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TWISTED CALIBRATIONS

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ABSTRACT. The methods of calibrated geometry are extended to include nonorientable submanifolds which can be oriented by some real Euclidean line bundle. Specifically, if there exists a line bundle-valued differential form of comass one which restricts to a submanifold to be a density, then the submanifold satisfies a minimizing property. The results are applied to show that the cone on the Veronese surface minimizes among a general class of comparison 3-folds.

INTRODUCTION

An important area of study in differential geometry concerns minimal and area-minimizing subvarieties of Riemannian manifolds. A particularly elegant technique known as calibrated geometry was formalized with the appearance of the fundamental paper of Harvey and Lawson [HL]. A calibrated geometry on a Riemannian manifold M is the study of the minimal varieties associated to a closed p-form of unit comass. Any submanifold on which such a p-form restricts to be a volume form is homologically area-minimizing, that is, the submanifold has area no greater than any submanifold (or even current) in the same homology class [H, HL, or Mg1]. Since a submanifold possessing a volume form is orientable, it might appear that orientability is an indispensable part of calibration theory. However, in this paper we show that an analogous theory of "twisted" calibrations holds for some nonorientable submanifolds of a Riemannian manifold M. Using differential forms with values in a flat real line bundle, a theory analogous to the theory of calibrated geometry is developed.

The utility of this generalized method becomes more apparent when it is applied to an example and gives a result about area-minimization not previously known. The second half of the paper gives a proof that the cone on the Veronese surface is twisted-calibrated and hence area-minimizing among a large class of comparison 3-folds.

In §1, we define differential forms with values in a real Euclidean line bundle L over a smooth manifold M. These forms are interpreted naturally as ordinary differential forms of odd type on the two-sheeted covering space of M determined by L.

Section 2 introduces the notion of an L-orientation of a (not necessarily orientable) submanifold N of M. The fundamental observation is that it may

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be possible for N to be "L-oriented" by some nontrivial line bundle over M which is different from the orientation bundle of M. That is, for some line bundle L over M, it may be that the orientation bundle of N and the restriction of L to N are isomorphic. This condition allows a natural pairing, via integration, between the L-oriented submanifolds of M and the L-valued (or twisted) forms. The pairing satisfies an L-valued analogue of Stokes's theorem, exhibiting the duality between the exterior derivative on twisted forms and the boundary operator on L-oriented submanifolds.

In §3, we show that the comass of a twisted p-form on a Riemannian manifold is well defined and hence that twisted-calibrations are well defined. The L-valued Stokes's theorem is applied to prove the Fundamental Theorem of Twisted-Calibrations and to show the area-minimizing property of twisted-calibrated submanifolds analogous to the area-minimizing property of calibrated submanifolds in calibrated geometries.

Sections 4 and 5 concern the application of the theory to the specific example of the cone on the Veronese surface, $\widehat{\Sigma} \subseteq \mathbf{R}^5$. Even though $\mathbf{E}^5$ is simply-connected, and so possesses no nontrivial line bundles, the open subset of $\mathbf{E}^5$ obtained by removing the cone on the *opposite* Veronese surface $-\widehat{\Sigma}$ *does* admit a nontrivial line bundle L. Furthermore, $\widehat{\Sigma} - \{0\}$ (the cone minus the vertex) is L-orientable. Finally, considering $\mathbf{E}^5$ as an irreducible SO(3) representation, we find an SO(3)-invariant twisted-calibration Φ on $\mathbf{E}^5 - (-\widehat{\Sigma})$ which twisted-calibrates $\widehat{\Sigma} - \{0\}$. This proves that $\widehat{\Sigma} - \{0\}$ minimizes area among all L-homologous 3-folds—a result complementing nicely the recent work of G. Lawlor [Lr] on area-minimizing cones.

In §6, analysis of the twisted-calibration enables us to prove that $\widehat{\Sigma} - \{0\}$ is the only 3-fold which is twisted-calibrated by the 3-form Φ.

This paper constitutes a summary of the results from author's Ph.D. thesis [M]. The author wishes to express his thanks to his advisor, Robert L. Bryant, for all his help and advice.

Preliminaries

To establish terminology and notation we begin by recalling some elementary facts about real line bundles on smooth manifolds.

Let M be a smooth manifold and let $\pi : L \to M$ be a smooth real line bundle over M. Recall that a (*smooth*) *real Euclidean line bundle* L *over* M is a smooth real line bundle equipped with a function $q : L \to \mathbf{R}$ which when restricted to each fiber $\pi^{-1}(x)$ of L is a positive definite quadratic form (such a function is a *Euclidean metric* on L). Note that every real Euclidean line bundle L possesses a canonical involution σ obtained from the linear involution on the fiber $x \to -x$. This involution plays an important role in the definition of "twisted" forms in §1.

Next, recall that a real Euclidean line bundle L over M is *flat* if there is some trivialization with locally constant transition functions. Since we may assume

that transition functions of any real Euclidean line bundle take values in the one-dimensional orthogonal group (which is just $\mathbf{Z}_2$, the group of integers modulo two), it follows that any real Euclidean line bundle over a smooth manifold is flat. Thus the set of real Euclidean line bundles over M is the same as the set of flat Euclidean line bundles over M.

The transition functions of a trivialization by unit length sections of a real Euclidean line bundle L are $\mathbf{Z}_2$-valued cocycles, so a real Euclidean line bundle is naturally identified with an element of the cohomology group $H^1(M, \mathbf{Z}_2)$. Furthermore, letting M_L denote the set of points of length one (in the fibers) of L, we see that M_L forms a smooth submanifold of L such that the projection $\pi : L \to M$ restricts to M_L to be a smooth two-sheeted covering of M. Clearly, the sheet-interchange involution is just the restriction of the involution σ to M_L.

Combining the above, we have the following

Proposition 1. *If M is a smooth manifold, then the following data are equivalent:*

(1) *A smooth Euclidean line bundle L over M.*

(2) *A cohomology class $[g_L] \in H^1(M, \mathbf{Z}_2)$.*

(3) *A smooth two-sheeted cover $\pi : M_L \to M$.*

(4) *A subgroup K of $\pi_1(M, x)$ of index at most two, where $\pi_1(M, x)$ denotes the fundamental group of M based at the point $x \in M$.*

(5) *A homomorphism $\phi : \pi_1(M, x) \to \mathbf{Z}_2$.*

The *orientation bundle* of a smooth manifold M, $\varepsilon(M)$ plays a major role in twisted-calibration theory. Recall that this Euclidean line bundle over M can be described by the data $(\{U_\alpha\}, g_{\alpha\beta})_{\alpha, \beta \in A}$ consisting of an open cover of M by contractible coordinate neighborhoods and transition functions

$$g_{\alpha\beta} = \operatorname{sgn}[\det(J_{\alpha\beta})],$$

where $J_{\alpha\beta}$ denotes the Jacobian of the coordinate overlap maps (i.e. the transition functions for the tangent bundle of M).

In the case when $\varepsilon(M)$ is not trivial, i.e. when M is not orientable, the double cover $M_{\varepsilon(M)}$ is the unique orientable (in fact canonically oriented, see [AMR, p. 452]) two-sheeted cover of M.

1. L-VALUED FORMS

Let M be a smooth manifold and let $\pi : L \to M$ be a smooth Euclidean line bundle. For an open subset U of M, the space of smooth *L-valued p-forms* on U, $\Omega^p(U, L)$, is the space of smooth sections of the bundle $\Lambda^p(T^*U) \otimes L$ over U.

For our purposes, an important example is when $L = \varepsilon(M)$, the orientation bundle of M. A section $\psi \in \Omega^m(M, \varepsilon(M))$, where m is the dimension of M, is a *density*. Densities play a central part in the formulation of Stokes's theorem for L-valued forms proved in §2.

In general, since L is Euclidean it possesses local (smooth) sections of unit length and so every smooth L-valued p-form φ can be expressed locally as

$$\varphi = \omega \otimes \varepsilon,$$

where ω is a smooth p-form on M (in the usual sense) and ε is a smooth unit length section of L. This shows that any L-valued p-form φ on M satisfies

$$1 \otimes \sigma(\varphi) = -\varphi,$$

where σ is the canonical involution of L. Furthermore, since L is flat, the exterior derivative d of (smooth) L-valued p-forms given by the local formula

$$d(\omega \otimes \varepsilon) = (d\omega) \otimes \varepsilon$$

is a well-defined global operator.

Giving closed and exact L-valued forms the obvious definitions, we let $Z^p(M, L)$ denote the vector space of closed L-valued p-forms. Then we form the *L-valued de Rham cohomology* (*in dimension* p):

$$H^p(M, L) = Z^p(M, L)/d(\Omega^{p-1}(M, L)).$$

These L-valued objects have a natural identification with forms and cohomology on M_L.

Proposition 2. *Let L be a smooth Euclidean line bundle over the smooth manifold M with Euclidean metric q and smooth L-valued p-forms, $\Omega^p(M, L)$. Let M_L be the associated two-sheeted covering space. Then*

(a) $\Omega^p(M_L) = \Omega^p_-(M_L) \oplus \Omega^p_+(M_L)$, *where $\Omega^p_-(M_L)$ (resp. $\Omega^p_+(M_L)$) is the space of σ^*-anti-invariant (resp. σ^*-invariant) p-forms.*

(b) $H^p(M_L, \mathbf{R}) = H^p_+(M_L, \mathbf{R}) \oplus H^p_-(M_L, \mathbf{R})$,

(c) $\Omega^p(M, L) \cong \Omega^p_-(M_L)$,

(d) $H^p(M, L) \cong H^p_-(M_L, \mathbf{R})$,

(e) *The map $\pi^* : H^p(M, \mathbf{R}) \to H^p_+(M_L, \mathbf{R})$ is an injection.*

Proof. (a) and (b) are clear, since exterior differentiation commutes with the decomposition into type. To prove (c), define a map (called tilde) $\sim: \Omega^p(M, L) \to \Omega^p_-(M_L)$ as follows. Let $\psi \in \Omega^p(M, L)$, and $v_1, v_2, \dots, v_p \in T_{\tilde{x}}(M_L)$, where $\tilde{x} \in M_L$ is such that $q(\tilde{x}) = 1$. Then the equation

$$\psi(\pi_*(v_1), \pi_*(v_2), \dots, \pi_*(v_p)) = \tilde{\psi}(v_1, v_2, \dots, v_p)\tilde{x}, \tag{1.1}$$

defines $\tilde{\psi} \in \Omega^p_-(M_L)$. Here π is the covering map and $x = \pi(\tilde{x})$. Note that both sides of (1.1) have values in L_x $(= \pi^{-1}(x))$. It is now easy to check that $\tilde{\psi} \in \Omega^p_-(M_L)$, so (1.1) gives a natural identification of the spaces $\Omega^p(M, L)$ and $\Omega^p_-(M_L)$. Also, it is easy to check that the map tilde commutes with exterior differentiation and so induces an isomorphism of cohomology, thus proving (d).

Finally, to prove part (e), observe that the pull-back map $\pi^*: \Omega^p(M) \to \Omega^p_+(M_L)$ also commutes with exterior differentiation. From this, we see that

the induced map on de Rham cohomology is injective since the covering map π is a local diffeomorphism. Q.E.D.

Remark. From the point of view of constructing L-valued forms, part (c) of Proposition 2 is especially useful since the L-valued forms are naturally interpreted as ordinary differential forms on M_L satisfying a "twisting" condition. Also, both of $\psi \in \Omega^p(M, L)$ and $\tilde{\psi} \in \Omega^p_-(M_L)$ are *twisted p-forms.* However, no confusion should arise since the meaning will be clear from the context.

2. Integration and L-manifolds

To have an analogue of the Fundamental Theorem of Calibrations (see [HL, Theorem 4.2 of Chapter II]), a version of Stokes's theorem for the integration of L-valued differential forms must hold. Proposition 2 shows that smooth L-valued p-forms on M are identified with smooth twisted p-forms on the double cover M_L. The latter can be integrated over oriented p-dimensional submanifolds of M_L. If a submanifold P of M_L is the orientation double cover of some submanifold of M, then Stokes's theorem applied to P gives geometric information about that submanifold of M. This idea leads to the

Definition. Let L be a smooth Euclidean line bundle over the smooth manifold M^m of dimension m. Suppose N^n is a smooth (embedded) submanifold of M of dimension n. Then N is an (*n-dimensional*) *L-orientable submanifold* of M, or *L-manifold*, if the orientation bundle of N is isomorphic to the Euclidean line bundle over N obtained by restricting L to N. An *L-orientation* of N is a choice of isomorphism between the restriction of L to N and the orientation bundle of N.

Remarks. (i) If a connected submanifold N is L-orientable, then any two isomorphisms between the orientation bundle of N and the restriction of L to N differ at most by a sign. Thus if N is given an L-orientation, we denote the oppositely L-oriented submanifold by $-N$.

(ii) An L-oriented submanifold N determines an embedded submanifold N_L of M_L which is canonically oriented (in the usual sense). In fact, N_L is the image of the manifold of unit length elements of $\varepsilon(N)$ under the embedding determined by the isomorphism between $\varepsilon(N)$ and the restriction of L to N.

By definition, the L-valued n-forms of M restrict to an n-dimensional L-oriented submanifold to be densities. Thus, if N^n is an L-oriented submanifold of M and $\psi \in \Omega^n(M, L)$, then we have

$$\int_N \psi = \frac{1}{2} \int_{N_L} \tilde{\psi} \tag{2.1}$$

where N_L is the canonically oriented double cover of N determined by L, $\tilde{\psi}$ is determined from ψ by (1.1), and the integral on the left-hand side of (2.1) is the integral of a density.

To prove an L-valued Stokes's theorem, the boundary operator for an L-manifold-with-boundary, i.e., the induced L-orientation for the boundary of an

L-oriented manifold, must be defined. Let N be an orientable submanifold-with-boundary of M. Recall that a "sign free" statement of Stokes's theorem for an orientable manifold-with-boundary is obtained when the boundary inherits the orientation induced by the "outward-pointing normal first" rule (for example, see [S, pp. 352–355]). In the language of real line bundles, the "outward-pointing normal first" rule is a canonical choice of isomorphism between the line bundles $\varepsilon(N)|_{\partial N}$ and $\varepsilon(\partial N)$. When N is not orientable, it must be shown that such a canonical choice can still be made. In view of the equivalence between real Euclidean line bundles over N and two-sheeted covers of N (cf. Proposition 1), such a choice is equivalent to showing that the double cover, $N_{\varepsilon(N)}$, of N determined by $\varepsilon(N)$ induces the orientation double cover of ∂N.

Proposition 3. $\partial N_{\varepsilon(N)} = \partial N_{\varepsilon(\partial N)}$.

Proof. Note that $(\partial N)_{\varepsilon(N)} = \partial(N_{\varepsilon(N)})$, so the expression $\partial N_{\varepsilon(N)}$ is unambiguous. Since $N_{\varepsilon(N)}$ is canonically oriented, the two points of $\partial N_{\varepsilon(N)}$ over a point of ∂N correspond to opposite orientations of N. However, $\partial N_{\varepsilon(N)}$ is also the boundary of an oriented manifold and can be oriented by the "outward-pointing normal first" rule. Taken in combination, these orientations show that the two points of $\partial N_{\varepsilon(N)}$ over a point of ∂N correspond to opposite orientations of ∂N. This last condition defines the orientation double cover of ∂N. Q.E.D.

Definition. Let N be an L-oriented submanifold-with-boundary of M and let $\lambda : \varepsilon(N) \to L$ be the isomorphism defining the L-orientation. The *induced L-orientation* of ∂N is the L-orientation determined by the isomorphism $\lambda \circ \alpha : \varepsilon(\partial N) \to L$, where α is the canonical choice of isomorphism between the line bundles $\varepsilon(\partial N)$ and $\varepsilon(N)|_{\partial N}$ determined by the "outward-pointing normal first rule."

Combining this definition of the induced L-orientation with the fact that an L-valued p-form restricted to an L-oriented submanifold has a natural interpretation as a density, we have

Theorem 1 (Stokes's theorem for L-manifolds). *Let L be a smooth Euclidean line bundle over a smooth manifold M and let N be a smooth L-oriented submanifold-with-boundary of dimension n. Then for $\psi \in \Omega^{n-1}(M, L)$,*

$$\int_{\partial N} \psi = \int_N d\psi. \tag{2.2}$$

where ∂N is given the induced L-orientation.

Proof. Everything has been defined so that the exterior derivative of an L-valued p-form is dual to the boundary operator defined for L-manifolds-with-boundary. That is, for an L-manifold-with-boundary, (2.2) is the statement of the usual Stokes's theorem for the oriented double covering manifold-with-boundary, N_L, of N. Q.E.D.

Remark. In the case when $M = N$ and $n = \dim M$, then the above theorem is just the well-known version of Stokes's theorem for densities (for example, see [AMR, pp. 485–489]).

3. Twisted-calibrations

Let (M, g) be a smooth Riemannian manifold of dimension m. We recall some basic definitions from the theory of calibrated geometry. The *comass* of a p-form φ on M at the point $x \in M$ is

$$\|\varphi\|_x^* = \sup\{\varphi(\xi) | \xi \in G(p, T_x M)\},$$

where $G(p, T_x M)$ is the Grassmannian of oriented unit simple p-vectors of $T_x M$. The comass of φ on a subset N of M is the supremum of the values of the comass of φ at points of N. A smooth closed p-form φ of comass one on M is a *calibration.* For a calibration φ we have the fundamental inequality

$$\varphi|_P \leq \mathrm{vol}_P,$$

for any p-dimensional submanifold P of M. A submanifold P of M of dimension p is a *φ-submanifold* (*in the calibrated sense*) if

$$\varphi|_P = \mathrm{vol}_P,$$

i.e. if φ restricts to P to be the volume form. We also say that φ *calibrates* P.

Now suppose L is a Euclidean line bundle over the Riemannian manifold (M, g) with the associated Riemannian double cover $\pi : M_L \to M$, i.e. give M_L the pull-back metric $\tilde{g} = \pi^*(g)$. Then π is a local isometry and the sheet-interchange involution is an isometry of M_L. Letting $\varphi \in \Omega^p(M, L)$ be an L-valued p-form, the *comass* of φ is defined as the comass of $\tilde{\varphi} \in \Omega^p_-(M_L)$, where $\tilde{\varphi}$ is given by Proposition 2. A closed L-valued p-form, φ, is an (L-)*twisted-calibration* if $\tilde{\varphi}$ is a calibration on $(M_L, \tilde{g})$. Finally, let N be a p-dimensional L-oriented submanifold of M and let $\varphi \in \Omega^p(M, L)$ be an L-valued form of comass one. Then N is a *φ-submanifold* (*in the twisted-calibrated sense*), or φ *twisted-calibrates* N, if the double cover N_L of N is a $\tilde{\varphi}$-submanifold of M_L in the calibrated sense.

Theorem 2. *Let $\varphi \in \Omega^p(M, L)$ be a twisted-calibration and let N be a φ-submanifold. Then N is a minimal submanifold of M.*

Proof. If N is a φ-manifold, where φ is a twisted-calibration, then the double cover N_L is calibrated and hence homologically mass minimizing in M_L. In particular, N_L is a minimal submanifold of M_L. Recalling that π is a local isometry, the mean curvature vector must be preserved (locally) and thus N must be a minimal submanifold of M. Q.E.D.

In fact, Stokes's theorem for L-manifolds shows that twisted-calibrated submanifolds satisfy a minimization property very similar to that for calibrated submanifolds.

Theorem 3 (The Fundamental Theorem of Twisted-Calibrations). *Let* $\varphi \in \Omega^p(M, L)$ *be a twisted-calibration and let* N *be an* p*-dimensional* φ*-submanifold. Suppose* N' *is a* p*-dimensional* L*-oriented submanifold of* M *such that* $N - N'$ *is the* L*-oriented boundary of an* $(p+1)$*-dimensional* L*-oriented manifold* P. *Then* $\mathrm{vol}(N) \leq \mathrm{vol}(N')$.

Proof. By (2.1),

(a) $\int_N \varphi = \frac{1}{2}\int_{N_L} \tilde{\varphi}$ and $\int_{N'} \varphi = \frac{1}{2}\int_{N'_L} \tilde{\varphi}$,

where N_L and N'_L are the associated double covers. Since N is a φ-manifold we have

(b) $\int_{N_L} \tilde{\varphi} = \mathrm{vol}(N_L)$.

By Stokes's theorem for L-manifolds we have

(c) $\int_{N-N'} \varphi = \int_{\partial P} \varphi = \int_P d\varphi = 0$,

since $N - N'$ is the L-oriented boundary of P^{p+1} and φ is closed. Thus

$$\int_{N-N'} \varphi = \int_N \varphi - \int_{N'} \varphi = 0,$$

or,

(d) $\int_N \varphi = \int_{N'} \varphi$.

Combining (a)–(d) we obtain

$$\begin{aligned}\mathrm{vol}(N) &= \frac{1}{2}\mathrm{vol}(N_L) = \frac{1}{2}\int_{N_L} \tilde{\varphi} = \int_N \varphi \\ &= \int_{N'} \varphi = \frac{1}{2}\int_{N'_L} \varphi \leq \frac{1}{2}\mathrm{vol}(N'_L) = \mathrm{vol}(N'),\end{aligned}$$

which proves the theorem. Q.E.D.

Remark. If N is orientable, then N_L is a disconnected set consisting of two connected components. Furthermore, each component is a submanifold of M_L which is mapped isometrically onto N by the covering map when N and (the components of) N_L are given the induced metric. Thus if N is twisted-calibrated, then each component of N_L is calibrated by $\tilde{\varphi}$.

One immediate consequence of Theorem 3 is

Theorem 4. *If* N *is a twisted-calibrated submanifold of* M, *then* N *is stable.*

Proof. Let N be an L-oriented submanifold of M and let $\Phi : (-\varepsilon, \varepsilon) \times N \to M$ be a compactly supported smooth variation. Then for ε sufficiently small, the submanifolds $N_t = \Phi(t, N)$ are all L-oriented. Since $N = N_0$ has a calibrated double cover, the fundamental theorem of calibrations shows that the double cover of N has area no greater than the area of the double covers of the N_t. Q.E.D.

Remark. If N is an L-oriented submanifold-with-boundary which is twisted-calibrated, then N is area-minimizing for variations which hold the boundary fixed.

The next two sections provide the proof that the cone on the Veronese surface is twisted-calibrated, hence stable and, moreover, area-minimizing among all comparison L-manifolds.

4. The cone on the Veronese surface

The *Veronese surface* Σ is a real projective plane which is minimally embedded in the Euclidean 4-sphere in $\mathbf{E}^5$ as an SO(3)-orbit. The *cone on the Veronese surface* $\widehat{\Sigma}$ is the subset of $\mathbf{E}^5$ which is the union of line segments of the form $\overline{0p}$, where 0 is the origin of $\mathbf{E}^5$ and $p \in \Sigma$. The origin is the *vertex* of the cone. The *extended cone on the Veronese surface* $\widehat{\Sigma}_e$ is the set of rays from the origin through points on Σ.

Remark. Our adherence to these distinctions will be weak, but the general philosophy is that all results about cones refer to extended cones, except those results involving manifolds-with-boundary.

Concretely, the Veronese surface is realized as an orbit of the irreducible linear action of SO(3) on $\mathbf{E}^5$. Let V denote the real inner product space of traceless symmetric 3-by-3 matrices. The left SO(3) action on V is given by

$$g \cdot m \equiv g m g^t,$$

where $g \in \mathrm{SO}(3)$ and $m \in V$. Endowing V with the inner product

$$\langle m_1, m_2 \rangle = \tfrac{1}{2}\operatorname{tr}(m_1 m_1),$$

for $m_1, m_2 \in V$, it is straightforward to check that $\mathbf{E}^5$ is isometric to $(V, \langle\ ,\ \rangle)$ via the isometry

$$\Psi(x_1, x_2, x_3, x_4, x_5) = \begin{bmatrix} \frac{1}{\sqrt{3}}x_1 + x_2 & x_3 & x_4 \\ x_3 & \frac{1}{\sqrt{3}}x_1 - x_2 & x_5 \\ x_4 & x_5 & -\frac{2}{\sqrt{3}}x_1 \end{bmatrix}.$$

Furthermore, it is clear that

$$\langle g \cdot m_1, g \cdot m_2 \rangle = \langle m_1, m_2 \rangle,$$

so SO(3) acts orthogonally on $(V, \langle\ ,\ \rangle)$.

Every matrix in V is O(3)-similar to a diagonal matrix, and in fact, SO(3)-similar to a matrix lying in the extended cone

$$\overline{W} = \{\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) \in V \mid \lambda_1, \lambda_2 \geq \lambda_3\}.$$

The intersection of $\overline{W}$ with the unit 4-sphere of V is a set homeomorphic to a closed interval [L, p. 24]. The endpoints of this interval correspond to the orbits of diagonal matrices having two equal eigenvalues. It is easy to check that the stabilizer of such a matrix is isomorphic to the full orthogonal group O(2), so that each orbit is an embedded real projective plane. The orbit of the matrix $m_0 = \operatorname{diag}(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$ is the Veronese surface Σ. The orbit of $-m_0$ is the *opposite Veronese surface* $-\Sigma$ (with corresponding cone $-\widehat{\Sigma}$). Note

that the antipodal map of S^4 is an equivariant isometry which interchanges the two (extended) cones. Finally, since these orbits are of singular type and are manifolds, they are minimal [HsL, p. 5].

Now, recall the well-known parameterization of $-\Sigma$. Let $y \in \mathbf{E}^3$ be a column vector. We define the map $\tilde{\mu} : \mathbf{E}^3 \to V$ by the formula

$$\tilde{\mu}(\mathbf{y}) = \mathbf{y}\mathbf{y}^t - \tfrac{1}{3}|\mathbf{y}|^2 I_3$$

where $\mathbf{y}^t = (y_1, y_2, y_3)$ and $|\mathbf{y}|^2 = y_1^2 + y_2^2 + y_3^2$.

If we let $\mathrm{SO}(3)$ act on the left on $\mathbf{E}^3$ in the standard way, then $\tilde{\mu}$ is equivariant. In matrix form:

$$\tilde{\mu}(\mathbf{y}) = \frac{1}{3}\begin{bmatrix} 2y_1^2 - y_2^2 - y_3^2 & 3y_1y_2 & 3y_1y_3 \\ 3y_1y_2 & 2y_2^2 - y_1^2 - y_3^2 & 3y_2y_3 \\ 3y_1y_3 & 3y_2y_3 & 2y_3^2 - y_1^2 - y_2^2 \end{bmatrix}.$$

A straightforward calculation shows

$$\langle \tilde{\mu}(\mathbf{y}), \tilde{\mu}(\mathbf{y}) \rangle = \tfrac{1}{3}|\mathbf{y}|^4,$$

so this map restricts to $\tilde{\mu} : S^2(\sqrt[4]{3}) \to S^4$. Obviously $\tilde{\mu}(-\mathbf{y}) = \tilde{\mu}(\mathbf{y})$, so $\tilde{\mu}$ descends to a map of the real projective plane which is easily checked to be one-to-one. Finally, since $\tilde{\mu}$ is equivariant and

$$\tilde{\mu}((\sqrt[4]{3}, 0, 0)^t) = \operatorname{diag}\left(\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right),$$

it follows that $\tilde{\mu}$ gives an embedding of the real projective plane with image $-\Sigma$.

Extend the map $\tilde{\mu}$ of $\mathbf{E}^3$ into V to the map $\mu : \mathbf{C}^3 \to V$ defined by

$$\mu(\mathbf{z}) = \tfrac{1}{2}(\bar{\mathbf{z}}\mathbf{z}^t + \mathbf{z}\bar{\mathbf{z}}^t) - \tfrac{1}{3}|\mathbf{z}|^2 I_3,$$

where $\mathbf{z}^t = (z_1, z_2, z_3) \in \mathbf{C}^3$ and $|\mathbf{z}|$ is the usual norm. If $\mathbf{z}$ is real (i.e. if $\mathbf{z} = \bar{\mathbf{z}}$), then $\mu(\mathbf{z}) = \tilde{\mu}(\mathbf{z})$, so μ is an extension of $\tilde{\mu}$ to $\mathbf{C}^3$.

Proposition 4. *The map μ satisfies the following properties*:

(i) $\mu(\lambda\mathbf{z}) = |\lambda|^2\mu(\mathbf{z})$.

(ii) $\mu(\mathbf{z}) = \mu(\bar{\mathbf{z}})$.

(iii) $\mu : \mathbf{C}^3 \to V$ *is* $\mathrm{SO}(3)$*-equivariant with respect to the complexified action on* $\mathbf{C}^3 = \mathbf{C} \otimes \mathbf{R}^3$.

(iv) μ *is surjective.*

(v) $\mu(\mathbf{z}) = \mu(\mathbf{w})$ *if and only if* $\mathbf{w} = e^{i\theta}\mathbf{z}$ *or* $\mathbf{w} = e^{i\theta}\bar{\mathbf{z}}$.

(vi) *The singular set of* μ *is* $N^4 = \{\lambda\mathbf{x} \in \mathbf{C}^3 | \lambda \in \mathbf{C}, \mathbf{x} \in \mathbf{R}^3\}$. *Furthermore, the image of* N^4 *is the extended cone on the opposite Veronese surface.*

Proof. (i), (ii), and (iii) are clear.

(iv) Since μ is $\mathrm{SO}(3)$-equivariant, it suffices to show that the map $\mu : \mathbf{C}^3 \to \overline{W} \subseteq V$ is surjective, where $\overline{W} = \{\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) \in V | \lambda_1, \lambda_2 \geq \lambda_3\}$, since

every element of V is equivalent, via the group action, to an element in this set. Thus, if $\delta_0 = \operatorname{diag}(\lambda_1^0, \lambda_2^0, \lambda_3^0) \in \overline{W}$, then $\mathbf{z}_0^t = (\sqrt{\lambda_1^0 - \lambda_3^0}, i\sqrt{\lambda_2^0 - \lambda_3^0}, 0)$ satisfies $\mu(\mathbf{z}_0) = \delta_0$, since $\lambda_1^0 + \lambda_2^0 + \lambda_3^0 = 0$.

(v) Sufficiency is clear. Conversely, suppose $\mu(\mathbf{z}) = \mu(\mathbf{w})$. Using the equivariance of μ, we may assume $\mu(\mathbf{z}) = \mu(\mathbf{w}) = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) \in \overline{W}$. Using part (i), choose $\theta_1, \theta_2 \in \mathbf{R}/2\pi\mathbf{Z}$ so that the first coordinates of both $e^{i\theta_1}\mathbf{z}, e^{i\theta_2}\mathbf{w}$ are real and positive. Since $\mu(\mathbf{z}) = \mu(\mathbf{w})$ is diagonal and $\lambda_1, \lambda_2 \geq \lambda_3$, the second coordinates of both $e^{i\theta_1}\mathbf{z}, e^{i\theta_2}\mathbf{w}$ must be purely imaginary, while the third must be zero. Thus $\mathbf{w} = e^{i\theta}\mathbf{z}$, where $\theta = \theta_1 - \theta_2$, or $\mathbf{w} = e^{i\theta}\overline{\mathbf{z}}$, where $\theta = -\theta_1 - \theta_2$.

(vi) Writing

$$\mu = \begin{bmatrix} f_1 & f_3 & f_4 \\ f_3 & f_2 & f_5 \\ f_4 & f_5 & -(f_1 + f_2) \end{bmatrix}$$

it suffices to find the singular set of the map $\mathbf{f} = (f_1, \dots, f_5)$ from $\mathbf{C}^3 \cong \mathbf{R}^6$ to $\mathbf{R}^5$. It is now straightforward to show that any real linear relation among the rows of the five-by-six matrix $D\mathbf{f}$ at $\mathbf{z} \in \mathbf{C}^3$ implies that the real and imaginary parts of $\mathbf{z}$ lie in the kernel of a nontrivial traceless real symmetric 3-by-3 matrix. Since such a matrix has kernel of dimension at most one, the real and imaginary parts of $\mathbf{z}$ are linearly dependent over $\mathbf{R}$, so $\mathbf{z}$ must be a complex multiple of a vector with real entries.

Since μ is SO(3)-equivariant and $\mu(\lambda\mathbf{x}) = |\lambda|^2\mu(\mathbf{x})$, the image of the singular set is the set of rays through the origin determined by points on the opposite Veronese surface. Q.E.D.

Proposition 4 shows that the set of regular values of μ defines the open 5-manifold $M^5 = \mathbf{E}^5 - (-\widehat{\Sigma})_e$ of $\mathbf{E}^5$. Moreover, by constructing a linear (hence SO(3)-equivariant) deformation retraction of $W = \{\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) \in V | \lambda_1, \lambda_2 > \lambda_3\}$ onto $C = \{\operatorname{diag}(\lambda, \lambda, -2\lambda) | \lambda \in \mathbf{R}^+\}$, it is easy to check that M^5 deformation retracts onto Σ. Thus $\pi_1(M^5) = \mathbf{Z}_2$ and M^5 admits a nontrivial Euclidean line bundle L with associated double cover M_L^5.

The two-sheeted cover, M_L^5, is described as follows. Letting S^1 act on $\mathbf{C}^3$ by multiplication by unit complex numbers, we obtain the fibration:

$$\begin{array}{ccc} S^1 & \longrightarrow & \mathbf{C}^3 - N^4 \\ & & \downarrow \\ & & M_L^5 \end{array}$$

where N^4 is defined in part (vi) of Proposition 4. Part (i) of Proposition 4 shows that the map μ factors through the projection map of the fibration to give a map $\hat{\mu} : M_L^5 \to M^5$. Parts (iv) and (v) of Proposition 4 combine to show that $\hat{\mu}$ is a two-sheeted covering map such that the sheet-interchange involution

is given as the map induced on M_L^5 by complex conjugation on $\mathbf{C}^3$. Finally, we observe that the smooth map from $\mathbf{C}^3 - N^4$ to $\mathbf{R}^+ \times (\mathbf{CP}^2 - \mathbf{RP}^2)$ given by

$$\mathbf{z} \to (|\mathbf{z}|^2, [\mathbf{z}]), \tag{4.1}$$

where $z \in \mathbf{C}^3 - N^4$, $|\mathbf{z}|$ is the usual norm, and $[\mathbf{z}]$ denotes homogeneous coordinates on $\mathbf{CP}^2$, is clearly invariant under the S^1 action on $\mathbf{C}^3 - N^4$. Thus the map (4.1) descends to a smooth map from M_L^5 to $\mathbf{R}^+ \times (\mathbf{CP}^2 - \mathbf{RP}^2)$ which is easily checked to have smooth inverse. Thus M_L^5 is diffeomorphic to $\mathbf{R}^+ \times (\mathbf{CP}^2 - \mathbf{RP}^2)$.

Proposition 5. *The cone of the Veronese surface, minus the vertex,* $\widehat{\Sigma} - \{0\}$ *is an L-manifold. Moreover, the locus of points in* M_L^5 *covering* $\widehat{\Sigma} - \{0\}$ *is diffeomorphic to* $\mathbf{R}^+ \times Q$, *where* Q *is a nonsingular conic in* $\mathbf{CP}^2 - \mathbf{RP}^2$ *given by* $Q = \{[z_1, z_2, z_3] \in \mathbf{CP}^2 | z_1^2 + z_2^2 + z_3^2 = 0\}$.

Proof. For notational simplicity, we use $\mathbf{z}$ to denote both a point in $\mathbf{C}^3 - N^4$ as well as its equivalence class in $\mathbf{M}_L^5$. Recalling that $\mu^{-1}(\widehat{\Sigma} - \{0\}) = \{\mathbf{z} \in \mathbf{C}^3 - N^4 | \mu(\mathbf{z})$ has repeated eigenvalues$\}$, the proof of part (iv) of Proposition 4 shows that if $\mu(\mathbf{z}_0) = \operatorname{diag}(\lambda, \lambda, -2\lambda)$, where $\lambda > 0$, then $\mathbf{z}_0^t = e^{i\theta}\sqrt{3\lambda}(1, \pm i, 0)$. In either case, we have $\mathbf{z}_0 \bullet \mathbf{z}_0 = z_1^2 + z_2^2 + z_3^2 = 0$. Furthermore, the condition $\mathbf{z}_0 \bullet \mathbf{z}_0 = 0$ is invariant under the actions of both S^1 and $\mathrm{SO}(3)$.

Letting $\overline{\widehat{\Sigma} - \{0\}}$ be the subset of M_L^5 determined by $\mu^{-1}(\widehat{\Sigma} - \{0\}) \subset \mathbf{C}^3 - N^4$, the mapping (4.1) shows that $\overline{\widehat{\Sigma} - \{0\}}$ is the inverse image of the cone over Q in $\mathbf{R}^+ \times (\mathbf{CP}^2 - \mathbf{RP}^2)$. Thus, $\mathbf{R}^+ \times Q$ is diffeomorphic to the orientable manifold $\mathbf{R}^3 - \{0\}$ (since a nonsingular conic is rational), proving that $\widehat{\Sigma} - \{0\}$ is an L-manifold. Q.E.D.

Our goal is to prove

Theorem 5. *The cone on the Veronese surface (minus the vertex) is twisted-calibrated, hence area minimizing among all L-orientable 3-folds in* M^5 *having the Veronese surface as boundary.*

5. Proof of theorem 5

The construction of a comass one twisted 3-form on M_L^5 which twisted-calibrates the cone on the Veronese surface is simplified greatly by introducing two auxiliary spaces. Previously, we defined $W = \{\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) \in V | \lambda_1, \lambda_2 > \lambda_3\}$. Introducing $Y = \{(a, ib, 0)^t \in \mathbf{C}^3 | a, b \in \mathbf{R}^+\}$, the proof of the following proposition reduces to straightforward calculations which are left to the reader.

Proposition 6. (a) *The map* $f_W : \mathrm{SO}(3) \times W \to M^5$ *given by* $f_W(g, \delta) = g\delta g^t$ *is surjective, generically eight-to-one and equivariant with respect to the* $\mathrm{SO}(3)$

action on $\mathrm{SO}(3) \times W$ *given by* $h \cdot (g, \delta) = (hg, \delta)$. *The set of singular values of* f_W *is precisely the cone on the Veronese surface.*

(b) *For* $m \in V$ *we have*

$$f_W^{-1}(m) = \begin{cases} \{R_h(g, \delta) | h \in D_4\} & \text{when } \delta \in W \text{ has distinct eigenvalues}, \\ \{R_h(g, \delta) | h \in T\} & \text{when } \delta \in W \text{ has repeated eigenvalues}, \end{cases}$$

where $m = f_W(g, \delta)$, R_h *denotes the right action of* $\mathrm{SO}(3)$ *on* $\mathrm{SO}(3) \times W$ *given by* $R_h(g, \delta) = (gh, h^t \delta h)$, D_4 *is the subgroup of* $\mathrm{SO}(3)$ *of order eight generated by the set*

$$\left\{ \varepsilon_1 = \mathrm{diag}(-1, -1, 1), \ \varepsilon_2 = \mathrm{diag}(1, -1, -1), \ \text{and } s = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\},$$

and

$$T = \left\{ \begin{bmatrix} A & 0 \\ 0 & \det A \end{bmatrix} \middle| A \in O(2) \right\}.$$

(c) *Letting* $\mathrm{SO}(3)$ *act on* $\mathrm{SO}(3) \times Y$ *by* $h \cdot (g, \mathbf{w}) = (hg, \mathbf{w})$, *the map* $f_Y : \mathrm{SO}(3) \times Y \to M_L^5$ *given by* $f_Y(g, \mathbf{w}) = [g\mathbf{w}]$ *is surjective, equivariant, and is four-to-one away from the singular set. The set of singular values corresponds precisely to the cone on the conic* Q *of Proposition* 5 *under the diffeomorphism* (4.1).

(d) *The map* $\nu : W \to Y$ *given by*

$$\nu(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)) = (\sqrt{\lambda_1 - \lambda_3}, i\sqrt{\lambda_2 - \lambda_3}, 0)^t$$

is a diffeomorphism.

(e) *The map* $\tilde{f}_W : \mathrm{SO}(3) \times W \to M_L^5$ *given by* $\tilde{f}_W(g, \delta) = f_Y(g, \nu(\delta)) = [g(\nu(\delta))]$ *is a left-equivariant lifting of* f_W *which is generically four-to-one. In fact, if* $\mathbf{z}_0$ *is a regular value of* $\tilde{f}_W$, *then* $\tilde{f}_W^{-1}(\mathbf{z}_0) = \{R_h(g, \delta) | h \in \mathbf{Z}_4\}$, *where* $\tilde{f}_W(g, \delta) = \mathbf{z}_0$ *and* $\mathbf{Z}_4$ *is the unique cyclic normal subgroup of order four in* D_4. *Moreover,* $\mathbf{Z}_4$ *is generated by* $s\varepsilon_2 \in D_4$, *where* s *and* ε_2 *are as in part* (b). *Finally, if* $\tilde{f}_W(g, \delta) = \mathbf{z}$, *then* $\tilde{f}_W(R_{\varepsilon_2}(g, \delta)) = \bar{\mathbf{z}}$, *i.e. the sheet interchange involution on* M_L^5 *is represented by* R_{ε_2} *on* $\mathrm{SO}(3) \times W$.

Remark. The results of Proposition 6 may be more easily understood by referring to the following commutative diagram

$$\begin{tikzcd} \mathrm{SO}(3) \times Y \arrow[r, "f_Y"] & M_L^5 \arrow[d, "\hat{\mu}"] \\ \mathrm{SO}(3) \times W \arrow[u, "(\mathrm{Id} \times \nu)"] \arrow[r, "f_W"'] \arrow[ur, "\tilde{f}_W"] & M^5 \end{tikzcd}$$

where all maps are (left) $\mathrm{SO}(3)$-equivariant and all subdiagrams are commutative.

To begin constructing the twisted-calibration, g_0, the pull-back to $\mathrm{SO}(3)\times W$ of the metric on M^5 is computed to be

$$\begin{aligned} g_0 = f_W^*(\langle\ ,\ \rangle) &= \tfrac{1}{2}\operatorname{tr}[d(g\delta g^t)\circ d(g\delta g^t)] \\ &= \tfrac{1}{2}\operatorname{tr}[(\omega\delta + d\delta + \delta\omega^t)\circ(\omega\delta + d\delta + \delta\omega^t)] \\ &= d\lambda_1\circ d\lambda_1 + d\lambda_1\circ d\lambda_2 + d\lambda_2\circ d\lambda_2 + (\lambda_1+2\lambda_2)^2\omega^1\circ\omega^1 \\ &\quad + (\lambda_2+2\lambda_1)^2\omega^2\circ\omega^2 + (\lambda_1-\lambda_2)^2\omega^3\circ\omega^3, \end{aligned}$$

where

$$\omega = g^{-1}dg = {}^t g\, dg = \begin{bmatrix} 0 & \omega^3 & -\omega^2 \\ -\omega^3 & 0 & \omega^1 \\ \omega^2 & -\omega^1 & 0 \end{bmatrix}$$

is the matrix of Maurer-Cartan forms for $\mathrm{SO}(3)$ and

$$\delta = \operatorname{diag}(\lambda_1, \lambda_2, -(\lambda_1+\lambda_2)).$$

Note that ω satisfies the structure equation

$$d\omega = \omega^t\wedge\omega = -\omega\wedge\omega.$$

Also, observe that this metric is singular precisely when $\lambda_1 = \lambda_2$, i.e. exactly on the f_W-inverse image of the cone on the Veronese surface.

Next, choosing the coordinates

$$u = \tfrac{3}{2}(\lambda_1+\lambda_2), \qquad v = \tfrac{1}{2}(\lambda_1-\lambda_2),$$

yields

$$g_0 = \tfrac{1}{3}du\circ du + dv\circ dv + (u-v)^2\omega^1\circ\omega^1 + (u+v)^2\omega^2\circ\omega^2 + 4v^2\omega^3\circ\omega^3. \tag{5.1}$$

In these coordinates $W = \{\operatorname{diag}(\frac{1}{3}u+v, \frac{1}{3}u-v, -\frac{2}{3}u) | u > |v|\}$ and the singular locus of the metric is just the set of points of $\mathrm{SO}(3)\times W$ with $v = 0$. Furthermore, (5.1) shows that v and u may be geometrically interpreted respectively as the shortest distance from a point $m\in M^5$ to a point $p\in(\widehat{\Sigma}-\{0\})_e$ and $\sqrt{3}$ times the distance from p to the vertex of the cone.

Away from the singular set of the metric, equation (5.1) shows that the following forms are an orthonormal coframing of $\mathrm{SO}(3)\times W$:

$$\begin{aligned} \eta^1 &= \frac{1}{\sqrt{3}}du, \qquad \eta^2 = dv, \qquad \eta^3 = (u-v)\omega^1, \\ \eta^4 &= (u+v)\omega^2, \qquad \eta^5 = 2v\omega^3. \end{aligned} \tag{5.2}$$

Thus, a natural set of 3-forms to consider are of the type

$$\Phi' = \sum_{i<j<k} a_{ijk}\eta^i\wedge\eta^j\wedge\eta^k,$$

where $a_{ijk}\in\mathbf{R}$. In order that there exist $\Phi\in\Omega^3(M_L^5)$ satisfying $\tilde{f}_W^*(\Phi) = \Phi'$, it is necessary that Φ' satisfy

$$R_h^*\Phi' = \Phi',$$

where $h \in D_4$, the group of order eight described in part (b) of Proposition 6. A straightforward calculation then shows that this requirement reduces the admissible Φ' to

$$\Phi' = \eta^1 \wedge (a_{125}\eta^2 \wedge \eta^5 + a_{134}\eta^3 \wedge \eta^4).$$

It is checked similarly that these admissible 3-forms automatically satisfy $R^*_{\varepsilon_2}\Phi' = -\Phi'$, i.e. the admissible 3-forms are already "twisted."

Proposition 7. *Φ' is closed if and only if $a_{134} = -a_{125}$.*

Proof. By (5.2) the expression for Φ' expands as

$$\begin{aligned}\Phi' &= \eta^1 \wedge (a_{125}\eta^2 \wedge \eta^5 + a_{134}\eta^3 \wedge \eta^4)\\ &= \frac{1}{\sqrt{3}}du \wedge (2a_{125}v\,dv \wedge \omega^3 + a_{134}(u^2 - v^2)\omega^1 \wedge \omega^2)\\ &= \frac{1}{\sqrt{3}}du \wedge (2a_{125}v\,dv \wedge \omega^3 + a_{134}(u^2 - v^2)\,d\omega^3).\end{aligned}$$

Note that the last equality uses the structure equation for ω. We easily compute

$$\begin{aligned}d\Phi' &= -\frac{1}{\sqrt{3}}du \wedge d(2a_{125}v\,dv \wedge \omega^3 + a_{134}(u^2 - v^2)\,d\omega^3)\\ &= \frac{1}{\sqrt{3}}(a_{125} + a_{134})du \wedge d(v^2) \wedge d\omega^3.\end{aligned}$$

Thus $d\Phi' = 0$ if and only if $a_{125} + a_{134} = 0$. Q.E.D.

Now assume Φ' is closed.

Proposition 8. *The comass of Φ' is $|a_{125}|$.*

Proof. Since the η^j are orthonormal away from the singular set, the comass of Φ' is the same as the comass of its Hodge dual 2-form $*\Phi' = a_{125}(\eta^{34} - \eta^{25})$, where we have used the orientation determined by η^{12345}. By renumbering the basis $\{\eta^j\}$, if necessary, we may assume that $a_{125} > 0$. For points not on the singular set of the metric, letting e_j denote the metric dual to η^j, it is easy to see that $*\Phi'$ is maximized (at a point p) among the unit simple 2-vectors of the vector space $V_p = \operatorname{span}\{e_2, e_3, e_4, e_5\}$. Since we clearly have

$$*\Phi'(e_2 \wedge e_5) = *\Phi'(e_4 \wedge e_3) = a_{125},$$

the comass of $*\Phi'$ is at least a_{125}. To show that the comass is no greater, suppose ζ is a simple unit 2-vector in $\Lambda^2(V_p)$. Writing ζ as

$$\begin{aligned}\zeta &= (a_1 + b_1)e_{23} + (a_2 + b_2)e_{24} + (a_3 + b_3)e_{25}\\ &= (a_1 - b_1)e_{45} + (a_2 - b_2)e_{53} + (a_3 + b_3)e_{34},\end{aligned}$$

we have

(a) ζ is simple $\Leftrightarrow \zeta \wedge \zeta = 0 \Leftrightarrow (a_1^2 + a_2^2 + a_3^2) - (b_1^2 + b_2^2 + b_3^2) = 0$, and

(b) ζ is unit $\Leftrightarrow 2(a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2) = 1$.

Now,

$$*\Phi'(\zeta) = a_{125}((a_3 - b_3) - (a_3 + b_3)) = -2a_{125}b_3,$$

and

$$2a_{125}b_3 \leq |a_{125}|,$$

since $b_1^2 + b_2^2 + b_3^2 = \frac{1}{4}$, i.e. $|b_3| \leq \frac{1}{2}$, when ζ is a simple unit 2-vector. Thus, the comass is a_{125} at points away from the singular set of the metric. However, on the singular set, $*\Phi'$ restricts to be the simple 2-form $a_{125}\eta^{34}$, which clearly has comass a_{125}. Q.E.D.

Remark. The proof of Proposition 8 is essentially the Wirtinger inequality (see the proof of the lemma in §6). However, the referee has indicated to us that the comass can be determined directly from a result of Morgan [Mg2, pp. 7–11].

Propositions 7 and 8 immediately show that the 3-form

$$\Phi' = \eta^1 \wedge (\eta^2 \wedge \eta^5 - \eta^3 \wedge \eta^4)$$

is closed and of unit comass on $\mathrm{SO}(3) \times W$. To complete the proof of Theorem 5, we find a 3-form Φ on M_L^5 which satisfies $\tilde{f}_W^*(\Phi) = \Phi'$. Then Φ will be closed and have comass one, i.e. Φ will be a calibration. Closure of Φ is immediate, while the statement about comass follows from the facts that both M_L^5 and $\mathrm{SO}(3) \times W$ have the pull-back metric and that the maps in our construction are local isometries (away from the singular locus of the metric in the case of $\mathrm{SO}(3) \times W$). On the singular locus of $\mathrm{SO}(3) \times W$, $\tilde{f}_W$ is a submersion such that $(\tilde{f}_W)_*$ is an isometry onto the complement of its kernel.

To construct Φ, use the maps $\tilde{f}_W : \mathrm{SO}(3) \times W \to M_L^5$ and $f_Y : \mathrm{SO}(3) \times Y \to M_L^5$ defined in Proposition 6 to obtain

(5.3)

$$\begin{cases} \text{(a) } \mathbf{z} \bullet \mathbf{z} = \bar{\mathbf{z}} \bullet \bar{\mathbf{z}} = 2v, \\ \text{(b) } \mathbf{z} \bullet \bar{\mathbf{z}} = |\mathbf{z}|^2 = 2u, \\ \text{(c) } \tau(\mathbf{z}) \equiv \sqrt{|\mathbf{z}|^4 - (\mathbf{z} \bullet \mathbf{z})(\bar{\mathbf{z}} \bullet \bar{\mathbf{z}})} = 2\sqrt{u^2 - v^2}, \\ \text{(d) } g^{-1}d\mathbf{z} = (g^{-1}d\bar{\mathbf{z}}) = \omega\mathbf{w} + d\mathbf{w}, \\ \text{(e) } \mathbf{z} \bullet d\bar{\mathbf{z}} = (g^{-1}\mathbf{z}) \bullet \overline{(g^{-1}d\mathbf{z})} \\ \qquad\qquad = du - 2i\sqrt{u^2 - v^2}\,\omega^3, \text{ where } \mathbf{z} = g(\sqrt{u+v}, i\sqrt{u-v}, 0)^t, \\ \text{(f) } 8(u^2 - v^2)\omega^3 = i\tau(\mathbf{z})(\mathbf{z} \bullet d\bar{\mathbf{z}} - \bar{\mathbf{z}} \bullet d\mathbf{z}). \end{cases}$$

Observe that only (b), (c), (e), and (f) are well defined on M_L^5. Next, recalling that

$$\begin{aligned} \Phi' &= \frac{1}{\sqrt{3}} du \wedge (d(v^2) \wedge \omega^3 - (u^2 - v^2)\, d\omega^3) \\ &= -\frac{1}{\sqrt{3}} du \wedge d((u^2 - v^2)\omega^3), \end{aligned}$$

then (5.3) shows that the 3-form

$$\Phi = \frac{1}{i8\sqrt{3}} d|\mathbf{z}|^2 \wedge d[\tau(\mathbf{z})(\mathbf{z} \bullet d\bar{\mathbf{z}} - \bar{\mathbf{z}} \bullet d\mathbf{z})]$$

satisfies $\tilde{f}_W^*(\Phi) = \Phi'$. Moreover, the sheet-interchange involution of M_L^5 is induced by complex conjugation on $\mathbf{C}^3$, so Φ is a twisted 3-form.

The $\tilde{f}_W$-inverse image of the double cover of the cone on the Veronese surface is the 4-manifold of points of $\mathrm{SO}(3) \times W$ having $v = 0$. At every point of this manifold, the three-dimensional subspace, $\mathrm{span}\{e_1, e_3, e_4\}$, of the tangent 4-plane, $\mathrm{span}\{e_1, e_3, e_4, e_5\}$, is a simple unit 3-plane maximizing Φ'. Thus Φ restricts to $\overline{\hat{\Sigma} - \{0\}}$ to be the volume form, so $\hat{\Sigma} - \{0\}$ is twisted-calibrated. Q.E.D.

6. Determining the Φ-submanifolds

In this section the cone of the Veronese surface is shown to be the unique 3-fold of M^5 which is twisted-calibrated by Φ. Employing a self-dual 2-form naturally associated to the twisted calibration Φ to define an almost complex structure on M_L^5, the Φ-submanifolds are proved to be cones having almost complex curves as "level" surfaces. Determining the cones satisfying this condition gives

Theorem 6. *Any 3-fold of M^5 which is twisted-calibrated by Φ is a subset of the cone on the Veronese surface.*

Proof. The Hodge star operator on $\mathrm{SO}(3) \times W$ induces a star operator on the space of forms generated by the four-dimensional subspace dual to e_1, i.e. $\mathrm{span}\{\eta^2, \eta^3, \eta^4, \eta^5\}$. This operator, denoted $*'$, is defined by the equation

$$*'\varphi = -\left(\sqrt{3}\frac{\partial}{\partial u}\right) \rfloor (*\varphi)$$

where φ is a smooth section of the $\Lambda^p(W)$-bundle for $0 \le p \le 4$, and $\rfloor$ denotes interior product. It is easy to check that $(*')^2 = (-1)^{p(4-p)}$, where we use the orientation given by η^{2345} (recall that we use the orientation given by η^{12345} for $*$). Since

$$\Phi' = \eta^1 \wedge *\Phi', \tag{6.1}$$

where the 2-form

$$*\Phi' = \eta^2 \wedge \eta^5 - \eta^3 \wedge \eta^4$$

is self-dual with respect to the $*'$ operator, it follows that $*\Phi'$ defines an almost-complex structure J on the subspace $\mathrm{span}\{e_2, e_3, e_4, e_5\} \subset T(\mathrm{SO}(3) \times W)$. Furthermore, it is easy to check that

$$\eta^2 \wedge \eta^5 - \eta^3 \wedge \eta^4 = \tilde{f}_W^*(\Omega),$$

where $\Omega = \frac{1}{8}d|\mathbf{z}|^2 \wedge [\frac{i}{2}(\mathbf{z}\bullet d\bar{\mathbf{z}} - \bar{\mathbf{z}}\bullet d\mathbf{z}) - d(\frac{i}{2}\tau(\mathbf{z})(\mathbf{z}\bullet d\bar{\mathbf{z}} - \bar{\mathbf{z}}\bullet d\mathbf{z}))]$ is a self-dual 2-form on M_L^5. Thus, the almost-complex structure J on $\mathrm{SO}(3) \times W$ corresponds to an almost-complex structure $\widetilde{J}_{|\mathbf{z}|}$ on $(\partial/\partial|\mathbf{z}|)^\perp = T(\mathbf{CP}^2 - \mathbf{RP}^2) \subset T(M_L^5)$ defined by Ω (here $\mathbf{CP}^2 - \mathbf{RP}^2$ denotes the subset of M_L^5 diffeomorphic to $\mathbf{CP}^2 - \mathbf{RP}^2$ under the map (4.1)). By definition, $(\tilde{f}_W)_*$ is complex linear with respect to these complex structures.

Lemma. *If X^3 is a Φ-manifold, then X^3 is double-covered by a subset, X_L^3, of M_L^5 diffeomorphic to a cone having level surfaces $\Sigma_t = \{\mathbf{z} \in X_L^3 | \, |\mathbf{z}| = t\}$ which are almost-complex curves with respect to the almost-complex structure $\widetilde{J}_{|\mathbf{z}|}$.*

Proof. It is clear that $\tilde{f}_W$ maps cones to cones, so the assertion holds on $\mathrm{SO}(3) \times W$ since $(\tilde{f}_W)_*$ is complex linear and at every point of $\mathrm{SO}(3) \times W$ equation (6.1) shows that Φ' is maximized on simple unit 3-vectors of the form $\partial/\partial u \wedge \zeta$ for some simple 2-vector ζ. Hence, every Φ'-threefold is a subset of a cone.

To complete the proof of the lemma, define the complex-valued 1-forms

$$\theta^0 = \eta^2 - i\eta^5, \qquad \theta^1 = \eta^3 + i\eta^4.$$

Then

$$*\Phi' = \frac{1}{2i}(\theta^0 \wedge \overline{\theta^0} + \theta^1 \wedge \overline{\theta^1}),$$

and the standard Wirtinger equality (see [L, p. 37]) applied to a simple unit 2-plane $\zeta \in \Lambda^2(\mathrm{span}\{e_2, e_3, e_4, e_5\})$, expressed in the form used in the proof of Proposition 8, becomes

$$*\Phi'(\zeta)^2 + |\theta^0 \wedge \theta^1(\zeta)|^2 = 1.$$

Thus, ζ is a complex line (with respect to J) if and only if $*\Phi'(\zeta) = 1$. Hence, X^3 is a Φ'-threefold if and only if it is a subset of a cone which has an almost-complex curve for each u-level surface. The proof of the lemma is complete.

To complete the proof of Theorem 6, observe that if a cone in $\mathrm{SO}(3) \times W$ with the property that each surface Σ_u, defined by $u = \text{constant}$, is an almost-complex curve, then the following holds:

$$L_{\partial/\partial u}(\theta^0 \wedge \theta^1) = \lambda(\theta^0 \wedge \theta^1), \tag{6.2}$$

where L denotes the Lie derivative and λ is a complex-valued function. Geometrically, (6.2) means that the flow in the "cone direction" must preserve the complex 2-form $\theta^0 \wedge \theta^1$.

A short calculation reveals

$$L_{\partial/\partial u}(\theta^0 \wedge \theta^1) = \frac{v}{u^2 - v^2}(\theta^0 \wedge \overline{\theta^1}) + \frac{u}{u^2 - v^2}(\theta^0 \wedge \theta^1).$$

Thus, the condition given by (6.2) is satisfied if and only if $v = 0$, i.e. the only cone satisfying (6.2) is the f_W-inverse image of the cone on the Veronese surface. Q.E.D.

Remark. Theorem 6 shows that any other 3-fold of M_L^5 having boundary Q, the double cover of the Veronese surface (cf. Proposition 5), must have volume *strictly* greater than the cone on Q. Thus the cone on the Veronese surface is absolutely area-minimizing among all L-manifolds with the Veronese surface as boundary. This class includes all compactly supported deformations of $\widehat{\Sigma} - \{0\}$ in M^5. This minimizing property strongly suggests that the cone on the Veronese surface is area-minimizing in $\mathbf{E}^5$, but we do not yet have a proof of this.

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