SHARP SQUARE-FUNCTION INEQUALITIES
FOR CONDITIONALLY SYMMETRIC MARTINGALES

GANG WANG

Abstract. Let $f$ be a conditionally symmetric martingale taking values in a
Hilbert space $H$ and let $S(f)$ be its square function. If $\nu_p$ is the smallest
positive zero of the confluent hypergeometric function and $\mu_p$ is the largest
positive zero of the parabolic cylinder function of parameter $p$, then the fol-
lowing inequalities are sharp:

$$
\|f\|_p \leq \nu_p \|S(f)\|_p \quad \text{if } 0 < p < 2,
$$

$$
\|f\|_p \leq \mu_p \|S(f)\|_p \quad \text{if } p \geq 3,
$$

$$
\nu_p \|S(f)\|_p \leq \|f\|_p \quad \text{if } p \geq 2.
$$

Moreover, the constants $\nu_p$ and $\mu_p$ for the cases mentioned above are also
best possible for the Marcinkiewicz-Paley inequalities for Haar functions.

1. Introduction

Let $W_t$, $0 \leq t < \infty$, be standard Brownian motion. It is known that there
exist positive constants $A_p$ and $a_p$ such that for any stopping time $T$ of $W_t$,

$$
\|W_T\|_p \leq A_p \|T^{1/2}\|_p, \quad \text{if } 0 < p < \infty,
$$

and

$$
a_p \|T^{1/2}\|_p \leq \|W_T\|_p, \quad \text{if } 1 < p < \infty \quad \text{and} \quad \|T^{1/2}\|_p < \infty.
$$

For the exponents $p > 1$, these follow from the inequalities of Burkholder in
[3]; see, for example, Millar [11]. Burkholder and Gundy in [6] extended (1.1)
to the exponents $0 < p \leq 1$. See the work of Novikov [13] for a different
method and [4] for more information and related results.

Davis in [7] obtained the best possible values for the constants $a_p$ and $A_p$.
For $p = 2n$, $n$ a positive integer, they are respectively $\nu_p$ and $\mu_p$, where $\nu_p$
and $\mu_p$ are the smallest and largest positive zeros of the Hermite polynomial
of order $2n$. When $p = 4$, this had been proven by Novikov in [12], and it is well known that the best values for $a_2$ and $A_2$ are 1. For more general $p$, things are little more complicated. Let $\nu_p$ be the smallest positive zero of $M_p$, the confluent hypergeometric function, and $\mu_p$ be the largest positive zero of $D_p$, the parabolic cylinder function of parameter $p$. We will define $M_p$ and $D_p$ in more detail later in $\S 2$. When $p = 2n$, both $M_p$ and $D_p$ become the Hermite polynomial of order $2n$. Then the best possible constants for $A_p$ are $\nu_p$ when $0 < p \leq 2$ and $\mu_p$ for $2 \leq p < \infty$. On the other hand, the best possible constants for $a_p$ are $\mu_p$ when $1 < p \leq 2$ and $\nu_p$ when $2 \leq p < \infty$.

Brownian motion is a continuous time martingale. In the analogues of (1.1) and (1.2) for discrete time martingales, less is known about the best possible values for the constants $a_p$ and $A_p$. Recall that $f = (f_1, f_2, \ldots, f_n, \ldots)$, a sequence of real integrable functions on a probability space $(\Omega, \mathcal{F}, P)$, is a real martingale if $d_{n+1}$ is orthogonal to $\varphi(dx, \ldots, d_n)$ for all real bounded continuous functions $\varphi$ on $\mathbb{R}^n$ and all $n \geq 1$, where $(d_1, d_2, \ldots, d_n, \ldots)$ is the difference sequence of $f$: $f_n = \sum_{k=1}^n d_k$. This is equivalent to

$$E(f_{n+1}|f_1, f_2, \ldots, f_n) = f_n \text{ a.e. for all } n \geq 1.$$ 

Let $S_n(f) = \left(\sum_{k=1}^n |d_k|^2\right)^{1/2}$. We also use the notations $\|f\|_p$ and $S(f)$ standing for $\sup_n (E|f_n|^p)^{1/p}$ and $\lim_{n \to \infty} S_n(f)$ respectively. The function $S(f)$ is called the square function of $f$.

If $H$ is a Hilbert space, we can define an $H$-valued martingale in a similar way: The integral of the product of the $H$-valued strongly integrable function $d_{n+1}$ with the scalar-valued function $\varphi(d_1, d_2, \ldots, d_n)$, where $\varphi$ is bounded and continuous on $\mathbb{R}^n$, is equal to the origin of $H$. If the norm of $H$ is denoted by $|\cdot|$, then $S_n(f)$, $S(f)$, and $\|f\|_p$ are defined as above.

Let $1 < p < \infty$. In [3], Burkholder showed that there exist positive constants $b_p$ and $B_p$ such that for all real-valued martingales $f$,

$$b_p \|S(f)\|_p \leq \|f\|_p \leq B_p \|S(f)\|_p \tag{1.3}$$

Recently Burkholder [5] proved the following extension of (1.3) and, at the same time, obtained some information about the best constants.

**Theorem A.** If $1 < p < \infty$, then, for any $H$-valued martingale $f$,

$$(p^* - 1)^{-1} \|S(f)\|_p \leq \|f\|_p \leq (p^* - 1) \|S(f)\|_p$$

where $p^* = \max\left(p, \frac{p}{p-1}\right)$. In particular,

$$\|f\|_p \leq (p - 1) \|S(f)\|_p \text{ if } p \geq 2 \tag{1.4}$$

and

$$(p - 1) \|S(f)\|_p \leq \|f\|_p \text{ if } 1 < p \leq 2 \tag{1.5}$$

Moreover, the constants in (1.4) and (1.5) are best possible.

Pittenger [15] proved part of (1.4): the special case in which $p \geq 3$ and $H = \mathbb{R}$. His proof can be modified to carry over to any Hilbert space, but
cannot be modified to carry over to $2 < p \leq 3$ even for $\mathbb{H} = \mathbb{R}$. The best possible constants are unknown for the cases not covered by (1.4) and (1.5).

Here we consider a class of special martingales: the class of conditionally symmetric martingales. A martingale $f = (f_1, f_2, \ldots)$ is conditionally symmetric if $d_{n+1}$ and $-d_{n+1}$ have the same conditional distribution given $d_1, \ldots, d_n$. To be precise, for any positive integer $n$ and any two bounded real continuous functions $\tau$ and $\chi$ on $\mathbb{H}$ and $\mathbb{H}^n$ respectively, the integral of the product of $\tau(d_{n+1})$ with $\chi(d_1, \ldots, d_n)$ is the same as that of $\tau(-d_{n+1})$ with $\chi(d_1, \ldots, d_n)$. In the real case, this is equivalent to $P(d_{n+1} > a|d_1, \ldots, d_n) = P(d_{n+1} < -a|d_1, \ldots, d_n)$ a.e. for each positive integer $n$ and each positive real number $a$. For example, let $\varphi_1, \varphi_2, \ldots$ be the complete orthonormal system of Haar functions on $[0, 1]$, and let $\lambda_1, \lambda_2, \ldots$ be elements of a Hilbert space $\mathbb{H}$. Then

$$f_n = \sum_{k=1}^{n} \lambda_k \cdot \varphi_k$$

defines a conditionally symmetric martingale $f = (f_1, f_2, \ldots)$. In fact, any dyadic martingale is conditionally symmetric.

For $\lambda_i \in \mathbb{R}$ and the exponents $p > 1$, Marcinkiewicz [10] proved (1.3) in the Haar case by using Paley's [14] work which gave an equivalent Walsh series form. Burkholder and Gundy in [6] proved the right-hand side of (1.3) for $f$ in the Haar case with real $\lambda_i$ and exponents $0 < p \leq 1$. Davis [7] found the best possible constants $B_p$ in (1.3) when $0 < p \leq 2$ and $b_p$ in (1.3) when $2 \leq p < \infty$ for real conditionally symmetric martingales. They are the same as those found for $A_p$ and $a_p$. He used Skorohod embedding but this does not work for $\mathbb{H}$-valued martingales.

In this paper we will find best possible constants of the right-hand side of (1.3) when $0 < p \leq 2$ and $p \geq 3$ and those of the left-hand side when $p \geq 2$ for Hilbert-space-valued conditionally symmetric martingales. We will show the following theorem:

**Theorem 1.** Let $f$ be an $\mathbb{H}$-valued conditionally symmetric martingale. Then

(1.6) $$\|f\|_p \leq \nu_p \|S(f)\|_p \quad \text{if } 0 < p \leq 2,$$

(1.7) $$\|f\|_p \leq \mu_p \|S(f)\|_p \quad \text{if } p \geq 3,$$

and

(1.8) $$\nu_p \|S(f)\|_p \leq \|f\|_p \quad \text{if } p \geq 2.$$

Moreover, the constants are best possible.

By the standard approximation argument we can assume that the sequence $(f_n)$ consists of simple functions and that $\mathbb{H} = \mathbb{R}^N$ for some positive integer $N$. Hence, Theorem 1 is implied by the following:
\textbf{Theorem 1'.} Let $f = (f_1, f_2, \ldots)$ be a Hilbert-space-valued conditionally symmetric martingale of simple functions. Then

\begin{align*}
(1.6)' & \quad \|f\|_p \leq \nu_p \|S(f)\|_p \quad \text{if } 0 < p \leq 2, \\
(1.7)' & \quad \|f\|_p \leq \mu_p \|S(f)\|_p \quad \text{if } p > 3, \\
\text{and} \\
(1.8)' & \quad \nu_p \|S(f)\|_p \leq \|f\|_p \quad \text{if } p \geq 2.
\end{align*}

Moreover, the constants are best possible.

The constants are also the best in the Haar case. This gives some new information about Marcinkiewicz-Paley inequalities.

Inequalities (1.6)'-(1.8)' are equivalent to: For any $n \geq 1$,

\begin{align*}
(1.9) & \quad \|f_n\|_p \leq \nu_p \|S_n(f)\|_p \quad \text{if } 0 < p \leq 2, \\
(1.10) & \quad \|f_n\|_p \leq \mu_p \|S_n(f)\|_p \quad \text{if } p > 3, \\
\text{and} \\
(1.11) & \quad \nu_p \|S_n(f)\|_p \leq \|f_n\|_p \quad \text{if } p \geq 2.
\end{align*}

The inequalities (1.6)'-(1.8)' imply (1.9)-(1.11) since for any $n \geq 1$, $(f_1, f_2, \ldots, f_n, 0, 0, \ldots)$ is a conditionally symmetric martingale. On the other hand, by taking $n \to \infty$, (1.9)-(1.11) imply (1.6)'-(1.8)'.

We also discuss what we know for the other cases, for the exponents not mentioned above.

The author would like to thank Professor Donald L. Burkholder for suggesting this problem and his continued guidance, encouragement, and support. He would like to express his gratitude to Professor Rodrigo Bañuelos for his invaluable discussions. He would also like to express his appreciation to the referee for the helpful suggestions which make the paper more concise and easy to read.

\textbf{2. Confluent hypergeometric functions and parabolic cylinder functions}

The confluent hypergeometric function $M_p$ is closely related to Kummer's function $M(a, b, z)$, which is a solution of the differential equation

\begin{equation}
zw''(z) + (b - z)w'(z) - aw(z) = 0.
\end{equation}

The explicit form of $M(a, b, z)$ is

\begin{equation}
M(a, b, z) = 1 + \frac{a \cdot z}{b} + \frac{(a)_2 \cdot z^2}{(b)_2 \cdot 2!} + \cdots + \frac{(a)_n \cdot z^n}{(b)_n \cdot n!} + \cdots
\end{equation}

where $(a)_n = a(a + 1) \cdots (a + n - 1)$, $(a)_0 = 1$. Let $M_p(x) = M(-\frac{x}{2}, \frac{1}{2}, \frac{x^2}{4})$, the function mentioned in §1. If $p = 2n$, then $M_p$ is a constant multiple of the
Hermite polynomial of order $2n$ where the constant depends on $n$. By (2.1) and (2.2), the function $M_p$ satisfies

$$U''(x) - xU'(x) + pU(x) = 0$$

and

$$U(0) = 1, \quad U'(0) = 0.$$  

Differentiating (2.3) twice, we see that $M_p''$ satisfies

$$U''(x) - xU'(x) + (p-2)U(x) = 0.$$  

From (2.2) it is clear that $M_p''(0) = -p$ and $M_p^{(3)}(0) = 0$. Thus, by the uniqueness of the solution of the differential equation (2.3) with initial conditions (2.4), we have

$$M_p''(x) = -pM_{p-2}(x).$$

As in §1, we denote the smallest positive zero of $M_p$ by $\nu_p$. Let $\nu_p = \infty$ if no such zero exists.

Parabolic cylinder functions are related to the confluent hypergeometric functions. They are solutions of the differential equation

$$Y''(x) + (ax^2 + bx + c)Y(x) = 0.$$  

Here we consider the solutions of the special case

$$Y''(x) - (\frac{1}{4}x^2 - p - \frac{1}{2})Y(x) = 0.$$  

Two linearly independent solutions are

$$y_1(x) = e^{-x^2/4}M(-\frac{p}{2}, \frac{1}{2}, \frac{x^2}{2})$$

and

$$y_2(x) = xe^{-x^2/4}M(-\frac{p}{2} + \frac{3}{2}, \frac{3}{2}, \frac{x^2}{2}).$$

The parabolic cylinder function $D_p$, which is also known as Whittaker’s function, is defined by

$$D_p(x) = Y_1(x) \cos \frac{p}{2}\pi + Y_2(x) \sin \frac{p}{2}\pi,$$

where

$$Y_1(x) = (2^{p/2}/\sqrt{\pi})\Gamma((p + 1)/2)y_1(x)$$

and

$$Y_2(x) = (2^{(p+1)/2}/\sqrt{\pi})\Gamma((p + 2)/2)y_2(x).$$

We shall sometimes write $U(-p - \frac{1}{2}, x)$ for $D_p(x)$. Let $h_p(x) = e^{x^2/4}D_p(x)$. An easy calculation yields that $h_p$ satisfies (2.3).

From 19.6.1 and 19.8.1 of [1], we have

$$h_p(x) \sim x^p \left\{ 1 - \frac{p(p-1)}{2x^2} + o(x^{-3}) \right\} \quad \text{as } x \to \infty,$$
and
\[ h'_p(x) = \frac{1}{2} e^{x^2/4} U(-p - \frac{1}{2}, x) + e^{x^2/4} U'(-p - \frac{1}{2}, x) \]
\[ = \frac{1}{2} e^{x^2/4} U(-p - \frac{1}{2}, x) + e^{x^2/4} \left\{ -\frac{x}{2} U(-p - \frac{1}{2}, x) + p U(-p + \frac{1}{2}, x) \right\} \]
\[ = ph_{p-1}(x). \]

Let \( \mu_p \) be the largest positive zero of \( h_p \). If \( h_p \) has no zero or no positive zero, let \( \mu_p = 0 \).

3. AN OUTLINE OF THE PROOFS

Recall that \( \nu_p \) and \( \mu_p \) are, respectively, the smallest positive zero of \( M_p \), and the largest positive zero of \( D_p \). We define functions \( v_p \), \( V_p \), \( \overline{v}_p \), and \( \overline{V}_p \) on \( \mathbb{H} \) by

\[ v_p(x) = |x|^p - \mu_p^p \quad \text{if } p \geq 2, \]
\[ = \mu_p^p - |x|^p \quad \text{if } 1 \leq p < 2, \]
\[ V_p(x, t) = t^{p/2} v_p(x/\sqrt{t}) \quad \text{if } t > 0, \]
\[ = \text{sgn}(p - 2)|x|^p \quad \text{if } t = 0. \]

(here \( \text{sgn}(0) = 1 \)), and

\[ \overline{v}_p(x) = |x|^p - \nu_p^p \quad \text{if } 0 < p < 2, \]
\[ = \nu_p^p - |x|^p \quad \text{if } p \geq 2, \]
\[ \overline{V}_p(x, t) = t^{p/2} \overline{v}_p(x/\sqrt{t}) \quad \text{if } t > 0, \]
\[ = -\text{sgn}(p - 2)|x|^p \quad \text{if } t = 0. \]

Let \( f = (f_1, f_2, \ldots) \) be a real conditionally symmetric martingale of simple functions. Then for \( 0 < p < 2 \),

\[ \int_{\mathbb{H}} |f_n|^p - \nu_p^p \|S_n(f)^p\|_p = E(\|f_n||^p - \nu_p^p S_n(f)^p) = E\overline{V}_p(f_n, S_n^2(f)). \]

So \( \|f_n\|_p \leq \nu_p \|S_n(f)\|_p \) is equivalent to \( E\overline{V}_p(f_n, S_n^2(f)) \leq 0 \) when \( 0 < p < 2 \). Similarly, when \( p \geq 2 \), \( \nu_p \|f_n\|_p \leq \|S_n(f)\|_p \) is equivalent to \( E\overline{V}_p(f_n, S_n^2(f)) \leq 0 \); and when \( p \geq 3 \), \( \|f_n\|_p \leq \mu_p \|S_n(f)\|_p \) is equivalent to \( E\overline{V}_p(f_n, S_n^2(f)) \leq 0 \). Thus (1.9)–(1.11) are equivalent to

\[ E\overline{V}_p(f_n, S_n^2(f)) \leq 0 \quad \text{for } p \geq 3, \]

and

\[ E\overline{V}_p(f_n, S_n^2(f)) \leq 0 \quad \text{for } p > 0. \]
Our method is to find functions $U_p(x, t)$ on $\mathbb{H} \times [0, \infty)$ for $p \geq 3$ and $\overline{U}_p(x, t)$ on $\mathbb{H} \times [0, \infty)$ for $p > 0$ such that

\begin{align*}
(3.3) & \quad V_p(x, t) \leq U_p(x, t), \\
(3.4) & \quad \frac{1}{2} \{ U_p(x + a, t + |a|^2) + U_p(x - a, t + |a|^2) \} - U_p(x, t) \leq 0 \\
\end{align*}

for $p \geq 3$ and all $x, a \in \mathbb{H}$, $t \in [0, \infty)$,

\begin{align*}
(3.5) & \quad \overline{V}_p(x, t) \leq \overline{U}_p(x, t), \\
(3.6) & \quad \frac{1}{2} \{ U_p(x + a, t + |a|^2) + \overline{U}_p(x - a, t + |a|^2) \} - U_p(x, t) \leq 0 \\
\end{align*}

for $p > 0$ and all $x, a \in \mathbb{H}$, $t \in [0, \infty)$; and

\begin{align*}
(3.7) & \quad U_p(a, |a|^2) \leq 0 \quad \text{and} \quad \overline{U}_p(a, |a|^2) \leq 0 \quad \text{for all} \quad a \in \mathbb{H}.
\end{align*}

We call (3.3) and (3.5) the majorization properties, (3.4) and (3.6) the averaging properties, and (3.7) the negativity property.

Once such functions are known, then for a conditionally symmetric martingale $f = (f_1, f_2, \ldots)$ with difference sequence $(d_1, d_2, \ldots)$ and with each $f_n$ simple,

\begin{align*}
(3.8) & \quad EV_p(f_n, S^2_n(f)) \leq EU_p(f_n, S^2_n(f)), \\
(3.9) & \quad EV_p(f_n, S^2_n(f)) \leq \overline{EU}_p(f_n, S^2_n(f)),
\end{align*}

from (3.3) and (3.5). For $m \geq 1$, on $\mathcal{D}_m = \{ d_1 = d', d_2 = d'_2, \ldots, d_m = d'_m \}$, let $f_m = \sum_{i=1}^{m} d'_i$, $S_m(f) = (\sum_{i=1}^{m} d_i^2)^{1/2}$. By conditional symmetry and (3.4),

\begin{align*}
\int_{\mathcal{D}_m} U_p(f_{m+1}, S^2_{m+1}(f)) & = \int_{\mathcal{D}_m} U_p(f_m + d_{m+1}, S_m(f) + |d_{m+1}|^2) \\
& = \frac{1}{2} \sum_a \left( \int_{\mathcal{D}_m, d_{m+1} = a} U_p(f_m + a, S^2_m(f) + |a|^2) \\
& \quad + \int_{\mathcal{D}_m, d_{m+1} = -a} U_p(f_m - a, S^2_m(f) + |a|^2) \right) \\
& = \frac{1}{2} \sum_a \left( P(\mathcal{D}_m, d_{m+1} = a) U_p(f_m + a, S^2_m(f) + |a|^2) \\
& \quad + P(\mathcal{D}_m, d_{m+1} = -a) U_p(f_m - a, S^2_m(f) + |a|^2) \right) \\
& = \frac{1}{2} \sum_a P(\mathcal{D}_m, d_{m+1} = a) U_p(f_m + a, S^2_m(f) + |a|^2) \\
& \quad + U_p(f_m - a, S^2_m(f) + |a|^2) \\
& \leq \sum_a P(\mathcal{D}_m, d_{m+1} = a) U_p(f_m, S^2_m(f)) \\
& = U_p(f_m, S^2_m(f)) \cdot P(\mathcal{D}_m) = \int_{\mathcal{D}_m} U_p(f_m, S^2_m(f)).
\end{align*}
Consequently, for $m \geq 1$,

$$E_U(p(f_{m+1}, S_{m+1}^2(f)) \leq E_U(p(f_m, S_m^2(f))).$$

Thus for $m \geq 1$,

$$E_U(p(f_m, S_m^2(f)) \leq E_U(p(f_1, S_1^2(f)) = E_U(p(f_1, |f_1|^2)).$$

But $E_U(p(f_1, |f_1|^2) \leq 0$ by (3.7), so combining (3.8) and (3.10), we have (3.1). Similarly (3.2) will follow if $U_p(x, t)$ is known.

Note. When $p > 0$, if we only find $U_p(x, t)$ defined on $\mathcal{F} = \{(x, t): x \in \mathcal{H}, t > 0\}$ satisfying

$$(3.5) V_p(x, t) \leq U_p(x, t),$$

$$(3.6) \frac{1}{2}\{U_p(x + a, t + |a|^2) + U_p(x - a, t + |a|^2)\} - U_p(x, t) \leq 0 \text{ for } a \in \mathbb{H},$$

and

$$(3.7) U_p(a, |a|^2) \leq 0 \text{ for all } a \in \mathbb{H}\setminus\{0\},$$

then (3.2) still holds. The reason is the following.

Extend $U_p(x, t)$ to $\mathcal{F} \cup \{(0, 0)\}$ by defining $U_p(0, 0) = 0$. Since $S_n(f) = 0$ implies $f_n = 0$ for all $n \geq 1$, in order to have

$$E V_p(f_n, S_n^2(f)) \leq E U_p(f_n, S_n^2(f)),
$$

and

$$E U_p(f_{n+1}, S_{n+1}^2(f)) \leq E U_p(f_n, S_n^2(f)),$$

we need only

$$V_p(x, t) \leq U_p(x, t),$$

$$(3.12) \frac{1}{2}\{U_p(x + a, t + |a|^2) + U_p(x - a, t + |a|^2)\} - U_p(x, t) \leq 0,$$

for all $a \in \mathbb{H}$, $(x, t) \in \mathcal{F} \cup \{(0, 0)\}$, and

$$(3.13) U_p(a, |a|^2) \leq 0 \text{ for all } a \in \mathbb{H}.$$

(3.11) and (3.13) follow clearly from the definition of $U_p$ and (3.5)'.

If $t > 0$, (3.12) follows from (3.6)'. If $t = 0$, then (3.12) becomes $U_p(a, |a|^2) \leq 0$ which follows from (3.13). Thus (3.5)’–(3.7)’ will ensure (3.2).

4. The existence of the function $U_p(x, t)$: Real case

Recall from §2 that $h_p(x) = e^{x^2/4}D_p(x) = e^{x^2/4}U(-p - \frac{1}{2}, x)$ satisfies the differential equation

$$U''(x) - xU'(x) + pU(x) = 0.$$
Moreover

\[ h_p(x) \sim x^p \left\{ 1 - \frac{p(p-1)}{2x^2} + o(x^{-3}) \right\} \quad \text{as } x \to \infty, \]

and

\[ h'_p(x) = ph_{p-1}(x). \]

As in §2, let \( \mu_p \) be the largest positive zero of \( h_p \). If \( h_p \) has no zero or no positive zero, let \( \mu_p = 0 \).

As we will show in Lemma 5.1 of §5, for \( p \geq 1 \), \( h_p(\mu_p) = 0 \). Since \( h_p(\mu_p) = 0 \) and \( h_p \) is not identically zero, \( h'_p(\mu_p) \neq 0 \) from the uniqueness of the solution of (4.1). Thus it is meaningful to let

\[ \alpha_p = \frac{v'_p(\mu_p)}{h'_p(\mu_p)} \quad \text{and} \quad w_p(x) = \alpha_p h_p(|x|). \]

We define for \( x \in \mathbb{R} \)

\[ u_p(x) = \begin{cases} v_p(x) & \text{if } 0 \leq |x| \leq \mu_p, \\ w_p(x) & \text{if } \mu_p < |x| < \infty, \end{cases} \]

and

\[ U_p(x, t) = \begin{cases} t^{p/2} u_p(x/\sqrt{t}) & \text{if } t > 0, \\ \alpha_p |x|^p & \text{if } t = 0. \end{cases} \]

If \( x \neq 0 \), then, by (4.2), \( V_p(x, \cdot) \) and \( U_p(x, \cdot) \) are continuous functions on \([0, \infty)\), where \( V_p(x, \cdot) \) is defined in §3.

In the following several sections, we shall show that

\[ U_p(x, t) \geq V_p(x, t) \quad \text{for } p > 1, t \geq 0, x \in \mathbb{R}, \]

\[ \frac{1}{2} \{ U_p(x + a, t + a^2) + U_p(x - a, t + a^2) \} - U_p(x, t) \leq 0 \]

for \( p \geq 3 \), \( t \geq 0 \), \( x \in \mathbb{R} \), \( a \in \mathbb{R} \), and

\[ U_p(a, a^2) \leq 0 \quad \text{for } p \geq 3 \text{ and } a \in \mathbb{R}. \]

The proof of (4.4)–(4.6) will complete the proof of the inequality (1.7) of Theorem 1.

We also show why (4.5) is not true for the exponents \( 1 < p < 2 \) and \( 2 < p < 3 \).

As before, we call (4.4) the majorization property, (4.5) the averaging property, and (4.6) the negativity property.

5. Proof of the majorization property

By the definition of \( U_p(x, t) \) and \( V_p(x, t) \) as well as the continuity, (4.4) is equivalent to

\[ w_p(x) \geq v_p(x) \quad \text{for all } x \geq \mu_p. \]
In fact, an even stronger property holds:

\[(5.2) \quad w_p(x) \geq v_p(x) \quad \text{for all } x \geq 0.\]

See Wang [17] for the proof.

First we show some lemmas.

**Lemma 5.1.** \(\mu_p \geq \mu_q\) and \(h_p(\mu_p) = 0\) for \(p \geq q \geq 1\).

**Proof.** Let \(W_t\) be standard Brownian motion. Consider the stopping time defined by

\[
S_a = \inf\{t > 0 : W_t = a\sqrt{t} - 1\}, \quad a > 0.
\]

Novikov proved in [12] that \(S_a\) satisfies

\[(5.3) \quad ES_a^p < \infty \quad \text{if } a > \mu_{2p}\]

and

\[(5.4) \quad ES_{\mu_{2p}}^p = \infty \]

for \(p > \frac{1}{2}\). Thus if there exist \(p\) and \(q\) such that \(p > q > 1\) and \(\mu_p < \mu_q\), then by (5.3) \(\|S_{\mu_q}^{1/2}\|_p < \infty\). The Liapounov inequality implies that \(\|S_{\mu_q}^{1/2}\|_q \leq \|S_{\mu_q}^{1/2}\|_p < \infty\), which is contrary to (5.4). Thus \(\mu_p \geq \mu_q\) if \(p > q > 1\).

Note that \(\mu_1 = h_1(\mu_1) = 0\) since \(h_1(x)\) is a constant multiplying \(x\). So if we can show \(\mu_p > 0\) for \(1 < p < 2\), then the lemma is proven.

By 19.3.3 of [1],

\[
h_p(0) = (2^{p/2}/\sqrt{\pi})\Gamma((p + 1)/2)\cos(p\pi/2).
\]

Hence \(h_p(0) < 0\) when \(1 < p < 2\). Using (4.2), we see that \(h_p(x)\) is positive when \(x\) is large. Therefore, from the fact that \(h_p(x)\) is continuous, it follows that \(\mu_p > 0\). 

**Lemma 5.2.** If \(p < 1\), then \(h_p(x) > 0\) on \([0, \infty)\). Moreover \(h_1(x) > 0\) on \((0, \infty)\).

**Proof.** We first show that \(h_p(x) > 0\) on \([0, \infty)\) if \(p \leq -\frac{1}{2}\). This is equivalent to \(U(-p - \frac{1}{2}, x) > 0\) on \([0, \infty)\) if \(p \leq -\frac{1}{2}\).

Recall from §2 that \(U(-p - \frac{1}{2}, x)\) satisfies

\[(5.5) \quad y'' - \left(\frac{1}{4}x^2 - \frac{1}{2} - p\right)y = 0.\]

By our condition on \(p\), \(\frac{1}{4}x^2 - \frac{1}{2} - p > 0\) for all \(x \in \mathbb{R}\setminus\{0\}\). Also, by 19.3.5 of [1],

\[(5.6) \quad U\left(-p - \frac{1}{2}, 0\right) = \frac{2^{p/2}\sqrt{\pi}}{\Gamma((1-p)/2)} > 0,\]

\[(5.7) \quad U'\left(-p - \frac{1}{2}, 0\right) = -\frac{2^{(p+1)/2}\sqrt{\pi}}{\Gamma(-p/2)} < 0\]

since \(-\frac{p}{2} \geq 0\).
Suppose $U$ has a zero in $(0, \infty)$. Let $z_1$ be its smallest positive zero. Then, by (5.6),

$$U(-p - \frac{1}{2}, x) > 0 \quad \text{on } [0, z_1),$$

and, by (5.5),

$$U''(-p - \frac{1}{2}, x) > 0 \quad \text{on } (0, z_1).$$

**Case (i).** $U'(-p - \frac{1}{2}, z_1) > 0$.

By (5.7), $U'$ has a zero in $(0, z_1)$. Let $z_2$ be its smallest one. The inequality (5.9) and the mean value theorem of calculus imply

$$U'(-p - \frac{1}{2}, x) > 0 \quad \text{on } (z_2, z_1).$$

The mean value theorem once more implies, by (5.8) and (5.10), that

$$U(-p - \frac{1}{2}, z_1) > 0$$

which is contrary to $z_1$ being a zero of $U$.

**Case (ii).** $U'(-p - \frac{1}{2}, z_1) = 0$.

Since $U(-p - \frac{1}{2}, z_1) = 0$, the uniqueness of the solution of (5.5) implies that $U(-p - \frac{1}{2}, x) \equiv 0$, contrary to (5.6).

**Case (iii).** $U'(-p - \frac{1}{2}, z_1) < 0$.

Because $U(-p - \frac{1}{2}, z_1) = 0$, the continuity of $U'$ at $z_1$ and the mean value theorem show that, for some $\varepsilon > 0$,

$$U(-p - \frac{1}{2}, x) < 0 \quad \text{on } (z_1, z_1 + \varepsilon).$$

By (4.2), $z_3 = \inf\{x > z_1 : U(-p - \frac{1}{2}, x) = 0\} < \infty$, and

$$U(-p - \frac{1}{2}, x) < 0 \quad \text{on } (z_1, z_3).$$

Hence, by (5.5),

$$U''(-p - \frac{1}{2}, x) < 0 \quad \text{on } (z_1, z_3).$$

Thus by the mean value theorem and $U'(-p - \frac{1}{2}, z_1) < 0$,

$$U'(-p - \frac{1}{2}, x) < 0 \quad \text{on } (z_1, z_3).$$

However, by Rolle's theorem, since $U(-p - \frac{1}{2}, z_1) = U(-p - \frac{1}{2}, z_3) = 0$, there exists a $z_4 \in (z_1, z_3)$, such that

$$U'(-p - \frac{1}{2}, z_4) = 0$$

which is contrary to (5.11). Thus the lemma is true when $p \leq -\frac{1}{2}$.

For the remaining $p$'s, let us first consider $-\frac{1}{2} < p \leq 0$. By (4.3)

$$h'_p(x) = ph'_{p-1}(x) \quad \text{on } [0, \infty).$$

Thus, using $h'_{p-1}(x) > 0$, the above line, and $p \leq 0$, we have

$$h'_p(x) \leq 0 \quad \text{on } [0, \infty).$$
Suppose there exists a $z_5 \geq 0$ such that $h_p(z_5) \leq 0$. Then, by the mean value theorem and (5.12),

$$h_p(x) \leq 0 \quad \text{on} \quad (z_5, \infty),$$

which is contrary to (4.2). So $h_p(x) > 0$ when $-\frac{1}{2} < p \leq 0$.

Finally when $0 < p \leq 1$, again, by (4.3) we have

$$h_p(x) > 0 \quad \text{on} \quad (0, \infty).$$

Using $h_p(0) = (2^{p/2}/\sqrt{\pi})\Gamma((p+1)/2)\cos(p\pi/2) > 0$ for $0 < p < 1$, $h_1(0) = 0$, and (5.13), we have $h_p(x) > 0$ on $[0, \infty)$ for $0 < p < 1$ and $h_1(x) > 0$ on $(0, \infty)$. This completes the proof. □

**Lemma 5.3.** (a) For $1 \leq p \leq 2$,

$$h_p(x) > 0 \quad \text{on} \quad (\mu_p, \infty),$$

$$h_p''(x) > 0, \quad h_p^{(3)}(x) \geq 0, \quad h_p^{(4)}(x) \leq 0 \quad \text{and} \quad h_p^{(5)}(x) \geq 0 \quad \text{on} \quad (0, \infty).$$

(b) For $2 < p \leq 3$,

$$h_p(x) > 0 \quad \text{on} \quad (\mu_p, \infty),$$

$$h_p'(x) > 0 \quad \text{on} \quad (\mu_{p-1}, \infty),$$

$$h_p''(x) > 0, \quad h_p^{(3)}(x) > 0 \quad \text{and} \quad h_p^{(4)}(x) \leq 0 \quad \text{on} \quad (0, \infty).$$

(c) For $p > 3$,

$$h_p(x) > 0 \quad \text{on} \quad (\mu_p, \infty),$$

$$h_p'(x) > 0 \quad \text{on} \quad (\mu_{p-1}, \infty),$$

$$h_p''(x) > 0 \quad \text{on} \quad (\mu_{p-2}, \infty),$$

$$h_p^{(3)}(x) > 0 \quad \text{on} \quad (\mu_{p-3}, \infty),$$

$$h_p^{(4)}(x) > 0 \quad \text{on} \quad (\mu_{p-4}, \infty).$$

In particular, $h_p^{(n)}(x) > 0$ on $(\mu_p, \infty)$ for $n = 0, 1, 2, 3, 4$.

Notice that by Lemma 5.2, if $p \leq n+1$, then $\mu_{p-n} = 0$ according to our definition.

**Proof.** Let us first show that $h_p(x) > 0$ on $(\mu_p, \infty)$ for all $p$. If $p \leq 1$, this follows from Lemma 5.2. Now consider the case $p > 1$. By Lemma 5.1, since $\mu_p$ is the largest positive zero of $h_p(x)$, $h_p(x)$ must keep the same sign in $(\mu_p, \infty)$. By (4.2),

$$h_p(x) > 0 \quad \text{on} \quad (\mu_p, \infty),$$

since $h_p(x) > 0$ when $x$ is large. Thus, using (4.3), we have

$$h_p''(x) = ph_p'(x) = p(p-1)h_p(x),$$

and

$$h_p^{(3)}(x) = p(p-1)(p-2)h_p(x).$$
and

\[ h_p^{(4)}(x) = p(p - 1)(p - 2)(p - 3)h_{p-4}(x). \]

The lemma now follows from (5.15)--(5.18), (4.3), Lemma 5.1, and Lemma 5.2. \( \square \)

**Proof of (5.1).** There are two cases.

**Case (i).** \( p \geq 2 \). We want to show that

\[ w'_p(x) \geq v'_p(x) \quad \text{on } [\mu_p, \infty). \]

Then by the mean value theorem and \( w_p(\mu_p) = v_p(\mu_p) = 0 \), we get

\[ w_p(x) \geq v_p(x) \quad \text{on } [\mu_p, \infty). \]

Consider \( B_p(x) = \frac{w'_p(x)}{v'_p(x)} = \frac{1}{p} \alpha_p h'_p(x) / x^{p-1} \). Then

\[ B'_p(x) = \frac{1}{p} \alpha_p \left( x^{p-1} h''_p(x) - (p - 1) x^{p-2} h'_p(x) \right) / x^{2p-2} \]

\[ = \frac{1}{p} \alpha_p \left( x h''_p(x) - (p - 1) h'_p(x) \right) / x^p. \]

Differentiating (4.1) once, we have

\[ U^{(3)}(x) - x U'''(x) + (p - 1) U'(x) = 0. \]

Thus, \( h_p^{(3)}(x) - x h_p''(x) + (p - 1) h_p'(x) = 0 \), or

\[ h_p^{(3)}(x) = x h_p''(x) - (p - 1) h_p'(x). \]

Using (5.21) in the last equality of \( B_p(x) \), (b) and (c) of Lemma 5.3, and the definition of \( \alpha_p \), we have that

\[ B'_p(x) = \frac{1}{p} \alpha_p h_p^{(3)}(x) / x^{p} \geq 0 \quad \text{on } (\mu_p, \infty). \]

Hence, the mean value theorem and \( B_p(\mu_p) = 1 \) imply (5.19).

**Case (ii).** \( 1 \leq p \leq 2 \). Again let \( B_p(x) = \frac{w'_p(x)}{v'_p(x)} = -\frac{1}{p} \alpha_p h'_p(x) / x^{p-1} \). As in Case (i), \( B'_p(x) = -\frac{1}{p} \alpha_p h_p^{(3)}(x) / x^p \). Since \( \alpha_p = -p \mu_p^{p-1} / h_p'(\mu_p) \leq 0 \) and \( h_p^{(3)}(x) \leq 0 \) on \((0, \infty)\) by Lemma 5.3(a), we have \( B'_p(x) \leq 0 \) on \((0, \infty)\). Now using the mean value theorem and \( B_p(\mu_p) = 1 \), we get (5.19) and (5.20), thus (5.1). \( \square \)

Before we go to the proof of (4.5), we show an important lemma which will be needed very often later.

**Lemma 5.4.** \( \mu_p^2 \geq p - 1 \) if \( p \geq 2 \) and \( \mu_p^2 \leq p - 1 \) if \( 1 \leq p \leq 2 \).

**Proof.** From (4.1) and \( h_p(\mu_p) = 0 \), we see \( h''_p(\mu_p) = \mu_p h'_p(\mu_p) \). Using (5.21) we have

\[ h_p^{(3)}(\mu_p) = \mu_p h''_p(\mu_p) - (p - 1) h'_p(\mu_p) = (\mu_p^2 - (p - 1)) h'_p(\mu_p). \]
Applying (5.17) and Lemmas 5.1 and 5.3, we see when \( p \geq 2 \),
\[
\mu_p^2 - (p - 1) = \frac{h_p^{(3)}(\mu_p)}{h_p'(\mu_p)} \geq 0,
\]
and when \( 1 \leq p \leq 2 \),
\[
\mu_p^2 - (p - 1) = \frac{h_p^{(3)}(\mu_p)}{h_p'(\mu_p)} \leq 0.
\]
This completes the proof. \( \square \)

6. Proof of the averaging property

We first notice that inequality (4.5) is equivalent to
\[
(6.1) \left\{ \frac{1}{2}(t + a^2)^{p/2} \{ u_p\left(\frac{x + a}{\sqrt{t + a^2}}\right) + u_p\left(\frac{x - a}{\sqrt{t + a^2}}\right)\} \right.
- \left. t^{p/2} u_p(x/\sqrt{t}) \right\} \leq 0 \quad \text{for } x \in \mathbb{R}, a \in \mathbb{R} \text{ and } t > 0.
\]
The case \( t = 0 \) is from the continuity of \( u_p \).

Without loss of generality we can set \( t = 1 \). Also, since \( u_p \) is an even function, we need to prove (6.1) only for \( x \geq 0 \) and \( a \geq 0 \). Thus (6.1) is equivalent to
\[
(6.1)' \left\{ \frac{1}{2}(1 + a^2)^{p/2} \{ u_p\left(\frac{x + a}{\sqrt{1 + a^2}}\right) + u_p\left(\frac{x - a}{\sqrt{1 + a^2}}\right)\} \right.
- \left. u_p(x) \right\} \leq 0 \quad \text{for } x \geq 0 \text{ and } a \geq 0.
\]
Denote
\[
G_x(a) = \frac{1}{2}(1 + a^2)^{p/2} \{ u_p\left(\frac{x + a}{\sqrt{1 + a^2}}\right) + u_p\left(\frac{x - a}{\sqrt{1 + a^2}}\right)\} - u_p(x)
\]
and let \( y = x/\sqrt{1 + a^2}, b = a/\sqrt{1 + a^2} \). Since \( G_x(0) = 0 \) for \( x \geq 0 \), inequality (6.1)' will follow if we can show
\[
(6.2) \quad G_x'(a) \leq 0 \quad \text{for all } a \geq 0.
\]
In view of the definition of \( u_p \), the proof of (6.2) is conveniently divided into six cases:

Case (1). \( 0 \leq x \leq \mu_p \), \( a \geq 0 \), and \( (x \pm a)^2 \leq \mu_p^2(1 + a^2) \).

Solving the above inequalities, we have
\[
(6.3) \quad \rho_1(x) = \frac{x - \sqrt{\mu_p^2(x^2 - \mu_p^2 + 1)}}{\mu_p^2 - 1}, \quad \rho_2(x) = \frac{x + \sqrt{\mu_p^2(x^2 - \mu_p^2 + 1)}}{\mu_p^2 - 1}
\]
or
\[
(\text{ii}) \quad 0 \leq x \leq \sqrt{\mu_p^2 - 1} \text{ and } a \geq 0.
\]
Differentiating $G_x(a)$, we have

$$G'_x(a) = \frac{1}{2} p a (1 + a^2)^{p-1} \left\{ \left( \frac{x + a}{\sqrt{1 + a^2}} \right)^p - \mu_p^p + \left( \frac{x - a}{\sqrt{1 + a^2}} \right)^p - \mu_p^p \right\}$$

$$+ \frac{1}{2} (1 + a^2)^{p/2} \left\{ \text{sgn}(a - x) p \left( \frac{x - a}{\sqrt{1 + a^2}} \right)^{p-1} \frac{1 + ax}{(\sqrt{1 + a^2})^3} \right\}$$

$$+ p \left( \frac{x + a}{\sqrt{1 + a^2}} \right)^{p-1} \frac{1 - ax}{(\sqrt{1 + a^2})^3}$$

$$= \frac{1}{2} (1 + a^2)^{(p-1)/2} \left\{ p \left( \frac{x - a}{\sqrt{1 + a^2}} \right)^{p-1} \left( \frac{a|x - a|}{1 + a^2} + \text{sgn}(a - x) \frac{1 + ax}{1 + a^2} \right) \right\}$$

$$+ \frac{1}{2} (1 + a^2)^{(p-1)/2} \left\{ \text{sgn}(a - x) p \left( \frac{x - a}{\sqrt{1 + a^2}} \right)^{p-1} \frac{1 - ax}{(\sqrt{1 + a^2})^3} - 2 \frac{p a \mu_p^p}{\sqrt{1 + a^2}} \right\}$$

$$= \frac{1}{2} (1 + a^2)^{(p-1)/2} \left\{ v'_p(y + b) + \text{sgn}(b - y) v'_p(|y - b|) - 2 p b \mu_p^p \right\}$$

$$= \frac{1}{2} (1 + a^2)^{(p-1)/2} C_y(b) \quad \text{if } 0 \leq b < y$$

$$= \frac{1}{2} (1 + a^2)^{(p-1)/2} \varphi_y(b) \quad \text{if } 0 \leq y < b,$$

where

$$C_y(b) = v'_p(y + b) - v'_p(y - b) - 2 p b \mu_p^p$$

under the condition

(6.3.1) \quad 0 \leq b \leq y, \quad y + b \leq \mu_p,$$

and

$$\varphi_y(b) = v'_p(y + b) + v'_p(b - y) - 2 p b \mu_p^p$$

under the condition

(6.3.2) \quad 0 \leq y \leq b, \quad y + b \leq \mu_p.$$

**Lemma 6.1.** Both $C_y(b)$ and $\varphi_y(b)$ are nonpositive on the domain on which they are defined.
Proof. Assume that (6.3.1) holds and recall that, for \( p \geq 3 \), we have \( v_p(x) = |x|^p - \mu_p^p \). So, if \( x \geq 0 \), then \( v''_p(x) = p(p - 1)x^{p-2} \). By (6.3.1) and Lemma 5.4,

\[ C'_y(b) = v''_p(y + b) + v''_p(y - b) - 2p\mu_p^p \]
\[ = p(p - 1)((y + b)^{p-2} + (y - b)^{p-2}) - 2p\mu_p^p \]
\[ \leq 2p(p - 1)\mu_p^{p-2} - 2p\mu_p^p \]
\[ = 2p\mu_p^{p-2}((p - 1) - \mu_p^2) \leq 0. \]

This, together with \( C_y(0) = 0 \), implies that \( C_y(b) \leq 0 \) on \([0, \mu_p - y]\).

Now assume that (6.3.2) holds. Then \( C'_y(b) \leq 0 \) on \([y, \mu_p - y]\) by an argument similar to the one above. By Lemma 5.4 and \( 2y \leq \mu_p \),

\[ C'_y(y) = v'_p(2y) - 2yp\mu_p^p \]
\[ = p(2y)^{p-1} - 2yp\mu_p^p \]
\[ \leq 2yp\mu_p^{p-2}(1 - \mu_p^2) \leq 0. \]

So \( C_y(b) \leq 0 \) on \([y, \mu_p - y]\). This completes the proof of Lemma 6.1. \( \square \)

Thus (6.2) is proven by Lemma 6.1 under Case (I).

Case (II). \( 0 \leq x \leq \mu_p \), \( a \geq 0 \), and \((x + a)^2 \geq \mu_p^2(1 + a^2)\), \((x - a)^2 \leq \mu_p^2(1 + a^2)\).

Solving the above inequalities, we have

\[
\begin{align*}
(i) \quad \rho_1(x) &\leq a \leq \rho_2(x), \text{ if } \sqrt{\mu_p^2 - 1} \leq x \leq \mu_p, \\
(ii) \quad \text{No solution if } x < \sqrt{\mu_p^2 - 1},
\end{align*}
\]

where \( \rho_1(x) \) and \( \rho_2(x) \) are defined in (6.3).

Notice that, by Lemma 5.1 and our assumption that \( 0 \leq x \leq \mu_p \),

\[
p \geq 3 \Rightarrow \mu_p^2 \geq \mu_3^2 = 3 \]
\[
\Rightarrow \mu_p^2 - 4 \geq -1 \]
\[
\Rightarrow (\mu_p^2 - 1)(x^2(\mu_p^2 - 4) + \mu_p^2) \geq 0 \]
\[
\Rightarrow (\mu_p^2 - 2)^2 x^2 \geq \mu_p^2(x^2 - \mu_p^2 + 1) \]
\[
\Rightarrow x \geq \rho_2(x). \]

So \( \rho_1(x) \leq \rho_2(x) \leq x \). Thus, by (6.4), \( x \geq a \).
Using (4.1),
\[ G'_x(a) = \frac{1}{2}p a (1 + a^2)^{p/2 - 1} \left\{ \alpha_p h_p \left( \frac{x + a}{\sqrt{1 + a^2}} \right) + v_p \left( \frac{x - a}{\sqrt{1 + a^2}} \right) \right\} \]
\[ + \frac{1}{2} (1 + a^2)^{p/2} \left\{ \alpha_p h'_p \left( \frac{x + a}{\sqrt{1 + a^2}} \right) \frac{1 - ax}{(1 + a^2)^3} - v'_p \left( \frac{x - a}{\sqrt{1 + a^2}} \right) \frac{1 + ax}{(1 + a^2)^3} \right\} \]
\[ = \frac{1}{2} (1 + a^2)^{(p-1)/2} \left\{ \frac{pa \alpha_p}{\sqrt{1 + a^2}} h_p \left( \frac{x + a}{\sqrt{1 + a^2}} \right) + \frac{pa}{\sqrt{1 + a^2}} v_p \left( \frac{x - a}{\sqrt{1 + a^2}} \right) \right\} \]
\[ + \alpha_p h'_p \left( \frac{x + a}{\sqrt{1 + a^2}} \right) \frac{1 - ax}{1 + a^2} - v'_p \left( \frac{x - a}{\sqrt{1 + a^2}} \right) \frac{1 + ax}{1 + a^2} \]
\[ = \frac{1}{2} (1 + a^2)^{(p-1)/2} \left\{ \frac{a \alpha_p}{\sqrt{1 + a^2}} \frac{1}{h'_p \left( \frac{x + a}{\sqrt{1 + a^2}} \right)} \right\} \]
\[ + \frac{x + a}{\sqrt{1 + a^2}} h'_p \left( \frac{x + a}{\sqrt{1 + a^2}} \right) \frac{1}{\sqrt{1 + a^2}} \]
\[ + \alpha_p h'_p \left( \frac{x + a}{\sqrt{1 + a^2}} \right) \frac{1 - ax}{1 + a^2} - \frac{pa \mu^p}{\sqrt{1 + a^2}} \]
\[ + p \left( \frac{x - a}{\sqrt{1 + a^2}} \right)^{p-1} \left[ \frac{a}{\sqrt{1 + a^2}} \frac{x - a}{\sqrt{1 + a^2}} - \frac{1 + ax}{1 + a^2} \right] \}
\[ = \frac{1}{2} (1 + a^2)^{(p-1)/2} \left\{ \alpha_p h'_p \left( \frac{x + a}{\sqrt{1 + a^2}} \right) - v'_p \left( \frac{x - a}{\sqrt{1 + a^2}} \right) \right\} \]
\[ - \frac{a}{\sqrt{1 + a^2}} \left( \frac{h''_p}{\sqrt{1 + a^2}} \left( \frac{x + a}{\sqrt{1 + a^2}} \right) + p \mu^p \right) \}.
\]

We will show \( G'_x(a) \leq 0 \) on \([\rho_1(x), \rho_2(x)]\) by the following lemma.

**Lemma 6.2.** Define
\[ D_y(b) = \alpha_p h'_p (y + b) - v'_p (y - b) - \alpha_p b h''_p (y + b) - p b \mu^p \]
on the union of
\[ (6.4.1) \quad 0 \leq y \leq \mu_p \leq 2y, \quad \mu_p - y \leq b \leq y, \]
and
\[ (6.4.2) \quad \mu_p \leq y, \quad y - \mu_p \leq b \leq y; \]
\[ \mathcal{D}_y(b) = \alpha_p h'_p (y + b) + v'_p (b - y) - \alpha_p b h''_p (y + b) - p b \mu^p \]
on the union of
\[ (6.4.3) \quad \mu_p \leq 2y, \quad y \leq b \leq y + \mu_p, \]
and

\[(6.4.4) \quad 2y \leq \mu_p, \quad \mu_p - y \leq b \leq y + \mu_p.\]

Then both \(D_y(b)\) and \(\mathcal{D}_y(b)\) are nonpositive on their respective domains.

Before we start the proof, we observe that the union of (6.4.1) and (6.4.2) is the set

\[(6.4.5) \quad 0 \leq b \leq y, \quad y + b \geq \mu_p, \quad \text{and} \quad y - b \leq \mu_p.\]

The union of (6.4.3) and (6.4.4) is the set

\[(6.4.6) \quad 0 \leq y \leq b, \quad y + b \geq \mu_p, \quad \text{and} \quad b - y \leq \mu_p.\]

**Proof.** Under (6.4.5), by Lemmas 5.3 and 5.4, we have

\[(6.4.7) \quad D_y'(b) = -\alpha_p b h_p^{(3)}(y + b) + v^{''}_p(y - b) - p\mu_p^p\]

\[\leq -\alpha_p b h_p^{(3)}(y + b) + p(p - 1)\mu_p^{p-2} - p\mu_p^p\]

\[\leq p\mu_p^{p-2}(p - 1) - \mu_p^2 \leq 0.\]

Similarly, under (6.4.6),

\[(6.4.8) \quad \mathcal{D}_y'(b) = -\alpha_p b h_p^{(3)}(y + b) + v^{''}_p(b - y) - p\mu_p^p \leq 0.\]

Assume (6.4.1) and fix a \(y \in [\mu_p/2, \mu_p]\),

\[D_y(\mu_p - y) = \alpha_p h_p^{(3)}(\mu_p) - v^{''}_p(\mu_p - 2b) - \alpha_p b h_p^{''}(\mu_p) - pb\mu_p^p,\]

where \(0 \leq b = \mu_p - y \leq \mu_p\). For \(b \in [0, \mu_p]\), let

\[d_1(b) = \alpha_p h_p^{(3)}(\mu_p) - v^{''}_p(\mu_p - 2b) - \alpha_p b h_p^{''}(\mu_p) - pb\mu_p^p.\]

Then by Lemma 5.4 and the fact that \(|v^{''}_p(x)| \leq p(p - 1)|x|^{p-2}\), \(|\mu_p - 2b| \leq \mu_p\), and \(p\mu_p^p = \alpha_p h_p^{''}(\mu_p)\), we have on \([0, \mu_p]\),

\[d_1(0) = \alpha_p h_p^{(3)}(\mu_p) - v^{''}_p(\mu_p) = 0,\]

\[d_1'(b) = 2v^{''}_p(\mu_p - 2b) - p\mu_p^p - \alpha_p h_p^{''}(\mu_p)\]

\[\leq 2p(p - 1)|\mu_p - 2b|^{p-2} - p\mu_p^p - \alpha_p h_p^{''}(\mu_p)\]

\[\leq 2p(p - 1)\mu_p^{p-2} - 2p\mu_p^p = 2p\mu_p^{p-2}(p - 1) - \mu_p^2 \leq 0.\]

Thus by the mean value theorem, \(d_1(b) \leq 0\) on \([0, \mu_p]\). Consequently, \(d_1(\mu_p - y) = D_y(\mu_p - y) \leq 0\). Hence by (6.4.7), \(D_y(b) \leq 0\) on \([\mu_p - y, y]\).

Assume (6.4.2) and fix a \(y \in [\mu_p, \infty)\); then

\[D_y(y - \mu_p) = \alpha_p h_p^{(3)}(\mu_p + 2b) - v^{''}_p(\mu_p) - \alpha_p b h_p^{''}(\mu_p + 2b) - pb\mu_p^p,\]

where \(0 \leq b = y - \mu_p\). For \(b \in [0, \infty)\), let

\[d_2(b) = \alpha_p h_p^{(3)}(\mu_p + 2b) - v^{''}_p(\mu_p) - \alpha_p b h_p^{''}(\mu_p + 2b) - pb\mu_p^p.\]
Then
\[ d_2(0) = \alpha_p h_p'(\mu_p) - v_p'(\mu_p) = 0, \]
\[ d_2'(b) = \alpha_p h_p''(\mu_p + 2b) - 2\alpha_p b h_p^{(3)}(\mu_p + 2b) - p\mu_p. \]

So, by \( \alpha_p h_p''(\mu_p) = p\mu_p^p \) and Lemma 5.3,
\[ d_2'(0) = \alpha_p h_p''(\mu_p) - p\mu_p = 0, \]
\[ d_2''(b) = -4\alpha_p b h_p^{(4)}(\mu_p + 2b) \leq 0. \]

Thus on \([0, \infty)\), \( d_2(b) \leq 0 \) by the mean value theorem. Consequently,
\[ D_y(y - \mu_p) = d_2(y - \mu_p) \leq 0. \]

Using (6.4.7), we have \( D_y(b) \leq 0 \) on \([y - \mu_p, y]\).

Assume (6.4.3), and fix \( y \in [\mu_p/2, \mu_p] \),
\[ D_y(y) = \alpha_p h_p'(2y) - \alpha_p y h_p''(2y) - py\mu_p. \]

For \( y \in [\mu_p/2, \infty) \), let
\[ \alpha_1'(y) = \alpha_p h_p'(2y) - \alpha_p y h_p''(2y) - py\mu_p. \]

Then, by essentially the same computation as above, we get
\[ \alpha_1'(\mu_p/2) = p\mu_p^{p-1}(1 - \mu_p^2) \leq 0, \]
\[ \alpha_1''(\mu_p/2) = -\mu_p\alpha_p h_p^{(3)}(\mu_p) \leq 0 \]
and on \([\mu_p/2, \infty)\),
\[ \alpha_1''(y) \leq 0. \]

Now using the mean value theorem, we have for \( y \in [\mu_p/2, \infty) \), \( \alpha_1'(y) \leq 0 \) and \( \alpha_1(y) \leq 0 \). Thus (6.4.8) and \( D_y(y) = \alpha_1(y) \) imply that
\[ D_y(b) \leq 0 \quad \text{on} \quad [y, \mu_p + y]. \]

Finally assume (6.4.4) and fix a \( y \in [0, \mu_p/2] \); then
\[ D_y(\mu_p - y) = \alpha_p h_p'(\mu_p) + v_p'(2b - \mu_p) - \alpha_p y h_p''(\mu_p) - pb\mu_p, \]
where \( \mu_p/2 \leq b = \mu_p - y \leq \mu_p \). For \( b \in [\mu_p/2, \mu_p] \), let
\[ \alpha_2'(b) = \alpha_p h_p'(\mu_p) + v_p'(2b - \mu_p) - \alpha_p y h_p''(\mu_p) - pb\mu_p. \]

Then again,
\[ \alpha_2'(\mu_p/2) = p\mu_p^{p-1}(1 - \mu_p^2) \leq 0 \]
and
\[ \alpha_2'(b) = 2p(p - 1)(2b - \mu_p)^{p-2} - 2p\mu_p \leq 0. \]
Thus by the mean value theorem, \( \alpha'_{2}(b) \leq 0 \) on \([\mu_{p}/2, \mu_{p}]\). So \( \mathcal{D}_{y}(\mu_{p} - y) = \alpha'_{2}(\mu_{p} - y) \leq 0 \), and by (6.4.8)
\[
\mathcal{D}_{y}(b) \leq 0 \quad \text{on} \quad [\mu_{p} - y, \mu_{p} + y].
\]
This finishes the proof. □

Now back to \( G'_{x}(a) \). Since
\[
G'_{x}(a) = \frac{1}{2}(1 + a^{2})^{(p-1)/2}D_{y}(b),
\]
so by Lemma 6.2
\[
G'_{x}(a) \leq 0 \quad \text{on} \quad [\rho_{1}(x), \rho_{2}(x)].
\]
That is, (6.2) is proven in this case.

Case (III). \( 0 \leq x < \mu_{p}, \quad a > 0 \), and \( (x \pm a)^{2} \geq \mu_{p}^{2}(1 + a^{2}). \)
We will show this is an impossible case.
Since \( 0 \leq x < \mu_{p}, \quad a > 0 \), and, by Lemma 5.4, \( \mu_{p} \geq 1 \) for \( p \geq 3 \),
\[
(x - a)^{2} = x^{2} - 2ax + a^{2} \leq x^{2} + a^{2} < \mu_{p}^{2} + \mu_{p}^{2}a^{2} = \mu_{p}^{2}(1 + a^{2})
\]
which is contrary to \( (x - a)^{2} \geq \mu_{p}^{2}(1 + a^{2}). \)

Case (IV). \( x \geq \mu_{p}, \quad a > 0 \), and \( (x \pm a)^{2} \leq \mu_{p}^{2}(1 + a^{2}). \)
As in Case (I),
\[
G'_{x}(a) = \frac{1}{2}(1 + a^{2})^{(p-1)/2}C_{y}(b), \quad \text{if} \quad 0 \leq a < x,
\]
\[
= \frac{1}{2}(1 + a^{2})^{(p-1)/2}\mathcal{D}_{y}(b), \quad \text{if} \quad 0 \leq x < a.
\]
Thus \( G'_{x}(a) \leq 0 \) by Lemma 6.1 in this case.

Case (V). \( x \geq \mu_{p}, \quad a > 0 \), and \( (x + a)^{2} \geq \mu_{p}^{2}(1 + a^{2}), \quad (x - a)^{2} \leq \mu_{p}^{2}(1 + a^{2}). \)
As in Case (III),
\[
G'_{x}(a) = \frac{1}{2}(1 + a^{2})^{(p-1)/2}D_{y}(b), \quad \text{if} \quad 0 \leq a < x,
\]
\[
= \frac{1}{2}(1 + a^{2})^{(p-1)/2}\mathcal{D}_{y}(b), \quad \text{if} \quad 0 \leq x < a.
\]
Thus \( G'_{x}(a) \leq 0 \) by Lemma 6.2 in this case.

Case (VI). \( x \geq \mu_{p}, \quad a > 0 \), and \( (x \pm a)^{2} \geq \mu_{p}^{2}(1 + a^{2}). \)
Solving the above inequalities, we have
\[
(6.5) \quad 0 \leq a \leq -\rho_{1}(x)
\]
where \( \rho_{1}(x) \) is defined in (6.3). Using the condition \( x \geq \mu_{p} \) and Lemma 5.1, we see
\[
p \geq 2 \Rightarrow \mu_{p}^{2} \geq 1
\]
\[
\Rightarrow (x^{2} + 1)(\mu_{p}^{2} - 1) \geq 0
\]
\[
\Rightarrow x\mu_{p}^{2} \geq \sqrt{\mu_{p}^{2}(x^{2} - \mu_{p}^{2} + 1)}
\]
\[
\Rightarrow x \geq -\rho_{1}(x).
\]
So when $2 \leq p$, $-\rho_1(x) \leq x$. Hence for $0 \leq a \leq -\rho_1(x)$, by (4.1),

$$G'_x(a) = \frac{1}{2} \alpha_p (1 + a^2)^{p/2 - 1} \alpha_p \left\{ h_p \left( \frac{x + a}{\sqrt{1 + a^2}} \right) + h_p \left( \frac{x - a}{\sqrt{1 + a^2}} \right) \right\} + \frac{1}{2} \alpha_p (1 + a^2)^{p/2} \left\{ h'_p \left( \frac{x + a}{\sqrt{1 + a^2}} \right) - \frac{1 - ax}{(1 + a^2)^3} h'_p \left( \frac{x - a}{\sqrt{1 + a^2}} \right) \right\}$$

$$= \frac{1}{2} (1 + a^2)^{(p-1)/2} \alpha_p \left\{ \frac{pa}{\sqrt{1 + a^2}} h_p \left( \frac{x + a}{\sqrt{1 + a^2}} \right) + \frac{pa}{\sqrt{1 + a^2}} h_p \left( \frac{x - a}{\sqrt{1 + a^2}} \right) + h'_p \left( \frac{x + a}{\sqrt{1 + a^2}} \right) \frac{1 - ax}{1 + a^2} - h'_p \left( \frac{x - a}{\sqrt{1 + a^2}} \right) \frac{1 + ax}{1 + a^2} \right\}$$

$$= \frac{1}{2} (1 + a^2)^{(p-1)/2} \alpha_p \left\{ \frac{a}{\sqrt{1 + a^2}} \left[ -h''_p \left( \frac{x + a}{\sqrt{1 + a^2}} \right) + \frac{x + a}{\sqrt{1 + a^2}} h'_p \left( \frac{x + a}{\sqrt{1 + a^2}} \right) - h''_p \left( \frac{x - a}{\sqrt{1 + a^2}} \right) + \frac{x - a}{\sqrt{1 + a^2}} h'_p \left( \frac{x - a}{\sqrt{1 + a^2}} \right) \right] \right\}$$

Define

$$E_y(b) = \alpha_p \{ h'_p(y + b) - h'_p(y - b) - b[h''_p(y + b) + h''_p(y - b)] \}$$

on $\mu_p \leq y$ and $b \in [0, y - \mu_p]$.

Note that (6.5.1) is equivalent to

(6.5.2) $0 \leq b \leq y$, $\mu_p \leq y + b$, $\mu_p \leq y - b$.

We will show $G'_x(a) \leq 0$ on $[0, -\rho_1(x)]$ by the following lemma.

**Lemma 6.3.** (a) $E_y(b)$ is nonpositive on the domain it is defined when $p \geq 3$.

(b) $E_y(b) \geq 0$ on the domain it is defined when $1 < p < 2$ and $2 < p < 3$.

**Proof.** (a) Assume (6.5.1) and fix $y \geq \mu_p$. By (6.5.2), Lemma 5.3, and the mean value theorem,

$$E'_y(b) = \alpha_p [-bh'^{(3)}_p(y + b) + bh'^{(3)}_p(y - b)] = -2b^2 \alpha_p h''_p(\xi) \leq 0,$$
where $\mu_p \leq \xi \in (y-b, y+b)$. Hence by the mean value theorem and $E_y(0) = 0$,
$$E_y(b) \leq 0 \text{ on } [0, y - \mu_p].$$

(b) By the mean value theorem, there exists a $\zeta \in (y-b, y+b)$ such that
$$E_y'(b) = -2b^2 \alpha_p h^{(4)}_p(\zeta).$$

From (6.5.2), $\mu_p \leq \zeta$. When $2 < p < 3$, Lemma 5.3(b) and the expression $\alpha_p$ imply that
$$\alpha_p > 0 \text{ and } E_y'(b) > 0 \text{ on } [0, y - \mu_p].$$

So the mean value theorem and $E_y(0) = 0$ give
$$E_y(b) \geq 0 \text{ on } [0, y - \mu_p].$$

When $1 < p < 2$, Lemma 5.3(a) implies
$$\alpha_p < 0 \text{ and } E_y'(b) > 0 \text{ on } [0, y - \mu_p].$$

So again we have
$$E_y(b) \geq 0 \text{ on } [0, y - \mu_p].$$

This concludes the proof. □

We turn again to $G_x'(a)$. Since
$$G_x'(a) = \frac{1}{2}(1 + a^2)(p-1)^{1/2}E_y(b),$$

So by Lemma 6.3 $G_x'(a) \leq 0$ on $[0, -\rho_1(x)]$. This completes the proof of (6.2).

7. Proof of the negativity property

From the definition of $U_p(x)$, (4.6) is equivalent to

(7.1) $u_p(1) \leq 0$ for $p > 1$.

By the definition of $u_p(x)$ and Lemma 5.4, (7.1) is equivalent to

(7.2) $v_p(1) \leq 0$ if $p \geq 2$,

and

(7.3) $\alpha_p h_p(1) \leq 0$ if $1 < p \leq 2$.

Now using Lemma 5.3(a) and Lemma 5.4, we have $v_p(1) \leq 0$ for $p \geq 2$, and $\alpha_p \leq 0$ and $h_p(1) \geq 0$ for $1 < p < 2$. So (7.2) and (7.3) both hold.

8. Remark on the case $p < 3$

We discuss here why (4.5) does not hold for $1 < p < 2$ and $2 < p < 3$. In fact we will show (6.1)' fails to be true when $x \geq \mu_p$, $a \geq 0$, and $(x \pm a)^2 \geq \mu_p(1 + a^2)$ in both cases. When $p = 1$ or $p = 2$, (4.5) is trivially true.
If $1 < p < 2$, solving $x \geq \mu_p$, $(x \pm a)^2 \geq \mu_p^2(1 + a^2)$, we have

$$0 \leq a \leq -\rho_1(x), \quad \text{or} \quad a \geq -\rho_2(x),$$

where $\rho_1(x)$ and $\rho_2(x)$ are defined in (6.3).

A similar argument as in Case (VI) shows that $x \leq -\rho_1(x)$ when $1 < p < 2$. Fix an $x \geq \mu_p$. For $a \in [0, x]$, let $y = x/\sqrt{1 + a^2}$ and $b = a/\sqrt{1 + a^2}$, then $y$ and $b$ satisfy (6.5.2). Hence, by the proof of Lemma 6.3(b),

$$G_x'(a) = \frac{1}{2}(1 + a^2)^{(p-1)/2}E_y(b) > 0 \quad \text{on } (0, x].$$

Thus, by the mean value theorem and $G_x(0) = 0$,

$$G_x(a) > 0 \quad \text{on } (0, x],$$

which is contrary to (6.1)'.

If $2 < p < 3$, then $\mu_p \leq x$, $a \geq 0$, $(x \pm a)^2 \geq \mu_p^2(1 + a^2)$ imply that

$$0 \leq a \leq -\rho_1(x),$$

where $\rho_1(x)$ is defined in (6.3), and $x \geq -\rho_1(x)$ as we have shown in Case (VI). Fix $x \geq \mu_p$. Thus $y = x/\sqrt{1 + a^2}$ and $b = a/\sqrt{1 + a^2}$ satisfy (6.5.3); we then have on $(0, -\rho_1(x)]$ by Lemma 6.3(b),

$$G_x'(a) = \frac{1}{2}(1 + a^2)^{(p-1)/2}E_y(b) > 0.$$

So $G_x(a) > 0$ on $(0, -\rho_1(x)]$ by the mean value theorem and $G_x(0) = 0$, which is contrary to (6.1)'.

9. The existence of the function $U_p(x, t)$: Hilbert space case

We want to generalize the above results to Hilbert spaces. Namely, we want to show that the analogues of (4.4)–(4.6) hold in Hilbert spaces. We will then have (1.7)' of Theorem 1'.

Let $\alpha_p$ and $u_p$ be as in §4. Define for $x \in \mathbb{H}$, $t \geq 0$,

$$U_p(x, t) = \begin{cases} \frac{1}{2} t^{p/2} u_p(|x|/\sqrt{t}) & \text{if } t > 0, \\ \alpha_p |x|^p & \text{if } t = 0, \end{cases}$$

for any inner product spaces $\mathbb{H}$. For any element $x \in \mathbb{H}$, denote $X = |x|$, $\cos \theta = (x \cdot a)/XA$, where $\cdot$ is the inner product. Then

$$|x + a| = \sqrt{X^2 + A^2 + 2XA \cos \theta}.$$ 

The analogue of (4.4) is

$$(9.1) \quad U_p(x, t) \geq V_p(x, t).$$

This is trivially true by the definition and (4.4).
The analogue of (4.5) is

\[(9.2) \quad \frac{1}{2} \{ U_p(x + a, t + A^2) + U_p(x - a, t + A^2) \} - U_p(x, t) \leq 0 \]

for \( p \geq 3 \) and all \( x \in \mathbb{H}, a \in \mathbb{H}, t \geq 0 \).

By our definition it is equivalent to

\[
\frac{1}{2} (1 + A^2)^{p/2} \left\{ u_p \left( \left( \frac{X^2 + A^2 + 2AX \cos \theta}{1 + A^2} \right)^{1/2} \right) \right. \\
+ \left. u_p \left( \left( \frac{X^2 + A^2 - 2AX \cos \theta}{1 + A^2} \right)^{1/2} \right) \right\} - u_p(X) \leq 0 \quad \text{for } X > 0, A > 0, \text{ and } \cos \theta \in [0, 1].
\]

Define

\[
\mathcal{R}_p(t) = \frac{1}{2} (1 + A^2)^{p/2} \left\{ u_p \left( \left( \frac{X^2 + A^2 + 2AXt}{1 + A^2} \right)^{1/2} \right) \right. \\
+ \left. u_p \left( \left( \frac{X^2 + A^2 - 2AXt}{1 + A^2} \right)^{1/2} \right) \right\} - u_p(X).
\]

Since \( \mathcal{R}_p(1) = G_x(A) \leq 0 \), it suffices to show \( \mathcal{R}_p'(t) \geq 0 \) on \([0, 1]\).

As in the real case, there are six cases according to the definition of \( u_p(x) \).

**Case (I).** \( 0 \leq X < \mu_p, A \geq 0, X^2 + A^2 \pm 2AXt \leq \mu_p^2(1 + A^2) \).

Since

\[ X^2 + A^2 + 2AXt \geq X^2 + A^2 - 2AXt \]

and \( p \geq 2 \), we have

\[
\mathcal{R}_p'(t) = \frac{1}{2} pXA(1 + A^2)^{p/2-1} \left\{ \left( \frac{X^2 + A^2 + 2AXt}{1 + A^2} \right)^{p/2-1} \right. \\
- \left. \left( \frac{X^2 + A^2 - 2AXt}{1 + A^2} \right)^{p/2-1} \right\} \geq 0
\]

in this case.

**Case (II).** \( 0 \leq X < \mu_p, A \geq 0, X^2 + A^2 + 2AXt \geq \mu_p^2(1 + A^2), \text{ and } X^2 + A^2 - 2AXt \leq \mu_p^2(1 + A^2) \).
In this case,

\[ s_p(t) = \frac{1}{2} AX(1 + A^2)^{p/2 - 1} \left\{ \alpha_p \left( \frac{X^2 + A^2 + 2AXt}{1 + A^2} \right)^{-1/2} \right. \]

\[ \times h_p' \left( \left( \frac{X^2 + A^2 + 2AXt}{1 + A^2} \right)^{1/2} \right) \]

\[ - p \left( \frac{X^2 + A^2 - 2AXt}{1 + A^2} \right)^{(p/2) - 1} \}

\[ \right\} \]

\[ \geq 0 \]

since \( pX^{p-1} \leq \alpha_p h_p'(X) \) on \([\mu_p, \infty)\) by (5.19).

Case (III). \( 0 \leq X < \mu_p, \ A \geq 0, \ X^2 + A^2 \pm 2AX \cos \theta \geq \mu_p^2(1 + A^2). \)

As in the real case in §6, this is impossible.

Case (IV). \( X \geq \mu_p, \ A \geq 0, \ X^2 + A^2 \pm 2AXt \leq \mu_p^2(1 + A^2). \)

Case (V). \( X \geq \mu_p, \ A \geq 0, \ X^2 + A^2 \pm 2AXt \geq \mu_p^2(1 + A^2), \) and \( X^2 + A^2 - 2AXt \leq \mu_p^2(1 + A^2). \)

\( s_p(t) \) in Cases (IV) and (V) has the same expression as in Cases (I) and (II). So it is nonnegative.

Case (VI). \( X \geq \mu_p, \ A \geq 0, \ X^2 + A^2 \pm 2AXt \geq \mu_p^2(1 + A^2). \)

In this case,

\[ s_p(t) = \frac{1}{2} AX \alpha_p(1 + A^2)^{p/2 - 1} \]

\[ \times \left\{ \left( \frac{X^2 + A^2 + 2AXt}{1 + A^2} \right)^{-1/2} h_p' \left( \left( \frac{X^2 + A^2 + 2AXt}{1 + A^2} \right)^{1/2} \right) \right. \]

\[ \left. - \left( \frac{X^2 + A^2 - 2AXt}{1 + A^2} \right)^{-1/2} h_p' \left( \left( \frac{X^2 + A^2 - 2AXt}{1 + A^2} \right)^{1/2} \right) \right\}. \]

Let \( C(X) = h_p'(X)/X \) for \( X \geq \mu_p \). Then,

\[ C'(X) = (Xh_p''(X) - h_p'(X))/X^2 \]

\[ = (h_p^{(3)}(X) + (p - 2)h_p'(X))/X^2 \geq 0 \]
by Lemma 5.3 and the fact that \( h_p \) satisfies differential equation (4.1). Thus if \( \mu_p \leq X_1 \leq X_2 \), then by the mean value theorem, \( C(X_1) \leq C(X_2) \). Hence if we let
\[
X_1 = \left( \frac{X^2 + A^2 - 2AXt}{1 + A^2} \right)^{1/2}, \quad X_2 = \left( \frac{X^2 + A^2 + 2AXt}{1 + A^2} \right)^{1/2},
\]
then, by the above inequality, \( \mathcal{R}_p'(t) \geq 0 \).
This completes the proof of (9.2).

The analogue of (4.6) is
\[
(9.12) \quad U_p(a, |a|^2) \leq 0 \quad \text{for } a \in \mathbb{R}.
\]
This is trivially true. Combining (9.1), (9.2), and (9.12), we see that the function \( U_p(x, t) \) satisfies the properties described in §3.

10. The existence of the function \( \overline{U}_p(x, t) \)

Recall from §2 that \( \nu_p \) is the smallest positive zero of \( M_p \). We define
\[
\overline{\alpha}_p = \overline{\nu}_p'(\nu_p)/M_p'(\nu_p) \quad \text{and} \quad \overline{w}_p(x) = \overline{\alpha}_p M_p(x).
\]
Also let
\[
\overline{u}_p(x) = \overline{w}_p(|x|) \quad \text{for } 0 \leq |x| \leq \nu_p,
= \overline{\nu}_p(|x|) \quad \text{for } \nu_p \leq |x| < \infty,
\]
\[
\overline{W}_p(x, t) = t^{p/2} \overline{w}_p(|x|/\sqrt{t}) \quad \text{for } t \neq 0,
\]
and
\[
\overline{U}_p(x, t) = t^{p/2} \overline{u}_p(|x|/\sqrt{t}) \quad \text{for } t > 0,
= - \text{sgn}(p - 2)|x|^p \quad \text{for } t = 0.
\]
Then functions \( \overline{U}_p(x, t) \) when \( p \geq 2 \) and \( \overline{W}_p(x, t) \) when \( 0 < p < 2 \) satisfy conditions (3.5)'–(3.7)' in §3. See Chapter 3 of Wang [17] for details. Thus the existence of the function \( \overline{U}_p(x, t) \) is ensured. When \( \mathbb{H} = \mathbb{R} \), this gives another proof of Davis’ [7] results.

11. The sharpness of the constants \( \nu_p \) and \( \mu_p \)

For the case \( \mathbb{H} = \mathbb{R} \), Davis [7] showed that \( \nu_p \) is the best possible constant in (1.6) and (1.8) of Theorem 1. The same procedure can be used to obtain a similar result for \( \mu_p \) in the case \( \mathbb{H} = \mathbb{R} \) and \( p \geq 3 \). The fact that (1.7) does hold in the real case is the main result of this paper. These inequalities, the inequalities (1.6), (1.7), and (1.8) of Theorem 1, which we have shown to be valid for any Hilbert space \( \mathbb{H} \), must therefore be sharp for \( \mathbb{H} \).
REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801
Current address: Department of Mathematics, DePaul University, Chicago, Illinois 60614

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use