HEEGAARD DIAGRAMS OF 3-MANIFOLDS

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Dedicated to Professor Fujitsugu Hosokawa on his sixtieth birthday

Abstract. For a 3-manifold $M(L)$ obtained by an integral Dehn surgery along an $n$-bridge link $L$ with $n$-components we define a concept of planar Heegaard diagrams of $M(L)$ using a link diagram of $L$. Then by using Homma-Ochiai-Takahashi’s theorem and a planar Heegaard diagram of $M(L)$ we will completely determine if $M(L)$ is the standard 3-sphere in the case when $L$ is a 2-bridge link with 2-components.

1. Introduction

It is well known that every closed connected orientable 3-manifold is a 3-manifold $M(L)$ obtained by integral Dehn surgery along an $n$-bridge link $L$ with $n$-components (see [5, 12, 6, 7]). Birman and Powell found the concept of a special Heegaard diagram of $M(L)$, making use of pure $2n$-plat representations of $L$ in [1 and 3]. In this paper, we will consider a method to directly construct 3-manifolds to make their Heegaard diagrams using the bridge diagrams of $n$-bridge links with $n$-components. The fundamental groups of such 3-manifolds have good presentations associated with the Heegaard diagrams (see Theorem 1). “Good” means that using such presentations, we can easily determine if $M(L)$ is the standard 3-sphere $S^3$ by Whitehead’s procedure. It is not known in general whether the Whitehead conjecture is true for planar Heegaard diagrams of $S^3$ (see [11, 16], and the last section in this paper).

The author proved in [9] that all 3-manifolds obtained by nontrivial Dehn surgery along nontrivial 2-bridge knots are not the standard 3-sphere $S^3$. Using Homma-Ochiai-Takahashi’s theorem [11] (see Theorem 2), a similar result will be proved in the case when surgery curves are 2-bridge links with 2-components, other than the torus link.

In this paper, we work in piecewise linear category. $S^n$ and $D^n$ denote $n$-sphere and $n$-disk, respectively. Let $X$ be a manifold and $Y$ a submanifold properly embedded in $X$. $N(X, Y)$ denotes a regular neighborhood of $Y$ in $X$. 

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X. Closure, interior and boundary over one symbol \( \cdot \) are denoted by \( \text{cl}(\cdot) \), \( \text{int}(\cdot) \) and \( \partial(\cdot) \), respectively.

2. HEEGAARD DIAGRAMS OBTAINED BY DEHN SURGERY ALONG LINKS

A Heegaard splitting \((H_1, H_2; F)\) of a 3-manifold \( M \) is a representation of \( M \) as \( H_1 \cup H_2 \), where \( H_1 \) and \( H_2 \) are handlebodies of some fixed genus \( n \) and \( H_1 \cap H_2 = \partial H_1 = \partial H_2 = F \), a Heegaard surface. A properly embedded 2-disk \( D \) in a handlebody \( H \) of genus \( n \) is called a meridian disk of \( H \) if \( \text{cl}(H - N(D, H)) \) is a handlebody of genus \( n - 1 \). Moreover a collection of mutually disjoint disks \( D_1, D_2, \ldots, D_n \) in \( H \) is called a complete system of meridian disks of \( H \) if \( \text{cl}(H - \bigcup N(D_i, H)) \) is a 3-ball. A collection of \( n \) mutually disjoint circles on the boundary of \( H \) is called a complete system of meridians of \( H \) (or \( \partial H \)) if it bounds a complete system of meridian disks of \( H \).

Let \( v = \{v_1, v_2, \ldots, v_n\} \) and \( w = \{w_1, w_2, \ldots, w_n\} \) be complete systems of meridians of \( H_1 \) and \( H_2 \), respectively. The triplet \( \{F; v, w\} \) is called a Heegaard diagram of a Heegaard splitting \((H_1, H_2; F)\) of \( M \) (or simply \( M \)).

Let \( H_1 \) be a handlebody of genus \( n \) standardly embedded in \( R^3 \). Let \( v = \{v_1, v_2, \ldots, v_n\} \) be a standard complete system of meridians of \( \partial H_1 = F \) and let \( x = \{x_1, x_2, \ldots, x_n\} \) be a collection of mutually disjoint circles on \( F \) such that \( x_i \) intersects \( v_j \) transversely at only one point if \( i = j \) and are disjoint from \( v_j \) if \( i \neq j \). Assume further that each \( x_i \) bounds a disk in the complement of \( H_1 \) in \( R^3 \). The pair \( \{v; x\} \) is called a standard meridian-longitude system of \( H_1 \). Let \( L \) be a collection of mutually disjoint circles included in \( F \subset R^3 \) (or \( S^3 \)). Then \( L \) is called an \( n \)-bridge link if each connected component of \( L \) always intersects some \( x_i \) transversely at only one point, and \( L \cap x \) is \( n \) points \( (i = 1, 2, \ldots, n) \).

Let \( L \) be an \( n \)-bridge link with \( n \)-components \( K_1, K_2, \ldots, K_n \). Then we can assume that each \( K_i \) intersects \( x_i \) in one point \( p_i \), that \( p_i \) is disjoint from \( v_j \cap x_i \), that \( K_i \) is disjoint from \( x_j \) if \( i \neq j \), and that \( K_{p_i} \cap (v_i \cup x_i) = p_i \), where \( K_{p_i} \) is the closure of the connected component including \( p_i \) among \( K_i - v \). We call such a link canonical. From now on, we assume that all bridge links to be considered are canonical. Let \( b(x_i, K_i) \) denote \( N(x_i \cup K_i, F) \) and \( u_i \) be \( \partial b(x_i, K_i) \). Then \( u_i \) is a circle on \( F \) and there exist \( n \) mutually disjoint disks \( U_1', U_2', \ldots, U_n' \) outside \( H_1 \) in \( S^3 \) such that \( u_i = \partial U_i' \).

Lemma 1. Let \( E \) be \( H_1 \cup (\bigcup U_i' \times I) \). Then \( \text{cl}(S^3 - E) \) consists of one 3-ball \( B^3 \) and \( n \) solid tori \( V_1', V_2', \ldots, V_n' \). Moreover \( E \cup B^3 \) is homeomorphic to \( S^3 - \text{int}(N(L, S^3)) \).

Proof. Without loss of generality, we can assume that the torus \( N(x_i \cup K_i, F) \cup (\bigcup U_i') \) does bound \( V_i' \). Then each \( x_i \) bounds a disk in \( V_i' \) and so we can assume that each \( V_i' \) is a regular neighborhood of \( K_i \) in \( S^3 \).
Let $w'_i$ be circles on $b(x_i, K_i)$ ($i = 1, 2, \ldots, n$) and $w' = \{w'_1, w'_2, \ldots, w'_n\}$. Then $w'_i$ gives a framing for $V'_i$. Let $L$ be an oriented link in $S^3$ and $M(L; w')$ denote the 3-manifold obtained by the Dehn surgery determined by the framing. Then we have

**Lemma 2.** $(F; v, w')$ gives a Heegaard diagram of the 3-manifold $M(L; w')$. Conversely all 3-manifolds obtained by Dehn surgery along $L$ have such Heegaard diagrams.

From now on, we denote $M(L; w')$ as $M(L; a_1/b_1, \ldots, a_n/b_n)$ when $w'_i$ gives surgery coefficients $a_i/b_i$ ($i = 1, \ldots, n$) and later we abbreviate $M(L; a_1/b_1, \ldots, a_n/b_n)$ to $M(L)$.

Let $H_2$ be another handlebody of genus $n$ standardly embedded in $R^3$ and $\{x; y\}$ be a standard meridian-longitude system of $H_2$. Then $M(L; w')$ has a Heegaard splitting $H_i \cup H_2$ such that $w'_i$ is identified with $x'_i$ ($i = 1, 2, \ldots, n$). It is well known that any closed connected 3-manifold can be obtained by integral Dehn surgery along some $n$-bridge link with $n$ components (see [5, 12, 6, 7]). And so, by Lemma 1 and Lemma 2, we have

**Proposition 1.** Any closed connected orientable 3-manifold has a Heegaard diagram $(F; v, w)$ such that

1. it is naturally obtained by integral Dehn surgery along an $n$-bridge link $L$ with $n$-components,
2. each meridian $w_i$ intersects $x_i$ transversely at only one point and it is disjoint from $x_j$, if $i \neq j$, and
3. each longitude $x_i$ is identified with $y_i$ ($i = 1, 2, \ldots, n$). Moreover, the dual diagram $(F; w, v)$ has the same property also.

It will be noted that Proposition 1 was found by Birman and Powell in [1 and 3], because every pure $2n$-plat gives an $n$-bridge link with $n$-components and that a theorem similar to Proposition 1 in the case when $L$ is a knot was also done by the author in [9].

Let $L$ be a canonical $n$-bridge link with $n$-components. Then the number of points in $L \cap v$ is called the complexity of $L$ or $(F; v, w)$, a Heegaard diagram mentioned above.

Next we introduce notations and defining terms for presentations of fundamental groups of 3-manifolds. Let $(F; v, w)$ be a Heegaard diagram of a 3-manifold $M$. Then orienting $F$ and all the circles in $v$ and $w$, we can get cyclic words $w_1(v), w_2(v), \ldots, w_n(v)$ from $w_1, w_2, \ldots, w_n$, respectively, by travelling each $w_i$ once (in the given orientation) and reading $v_k$ or $v_k^{-1}$ for each crossing with $v_k$ when $w_i$ does cross $v_k$ from left to right or from right to left, respectively. Then we have the following theorem.

**Theorem 1.** Let $M$ be a closed connected orientable 3-manifold and $G$ be the fundamental group of $M$. Then $G$ has the following presentation:
generators: $v_1, v_2, \ldots, v_n$
relators: $v_1^{s_1} = v_1^{e_{i1}} \cdot v_2^{e_{i2}} \cdots v_n^{e_i q_i}$

where $s_1, s_2, \ldots, s_n$ are integers, $e_{ij}$ is 1 or $-1$, $v_{ij}$ is some $v_k$ $(k = 1, 2, \ldots, n)$ for all $i$ and $j$ $(i = 1, 2, \ldots, n; \ j = 1, 2, \ldots, q_i)$, and $v_{ij}$ is different from $v_{ij+1}$ $(j = 1, 2, \ldots, q_i - 1)$ and both $v_{i1}$ and $v_{iq_i}$ are different from $v_j$ for all $j$ $(i = 1, 2, \ldots, n)$.

Proof. Let $(F; v, w)$ be a Heegaard diagram of $M(L; w')$ given by Proposition 1. Then each $w_i$ induces the cyclic word $w_i(v)$ which gives the $i$th relator of $G$. Let us suppose that $(F; v, w)$ is a Heegaard diagram with the minimal complexity among Heegaard diagrams given by Proposition 1 and that both $v_{cj}$ and $v_{cj+1}$ are the same as $v_k$ for some $k$. Then two cases happen.

Case 1. $e_{cj} e_{cj+1} = -1$; This case is illustrated in Figure 1. Let $\tau_1$ be the $(v_{cj}, v_{cj+1})$-section of $w_i$. Then $\text{int}(\tau_1)$ is disjoint from $v$ and so along $\tau_1$ $v_k$ can be changed to the new meridian $v'_k$ and the new Heegaard diagram obtained by the change of meridians has smaller complexity than $(F; v, w)$ (see Figure 1).

Case 2. $e_{cj} e_{cj+1} = 1$; This case is illustrated in Figure 2. The $(v_{cj}, v_{cj+1})$-section $S$ of $v_k$, which does not include the point $p_k$, always includes at least two crossing points, because the section $S$ must intersect $w_i$ and $w_k$. Let $\tau_2$ be the $(v_{cj}, v_{cj+1})$-section of $w_i$. Then $\text{int}(\tau_2)$ is disjoint from $v$ and so along $\tau_2$ $v_k$ can be changed to the new meridian $v'_k$ and the new Heegaard diagram obtained by the change of meridians has the complexity which is at least one less than the one of $(F; v, w)$ (see Figure 2).
In the discussion mentioned above, every word \( w_i(v) \) may change to a new word if \( w_i \cap v_k \neq \emptyset \) but all \( w_1, w_2, \ldots, w_n \) remain to be fixed. Since both \( \text{int}(\tau_1) \) and \( \text{int}(\tau_2) \) are disjoint from all \( x_i \) and \( v_j \) \( (i, j = 1, 2, \ldots, n) \), the new Heegaard diagram obtained by the change of meridians induces a link diagram of \( L \) with smaller crossing points.

Thus all the meridians in \( w \) have no such arcs \( \tau_1 \) and \( \tau_2 \), since \( (F; v, w) \) has the minimal complexity.

Note that by a similar method, we can verify that all meridians in \( v \) have no arcs similar to such \( \tau_1 \) and \( \tau_2 \).

Later in group \( G \) with the presentation is called the characteristic group of \( M(L; w') \) and let \( G^* \) denote the group given by the dual diagram \( (F; w', v) \) from Theorem 1.

It will be noticed that the work \( v_i^{e_i} \) is induced from the part of \( w_i \) in \( N(x_i, F) \) and the one \( v_i^{e_i} \cdot v_i^{e_i} \cdot \ldots \cdot v_i^{e_i} \) is induced from the part of \( w_i \) in \( w_i - N(x_i, F) \) \( (i = 1, 2, \ldots, n) \) and that from now on we can assume that all Heegard diagrams to be considered, including their dual diagrams, satisfy these conditions.

Let \( \tau \) be an (oriented) arc on \( F \) such that \( \text{int}(\tau) \cap (v \cup w) = \emptyset \), \( \partial \tau \cap (v \cup w - v \cap w) = a \cup b \) are two disjoint points, and \( (a \cup b) \subset v_i \) (resp. \( w_i \)). Then \( \tau \) is called a wave of type 0 for \( v_i \) (resp. \( w_i \)) if it intersects \( v_i \) (resp. \( w_i \)) with the opposite sign and is also called a wave of type 1 for \( v_i \) (resp. \( w_i \)) if it intersects \( v_i \) (resp. \( w_i \)) and \( \tau \subset F - \cup N(x_i, F) \). Note that \( F - \cup N(x_i, F) \) is a planar surface and we can assume that \( N(x_i, F) = N(y_i, F) \). If a Heegaard diagram has such arcs as \( \tau_1 \) and \( \tau_2 \) mentioned in the proof of Theorem 1, then it also has a wave of type 1 at these arcs. Later a wave of type 0 is simply called a wave (see the definition in [11]).

Let \( M \) be \( M(L; w') \) and \( A \) be the relation matrix for \( H_1(M, Z) \) induced
from $G$ by abelianizing the relators of $G$ given by Theorem 1. Then we have

**Corollary 1.1.** The matrix $A = (a_{ij})$ is a symmetric, integer matrix. If $M$ is $M(L; b_1, b_2, \ldots, b_n)$, then $a_{ii} = b_i$ ($i = 1, 2, \ldots, n$).

**Proof.** If $n = 2$, then $L$ is a 2-bridge link and so $A$ is symmetric (see the next section). Otherwise, $K_1 \cup K_2$ is a 2-bridge link and so $a_{ij} = a_{ji}$, where $i \neq j$ and $i, j = 1, 2, \ldots, n$. Moreover $a_{ii} = b_i$, because $b_i = 1$ if and only if $e(v_i) = 1$, where $e(v_i)$ is the exponent sum of $v_i$ in $w_i(v)$ ($i = 1, 2, \ldots, n$).

### 3. Dehn surgery along a 2-bridge link along 2-components

Let $L$ be a 2-bridge link with 2-components, $L = K_1 \cup K_2$, let $w' = w_1' \cup w_2'$ be frame curves of a Dehn surgery along $L$ and let $(F; v, w)$ be a Heegaard diagram of genus 2 given by Proposition 1 from $w'$. Furthermore let $M$ be $M(L; w')$ and $G$ be the characteristic group. Then by Theorem 1, $G$ has the following presentation:

$$G = \{a, b; a^m = b^{e_1} a^{e_2} \cdots b^{e_{q_1}}, b^n = a^{e_2} b^{e_3} \cdots a^{e_{q_1}}\}.$$ 

Since $L$ is a 2-bridge link, $L$ has a Schubert's normal form of type $(p, q)$ as a 2-bridge diagram, where $p$ is even, $q$ is odd with $0 < q < p$ and $p, q$ are relatively prime (see [13]). Thus $q_1 = q_2$ and $e_{1i} = e_{2i}$ ($i = 1, 2, \ldots, q_1$). Moreover, there exists an involution $f$ on $F$ such that $f(v_i) = v_i$, $f(x_i^*) = x_i^*$, $f(w'_i) = w'_i$, and that $p_i$ is a fixed point among the six fixed points of $f$, where $p_i = x_i \cap w'_i$ and $i = 1, 2$. Thus we have that $e_i = e_{r-i+1}$, where $i = 1, 2, \ldots, \lfloor r/2 \rfloor$. Then $G$ is as follows:

$$G = \{a, b; a^m = b^{e_1} a^{e_2} \cdots b^{e_{r-1}} a^{e_r}, b^n = a^{e_2} b^{e_3} \cdots a^{e_{r-1}} a^{e_r} \}.$$ 

$r$ is an odd integer, $e_i$ is $\pm 1$, and $e_i = e_{r-i+1}$ ($i = 1, 2, \ldots, \lfloor r/2 \rfloor$).

Moreover the dual presentation $G^*$ of $G$ given by $(F; w, v)$ is as follows:

$$G^* = \{c, d; c^m = d^{f_1} c^{f_2} d^{f_3} \cdots d^{f_{r-1}} c^{f_r}, d^n = c^{f_2} d^{f_3} \cdots d^{f_{r-1}} c^{f_r} \}.$$ 

$r$ is an odd integer, $f_i$ is $\pm 1$, and $f_i = f_{r-i+1}$ ($i = 1, 2, \ldots, \lfloor r/2 \rfloor$).

Let us suppose that $1 < q < p/2$ and let $s$ be the smallest integer which satisfies $sq < p < (s + 1)q$. Then $s \geq 2$. Thus, if $1 < q < p/2$ (or even if $p/2 < q < p - 1$), then we can assume that $e_1 = 1$, $e_2 = 1$, $\ldots$, $e_s = 1$ and $e_{s+1} = -1$.

Here we study the case when $M$ is $S^3$. Let $r = 2k-1$ and let us suppose that all $e_1, e_2, \ldots, e_r$ are $+1$. In this case, we have that $f_i = +1$ ($i = 1, 2, \ldots, r$).
and \( L \) is a \((k, \pm 1)\) torus link with 2-components because either \( q = 1 \) or \( q = p - 1 \).

**Case 1.1.** \( r = 1 \); \( L \) is the Hopf link. Then \( M \) is \( S^3 \) if and only if either \( mn = 0 \) or \( mn = 2 \).

**Case 1.2.** \( r \geq 3 \); Let \( q_1, q_2, q_3 \) and \( q_4 \) (resp. \( q'_1, q'_2, q'_3 \) and \( q'_4 \)) be non-negative integers which give the numbers of edges of the Whitehead graph of \((F; v, w)\) (resp. \((F; w, v)\)) illustrated in Figure 3. If either \( m \geq 1 \) or \( n \geq 1 \), then all \( q_1, q_2, q'_3, q'_4 \) and \( q'_2 \) are positive. If \( m, n \leq -2 \), then all \( q_1, q_3, q_4, q'_1, q'_3, q'_4 \) are positive. Thus if either \( m, n \leq -2 \) or \( m \geq 1 \) or \( n \geq 1 \), then \((F; v, w)\) has no waves. Then \( M \) is not \( S^3 \) by Homma-Ochiai-Takahashi's theorem in [11]. If \( m = 0 \), then \( a = b^{n+1} \) and \( G = \{ b; b^{k(n+2)-(n+1)} = 1 \} \). Thus \( G \) is trivial if and only if either \( n = -2 \) or \( k = 2 \), \( n = -4 \) or \( k = 3 \), \( n = -3 \). If \( m = -1 \), then \( G = \{ a, b; (ab)^k = b^{n+1} = 1, k > 1 \} \) and so \( H_1(M, \mathbb{Z}) \) is nontrivial. Thus \( M \) is \( S^3 \) in the case when \( m = 0 \) and either \( n = -2 \) or \( k = 2 \), \( n = -4 \) or \( k = 3 \), \( n = -3 \). It will be noted that to verify \( M \) to be \( S^3 \) we can use the reduction procedure of Heegaard diagrams through waves (see [11]).

Next let us suppose that some \( e_i \) is \(-1\) \((i = 1, 2, \ldots, r)\). If all \( e_i \) \((i = 1, 2, \ldots, r)\) are \(-1\), then this case reduces to the above-mentioned case. And so another \( e_j \) \((j \neq i)\) is \(+1\). Since the case when either \( e_{2i-1} = 1, e_{2i} = -1 \) or \( e_{2i-1} = -1, e_{2i} = 1 \) \((i = 1, 2, \ldots, k + 1)\) reduces to the first case, we can assume that either \( e_1 = 1, e_2 = 1, \ldots, e_s = 1, e_{s+1} = -1 \) or \( e_1 = -1, e_2 = -1, \ldots, e_s = -1, e_{s+1} = 1 \), where \( s \geq 2 \). Then \((F; v, w)\) (resp. \((F; w, v)\)) has as the Whitehead graph the graph illustrated in Figure 3, where \( q_1, q_2 > 0 \) (resp. \( q'_1, q'_2 > 0 \)), and so it has no waves. By the theorem in [11], \( M \) is not \( S^3 \). Hence, by Corollary 1.1 we have

**Theorem 2.** Let \( M \) be a closed 3-manifold obtained by nontrivial integral Dehn surgery along a nontrivial 2-bridge link with 2-components. Then \( M \) is not \( S^3 \).
if $L$ is not a $(k, \pm 1)$-torus link with 2-components. If $M$ is $S^3$, then we have the following cases:

1. $L$ is a Hopf link, and either $M = M(L; m, 0)$ or $M = M(L; 2, 1)$ or $M = M(L; -2, -1)$.
2. $L$ is a $(2, \pm 1)$-torus link, and $M = M(L; 1, 5)$.
3. $L$ is a $(3, \pm 1)$-torus link, and $M = M(L; 2, 5)$.
4. $L$ is a $(k, \pm 1)$-torus link $(k > 1)$, and $M = M(L; k - 1, k + 1)$.

In particular, if $L$ is a nontrivial 2-bridge link, $M(L; \pm 1, \pm 1)$ is not $S^3$.

Let $G_1$ be the characteristic group of $M(L_1 \cup L_2; m_1/n_1, m_2/n_2)$ and $G_1^*$ be the dual presentation of $G_1$. Then $G_1$ and $G_1^*$ are as follows (see the presentations of $G$ and $G^*$);

\[
G_1 = \{a, b; a^{-i_2} = Aa^{i_1}A \cdots a^{i_{n_1}}A, b^{-j_1} = Bb^{j_1}B \cdots b^{j_{n_2}}B, A = b^{i_1}a^{i_1}b^{i_1}a^{i_1} \cdots b^{i_1}a^{i_1}B, B = a^{i_1}b^{i_1}a^{i_1}b^{i_1}a^{i_1} \cdots b^{i_1}a^{i_1}B, G_1^* = \{c, d; c^{-i_0} = d^{e_{n_1}}c^{e_{n_1}}d^{e_{n_1}} \cdots d^{e_{n_1}}c^{e_{n_1}}d^{e_{n_1}}, d^{-j_0} = c^{e_{n_1}}d^{e_{n_1}}c^{e_{n_1}}d^{e_{n_1}} \cdots d^{e_{n_1}}c^{e_{n_1}}d^{e_{n_1}}\right\}
\]

where $i_2 = i_3 = \cdots = i_{n_1}$, $i_1 = i_2 \pm 1$, $j_2 = j_3 = \cdots = j_{n_2}$, $j_1 = j_2 \pm 1$, $i_0 = i_1 + i_2 + \cdots + i_{n_1}$ and $j_0 = j_1 + j_2 + \cdots + j_{n_2}$. Let $\theta$ be a 9-tuple $(k, n_1, i_0, i_1, i_2, n_2, j_0, j_1, j_2)$. Then we have the following theorem:

**Theorem 3.** Let $M_1$ be $M(L_1 \cup L_2; m_1/n_1, m_2/n_2)$ and let $n_1 \geq 2$ and $n_2 \geq 1$. Then $M_1$ is not $S^3$, if $L = L_1 \cup L_2$ is a $(k, \pm 1)$-torus link. If $M_1$ is $S^3$, then $L$ is a $(k, \pm 1)$-torus link and $\theta$ is as follows:

**Case 1.** $k = 1$, $n_1 \geq 2$, $n_2 = 1$:

$(1, n_1, n_1 + 1, 2, 1, 1, 1, 1, 1, *), (1, n_1, 1, 1, 0, 1, n_1 + 1, n_1 + 1, *), (1, 3, -2, 0, -1, 1, -2, -2, *), (1, 2, -1, 0, -1, 1, -3, -3, *), (1, 2, -3, -1, -2, 1, -1, -1, *), (1, 2, 1, 0, 1, 1, 1, 1, *), (1, 3, 2, 0, 1, 1, 2, 2, *), (1, 2, 1, 0, 1, 1, 3, 3, *), (1, n_1, -1, -1, 0, 1, -n_1 - 1, -n_1 - 1, *), (1, n_1, -n_1 - 1, -2, -1, 1, -1, -1, *), (1, n_1, 1, 1, 0, 1, n_1 - 1, n_1 - 1, *), (1, n_1, -n_1 - 1, 0, -1, 1, -1, -1, *), (1, n_1, n_1 - 1, 0, 1, 1, 1, 1, *), (1, n_1, -1, -1, 0, 1, -n_1 + 1, -n_1 + 1, *).

**Case 2.** $k = 1$, $n_1 \geq 2$, $n_2 \geq 2$:

$(1, n_1, 1, 1, 0, 2, 2n_1 + 1, n_1, n_1 + 1), (1, n_1, 1, 1, 0, 2, 2n_1 - 1, n_1, n_1 - 1), (1, n_1, 1, 1, 0, n_2, n_1n_2 + 1, n_1 + 1, n_1), (1, n_1, 1, 1, 0, n_2, n_1n_2 + 1, n_1 + 1, n_1).$
Case 3. \( k \geq 2, n_1 \geq 2, n_2 = 1; \)
\[
(2, n_1, 3n_1 + 1, 4, 3, 1, 0, 0, *), (k, k-2, 2k-3, 3, 2, 1, 0, 0, *), \\
(3, 3, 8, 2, 3, 1, 0, 0, *), (k, k-2, 2k-1, 3, 2, 1, 0, 0, *), \\
(k, k-1, 1, 0, 1, 2, 2, *), (k, k+2, 1, 0, 1, 2, 2, *), \\
(2, 2, 1, 0, 1, 1, 2, 2, *).
\]

Case 4. \( k \geq 2, n_1 \geq 2, n_2 \geq 2; \)
\[
(k, 2, 1, 1, 0, 2, 3, 2, 1), (3, 2, 1, 1, 0, 3, 5, 1, 2), \\
(2, 2, 1, 0, 4, 7, 1, 2), (3, 2, 1, 0, 1, 3, 5, 1, 2), \\
(k, 2, 1, 0, 2, 3, 1, 2), (2, 3, 2, 0, 1, 2, 3, 1, 2), (k, n_1, n_1 - 1, 0, 1, n_1, n_1 + 1, 2, 1), \\
(2, n_1 + 1, n_2, 0, 1, n_2, n_2 + 1, 2, 1), (k, n_1, n_1 + 1, 2, 1, n_1, n_1 - 1, 0, 1), \\
(2, n_1, n_1 + 1, 2, 1, n_1, n_1 + 1, 0, 1).
\]

Proof. If \( 1 < q < p - 1, \) then by the proof of Theorem 2 all \( q_1, q_2, q'_1 \) and \( q'_2 \)
are positive and so \( M(L_1 \cup L_2; m_1/n_1, m_2/n_2) \) is not \( S^3. \) Let us suppose that
either \( q = 1 \) or \( q = p - 1. \) Then \( e_1 = e_2 = \cdots = e_r = 1 \) and there exist four
cases: Case 1, Case 2, Case 3, Case 4. Let \( \lambda \) be the determinant of the relation
matrix of \( G_1. \)

Case 1. \( \lambda = i_0j_0 - n_1; \) \( \lambda = \pm 1 \) and \( i_0 = i_2n_1 \pm 1. \) Then there are the four
following cases:

1.1. \( \lambda = 1; \) \( i_0 = i_2n_1 + 1. \)

1.2. \( \lambda = 1; \) \( i_0 = i_2n_1 - 1. \)

1.3. \( \lambda = -1; \) \( i_0 = i_2n_1 + 1. \)

1.4. \( \lambda = -1; \) \( i_0 = i_2n_1 - 1. \)
In the case 1.1, we have that \( n_1(i_2j_0 - 1) = 1 - j_0 \). If \( i_2 = 0 \), then \( j_0 = 1 \), \( i_1 = i_2 + 1 = 1, \) \( i_0 = 1 \). If \( i_2 \neq 1 \), then \( i_2j_0 > 1 \), \( 1 > j_0 \) and so \( 0 > j_0 \), \( 0 > i_2 \). Thus \( \theta \) is as follows:

\[
(1, n_1, n_1 + 1, 1, 1, 1, 1, 1, *,) \quad (1, n_1, 1, 1, 0, 1, n_1 + 1, n_1 + 1, *)
\]

\[
(1, 3, -2, 0, -1, 1, -2, -2, *) \quad (1, 2, -1, 0, -1, 1, -3, -3, *)
\]

\[
(1, 2, -3, -1, -2, 1, -1, -1, *)
\]

A similar method mentioned above is applied to the cases 1.2, 1.3 and 1.4.

**Case 2.** \( \lambda = i_0j_0 - n_1n_2 \); \( \lambda = \pm 1 \), \( i_0 = i_2n_1 \pm 1 \). Since \( i_0j_0 > 1 \), we can assume that \( |i_0| \leq n_1 \). If \( i_2 = 0 \), then \( i_0 = \pm 1 \). If \( i_2 > 0 \), then \( i_0 > 0 \) and so \( i_2 = 1, \) \( i_0 = n_1 - 1 \). If \( i_2 < 0 \), then \( i_0 < 0 \) and so \( i_2 = -1, \) \( i_0 = -n_1 + 1 \).

1. \( i_2 = 0, \) \( i_0 = 1 \),
2. \( i_2 = 0, \) \( i_0 = -1 \),
3. \( i_2 = 1, \) \( i_0 = n_1 - 1 \),
4. \( i_2 = -1, \) \( i_0 = -n_1 + 1 \).

In the case 2.1, \( \theta \) is as follows:

\[
(1, n_1, 1, 1, 0, n_2, n_1n_2 + 1, n_1 + 1),
(1, n_1, 1, 1, 0, n_2, n_1n_2 - 1, n_1 - 1),
(1, n_1, 1, 1, 0, 2, 2n_1 + 1, n_1 + 1),
(1, n_1, 1, 1, 0, 2, 2n_1 - 1, n_1 - 1).
\]

In the case 2.3, if \( \lambda = j_0(n_1 - 1) - n_1n_2 = 1 \), then \( j_0 > 0 \). Let \( s = j_0 - n_2 \) (> 0). Then \( j_0 = n_1s - 1, \) \( n_2 = n_1s - s - 1 \). Since \( n_1 \geq 2, \) \( n_2 \geq s - 1 \). If \( j_0 = n_2 + s = n_2j_2 + 1, \) then \( n_2(j_2 - 1) = s - 1 \). If \( s = 1, \) then \( j_2 = 1, \) \( j_1 = 2, \) \( j_0 = n_2 + 1, \) \( n_2 = n_1 - 2 \). If \( s > 1, \) then \( n_2 \leq s - 1 \) and so \( n_2 = s - 1 \). Thus \( j_0 = 2n_2 + 1, \) \( j_1 = 3, \) \( n_1 = 2 \). If \( j_0 = n_2 + s = n_2j_2 - 1, \) then \( n_2(j_2 - 1) = s + 1 \). Since \( s > 0, \) then \( n_2 \leq s + 1 \). If \( n_2 = s - 1, \) then \( n_1 = 2 \) and so \( 2s = j_0(s - 1) \). Since \( s > 2, \) \( s = 3 \) and so \( j_2 = 3, \) \( j_1 = 2, \) \( n_2 = 2 \). If \( n_2 = s, \) then \( 2s + 1 = n_1s \). But this is impossible, because \( s > 1 \). If \( n_2 = s + 1, \) then \( n_1s = 2s + 2 \). Since \( s > 0, \) \( s = 1 \) or \( s = 2 \). If \( s = 1, \) then \( n_1 = 4, \) \( n_2 = 2, \) \( j_0 = 3, \) \( j_1 = 1, \) \( j_2 = 2 \). If \( s = 2, \) then \( n_1 = 3, \) \( n_2 = 3, \) \( j_0 = 5, \) \( j_1 = 1, \) \( j_2 = 2 \). A similar method mentioned above is applied to the case of \( \lambda = -1 \).

**Case 3.** \( \lambda = \{(k - 1) + j_0\}\{(k - 1)n_1 + i_0\} - k^2n_1 \): In this case, if either \( i_2 \leq -1 \) or \( j_0 \leq -1 \) or \( i_2 \geq 2, \) \( j_0 \geq 2 \), then by the Main Theorem in [11] \( \mathcal{M}_1 \) is not \( S^3 \). Thus we have the following cases:

1. \( i_2 = 0 \),
2. \( i_2 = 1 \),
3. \( j_0 = 0 \),
4. \( j_0 = 1 \).

Since \( k \geq 2 \) and \( n_2 = 1 \), the case 3.4 is impossible. A similar method mentioned in Case 2 can be applied to the other cases. Note that if \( j_0 \) (= \( j_1 \)) = 2,
then $M_1$ is a 3-manifold of genus 1 but that $M_1$ is not a homology 3-sphere if $i_2 \geq 2$.

**Case 4.** $\lambda = \{(k - 1)n_2 + j_0\} \{(k - 1)n_1 + i_0\} - k^2n_1n_2$ : In this case, if either $i_2 \leq -1$ or $j_2 \leq -1$ or $i_2 \geq 2$, $j_2 \geq 2$, then by the Main Theorem in [11] $M_1$ is not $S^3$. A similar method mentioned in Case 3 can be applied to the case.

### 4. Dehn surgery along an $n$-bridge link with $n$-components

Let $L$ be an $n$-bridge link with $n$-components, $M$ be $M(L)$ and $G$ be the characteristic group of $M(L)$. Moreover, let $A = (a_{ij})$ be the relation matrix for $H_1(M, \mathbb{Z})$ induced from $G$.

**Proposition 2.** Suppose $M = M(L)$ is a homology 3-sphere. If the matrix $A$ satisfies $a_{ij} = 0$ if $i = n$ and $j = 1, 2, \ldots, n - 1$, or if $j = n$ and $i = 1, 2, \ldots, n - 1$, and that $a_{nn} = p$, and $v_{nj}$ $(j = 1, 2, \ldots, q_n)$ are different from $v_n$, then $M$ is $M(L')$, where $L'$ is an $(n - 1)$-bridge link with $(n - 1)$-components.

**Proof.** Since $H_1(M, \mathbb{Z}) = 0$, we have that $p = \pm 1$. Then $w_n$ intersects $v_n$ transversely at only one point and so we can cancel all the intersections of $v_n$ and $w_i$ $(i = 1, 2, \ldots, n - 1)$ using band sum operations (see [11, 15]) and later we get that $v_n \cap w = v_n \cap w_n$ is one point. Thus we can cancel the 3-ball $N(D_n \cup D'_n, M)$, where $D_n$ (resp. $D'_n$) is a meridian disk in $H_1$ (resp. $H_2$) with $v_n = \partial D_n$ (resp. $w_n = \partial D'_n$) and so get a Heegaard diagram of genus $n - 1$.

![Figure 4](https://www.ams.org/journal-terms-of-use)
It will be noted that if $M$ is not a homology 3-sphere, Proposition 2 is false, because all the 3-manifolds obtained by Dehn surgery along the Borromian link $6_2^3$ satisfy the condition of Proposition 2 and the class of such the manifolds contains the 3-dimensional torus (see Figure 4) but it has no Heegaard diagram of genus 2.

**Proposition 3.** Let $M$ be $M(L)$, where $L$ is $6_2^3$ or $8_5^3$. Then $M$ is not $S^3$.

*Proof.* Let us suppose that $M$ is a homology 3-sphere. Then $M$ satisfies the hypothesis of Proposition 2, because $6_2^3$ and $8_5^3$ are 3-bridge links with 3-components, and $G$ is as follows:

$$G = \{a, b, c; a^m = bcb^{-1}c^{-1}, b^n = c^{-1}a^{-1}ca, c^1 = ab^{-1}a^{-1}b\} \text{ if } L \text{ is } 6_2^3,$$

$$G = \{a, b, c; a^m = b^{-1}a^{-1}cbc^{-1}b^{-1},$$

$$b^n = a^{-1}b^{-1}a^{-1}bc^{-1}b^{-1}abc,\text{ and } c^1 = abab^{-1}a^{-1}b^{-1}a^{-1}b\} \text{ if } L \text{ is } 8_5^3.$$  

Then $M$ has a Heegaard diagram of genus 2 $(F_2; v, w)$ by Proposition 2 and has $G^2$ as the characteristic group of the diagram such that

$$G^2 = \{a, b; a^m = bab^{-1}a^{-1}b^{-1}ab, b^n = ab^{-1}a^{-1}b^{-1}ba\}$$

if $L$ is $6_2^3$ and $p = 1$,

$$G^2 = \{a, b; a^m = b^{-1}a^{-1}b^{-1}babab^{-1}a^{-1}b^{-1},$$

$$b^n = a^{-1}b^{-1}a^{-1}babab^{-1}a^{-1}b^{-1}a^{-1}\}$$

if $L$ is $8_5^3$ and $p = 1$.

Note that in the case of $p = -1$ $G^2$ has also the similar presentation to that mentioned above. And so by Theorem 2, $M$ is never $S^3$.

Next let $L$ be an $n$-bridge $(k, 1)$-torus link with $n$-components, $M$ be $M(L)$, and $A = (a_{ij})$ be the relation matrix.

**Proposition 4.** Suppose that $a_{ij} = k$ if $i \neq j$ $(i, j = 1, 2, \ldots, n)$ and that either $a_{i1} = 1$, $a_{ii} = k + 1$ if $i$ is even and $a_{ii} = k - 1$ if $i$ is odd $(1 < i \leq n)$, in the case when $n$ is odd, or $a_{i1} = k + 1$ if $i$ is odd, and $a_{ii} = k - 1$ if $i$ is even $(1 \leq i \leq n)$, in the case when $n$ is even. Then $M$ is $S^3$.

*Proof.* Since the Heegaard diagram $(F; v, w)$ has always a wave of type 0 (but not type 1) we can reduce it to another one which has Heegaard genus $n - 1$ by cancelling a handle through the wave and continuing this process (see Figure 5). Finally we get a Heegaard diagram of genus one and it is easily seen that $G$ is trivial and so $M$ is $S^3$.

Note that if the matrix $A$ does not satisfy the hypothesis of Proposition 4, then many nontrivial homology 3-spheres may be obtained (see [4]).

Next let $L(t_1, t_2, \ldots, t_{[n/2]})$ be an $n$-bridge $(k + 1, 1)$-torus link with $n$-components with full twists $t_1, t_2, \ldots, t_{[n/2]}$ (see Figure 6) and $M$ be
HEEGAARD DIAGRAMS OF 3-MANIFOLDS

Figure 5

$M(L(t_1, t_2, \ldots, t_{\lfloor n/2 \rfloor}))$, where $n \geq 3$ and all of the $t_1, t_2, \ldots$, and $t_{\lfloor n/2 \rfloor}$ are integer. Then $G$ is as follows:

$G = \{ a, b, c, d ; a^{m_1} = bcd(abcd)^k(ab)^{t_1}, b^{m_2} = cd(abcd)^k a(ba)^{t_1}, c^{m_3} = d(abcd)^k ab(cd)^{t_2}, d^{m_4} = (ab) (abcd)^k abc(d)(de)^{t_2} \} \text{ if } n = 4,$

$G = \{ a, b, c, d, e ; a^{m_1} = bcde(abcd)^k, b^{m_2} = cde(abcd)^k a(bc)^{t_1}, c^{m_3} = de(abcd)^k ab(cb)^{t_1}, d^{m_4} = e(abcd)^k abc(de)^{t_2}, e^{m_5} = (abcd)^k abcde(ed)^{t_2} \} \text{ if } n = 5.$
Case 1: \( n = 4 \) (\( n \) is even). In this case, if \( m_1 = -2, m_2 = 0, m_3 = -2, \) and \( m_4 = 0, \) then \( M \) is \( S^3 \) (see the proof of Proposition 4).

Case 2: \( n = 5 \) (\( n \) is odd). In this case, if \( m_1 = k \pm 1, m_2 = -2, m_3 = 0, m_4 = -2, \) and \( m_5 = 0, \) then \( M \) is \( S^3 \) (see the proof of the Proposition 4).

Then by Corollary 1.1 we have

**Proposition 5.** If \( t_1 = t_2 = \cdots = t_{[n/2]}, \) and \( k = -t_1 - 1, \) then
\[
M(L(t_1, t_2, \ldots, t_{[n/2]}); \pm 1, 1, -1, 1, -1, \ldots, -1, 1)
\]
is \( S^3 \) in the case when \( n \) is odd, and
\[
M(L(t_1, t_2, \ldots, t_{[n/2]}); 1, -1, 1, -1, \ldots, 1, -1)
\]
is also \( S^3 \) in the case when \( n \) is even.

By the way, we have examined many Heegaard diagrams of \( S^3 \) given by Proposition 1 and cannot find the nontrivial one without waves. Heegaard diagrams of 3-manifolds which have the properties (2) and (3) described in Proposition 1 are called planar Heegaard diagrams. Finally, we will propose the following conjecture: "Nontrivial planar Heegaard diagrams of \( S^3 \) always have waves." This conjecture probably has the affirmative answer in the case when the bridge number is three. It will be noticed that the conjecture is an alternative version of Whitehead's conjecture with restricted conditions and that the original one has counterexamples (see [14, 8, 10, 16]). Let \( L^1, L^2 \) and \( L(t_i) \) be the 3-bridge links with 3-components as illustrated in Figure 7. Then we will conjecture that if \( L \) is a nontrivial 3-bridge link with 3-components and \( M(L; \pm 1, \pm 1, \pm 1) \) is \( S^3, \) then \( L \) is one of the three links \( L^0, L^1 \) and \( L(t_i). \) Notice that for the 3-bridge link \( L, \) illustrated in Figure 8 with \( k \) full twists, \( M(L; k, k+1, k+2) \) is \( S^3 \) but \( M(L; \pm 1, \pm 1, \pm 1) \) is not a homology 3-sphere if \( k > 0. \)
FIGURE 8

REFERENCES


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