ON TOPOLOGICAL CLASSIFICATION OF FUNCTION SPACES $C_p(X)$ OF LOW BOREL COMPLEXITY

T. DOBROWOLSKI, W. MARCISZEWSKI AND J. MOGILSKI

Abstract. We prove that if $X$ is a countable nondiscrete completely regular space such that the function space $C_p(X)$ is an absolute $F_{\sigma\delta}$-set, then $C_p(X)$ is homeomorphic to $\sigma^\infty$, where $\sigma = \{(x_i) \in \mathbb{R}^\infty : x_i = 0$ for all but finitely many $i\}$. As an application we answer in the negative some problems of A. V. Arhangel'skii by giving examples of countable completely regular spaces $X$ and $Y$ such that $X$ fails to be a $b_R$-space and a $k$-space (and hence $X$ is not a $k_\omega$-space and not a sequential space) and $Y$ fails to be an $\aleph_0$-space while the function spaces $C_p(X)$ and $C_p(Y)$ are homeomorphic to $C_p(\mathbb{R})$ for the compact metric space $\mathbb{I} = \{0\} \cup \{n^{-1} : n = 1, 2, \ldots\}$.

1. Introduction

For a space $X$ we define $C_p(X)$ to be the set of all continuous real valued functions on $X$ endowed with the topology of pointwise convergence. The subspace of $C_p(X)$ consisting of all bounded functions is denoted by $C^*_p(X)$. This paper is devoted to the topological classification of $C_p(X)$ and $C^*_p(X)$ for countable completely regular spaces $X$. Let us note that if $X$ is nondiscrete, then $C_p(X)$ is a dense linear subspace of the countable cartesian product of real lines $\mathbb{R}^X$ (identified with $\mathbb{R}^\infty$), otherwise $C_p(X) = \mathbb{R}^\infty$ or $\mathbb{R}^k$. In [DGM] it was proved that for every countable metrizable nondiscrete space $X$ the spaces $C_p(X)$ and $C^*_p(X)$ are homeomorphic to $\sigma^\infty$, where $\sigma = \{(x_i) \in \mathbb{R}^\infty : x_i = 0$ for all but finitely many $i\}$ (cf. [vM, BGvM, BGvMP]). Extending the work of [DGM] we focus on the case when $C_p(X)$ is an absolute Borel set. The main result of this paper is the following

1.1. Theorem. Let $X$ be a countable nondiscrete completely regular space such that the function space $C_p(X)$ is an absolute $F_{\sigma\delta}$-set. Then $C_p(X)$ and $C^*_p(X)$ are homeomorphic to $\sigma^\infty$.

Since, for a countable metrizable space $X$, the space $C_p(X)$ is an absolute $F_{\sigma\delta}$-set Theorem 1.1 generalizes the result of [DGM]. According to [DGLvM], $C_p(X)$ cannot be an absolute $G_{\delta\sigma}$-set, provided that $X$ is nondiscrete. Thus

Received by the editors September 15, 1989.
1980 Mathematics Subject Classification (1985 Revision). Primary 57N17, 57N20, 54C35.
Key words and phrases. Function space, pointwise convergence topology, Borelian filter, $k$-space, $k_\omega$-space, $b_R$-space, $\aleph_0$-space.

©1991 American Mathematical Society
0002-9947/91 $1.00 + .25$ per page

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Theorem 1.1 gives a complete topological classification of spaces \( C_p(X) \) which are absolute Borel sets of the class not higher than 2. To the best of our knowledge there is no classification result for spaces \( C_p(X) \) of higher Borel complexity. Let us mention that all multiplicative classes of Borel sets \( \mathcal{M}_\alpha \), where \( \alpha \geq 1 \), are represented among spaces \( C_p(X) \) (see [LvMP, Ca]). We conjecture that the Borel class determines the topological type of a space \( C_p(X) \).

The essential step in classifying spaces \( C_p(X) \) is the case of countable spaces \( X \) which have exactly one nonisolated point. Such \( X \) are precisely the spaces \( \mathbb{N}_F = \{\infty\} \cup \{0, 1, 2, \ldots\} \) topologized by isolating the points of \( \mathbb{N} = \{0, 1, 2, \ldots\} \) and by using the family \( \{A \cup \{\infty\}: A \in F\} \) as a neighborhood base at \( \infty \), where \( F \) is a filter on \( \mathbb{N} \). We recall that a family \( F \subseteq 2^\mathbb{N} \) is a filter on a set \( Y \) if \( \emptyset \notin F \), \( A \cap B \in F \) provided \( A, B \in F \) and \( A \subseteq C \subseteq Y \), \( A \in F \) implies \( C \in F \). We say that filters \( F \) on a set \( Y \) and \( G \) on a set \( Z \) are isomorphic if there exists a bijection \( \alpha: Y \to Z \) such that \( A \in F \) iff \( \alpha(A) \in G \). By \( \mathfrak{F} \) we denote the filter on \( \mathbb{N} \) consisting of all cofinite subsets of \( \mathbb{N} \). Obviously, the space \( \mathbb{N}_F \) is homeomorphic to the space \( \mathfrak{X} = \{0\} \cup \{n^{-1}: n = 1, 2, \ldots\} \subseteq \mathbb{R} \). The spaces \( C_p(\mathbb{N}_F) \) can have arbitrarily high Borel complexity; furthermore they may not be Borelian (see [LvMP]). Corollary 3.6 and Theorem 8.8 seem to be useful in classifying general spaces \( C_p(\mathbb{N}_F) \) and they are motivated by the \( F_{\sigma\delta} \)-case.

Here, similarly as in [DGM], we employ the method of absorbing sets but we do it in a more implicit way. We also explain basic facts on first category filters.

We also discuss two examples of filters \( F \) and \( G \) such that \( \mathbb{N}_F \) fails to be a \( b_R \)-space and a \( k \)-space (and hence \( \mathbb{N}_F \) is not a \( k_\omega \)-space and not a sequential space) and \( \mathbb{N}_G \) fails to be an \( N_0 \)-space while the function spaces \( C_p(\mathbb{N}_F) \) and \( C_p(\mathbb{N}_G) \) are absolute \( F_{\sigma\delta} \)-sets and according to Theorem 1.1 are homeomorphic to \( C_p(\mathbb{N}_\mathfrak{F}) \), for the compact metric space \( \mathbb{N}_\mathfrak{F} \). This solves in the negative several problems of A. V. Arhangel’skii from [Ar1, 2].

2. First category filters

As usual \( 2^\mathbb{N} \) denotes the set of all subsets of \( \mathbb{N} \). If \( A, S \in 2^\mathbb{N} \), then we write \( V(A, S) = \{C \in 2^\mathbb{N}: C \cap S = A \cap S\} \). We will consider the space \( 2^\mathbb{N} \) endowed with the topology generated by all sets \( V(A, S) \) for finite \( S \). Obviously, \( 2^\mathbb{N} \) can be identified with the Cantor set \( \{0, 1\}^\mathbb{N} \). In this notation \( \mathbb{N} \) can be replaced by any infinite countable set.

The following two lemmas are inspired by [LM, Theorems 4.6, 5.1, and 6.3]. Their proofs presented below are slight modifications of the reasoning from [LM] and they do not use the language of the game theory employed there.

2.1. Lemma. Let \( \{G_k\} \) be a decreasing sequence of open dense subsets of \( 2^\mathbb{N} \). Then for each finite tuple \((i_1, i_2, \ldots, i_k)\) of elements of \( \mathbb{N} \) we can assign a finite subset \( S(i_1, i_2, \ldots, i_k) \) of \( \mathbb{N} \) such that
TOPOLOGICAL CLASSIFICATION OF FUNCTIONS SPACES $C_p(X)$

(1) the families \( \{S(j)\}_{j=1}^{\infty} \) and \( \{S(i_1, i_2, \ldots, i_k, j)\}_{j=1}^{\infty} \) are pairwise disjoint,

(2) for every sequence \( \{i_k\}_{k=1}^{\infty} \) there exists \( C \in \bigcap_{n=1}^{\infty} G_n \) such that \( C \supseteq \mathbb{N} \setminus \bigcup_{k=1}^{\infty} S(i_1, i_2, \ldots, i_k) \).

Proof. We will construct the required sequence of finite sets \( \{S(i_1, i_2, \ldots, i_k)\} \) inductively on \( k \). Simultaneously, we will construct a sequence of subsets of \( \mathbb{N} \), \( \{C(i_1, i_2, \ldots, i_k)\} \). We will use the following notation:

\[
R(i_1, i_2, \ldots, i_k) = \bigcup_{j=0}^{i_k} S(i_1, i_2, \ldots, i_{k-1}, j)
\]

and

\[
T(i_1, i_2, \ldots, i_k) = R(i_1) \cup R(i_1, i_2) \cup \cdots \cup R(i_1, i_2, \ldots, i_k).
\]

We require that the sequences \( \{S(i_1, i_2, \ldots, i_k)\} \), \( \{R(i_1, i_2, \ldots, i_k)\} \), and \( \{T(i_1, i_2, \ldots, i_k)\} \) together with \( \{C(i_1, i_2, \ldots, i_k)\} \) satisfy the following conditions indexed by \( k \):

(a) \( S(0) = \emptyset \) and \( C(0) = \mathbb{N} \),

(b) \( V(C(i_1), R(i_1)) \subseteq G_1 \cap V(C(0), R(i_1 - 1)) \) and \( S(i_1) \cap R(i_1 - 1) = \emptyset \) for \( k = 1 \) and \( i_1 = 1, 2, \ldots \), and

(a_k) the family \( S(0), S(1), \ldots, S(i_1), S(i_1, 0), S(i_1, 1), \ldots, S(i_1, i_2), \ldots, S(i_1, i_2, \ldots, i_{k-1}, 0), S(i_1, i_2, \ldots, i_{k-1}, 1), \ldots, S(i_1, i_2, \ldots, i_k) \) is pairwise disjoint,

(b_k)

\[
V(C(i_1, i_2, \ldots, i_{k-1}, 0), T(i_1, i_2, \ldots, i_{k-1}, 0)) \subseteq G_k \cap V(C(i_1, i_2, \ldots, i_{k-1}), T(i_1, i_2, \ldots, i_{k-1})),
\]

(c_k)

\[
V(C(i_1, i_2, \ldots, i_{k-1}, i_k), T(i_1, i_2, \ldots, i_{k-1}, i_k)) \subseteq G_k \cap V(D, T(i_1, i_2, \ldots, i_{k-1}, i_k - 1)),
\]

where \( D = C(i_1, i_2, \ldots, i_{k-1}) \cup R(i_1, i_2, \ldots, i_{k-1}, i_k) \) if \( i_k > 0 \) for \( k > 1 \) and for every \( k \)-tuple \( (i_1, i_2, \ldots, i_k) \).

Let us assume that our construction is completed for some \( k \). Now we shall describe the construction for an arbitrary \( (k + 1) \)-tuple. Fix a \( k \)-tuple \( (i_1, i_2, \ldots, i_k) \). First we choose \( S(i_1, i_2, \ldots, i_k, 0) \) and \( C(i_1, i_2, \ldots, i_k, 0) \) in \( 2^\mathbb{N} \) such that \( S(i_1, i_2, \ldots, i_k, 0) \) is finite and the conditions (a_k+1) and (b_k+1) are satisfied. The \( S(i_1, i_2, \ldots, i_k, 0) \) and \( C(i_1, i_2, \ldots, i_k, 0) \) can be found to satisfy (a_k+1) and (c_k+1).

Now, for every sequence \( \{i_k\}_{k=1}^{\infty} \) we have

\[
C(i_1, i_2, \ldots, i_n) \cap T(i_1, i_2, \ldots, i_n) = C(i_1, i_2, \ldots, i_m) \cap T(i_1, i_2, \ldots, i_n)
\]

for \( n = 1, 2, \ldots \) and \( m \geq n \). By our construction, if

\[
C = \left( \mathbb{N} \setminus \bigcup_{k=1}^{\infty} S(i_1, i_2, \ldots, i_k) \right) \cap \bigcup_{k=1}^{\infty} (S(i_1, i_2, \ldots, i_k) \cap C(i_1, i_2, \ldots, i_k)),
\]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
then
\[ C \cap \left( \bigcup_{k=1}^{\infty} T(i_1, i_2, \ldots, i_k) \right) = \bigcup_{k=1}^{\infty} (C(i_1, i_2, \ldots, i_k) \cap T(i_1, i_2, \ldots, i_k)) \]
and hence the set \( C \) satisfies condition (2).

2.2. Lemma. Let \( F \) be a family of subsets of \( N \) with the property that \( A \subseteq B \), \( A \in F \) implies \( B \in F \). Then the following conditions are equivalent:

(a) \( F \) is a first category subset of \( 2^N \).

(b) There exists a matrix \( \{A(n, m): n, m = 1, 2, \ldots\} \) of finite subsets of \( N \) such that each row \( \{A(n, m): m = 1, 2, \ldots\} \) is pairwise disjoint and for every sequence \( \{m(n): n = 1, 2, \ldots\} \) and every \( A \in F \) we have \( A \cap \bigcup_{n=1}^{\infty} A(n, m(n)) \neq \emptyset \).

(c) There exists a matrix \( \{A(n, m): n, m = 1, 2, \ldots\} \) of pairwise disjoint finite subsets of \( N \) such that for every sequence \( \{m(n): n = 1, 2, \ldots\} \) and every \( A \in F \) we have \( A \cap \bigcup_{n=1}^{\infty} A(n, m(n)) \neq \emptyset \).

Proof. (a) \( \Rightarrow \) (b). Since \( F \) is a first category subset of \( 2^N \) there exists a decreasing sequence of open dense subsets \( \{G_n\}_{n=1}^{\infty} \) of \( 2^N \) such that \( \bigcap_{n=1}^{\infty} G_n \cap F = \emptyset \). Let \( \{S(i_1, i_2, \ldots, i_k)\} \) be the family of finite sets satisfying (1) and (2) of Lemma 2.1. The entries of a required matrix will be just the sets \( S(i_1, i_2, \ldots, i_k) \). We set \( A(1, m) = S(m) \) and let each family
\[ \{S(i_1, i_2, \ldots, i_k, m): m = 1, 2, \ldots\} \]
form a row \( \{A(n, m): m = 1, 2, \ldots\} \). Assume that the matrix \( \{A(n, m): n, m = 1, 2, \ldots\} \) fails to satisfy (b). Then there exists a sequence \( \{i_k\} \) and \( A \in F \) such that \( A \cap \bigcup_{n=1}^{\infty} S(i_1, i_2, \ldots, i_k) = \emptyset \). By (2) of Lemma 2.1, there exists \( C \in \bigcap_{n=1}^{\infty} G_n \) such that \( N \setminus \bigcup_{n=1}^{\infty} S(i_1, i_2, \ldots, i_k) \subseteq C \). Therefore \( A \subseteq C \), yielding \( C \in F \cap \bigcap_{n=1}^{\infty} G_n \), a contradiction.

(b) \( \Rightarrow \) (c). It is enough to find sequences \( \{m(n, j): j = 1, 2, \ldots\} \) such that the family \( \{A(n, m(n, j)): n, j = 1, 2, \ldots\} \) is pairwise disjoint. Set \( m(1, 1) = 1 \) and assume \( \{m(n, j): n + j \leq p\} \) has been constructed. For \( l \in N \) such that \( 1 \leq l \leq p \) we pick \( m(l, p + 1 - l) \) to be the first index such that \( A(l, m(l, p + 1 - l)) \) is disjoint with the following finite set:
\[ \bigcup_{n=1}^{\infty} \{A(n, m(n, j)): n < 1 \text{ and } n + j = p + 1\} \cup \bigcup_{n=1}^{\infty} \{A(n, m(n, j)): n + j \leq p\}. \]

(c) \( \Rightarrow \) (a). Condition (c) is equivalent to the following one: for every \( A \in F \) there exists \( n \) such that \( A(n, m) \cap A \neq \emptyset \) for \( m = 1, 2, \ldots \). Write \( X_n = \{C \in 2^N : \forall_m A(n, m) \cap C \neq \emptyset \} \). It follows that \( F \subseteq \bigcup_{n=1}^{\infty} X_n \). Moreover, each set \( X_n \) is a closed boundary subset of \( 2^N \).

Let us recall that a filter \( F \) on \( N \) is said to be free if \( \bigcap_{A \in F} A = \emptyset \). Obviously, a filter \( F \) is free iff \( F \) is dense in \( 2^N \), and iff \( \exists \subseteq F \).

2.3. Lemma. Let \( F \) be a free filter on \( N \). Then the following conditions are equivalent:
(i) $F$ is an element of the sigma-algebra generated by the open subsets and the first category subsets of $2^\mathbb{N}$.

(ii) $F$ is a first category subset of $2^\mathbb{N}$.

Proof. It is enough to show the implication (i) $\Rightarrow$ (ii). By the assumption $F = (U \setminus X) \cup Y$, where $U$ is an open subset of $2^\mathbb{N}$ and both $X$ and $Y$ are first category subsets of $2^\mathbb{N}$. Assume that $F$ is not a first category subset of $2^\mathbb{N}$. Then $U \neq \emptyset$, and hence there exists $C \in U$ and $l \in \mathbb{N}$ such that $V(C, S) \subseteq U$, where $S = \{1, 2, \ldots, l\}$. Write $N_0 = \mathbb{N} \setminus S$ and let $F_0 = \{A \cap N_0 : A \in F\}$. Since $F$ is dense in $2^\mathbb{N}$, $F_0$ is a dense filter on $N_0$. Moreover, since $F$ contains a dense $G_\delta$ subset of $U$, $F_0$ contains a dense $G_\delta$ subset of $2^{N_0}$. Let $\xi: 2^{N_0} \to 2^{N_0}$ be the homeomorphism assigning to each $C$ the set $N_0 \setminus C$. By the filter property we get $\xi(F_0) \cap F_0 = \emptyset$. Consequently, $2^{N_0}$ contains two disjoint dense $G_\delta$ subsets which contradicts the Baire category theorem.

2.4. Proposition. For every filter $F$ on $\mathbb{N}$ and every decomposition $\mathbb{N} = \bigcup_{i=1}^\infty N_i$ of $\mathbb{N}$ into infinite pairwise disjoint sets $N_i$ we write $F_i = \{A \cap N_i : A \in F\}$. Then we have

1. for every $A_i \in F_i$, $i = 1, 2, \ldots, l$, $\bigcup_{i=1}^j A_i \cup \bigcup_{i>j} N_i \in F$,
2. $F_i$ embeds as a closed subset of $F$,
3. if, in addition, $F$ is a free first category filter, then there exists a decomposition $\mathbb{N} = \bigcup_{i=1}^\infty N_i$ such that each $F_i$ is a free filter on $N_i$.

Proof. To verify (1), observe that $\bigcap_{i=1}^j \tilde{A}_i \subseteq \bigcup_{i=1}^j A_i \cup \bigcup_{i>j} N_i$, where $\tilde{A}_i \in F$ and $\tilde{A}_i \cap N_i = A_i$. The map which assigns to each $A \in F_i$ the set $A \cup (N \setminus N_i)$ is a closed embedding of $F_i$ into $F$. To prove the last assertion, pick from Lemma 2.2(c) a matrix $\{A(n, m) : n, m = 1, 2, \ldots\}$ of pairwise disjoint finite sets such that $\bigcup\{A(n, m) : n = 1, 2, \ldots\} \cap A \neq \emptyset$ for $A \in F$ and $m = 1, 2, \ldots$. We let $N_i = \bigcup_{m=1}^\infty A(m, i)$ for $i \geq 2$ and $N_1 = N \setminus \bigcup_{i=2}^\infty N_i$.

3. Z$_\sigma$-PROPERTY OF FUNCTION SPACES $C_p(X)$

We recall that a closed subset $X$ of an absolute neighborhood retract $M$ is a Z-set if every map $f: K \to M$ of a compactum $K$ into $M$ can be approximated by maps $\tilde{f}: K \to M \setminus \overline{X}$. A space which is a countable union of its own Z-sets is called a Z$_\sigma$-space. In this section we prove that some spaces $C_p(X)$ (and their subspaces) are Z$_\sigma$-spaces. We will need the following well-known fact about Z$_\sigma$-spaces.

3.1. Fact. Let $X$ and $Y$ be dense linear subspaces of $\mathbb{R}^\infty$ such that $X \subseteq Y$ and $Y$ is a Z$_\sigma$-space. Then $X$ is a Z$_\sigma$-space.

3.2. Lemma. For every completely regular infinite countable space $X$ the function space $C_p(X)$ is a Z$_\sigma$-space.

Proof. We identify $X$ with $\mathbb{N}$. Then $C_p(X)$ is a dense linear subspace of $\mathbb{R}^\infty$ which is contained in the subspace $\mathbb{R}^\infty_{bd} = \{(x_n) \in \mathbb{R}^\infty : \sup |x_n| < \infty\}$. The space $\mathbb{R}^\infty_{bd}$ is a Z$_\sigma$-space. By 3.1 the space $C_p(X)$ is a Z$_\sigma$-space.
Throughout the paper we use the following subspaces of $\mathbb{R}^\infty$. If $F \subseteq 2^\mathbb{N}$, then
\[
c_F = \left\{ (x_n) \in \mathbb{R}^\infty : \forall \varepsilon > 0 \exists A \subseteq F \left| x_n \right| \leq \varepsilon \text{ for all } n \in A \right\},
\]
\[
c_F^* = \left\{ (x_n) \in c_F : \sup_{n \geq 1} |x_n| < \infty \right\}, \text{ and}
\]
\[
B_F(r) = \left\{ (x_n) \in c_F : \sup_{n \geq 1} |x_n| \leq r \right\}, \text{ where } r > 0.
\]

3.3. **Proposition.** Let $F$ be a family of subsets of $\mathbb{N}$ such that $\emptyset \subseteq F$ and $A \subseteq B$, $A \in F$ implies $B \in F$. If $F$ as a subset of $2^\mathbb{N}$ is of the first category, then the sequence spaces $c_F$, $c_F^*$, and $B_F(r)$ are $Z_\sigma$-spaces.

**Proof.** By (b) of Lemma 2.2 there exists a matrix $\{A(n, m) : n, m = 1, 2, \ldots \}$ of pairwise disjoint finite subsets of $\mathbb{N}$ so that for every $A \in F$ there exists $n$ such that $A(n, m) \cap A \neq \emptyset$ for all $m$. Fix $r > 0$. Let $X_n(r) = \left\{ (x_i) \in c_F : \sup_{k \in A(n, m)} |x_k| \leq r/2 \right\}$ for $n = 1, 2, \ldots$. Clearly, each $X_n(r)$ is a closed subset of $c_F$ and $c_F = \bigcup_{n=1}^{\infty} X_n(r)$, $c_F^* = \bigcup_{n=1}^{\infty} X_n(r) \cap c_F^*$ and $B_F(r) = \bigcup_{n=1}^{\infty} X_n(r) \cap B_F(r)$. Fix $n \in \mathbb{N}$. We shall show that the sets $X_n(r)$, $X_n(r) \cap c_F^*$, and $X_n(r) \cap B_F(r)$ are $Z$-sets in the suitable spaces. Let $f : K \to c_F$ be a map of a compactum $K$. Let, for $m = 1, 2, \ldots$, $g_m : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the map defined by $g_m((x_i)) = (y_i)$, where $y_i = 0$ for $i > \max A(n, m)$, $y_i = r$ for $i \in A(n, m)$, and $y_i = x_i$ otherwise. If $m$ is sufficiently large then the map $\overline{f} = g_m f : K \to \mathbb{R}^\infty$ closely approximates $f$ and satisfies $\overline{f}(K) \cap X_n(r) = \emptyset$. Since $\emptyset \subseteq F$ we additionally have $\overline{f}(K) \subseteq c_F^*$; moreover, if $f(K) \subseteq c_F^*$ or $f(K) \subseteq B_F(r)$ then also $\overline{f}(K) \subseteq c_F^*$ or $\overline{f}(K) \subseteq B_F(r)$, respectively. The proof is complete.

3.4. **Corollary.** For every free Borelian filter on $\mathbb{N}$ the spaces $c_F$, $c_F^*$, and $B_F(r)$, $r > 0$, are $Z_\sigma$-spaces.

**Proof.** Follows immediately from 2.3 and 3.3.

It is standard that for every filter $F$ the spaces $C_p(\mathbb{N}_F)$ and $C_p^*(\mathbb{N}_F)$ are linearly isomorphic to the products $\mathbb{R} \times c_F$ and $\mathbb{R} \times c_F^*$, respectively.

3.5. **Corollary.** For every free first category filter $F$ on $\mathbb{N}$ the space $C_p(\mathbb{N}_F)$ is a $Z_\sigma$-space.

**Proof.** By Proposition 3.3 the space $c_F$ and consequently the product $\mathbb{R} \times c_F$ are $Z_\sigma$-spaces. Thus $C_p(\mathbb{N}_F)$, being homeomorphic to $\mathbb{R} \times c_F$, is a $Z_\sigma$-space.

3.6. **Corollary.** Let $X$ be a countable nondiscrete completely regular space such that the space $C_p(X)$ is analytic (i.e., a continuous image of the space of irrationals). Then $C_p(X)$ is a $Z_\sigma$-space.

**Proof.** Let $a \in X$ be an accumulation point, $Y = X \setminus \{a\}$, and $F = \{A \subseteq Y : a$ is an interior point of $A \cup \{a\}\}$. Then $F$ is a free filter on $Y$. We shall prove that $F$ is an analytic subset of $2^Y$. Let us recall that a set which is simultaneously closed and open is called clopen. First observe that the set $G = \{B \subseteq X : B$ is clopen in $X$ and $a \in B\}$ is analytic in $2^X$ since it can be identified with a closed subset $\{f \in C_p(X) : f(X) \subseteq \{0, 1\}$ and $f(a) = 0\}$ of
The set \( G = \{(A, B) \in 2^Y \times 2^X : B \subseteq A \cup \{a\}\} \) is analytic in \( 2^Y \times 2^X \) and because \( X \) is zero-dimensional \( F \) is an image of \( G \) by the projection onto the first axis. Thus \( F \) is analytic in \( 2^Y \).

Now, by [Ku, §39] and Lemma 2.3, \( F \) is the first category subset of \( 2^Y \). Obviously, \( C_p(X) \subseteq E \), where

\[
E = \{f \in \mathbb{R}^X : \forall \varepsilon > 0 \exists A \in F \forall x \in A |f(a) - f(x)| < \varepsilon\}.
\]

The space \( E \) can be identified with \( C_p(N_F) \). Hence, by Corollary 3.5, the space \( E \) is a \( Z_\sigma \)-space. By 3.1 the space \( C_p(X) \) is a \( Z_\sigma \)-space.

4. BOREL COMPLEXITY OF FUNCTION SPACES \( C_p(X) \)

For a countable ordinal \( \alpha \), \( \mathcal{M}_\alpha \) and \( \mathcal{A}_\alpha \) denote the class of all absolute Borel sets of the multiplicative and additive class \( \alpha \), respectively. If \( \alpha \geq 2 \), then there exists a filter on \( \mathbb{N} \) which belongs to \( \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha \) (see [LvMP], cf. [Ca2]). The filter \( \mathcal{F} \) is in the class \( \mathcal{A}_1 \setminus \mathcal{M}_1 \). It is an easy observation that there exist no filters in the class \( \mathcal{M}_1 \setminus \mathcal{A}_1 \) (see [Ca]). For every nonempty subset \( A \subseteq \mathbb{N} \) the filter \( F(A) = \{B \subseteq \mathbb{N}: A \subseteq B\} \) is a compact set in \( 2^\mathbb{N} \) and hence \( F(A) \in \mathcal{M}_0 \). Moreover, every compact filter is of the form \( F(A) \) for \( \emptyset \neq A \subseteq \mathbb{N} \).

A filter \( F \) is an absolute Borel set (shortly a Borelian filter) iff the space \( c_F \) is an absolute Borel set. Moreover, Borel complexity of \( c_F \) heavily depends on \( F \) and vice versa. Namely, we have

4.1. Lemma. For every filter \( F \) there exists a closed embedding of \( F \) into the space \( c_F \).

Proof. The map sending each \( A \in F \) onto \( \kappa_A \in c_F \), where \( \kappa_A(i) = 0 \) for \( i \in A \) and \( \kappa_A(i) = 1 \) for \( i \notin A \), is a required closed embedding.

4.2. Lemma. Let \( F \) be a filter on \( \mathbb{N} \) and let \( \alpha \) be a countable ordinal, \( \alpha \geq 1 \). Then:

(1) if \( F \in \mathcal{M}_\alpha \), then \( c_F \in \mathcal{M}_\alpha \),

(2) if \( F \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha \), then \( c_F \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha \),

(3) if \( F \in \mathcal{M}_\alpha \cap \mathcal{A}_\alpha \setminus \bigcup_{\beta < \alpha} (\mathcal{A}_\beta \cup \mathcal{M}_\beta) \), then \( c_F \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha \),

(4) if \( F \in \mathcal{A}_\alpha \setminus \mathcal{M}_\alpha \), then \( c_F \in \mathcal{M}_{\alpha+1} \setminus \mathcal{A}_{\alpha+1} \).

Proof. The assertions of this lemma (except for (3)) were proved in [Ca1, 2]. For the sake of completeness we include our proof of Lemma 4.2.

(1) We will present here a slight simplification of the original proof of (1). Suppose \( F \in \mathcal{M}_\alpha \) for some countable ordinal \( \alpha \). Write

\[
T = \mathbb{R} \setminus \left( \bigcup_{n=1}^{\infty} \left( \frac{1}{n} - 4^{-n}, \frac{1}{n} + 4^{-n} \right) \cup \bigcup_{-n=1}^{\infty} \left( \frac{1}{n} - 4^n, \frac{1}{n} + 4^n \right) \right)
\]

and let \( f_k : T^\infty \to 2^\mathbb{N} \) be defined by \( f_k((x_i)) = \{i \in \mathbb{N} : |x_i| < 1/k\} \) for \( (x_i) \in T^\infty \) and \( k = 1, 2, \ldots \). Then the maps \( f_k \) are continuous and \( T^\infty \cap c_F = \).
Thus $T^\infty \cap c_F \in \mathcal{M}_\alpha$. Let $g: T \to \mathbb{R}$ be a linear extension of the map sending $\frac{1}{n} - 4^{-n}$, $\frac{1}{n} + 4^{-n}$ onto $\frac{1}{n}$ for $n = 1, 2, \ldots$, and $\frac{1}{n} - 4^{-n}$, $\frac{1}{n} + 4^{-n}$ onto $\frac{1}{n}$ for $-n = 1, 2, \ldots$. Then the map $g^\infty: T^\infty \to \mathbb{R}^\infty$ defined by $g^\infty((x_i)) = (g(x_i))$ is a proper surjection with $(g^\infty)^{-1}(c_F) = T^\infty \cap c_F$ (let us recall that a map $f: X \to Y$ is proper if $f^{-1}(K)$ is compact whenever $K$ is a compact subset in $Y$). Now, the result of [SR] implies that $c_F \in \mathcal{M}_\alpha$.

(2) This is a consequence of (1) and Lemma 4.1.

(3) We shall use Calbrix's argument of [Ca, 2]. Let $F$ be a filter on $\aleph_0$ such that $F \in \mathcal{M}_\alpha \cap \mathfrak{A} \setminus \bigcup_{\beta < \alpha} (\mathfrak{A}_\beta \cup \mathfrak{M}_\beta)$. Express $\aleph_0$ as a union of pairwise disjoint families of infinite sets $N_i = \{n_{i,1}, n_{i,2}, \ldots\}$. Define a new filter $F^\infty$ consisting of all sets of the form $\bigcup_{i=1}^{\infty} A_i$, where $A_i \in F_i$ and $F_i$ is an isomorphic copy of $F$ on $N_i$. First we show that $F^\infty \in \mathcal{M}_\alpha \setminus \mathfrak{A}_\alpha$. It is enough to observe that for every $C \subseteq 2^{\aleph_0}$ such that $C \in \mathcal{M}_\alpha$ there exists a continuous map $f: 2^{\aleph_0} \to \prod_{i=1}^{\infty} 2^{N_i} = 2^{\aleph_0}$ with $f^{-1}(F^\infty) = C$. Let $C = \bigcap_{i=1}^{\infty} C_i$, where $C_i \in \bigcup_{\beta < \alpha} \mathfrak{A}_\beta$ for $i = 1, 2, \ldots$. Fix $i \geq 1$. By the "Wadge lemma" either there exists a continuous map $f_i: 2^{\aleph_0} \to 2^{\aleph_0}$ such that $f_i^{-1}(F_i) = C_i$ or there exists a continuous map $g_i: 2^{\aleph_0} \to 2^{\aleph_0}$ such that $g_i^{-1}(2^{\aleph_0} \setminus C_i) = F_i$. Since $F_i \in \mathcal{M}_\alpha \cap \mathfrak{A} \setminus \bigcup_{\beta < \alpha} (\mathfrak{A}_\beta \cup \mathfrak{M}_\beta)$ and $C_i \in \bigcup_{\beta < \alpha} \mathfrak{A}_\beta$ the map $g_i$ cannot exist. Let $f: 2^{\aleph_0} \to \prod_{i=1}^{\infty} 2^{N_i}$ be the map defined by $f(x) = (f_1(x), f_2(x), \ldots)$, where $f_i: 2^{\aleph_0} \to 2^{\aleph_0}$ satisfies $f_i^{-1}(F_i) = C_i$ for $i = 1, 2, \ldots$. Now, by (2), we get $c_F^\infty \in \mathcal{M}_\alpha \setminus \mathfrak{A}_\alpha$. The map $\phi: \mathbb{R}^\infty \to \mathbb{R}^\infty$ given by the formula $\phi((x_{n_{i,j}})) = (\sum_{i=1}^{\infty} 2^{-i-1}|x_{n_{i,j}}|(1+|x_{n_{i,j}}|)^{-1})_{j=1}^{\infty}$ has the property that $\phi^{-1}(c_F) = c_F^\infty$. Hence $c_F \notin \mathfrak{A}_\alpha$. By (1), $c_F \in \mathcal{M}_\alpha$.

(4) We can repeat the proof of (3) (cf., [Ca, 2]).

Let $X$ be a countable completely regular space and let $a$ be an accumulation point of $X$. Then

$$F_a = \{A \in 2^{X \setminus \{a\}}: a \text{ is an interior point of } A \cup \{a\}\}$$

is a free filter on $X \setminus \{a\}$.

4.3. Lemma. Let $X$ be a countable completely regular space such that for every accumulation point $a \in X$ the filter $F_a \in \mathcal{M}_\alpha$. Then $C_p(X) \in \mathcal{M}_\alpha$.

Proof. Let $a$ be an accumulation point of $X$ and let $X_{F_a}$ denote the space $X$ topologized by isolating points of $X \setminus \{a\}$ and by using the family $\{A \cup \{a\}: A \in F_a\}$ as a neighborhood base at $a$. The spaces $C_p(X)$ and $C_p(X_{F_a})$ are linear dense subspaces of $\mathbb{R}^X$. By Lemma 4.2 $C_p(X_{F_a}) \in \mathcal{M}_\alpha$. Since

$$C_p(X) = \bigcap\{C_p(X_{F_a}): a \text{ is an accumulation point of } X\}$$

we have $C_p(X) \in \mathcal{M}_\alpha$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
5. Sequence spaces related to $F_{\delta}$-filters

In this section we give a complete topological classification of sequence spaces $c_F$ which are absolute $F_{\delta}$-sets (i.e., $c_F \subseteq \mathcal{M}_2$). Since for a compact filter $F(C)$, $C \subseteq \mathbb{N}$, $c_{F(C)} = \{(x_i) \in \mathbb{R}^\infty : x_i = 0 \text{ for } i \in C\}$ and for an arbitrary filter $F$, $c_F \subseteq \{(x_i) \in \mathbb{R}^\infty : x_i = 0 \text{ for } i \in \bigcap_{A \in F} A\}$ we can reduce our classification to the case when the filter $F$ is free. Obviously, a filter $F$ is free iff the space $\mathbb{N}_F$ is completely regular.

5.1. Proposition. Let $F$ be a free filter which is an absolute $F_{\delta}$-set. Then the sequence space $c_F$ is homeomorphic to $\sigma^\infty$.

To prove Proposition 5.1 it is enough to verify that the space $c_F$ satisfies the assumptions of the following lemma:

5.2. Lemma. Let $\{X_i\}$ be a sequence of $F_{\delta}$-absolute retracts such that, for each $i$, $\sigma^\infty$ is embeddable onto a closed subset of $X_i$. Fix $p_i \in X_i$, $i = 1, 2, \ldots$. Then every $F_{\delta}$-space $X$ which is a $Z_\sigma$-space and satisfies

$$W(X_i, p_i) = \left\{(x_i) \in \prod_{i=1}^\infty X_i : x_i = p_i \text{ for all but finitely many } i \right\} \subseteq X \subseteq \prod_{i=1}^\infty X_i$$

is homeomorphic to $\sigma^\infty$.

Proof. Lemma 5.2 is a slight modification of the characterization of $\sigma^\infty$, used in [DGM, DM], which follows easily from Lemma 2.3 of [DM] and Theorem 6.5 of [BM].

We will also need the following fact proved in [DM, Corollary 2.5]:

5.3. Lemma. Let $X_i$, for $i = 1, 2, \ldots$, be an absolute retract which is a $Z_\sigma$-space. Then the product $\prod_{i=1}^\infty X_i$ contains a closed copy of $\sigma^\infty$.

Below we present a new, more elementary, proof of Lemma 5.3 which is a consequence of the following general observation:

5.4. Lemma. If $X$ is an absolute retract which is a $Z_\sigma$-space, then for each $\sigma$-compact space $A$ there exists a proper map $f : A \to X$.

Proof. Assume that $A$ is a subset of the Hilbert cube $I^\infty$, and $A = \bigcup_{n=1}^\infty A_n$, where $A_n$ is compact and $A_n \subseteq A_{n+1}$ for $n = 1, 2, \ldots$. Let $Y$ be a complete absolute retract such that $X \subseteq Y$ and $Y \setminus X$ is locally homotopy negligible in $Y$, i.e., for every open family $\mathcal{U}$ of $Y$ and for every map $f : I^\infty \to Y$ there exists a map $g : I^\infty \to Y$ which is $\mathcal{U}$-close to $f$ and such that $g(f^{-1}(\bigcup \mathcal{U})) \subseteq X$, see [T]. We shall construct a map $f : I^\infty \to Y$ with $f^{-1}(X) = A$. Then the restricted map $f|A$ is a proper map of $A$ into $X$. Since $X$ is a $Z_\sigma$-space, we can find $Z$-sets $X_n$ in $Y$ for $n = 1, 2, \ldots$ such that $X \subseteq \bigcup_n^\infty X_n$ and $X_1 \subseteq X_2 \subseteq \cdots$. Fix a complete metric $d$ on $Y$ which is bounded by 1. Let $f_0 : I^\infty \to Y \setminus \tilde{X}$ be a map and let $A_0 = X_0 = \emptyset$. We will inductively construct
a sequence of maps \( f_n: I^\infty \to Y \) satisfying for \( n = 1, 2, \ldots \) the following conditions:

(i) \( f_n(A_n) \subset X \),
(ii) \( f_n(I^\infty \setminus A_n) \subset Y \setminus \tilde{X} \),
(iii) \( f_n|A_{n-1} = f_{n-1}|A_{n-1} \),
(iv) \( d(f_n(x), f_{n-1}(x)) \leq 4^{-n}d(f_{n-1}(x), X_{n-1}) \).

Assume that the maps \( f_i: I^\infty \to Y \) satisfying the conditions (i)–(iv) have been already constructed for \( 0 < i < n \). By the local homotopy negligibility of \( Y \setminus X \), there exists a map \( g: I^\infty \to X \) such that \( g|A_n = f_n|A_n \) and \( d(g(x), f_n(x)) \leq 4^{-n-2}d(f_n(x), X_n) \). Since \( \tilde{X} \) is a countable union of \( Z \)-sets in a complete absolute retract \( Y \) there exists a homotopy \( h_t: Y \to Y \) such that \( h_t = id_Y, h_t|X_n = id_{X_n} \) for \( 0 \leq t \leq 1 \), \( h_t(Y \setminus X_n) \subset Y \setminus \tilde{X} \) for \( t > 0 \), and \( \text{diam}(h_t(y); 0 \leq t \leq 1) \leq 4^{-n-2}d(y, X_n) \). We let \( f_{n+1}(x) = h_{\lambda(x)}(g(x)) \), where \( \lambda: I^\infty \to [0, 1] \) is a continuous function with \( \lambda^{-1}(0) = A_{n+1} \). If \( x \in A_{n+1} \), then \( \lambda(x) = 0 \) and consequently \( f_{n+1}(x) = h_0(g(x)) = g(x) \); in particular, \( f_{n+1}(x) = g(x) = f_n(x) \) for \( x \in A_n \). If \( x \in I^\infty \setminus A_{n+1} \), then \( g(x) \notin X \) and \( \lambda_n(x) > 0 \); hence \( f_{n+1}(x) = h_{\lambda(x)}(g(x)) \in Y \setminus \tilde{X} \). Thus \( f_{n+1} \) satisfies the conditions (i)–(iii). To show (iv) we use the following inequalities:

\[
\begin{align*}
d(f_n(x), f_{n+1}(x)) &\leq d(f_n(x), g(x)) + d(g(x), h_{\lambda(x)}(g(x))) \\
&\leq 4^{-n-2}d(f_n(x), X_n) + 4^{-n-2}d(g(x), X_n) \\
&\leq (4^{-n-2} + 4^{-n-2}(1 + 4^{-n-2}))d(f_n(x), X_n) \\
&\leq 4^{-n-1}d(f_n(x), X_n).
\end{align*}
\]

Now, by (iv), the sequence \( \{f_n\} \) uniformly converges to a map \( f: I^\infty \to Y \). By (i) and (iii) we get \( f(A) \subset X \). To show that \( f(I^\infty \setminus A) \subset Y \setminus \tilde{X} \), we first observe that for \( n > k \),

\[
d(f_n(x), X_k) \geq (1 - 4^{-n})d(f_{n-1}, X_k) \\
\geq \cdots \geq (1 - 4^{-n})(1 - 4^{-n+1})\cdots (1 - 4^{-k-1})d(f_k(x), X_k).
\]

Hence, we have \( d(f(x), X_k) \geq \prod_{n=k+1}^{\infty} (1 - 4^{-n})d(f_k(x), X_k) \). If \( x \in I^\infty \setminus A \) then \( d(f_k(x), X_k) > 0 \). Since \( \prod_{n=k+1}^{\infty} (1 - 4^{-n}) > 0 \), \( k \geq 0 \), we obtain \( d(f(x), X_k) > 0 \) for \( x \in I^\infty \setminus A \) and \( k = 1, 2, \ldots \); consequently \( f(x) \notin X_k \) for \( k = 1, 2, \ldots \).

**Proof of 5.3.** We write \( N = \bigcup_{i=0}^{\infty} N_i \), where the \( N_i \) are infinite and pairwise disjoint for \( i = 1, 2, \ldots \). Since

\[
\prod_{i=1}^{\infty} X_i = \prod_{i=1}^{\infty} \left( \prod_{n \in \mathbb{N}_{2i-1}} X_n \times \prod_{n \in \mathbb{N}_{2i}} X_n \right)
\]
it is enough to show that for $i = 1, 2, \ldots$ there exists a closed embedding $v_i: \sigma \to \prod_{n \in \mathbb{N}_{2^i-1}} X_n \times \prod_{n \in \mathbb{N}_{2^i}} X_n$. First let us observe that each nontrivial absolute retract contains the interval $[0, 1]$ and the infinite product of such absolute retracts contains the Hilbert cube $I^\infty$. To obtain $v_i$ we choose any embedding $u_i: \sigma \to \prod_{n \in \mathbb{N}_{2^i}} X_n$ and a proper map from Lemma 5.4 $f_i: \sigma \to \prod_{n \in \mathbb{N}_{2^i}} X_n$ and set $v_i = u_i \times f_i$.

We will also employ the following

5.6. Lemma. For any filter $F$ on $\mathbb{N}$, any decomposition $N = \bigcup_{i=1}^{\infty} N_i$ into pairwise disjoint infinite sets $N_i$, and for the natural isomorphism $h: \mathbb{R}^N \to \prod_{i=1}^{\infty} \mathbb{R}^{N_i}$ we have
\begin{enumerate}
  \item $W(c_{F_i}, 0) \subset h(c_F) \subset \prod_{i=1}^{\infty} c_{F_i}$,
  \item $W(B_{F_i}(r), 0) \subset h(B_F(r)) \subset \prod_{i=1}^{\infty} b_{F_i}(r)$, for $r > 0$, and
  \item $\prod_{i=1}^{\infty} B_{F_i}(\frac{1}{r}) \subset h(B_F(1))$,
\end{enumerate}
where $F_i = \{A \cap N_i: A \in F\}$.

Proof. The inclusions are easy consequences of the observation that $A_1 \cup A_2 \cup \ldots \cup A_k \cup N_{k+1} \cup N_{k+2} \cup \ldots$ belongs to $F$ for every $A_i \in F_i$ and for arbitrary $k$; cf. Proposition 2.4.

Proof of Proposition 5.1. By Lemma 4.2 the space $c_F$ is an absolute $F_{\sigma\delta}$-set. Corollary 3.4 implies that $c_F$ is a $Z_{\sigma}$-space. Since $F$ is free on $\mathbb{N}$, by (3) of Proposition 2.4, there exists a decomposition $N = \bigcup_{i=1}^{\infty} N_i$ such that each $F_i = \{A \cap N_i: A \in F\}$ is a free $F_{\sigma\delta}$-filter on $N_i$. Now, by (1) of 5.6 and 5.2, it is enough to show that $c_{F_i}$ (equivalently, $c_F$) contains $\sigma^\infty$ as a closed subset. The last follows from (3) of Lemma 5.6, Lemma 5.3, and Corollary 3.4.

Similarly we can prove the following

5.7. Proposition. For every noncompact $F_{\sigma\delta}$-filter $F$ on $\mathbb{N}$ the spaces $c^*_F$ and $B_F(r)$, $r > 0$, are homeomorphic to $\sigma^\infty$.

6. TOPOLOGICAL CLASSIFICATION OF FUNCTION SPACES $C_p(X)$

OF TYPE $F_{\sigma\delta}$

In this section we prove Theorem 1.1. We start with the following general fact.

6.1. Proposition. Let $X$ be a countable nondiscrete completely regular space. Then one of the following conditions holds:
\begin{enumerate}
  \item there exists a clopen subset $Y$ of $X$ with exactly one accumulation point.
  \item there exists a decomposition $X = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}_{n=1}^{\infty}$ is a pairwise disjoint sequence of nondiscrete clopen sets.
\end{enumerate}

Proof. Suppose that (i) does not hold. Then by induction, we construct the decomposition $X = \bigcup_{n=1}^{\infty} X_n$ of (ii). Let $X = \{x_1, x_2, \ldots\}$ and $X_0 = \emptyset$.
Assume that we have constructed pairwise disjoint nondiscrete clopen subsets $X_1, X_2, \ldots, X_n$ of $X$ such that $Y_n = X \setminus \bigcup_{i=1}^n X_i$ is also nondiscrete and \{x_1, x_2, \ldots, x_n\} $\subseteq \bigcup_{i=1}^n X_i$. By our assumption $Y_n$ contains at least two accumulation points. Using the fact that $X$ is zero-dimensional, we can divide $Y_n$ into two nondiscrete clopen sets. We choose one of them as $X_{n+1}$ in such a way that \{x_1, x_2, \ldots, x_{n+1}\} is contained in $\bigcup_{i=1}^{n+1} X_i$.

The next proposition summarizes the results of the previous section.

6.2. Proposition. Let $F$ be a filter on $\mathbb{N}$. Then the following conditions are equivalent:

(a) $C_p(\mathbb{N}_F)$ is homeomorphic to $\sigma^\infty$,
(b) $C_p(\mathbb{N}_F)$ is an absolute $F_{\sigma\delta}$-set and not a $G_\delta$-set,
(c) The filter $F$ is a noncompact $F_{\sigma\delta}$-subset of $2^\mathbb{N}$.

The same is true for the space $C^*(\mathbb{N}_F)$.

Proof. The implication (a) $\Rightarrow$ (b) is well known, (b) $\Rightarrow$ (c) follows from Lemma 4.1. Finally (c) $\Rightarrow$ (a) follows from Proposition 5.1 and the fact that $C_p(\mathbb{N}_F)$ is linearly isomorphic to $\mathbb{R} \times c_F$.

Proof of Theorem 1.1. We only present a proof for the space $C_p(X)$ (the proof for the space $C^*(X)$ is the same). We shall consider two cases:

(1) The space $X$ satisfies (i) of Proposition 6.1. Let $Y$ be a clopen subset of $X$ with exactly one accumulation point. The space $Y$ is homeomorphic to $\mathbb{N}_F$, where $F$ is a noncompact filter on $\mathbb{N}$. Moreover, the space $C_p(X)$ is linearly homeomorphic to $C_p(Y) \times C_p(X\setminus Y)$. By Proposition 6.2, $C_p(Y)$ is homeomorphic to $\sigma^\infty$. Hence, by Corollary 5.4 of [BM], it follows that $C_p(X)$ is homeomorphic to $\sigma^\infty$.

(2) The space $X$ satisfies (ii) of Proposition 6.1. Let $X = \bigcup_{n=1}^\infty X_n$ be a decomposition of $X$ into pairwise disjoint nondiscrete clopen sets. Now, the space $C_p(X)$ is homeomorphic to the product $\prod_{n=1}^\infty C_p(X_n)$, where all spaces $C_p(X_n)$ and $C_p(X)$ are $F_{\sigma\delta}$-absolute retracts which, according to Corollary 3.6, are $Z_{\sigma}$-spaces. From Lemmas 5.2 and 5.3 it follows that $C_p(X)$ is homeomorphic to $\sigma^\infty$.

7. EXAMPLES OF SPECIAL $F_{\sigma\delta}$-FILTERS

In this section we apply Theorem 1.1 to answer in the negative several questions posed by A. V. Arhangel’skiı and related to the following general problem: how close do the properties of the spaces $X$ and $Y$ have to be if $C_p(X)$ and $C_p(Y)$ are homeomorphic? We will discuss the properties of the spaces $X$ and $Y$ listed below. A Hausdorff space $X$ is a $k$-space if for each $A \subseteq X$, the set $A$ is closed in $X$ provided that the intersection of $A$ with any compact subspace $K$ of $X$ is closed in $K$. A $k$-space $X$ is a $k_\omega$-space if there exists a countable family $\mathcal{K}$ of compact subsets of $X$ such that $\bigcup \mathcal{K} = X$ and for every
compact subspace $K$ of $X$ there exists $L \in \mathbb{R}$ such that $K \subseteq L$. A topological space $X$ is called a sequential space if a set $A \subseteq X$ is closed if and only if together with any sequence it contains all its limits. A space $X$ is called an $\aleph_0$-space if there exists a countable family $\mathcal{G}$ of subsets of $X$ such that for every compact subset $K \subseteq X$ and for every neighborhood $V$ of $K$ in $X$ one can find $P \in \mathcal{G}$ with $K \subseteq P \subseteq V$. A subset $A$ of a space $X$ will be called $R$-bounded in $X$, if every function $f \in C_p(X)$ is bounded on $A$. A function $f: X \to \mathbb{R}$ is called strictly $b$-continuous if for every $R$-bounded subset $A \subseteq X$ there exists a map $g \in C_p(X)$ such that $f|A = g|A$. A space $X$ is said to be a $b_R$-space if every strictly $b$-continuous function $f: X \to \mathbb{R}$ is continuous.

Recall that $\mathcal{X} = \{0\} \cup \{n^{-1}: n = 1, 2, \ldots\}$ and that $\mathcal{X}$ can be identified with the space $\mathbb{N}_\delta$.

7.1. Example. There exists a free $F_{\sigma\delta}$-filter $F$ on $\mathbb{N}$ such that:

(a) the space $\mathcal{N}_F$ is countable and completely regular,
(b) the function spaces $C_p(\mathcal{N}_F)$ and $C_p(\mathcal{X})$ are homeomorphic,
(c) the space $\mathcal{N}_F$ is not a $k$-space, while $\mathcal{X}$ is compact metric,
(d) the space $\mathcal{N}_F$ is not a $k_\omega$-space,
(e) the space $\mathcal{N}_F$ is not a sequential space,
(f) the space $\mathcal{N}_F$ is not a $b_R$-space.

Proof. Let $F$ be a filter of sets of density 1, i.e.,

$$F = \left\{ A \subseteq \mathbb{N}: \lim_{n \to \infty} n^{-1}\text{card}(A \cap \{1, 2, \ldots, n\}) = 1 \right\},$$

where card($B$) denotes the cardinality of a set $B$ (cf. [AU, p. 119; V, p. 98]). Since $F = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \{ A \subseteq \mathbb{N}: k^{-1}\text{card}(A \cap \{1, 2, \ldots, k\}) \geq 1 - n^{-1} \}$, $F$ is a free $F_{\sigma\delta}$-filter on $\mathbb{N}$. The assertion (a) is obvious and (b) follows from Lemma 4.2 and Theorem 1.1. Since for every infinite $M \subseteq \mathbb{N}$ there is $A \in F$ such that $M \setminus A$ is infinite the space $\mathcal{N}_F$ does not contain any nontrivial convergent sequence. Moreover, all compact subspaces of $\mathcal{N}_F$ are finite. Hence $\mathcal{N}_F$ is not a $k$-space and (c) holds. The assertions (d) and (e) are immediate consequences of (c). One can easily observe that all $R$-bounded subsets of $\mathcal{N}_F$ are finite. Hence every function from $\mathcal{N}_F$ into $\mathbb{R}$ is strictly $b$-continuous. Consequently $\mathcal{N}_F$ is not a $b_R$-space.

Example 7.1 answers the problems 12, 24, 25, 26 of [Ar₁] and 6, 7, 26 of [Ar₂].

7.2. Example. There exists a free $F_{\sigma\delta}$-filter $G$ on $\mathbb{N}$ such that:

(a) the space $\mathcal{N}_G$ is countable and completely regular,
(b) the function spaces $C_p(\mathcal{N}_G)$ and $C_p(\mathcal{X})$ are homeomorphic,
(c) the space $\mathcal{N}_G$ is not an $\aleph_0$-space while $\mathcal{X}$ is compact metric.

Proof. As a filter $G$ we take one of the filters described in [LvMP]. Let $2^n$ be the set of all functions from $\{0, 1, \ldots, n-1\}$ into $\{0, 1\}$ for $n = 1, 2, \ldots$. Let
us put $T = \bigcup_{n=1}^\infty 2^n$. For each function $x: N \to \{0, 1\}$ we define $B_x = \{x|n \in 2^n: n = 1, 2, \ldots\}$ to be a branch in $T$, where $x|n$ denotes the restriction of the function $x$ to the set $\{0, 1, \ldots, n-1\}$. The filter $G$ on the countable set $T$ is generated by the family $\{T \setminus (B_{x_1} \cup B_{x_2} \cup \cdots \cup B_{x_{n}} \cup S): n \geq 1, x_i \in \{0, 1\}^N \text{ and } S \text{ is a finite subset of } T\}$. We identify $T$ with $N$ and consider $G$ as a filter on $N$. Obviously, the filter $G$ is free. By [Ca,] $G$ is an $F_\sigma$-subset of $2^N$. The assertion (b) follows from Lemma 4.2 and Theorem 1.1. Now we shall verify (c). We identify $N_G$ with $T \cup \{\infty\}$. Let $\mathcal{G} = \{P_n: n = 1, 2, \ldots\}$ be a family of subsets of $T \cup \{\infty\}$. We will construct a compact set $K$ and an open set $V$ in $T \cup \{\infty\}$ such that $K \subseteq V$ and for every $n$ the set $K$ is not contained in $P_n$ or the set $P_n$ is not contained in $V$. By induction one can easily define a sequence $\{t_n\}_{n=1}^\infty$, where $t_n \in 2^n$ and such that

\begin{align}
(*)& t_n|\{0, 1, \ldots, n-2\} = t_{n-1}, \text{ if } n > 1, \\
(**) & t_n \in P_n \text{ or } s_n \notin P_n, \text{ where } s_n \in 2^n \text{ is defined by } s_n|\{0, 1, \ldots, n-2\} = t_n|\{0, 1, \ldots, n-2\}, \text{ if } n > 1, \text{ and } s_n(n-1) \neq t_n(n-1) \text{ if } n > 1.
\end{align}

Now, let $x \in \{0, 1\}^N$ be such that $x|\{0, 1, \ldots, n-1\} = t_n$. We set $V = (T \cup \{\infty\}) \setminus B_x$ and $K = \{s_n: n = 1, 2, \ldots\} \cup \{\infty\}$. Let us observe that $B_x \cap K = \emptyset$ and for every $y \in \{0, 1\}^N$, $y \neq x$, $B_y \cap K$ is finite (if $n$ is such that $x|n \neq y|n$, then $s_k \notin B_y$ for $k > n + 1$). Hence the set $K \subseteq V$ is compact and, by (**), the set $P_n$ is not contained in $V$ or the set $K$ is not contained in $P_n$ for every $n = 1, 2, \ldots$.

Example 7.2 answers the problem 34 of [Ar,] (cf. also problem 36 of [Ar2]).

8. Sequence spaces of higher Borel complexity

According to [LvMP, Ca,] and Lemma 4.2 for every countable ordinal $\alpha \geq 2$ there exists a filter $F$ on $N$ such that $c_F \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$. In [BM] it was shown that in each class $\mathcal{M}_\alpha$ there exists a maximal object $\Omega_\alpha$ which can be characterized as follows:

8.1. Proposition. A space $X$ is homeomorphic to $\Omega_\alpha$ iff $X$ satisfies the following conditions:

1. $X$ is an absolute retract,
2. $X \in \mathcal{M}_\alpha$,
3. $X$ is a $Z_\sigma$-space,
4. $X$ is homeomorphic to $X^\infty$.
5. $X$ is $\mathcal{M}_\alpha$-universal, i.e., each $Y \in \mathcal{M}_\alpha$ is embeddable onto a closed subset of $\Omega_\alpha$.

Let us note that $\Omega_2$ is just $\sigma^\infty$ and in §6 we have proved that if $X$ is a countable completely regular space such that $C_p(X) \in \mathcal{M}_2 \setminus \mathcal{A}_2$, then $C_p(X)$ is homeomorphic to $\sigma^\infty$. It suggests the following

8.2. Conjecture. For every countable completely regular space $X$ such that $C_p(X) \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$, $C_p(X)$ is homeomorphic to $\Omega_\alpha$. 
Now, we focus on the spaces $c_F$ for Borelian filters $F$ on $\mathbb{N}$. For spaces $c_F$ the condition (1) is clear. According to [Ca,] (see Lemma 4.2) $c_F \in \mathcal{M}_\alpha$ provided $F \in \mathcal{M}_\alpha$. The condition (3) is a consequence of Corollary 3.4. The conditions (4) and (5) are the major obstacles in order to confirm Conjecture 8.2 for higher Borelian classes.

8.3. Problem. Let $F$ be a filter on $\mathbb{N}$ such that $F \in \mathfrak{A}_\alpha \cup \mathcal{M}_\alpha$, where $\alpha > 1$. Is $c_F$ homeomorphic to $(c_F)^\infty$?

8.4. Problem. Let for a filter $F$ on $\mathbb{N}$ the space $c_F \in \mathcal{M}_\alpha \setminus \mathfrak{A}_\alpha$. Can every $X \in \mathfrak{A}_\alpha$ be embeddable onto a closed subset of $c_F$?

For $\alpha = 2$ the condition (5) is a consequence of the remaining four conditions. For higher $\alpha$, we ask

8.5. Problem. Let $X$ satisfy the following conditions:

1. $X$ is an absolute retract,
2. $X \in \mathcal{M}_\alpha \setminus \mathfrak{A}_\alpha$,
3. $X$ is a $Z_\sigma$-space,
4. $X$ is homeomorphic to $X^\infty$.

Is $X$ homeomorphic to $\Omega_\alpha$?

8.6. Remark. We say that a set $X$ in a compact space $M$ is Wadge $\mathcal{M}_\alpha$-maximal in $M$ if $X \in \mathcal{M}_\alpha$ and for a subset $Y$ of $M$, with $Y \in \mathcal{M}_\alpha$, there exists a map $f: M \to M$ such that $f^{-1}(X) = Y$. An inspection of the proof of Lemma 5.3 yields the fact that a space $X$ satisfying the conditions (1)-(4) is Wadge $\mathcal{M}_2$-maximal in a topological copy of the Hilbert cube $I^\infty$. The last result can be considered as a Hilbert cube counterpart of the fact that in the Cantor set each set $A \in \mathfrak{A}_\alpha \setminus \mathfrak{A}_\alpha$ is Wadge $\mathcal{M}_\alpha$-maximal (see [W]). To answer 8.5 in the positive, it is enough to show that a space $X$ satisfying (1')-(4') is Wadge $\mathcal{M}_\alpha$-maximal in a topological copy of the Hilbert cube.

The condition (4) for spaces $c_F$ is closely related to decomposability of filters $F$ described in Proposition 2.4. If a space $c_F \in \mathcal{M}_2 \setminus \mathfrak{A}_2$, then the restrictions $F_i$ of the filter $F$ of Proposition 2.4 are in the class $(\mathcal{M}_2 \setminus \mathfrak{A}_2) \cup (\mathfrak{A}_1 \setminus \mathcal{M}_1)$ and consequently $c_{F_i} \in \mathcal{M}_2 \setminus \mathfrak{A}_2$. This was the crucial step in verifying the condition (4). For higher $\alpha$ the Borel type of spaces $c_{F_i}$ for restricted filters $F_i$ can be essentially lowered. That is why we introduce the following definition. A filter $F$ on $\mathbb{N}$ is decomposable if there exist infinite, disjoint sets $N_1$ and $N_2$ such that $N = N_1 \cup N_2$ and $F_i = \{A \cap N_i: a \in F\}$ is a filter on $N_i$ which is isomorphic to $F$ for $i = 1, 2$. Then we have $F = \{A_1 \cup A_2: A_1 \in F_1$ and $A_2 \in F_2\}$ and we write $F = F_1 \times F_2$. By an easy induction we obtain

8.7. Lemma. If $F$ is a decomposable filter on $\mathbb{N}$, then there exists a sequence $\{N_i\}$ of infinite pairwise disjoint subsets of $\mathbb{N}$ with $\mathbb{N} = \bigcup_{i=1}^{\infty} N_i$, and such that each $F_i = \{A \cap N_i: a \in F\}$ is a filter on $N_i$ which is isomorphic to $F$.

The main result of this section is the following:
8.8. **Theorem.** Let $F$ be a first category filter on $\mathbb{N}$ which is free and decomposable. Then the sequence space $c_F$ is homeomorphic to $(c_F)\,^\infty$.

The proof of Theorem 8.8 is based on the following lemma which is a standard fact about absorbing sets (see [BM]):

8.9. **Lemma.** Let $X$ and $Y$ be absolute retracts which are $Z_\sigma$-spaces. Assume that there are noncompact absolute retracts $M$ and $N$ and $p \in M$, $q \in N$ satisfying $W(M, p) \subseteq X \subseteq M^\infty$ and $W(N, q) \subseteq Y \subseteq N^\infty$, where for a space $Z$ and $z \in Z$ we write

$$W(Z, z) = \{ (z_i) \in Z^\infty : z_i = z \text{ for all but finitely many } i \}.$$ 

If $X = \bigcup_{i=1}^\infty X_i$ and $Y = \bigcup_{i=1}^\infty Y_i$, where $X_i$ is closed in $X$ and $Y_i$ is closed in $Y$ for $i = 1, 2, \ldots$ and moreover each $X_i$ embeds onto a closed subset of $N$ and each $Y_i$ embeds onto a closed subset of $M$, then $X$ and $Y$ are homeomorphic.

**Proof of Theorem 8.8.** In the proof we will use the spaces $c_F$, $B_F(1)$, and $c_F^*$ and their products. These spaces are noncompact absolute retracts. Since $F$ is a first category filter, Proposition 3.3 implies that $c_F$, $B_F(1)$, and $c_F^*$ are $Z_\sigma$-spaces. Consider a decomposition $\mathbb{N} = \bigcup_{i=1}^\infty N_i$ into infinite pairwise disjoint sets $N_i$ so that the restricted filters $F_i$ are isomorphic to $F$ (see Lemma 8.7). Thus the spaces $c_{F_i}$, $B_{F_i}(1)$, and $c_{F_i}^*$ are linearly isomorphic to $c_F$, $B_F(1)$, and $c_F^*$, respectively. This together with Lemma 5.6 gives a homeomorphism $h: \mathbb{R}^\infty \to \mathbb{R}^\infty$ satisfying:

$$W(c_F, 0) \subseteq h(c_F) \subseteq (c_F)\,^\infty,$$

$$\prod_{i=1}^\infty B_F \left( \frac{1}{i} \right) \cup W(B_F(1), 0) \subseteq h(B_F(1)) \subseteq (B_F(1))\,^\infty,$$

$$W(c_F^*, 0) \subseteq h(c_F^*) \subseteq (c_F^*)\,^\infty.$$ 

Now, Theorem 8.8 follows from Lemma 8.9 applied for $M = N = c_F$, $X = h(c_F)$, and $Y = (c_F)\,^\infty$ and from the following fact:

(i) $(c_F)\,^\infty$ embeds onto a closed subset of $c_F$.

The last is a consequence of (ii)–(iv) below.

(ii) $B_F(1)$ is homeomorphic to $(B_F(1))\,^\infty$.

By the obvious fact that $B_F(r)$ is homeomorphic to $B_F(1)$, for $r > 0$, and by Lemma 5.6(3), the product $(B_F(1))\,^\infty$ embeds as a closed subset of $B_F(1)$.

Now, we apply Lemma 8.9, with $M = N = B_F(1)$, $X = h(B_F(1))$, and $Y = (B_F(1))\,^\infty$.

(iii) $c_F^*$ is homeomorphic to $B_F(1)$.

First let us observe that $c_F^* = \bigcup_{n=1}^\infty B_F(n)$. Now, (iii) follows from Lemma 8.9 applied for $M = c_F^*$, $N = B_F(1)$, $X = h(c_F^*)$, and $Y = h(B_F(1))$.

(iv) $c_F$ embeds onto a closed subset of $(c_F^*)\,^\infty$.

Let $r: \mathbb{R} \to [-1, 1]$ be the retraction defined by

$$r(x) = (\text{sgn } x) \min(|x|, 1).$$
Write for \((x_i) \in \mathbb{R}^\infty\),
\[
f_n(x_1, x_2, \ldots) = (x_1, x_2, \ldots, x_n, r(x_{n+1}), r(x_{n+2}), \ldots).
\]
Then \(f = (f_1, f_2, \ldots)\) defines a closed embedding of \(\mathbb{R}^\infty\) into \((\mathbb{R}^{\text{bd}})^\infty\) (recall that \(\mathbb{R}^{\text{bd}} = \{(x_n) \in \mathbb{R}^\infty : \sup |x_n| < \infty\}\)). Moreover, we have \(f^{-1}(\mathbb{C}_F)^\infty = c_F\). Thus \(f|c_F\) is an embedding of \(c_F\) onto a closed subset of \((\mathbb{C}_F)^\infty\).

**8.10. Remark.** For every countable ordinal \(\alpha \geq 2\) there exists a filter \(F\) such that \(c_F \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha\) and \(c_F\) is homeomorphic to \((c_F)^\infty\).

**Proof.** Let \(F \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha\) be a filter on \(\mathbb{N}\) and let \(F^\infty\) be a filter defined in the proof of Lemma 4.2(3). Obviously, the space \(c_{F^\infty}\) is homeomorphic to \((c_F)^\infty\) and consequently to \((c_{F^\infty})^\infty\).

**Added in proof.** The authors have just learned that an equivalent version of Lemma 2.2 for filters is contained in Theorem 21 of [Ta].

**References**


Institute of Mathematics, Warsaw University, PKiN IX p, 00-091 Warsaw, Poland

Department of Mathematics, The University of Kansas, Lawrence, Kansas 66045

Department of Mathematics, University of Alabama, Tuscaloosa, Alabama 35486

Current address: Department of Mathematics, Bradley University, Peoria, Illinois 61625