$C_p^*(T) \sim C_p^*\left(\sum \mu^n_i\right) \times \left(\prod C_p^*\left(\sum \mu^n_i\right)\right) \sim \left(\prod C_p^*\left(\sum \mu^n_i\right)\right) \sim \left(\prod C_p^*\left(\sum \mu^n_i\right)\right).$$

Finally we get

$$C_p^*\left(\sum \mu^n_i\right) \sim C_p^*(T) \times C_p^*\left(\sum \mu^n_i ; T\right) \sim \left(\prod C_p^*\left(\sum \mu^n_i\right)\right) \times C_p^*(T) \times C_p^*\left(\sum \mu^n_i ; T\right) \sim \left(\prod C_p^*\left(\sum \mu^n_i\right)\right) \times C_p^*\left(\sum \mu^n_i\right) \sim \left(\prod C_p^*\left(\sum \mu^n_i\right)\right).$$

3.12 **Lemma.** Suppose $p$ is a mapping from a space $X$ onto a space $Y$ such that for every compact subset $K$ of $Y$ the preimage $p^{-1}(K)$ is also compact.
Let \( p \) admit a regular averaging operator \( u: C_p(X) \to C_p(Y) \). Then \( C_p^*(X) \sim C_p^*(Y) \times (\prod E_i)_c^* \), where \( E_i = \{ g - u(g) \circ p : g \in C_p^*(X) \} \).

Proof. Consider the mapping \( r: Y \to \mathcal{P}_\infty(X) \) defined by \( r(y)(g) = u(g)(y) \) for all \( g \in C_p(X) \). We have \( \text{supp}(r(y)) \subset p^{-1}(y) \) for each \( y \in Y \). The last implies that \( \|u(g)\|_K \leq \|g\|_{p^{-1}(K)} \) for every \( g \in C_p^*(X) \) and \( K \subset Y \). Hence, \( u(C_p^*(X)) = C_p^*(Y) \) and the mapping \( v(g) = (u(g), g - u(g) \circ p) \) is a linear homeomorphism from \( C_p^*(X) \) onto \( C_p^*(Y) \times E_1 \). Next, let \( (g_1, \ldots, g_n, \ldots) \in \left( \prod C_p^*(X) \right)_c^* \) and \( K \) be a compact subset of \( Y \). Since, \( \|u(g_n)\|_K \leq \|g_n\|_{p^{-1}(K)} \) and \( p^{-1}(K) \) is compact, we have \( (u(g_1), \ldots, u(g_n), \ldots) \in \left( \prod C_p^*(Y) \right)_c^* \) and \( (g_1 - u(g_1) \circ p, \ldots, g_n - u(g_n) \circ p, \ldots) \in \left( \prod E_i \right)_c^* \). Obviously, \( (g_1, \ldots, g_n, \ldots) \in \left( \prod C_p^*(X) \right)_c^* \) if \( (u(g_1), \ldots, u(g_n), \ldots) \in \left( \prod C_p^*(Y) \right)_c^* \) and \( (g_1 - u(g_1) \circ p, \ldots, g_n - u(g_n) \circ p, \ldots) \in \left( \prod E_i \right)_c^* \). Thus, the mapping
\[
v_0(g_1, \ldots, g_n, \ldots) = ((u(g_1), \ldots, u(g_n), \ldots), (g_1 - u(g_1) \circ p, \ldots, g_n - u(g_n) \circ p, \ldots))
\]
is a linear homeomorphism from \( \left( \prod C_p^*(X) \right)_c^* \) onto \( \left( \prod C_p^*(Y) \right)_c^* \times \left( \prod E_i \right)_c^* \).

3.13 Proposition. Let \( \{U_i : i \in N\} \) be an infinite locally finite functionally open cover of a space \( X \). Suppose there is a space \( Y \) such that \( C_p^*(Y) \sim C_p^*(\sum \text{cl}_X(U_i)) \sim \left( \prod C_p^*(\sum \text{cl}_X(U_i)) \right)_c^* \). Then \( C_p^*(X) \sim C_p^*(Y) \) if \( X \) contains an \( l^* \)-embedded copy of \( Y \).

Proof. There exists a natural mapping \( p \) from \( \sum \text{cl}_X(U_i) \) onto \( X \) such that \( p^{-1}(K) \) is compact for every compact subset \( K \) of \( X \). As in the proof of Proposition 2.13 we conclude that \( p \) admits a regular averaging operator
\[
u: C_p^*(\sum \text{cl}_X(U_i)) \to C_p^*(X).
\]
By Lemma 3.12, \( \left( \prod C_p^*(\sum \text{cl}_X(U_i)) \right)_c^* \sim \left( \prod C_p^*(X) \right)_c^* \times \left( \prod E_i \right)_c^* \), where \( E_i = \{ g - u(g) \circ p : g \in C_p^*(\sum \text{cl}_X(U_i)) \} \). Since \( Y \) is \( l^* \)-embedded in \( X \), \( C_p^*(X) \sim C_p^*(Y) \sim C_p^*(Y) \times C_p^*(X; Y) \). Then we have
\[
C_p^*(X) \sim C_p^*(Y) \times C_p^*(X; Y) \sim \left( \prod C_p^*(\sum \text{cl}_X(U_i)) \right)_c^* \times C_p^*(X; Y)
\]
\[
\sim \left( \prod C_p^*(\sum \text{cl}_X(U_i)) \right)_c^* \times C_p^*(X) \sim \left( \prod C_p^*(\sum \text{cl}_X(U_i)) \right)_c^* \times C_p^*(X)
\]
\[
\sim \left( \prod C_p^*(X) \right)_c^* \times \left( \prod E_i \right)_c^* \times C_p^*(X) \sim \left( \prod C_p^*(X) \right)_c^* \times \left( \prod E_i \right)_c^*
\]
\[
\sim \left( \prod C_p^*(\sum \text{cl}_X(U_i)) \right)_c^* \sim C_p^*(Y).
\]
3.14 Theorem. Suppose $X$ is a noncompact $Y$-manifold, where $Y$ is one of the spaces $Q$, $I^n$, $\mu^n$, $I_2$. Then $C^*_p(X) \sim C^*_p(\Sigma Y)$.

Proof. Let $\{U_i ; i \in N\}$ be an infinite locally finite open cover of $X$ such that each $cl_X(U_i)$ is regularly closed subset of $Y$. By Corollary 3.5, Proposition 3.11 and Theorem 3.8 we have $C^*_p(\Sigma Y) \sim (\prod C^*_p(\Sigma Y))_c^*$. Since each set $cl_X(U_i)$ is closed in $Y$ and contains a closed copy of $Y$, it follows from Corollary 3.7 that $(\prod C^*_p(\Sigma cl_X(U_i)))_c^* \sim C^*_p(\Sigma Y)$. Obviously $X$ contains a closed copy of $\Sigma Y$. Thus, by Proposition 3.13, $C^*_p(X) \sim C^*_p(\Sigma Y)$.

3.15 Theorem. Let $U$ be a functionally open subset of $I^I$ and $\tau$ be an uncountable cardinal. Then $C^*_p(U) \sim C^*_p(\Sigma I^I)$.

Proof. Take a projection $p$ from $I^I$ onto a countable face $I^\omega$ of $I^I$ such that $p^{-1}(p(U)) = U$ (for the existence of a such projection see [PP]). Now, let $\{U_i ; i \in N\}$ be a locally finite open cover of $p(U)$ such that $cl_{I^\omega}(U_i) \subset p(U)$ for each $i \in N$. Then $\{p^{-1}(U_i) ; i \in N\}$ is an infinite locally finite functionally open cover of $U$ with $cl_{I^\omega}(p^{-1}(U_i)) \subset U$ for every $i \in N$. Since $p$ is an open mapping we have $cl_{I^\omega}(p^{-1}(U_i)) = p^{-1}(cl_{I^\omega}(U_i))$. Thus, by Lemma 2.1, each set $cl_{I^\omega}(p^{-1}(U_i))$ is strongly $l$-embedded in $I^I$ and contains a strongly $l$-embedded copy of $I^I$. Hence, it follows from Corollary 3.5 and Corollary 3.7 that $C^*_p(\Sigma cl_{I^\omega}(p^{-1}(U_i))) \sim C^*_p(\Sigma I^I)$. On the other hand $U$ contains an $l^*$-embedded copy of $\Sigma I^I$ (see the proof of Theorem 2.15). Therefore, by Proposition 3.13, $C^*_p(U) \sim C^*_p(\Sigma I^I)$.

3.16 Theorem. Let $Y$ be one of the spaces $Q$, $I^n$, $\mu^n$ and $X$ be a locally compact subset of a $Y$-manifold. Then $C^*_p(X) \sim C^*_p(\Sigma Y)$ if $X$ contains a closed copy of $\Sigma Y$.

Proof. Let $X$ be a locally compact subspace of a $Y$-manifold $Z$ and let $X$ contain a closed copy of $\Sigma Y$. Then $C^*_p(X) \sim C^*_p(\Sigma Y) \times C^*_p(X ; \Sigma Y)$. Take an infinite locally finite open cover $\{V_i ; i \in N\}$ of $X$ such that each set $cl_X(V_i)$ is compact and $cl_X(V_i) \subset U_i$, where $U_i$ is an open subset of $Y$. Thus, each $cl_X(V_i)$ is contained in a copy $Y_i$ of $Y$. Let $u: C_p(\Sigma cl_X(V_i)) \to C_p(X)$ be a regular averaging operator for the natural mapping $p: \Sigma cl_X(V_i) \to X$. As in the proof of Proposition 3.13, we get $(\prod C^*_p(\Sigma cl_X(V_i)))_c^* \sim (\prod C^*_p(X))_c^* \times (\prod E)_c^*$, where $E$ is a linear subspace of $C^*_p(\Sigma cl_X(V_i))$. Since $\Sigma cl_X(V_i)$ is a closed subset of $\Sigma Y_i$, by Corollary 3.3 we have $(\prod C^*_p(\Sigma Y_i))_c^* \sim (\prod C^*_p(\Sigma cl_X(V_i)))_c^* \times (\prod G)_c^*$, where $G = C^*_p(\Sigma Y_i ; \Sigma cl_X(V_i))$. Thus,

$$\left(\prod C^*_p(\Sigma Y_i)\right)_c^* \sim \left(\prod C^*_p(X)\right)_c^* \times \left(\prod E\right)_c^* \times \left(\prod G\right)_c^*.$$

Then

$$C^*_p(X) \sim C^*_p(\Sigma Y) \times C^*_p(X ; \Sigma Y) \sim \left(\prod C^*_p(\Sigma Y)\right)_c^* \times C^*_p(X ; \Sigma Y).$$
because $C_p^*(\sum Y) \sim (\prod C_p^*(\sum Y))^*_{c}$ (see Corollary 3.5 and Proposition 3.11). Hence

\[
C_p^*(X) \sim \left(\prod C_p^*(\sum Y)\right)^*_{c} \times C_p^*(X; \sum Y) \\
\sim \left(\prod C_p^*(\sum Y)\right)^*_{c} \times C_p^*(\sum Y) \times C_p^*(X; \sum Y) \\
\sim \left(\prod C_p^*(\sum Y)\right)^*_{c} \times C_p^*(X) \\
\sim C_p^*(X) \times \left(\prod C_p^*(X)\right)^*_{c} \times \left(\prod E\right)^*_{c} \times \left(\prod G\right)^*_{c} \\
\sim \left(\prod C_p^*(\sum Y)\right)^*_{c} \sim C_p^*(\sum Y).
\]

**Added in proof.** After this paper was submitted for publication Arhangel'skii [A4] introduced the notion of an $S$-stable space. A space $X$ is $S$-stable if $C_p(X) \sim C_p(X \times S)$, where $S = \{0, 1/n, n \in \mathbb{N}\}$. Obviously, if $X \times S$ is a $k_r$-space, then $X$ is $S$-stable iff $C_p^*(X)^*_{c} \sim C_p^*(X)$. An elementary proof of the $S$-stability of $\mu^n$ (without using Dranishnikov's results, see the proof of this fact in our Theorem 2.9) is given in [A4]. Arhangel'skii [A4] generalized our Theorem 2.8(ii) by proving that if a compact metric space $X$ contains a subspace $Y$ with $C_p(Y) \sim C_p(Q)$ then $C_p(X) \sim C_p(Q)$.

**References**


[KO] A. Koyama and T. Okada, On compacta which are $l$-equivalent to $I^n$, Tsukuba J. Math. 11, 1 (1987), 147–156.


Department of Mathematics, Sofia State University, 1126 Sofia, A. Ivanov 5, Bulgaria