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THE MODULAR REPRESENTATION THEORY OF q-SCHUR ALGEBRAS

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ABSTRACT. We developed some basic theory of characteristic zero modular representations of q-Schur algebras. We described a basis of the q-Schur algebra in terms of the relative norm which was first introduced by P. Hoefsmit and L. Scott, and studied the product of two such basis elements. We also defined the defect group of a primitive idempotent in a q-Schur algebra and showed that such a defect group is just the vertex of the corresponding indecomposable $\mathscr{H}_F$-module.

INTRODUCTION

The representation theory of classical Schur algebra is equivalent to the theory of polynomial representations of general linear groups (see Green's book [G1]). In [DJ4], q-analogues of classical Schur algebras are defined and the main results which appear in [G1] are generalized. On the other hand, there are some q-analogues of universal enveloping algebra of complex semisimple Lie algebras, called *quantum groups* (see [Dr, L] and the bibliography therein). It is natural to ask what relations between the representation theory of q-Schur algebras and the representation theory of quantum groups. It is also natural to compare the characteristic 0 modular representation theory of q-Schur algebras with the characteristic p modular representation theory of the classical Schur algebras.

In this paper we laid a foundation for the characteristic 0 modular representation theory of q-Schur algebras and generalized some interesting results in characteristic p situation (see [F, S]). For example, the *defect group* of a primitive idempotent and the vertex of an indecomposable module are discussed. In a future paper, we will generalize a fundamental result of Scott in 1973 (see [S]).

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1. Preliminaries

Let $\mathbf{Q}[u^{1/2}]$ be the polynomial ring of the indeterminate $u^{1/2}$ over $\mathbf{Q}$.

Let R be a $\mathbf{Q}[u^{1/2}]$-algebra, possibly of infinite rank, in which $u^{1/2}$ is invertible. Let (W, S) be the symmetric group on r letters where S is the set of basic transpositions. Then the Hecke algebra $\mathscr{H}_R$ corresponding to W is a free R-module with basis $\{T_w ; w \in W\}$ and the multiplication is defined by the rules:

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } l(ws) > l(w), \\ (u-1)T_w + uT_{ws}, & \text{otherwise}. \end{cases}$$

Here $l(x)$ denotes the length of the permutation $x \in W$ and $s \in S$.

Let

$$\widetilde{T}_w = (-u^{-1/2})^{l(w)} T_w .$$

Obviously, $\{\widetilde{T}_w ; w \in W\}$ is a basis of $\mathscr{H}_R$ and

$$\widetilde{T}_w \widetilde{T}_s = \begin{cases} \widetilde{T}_{ws}, & \text{if } l(ws) > l(w), \\ (u^{-1/2} - u^{1/2})\widetilde{T}_w + \widetilde{T}_{ws}, & \text{otherwise}. \end{cases}$$

Let λ be a composition of r. (A composition λ of r, denoted $\lambda \vDash r$, is a finite sequence $(\lambda_1, \lambda_2, \dots, \lambda_n)$ of nonnegative integers whose sum is r.) Then the standard Young (or the parabolic) subgroups W_λ of W consists of those permutations of $\{1, 2, \dots, r\}$ which leave invariant the following sets of integers $\{1, 2, \dots, \lambda_1\}, \{\lambda_1+1, \lambda_1+2, \dots, \lambda_1+\lambda_2\}, \{\lambda_1+\lambda_2+1, \dots\}, \dots$. If H is a parabolic subgroup of W, we denote by $\mathscr{D}_H$ the set of all distinguished coset representatives of right cosets of H in W and set $\mathscr{D}_\lambda = \mathscr{D}_H$ if $H = W_\lambda$. Let $\mathscr{D}_{\lambda\mu} = \mathscr{D}_\lambda \cap \mathscr{D}_\mu^{-1}$. Then $\mathscr{D}_{\lambda\mu}$ is the set of distinguished $W_\lambda - W_\mu$ double coset representatives.

If W' is a parabolic subgroup of W, then R-module $\sum_{w \in W'} RT_w$ is a subalgebra of $\mathscr{H}_R$, which is called the parabolic subalgebra of $\mathscr{H}_R$, denoted by $\mathscr{H}_{W'}$. We will use the abbreviation $\mathscr{H}_\lambda$ instead of $\mathscr{H}_{W_\lambda}$.

Let H be a parabolic subgroup of W and let d_H denote the Poincaré polynomial of H, i.e.

$$d_H = \sum_{w \in H} u^{l(w)} .$$

If $H = W$, then

$$d_W = \prod_{i=1}^{r-1} (1 + u + \cdots + u^i) . \tag{1.a}$$

Let l be a positive number and $l \le r$. Let $\Phi_l = \Phi_l(u)$ denote the lth cyclotomic polynomial. A parabolic subgroup W_λ is called *l-parabolic* if all parts λ_i of λ are 1 or l. In other words, $d_{W_\lambda} = (d_l)^m$ where m is the number of parts l of λ and

$$d_l = \prod_{i=1}^{l-1} (1 + u + \cdots + u^i) .$$

Let P_λ denote the maximal l-parabolic subgroup of W_λ.

1.1 **Lemma.** *Let λ be a composition of r and $x \in W$. Then*
(a) *If W_λ^x is parabolic then $d_{W_\lambda^x} = d_{W_\lambda}$.*
(b) $\Phi_l \nmid (d_{W_\lambda}/d_{P_\lambda})$.

Proof. (a) Let $W_\lambda^x = W_\mu$ for some μ and write

$$W_\lambda x W_\mu = W_\lambda d W_\mu \quad \text{where } d \in \mathscr{D}_{\lambda\mu}.$$

Then $x = ydz$ for some $y \in W_\lambda$, $z \in W_\mu$ and so $W_\lambda^d = W_\mu$.

Let $w = d^{-1}vd \in W_\mu$ where $v \in W_\lambda$. Then $dw = vd$ and

$$l(d) + l(w) = l(dw) = l(vd) = l(d) + l(v)$$

since $d \in \mathscr{D}_{\lambda\mu}$ and $v \in W_\lambda$. Hence, $l(w) = l(v)$ and consequently $d_{W_\lambda} = d_{W_\mu}$.

(b) It suffices to prove the case when $W_\lambda = W$. Let $r = kl + s$, $s < l$. Then $d_{P_\lambda} = (d_l)^k$. We claim that $\Phi_l | (1 + u + \cdots + u^{i-1})$ if and only if $l | i$.

Indeed, if $l | i$ then $i = li'$ for some i'. Thus

$$u^i - 1 = (u^l)^{i'} - 1 = (u^l - 1)(u^{l(i'-1)} + \cdots + u^l + 1).$$

Hence $\Phi_l | 1 + u + \cdots + u^{i-1}$. Conversely, let ω be an lth primitive root of unity, then $\Phi_l(\omega) = 0$ and hence $1 + \omega + \cdots + \omega^{i-1} = 0$, i.e., $\omega^i = 1$. This implies that $l | i$.

By the claim and (1.a) we have $\Phi_l^k | d_W$ but $\Phi_l^{k+1} \nmid d_W$. Hence $\Phi_l \nmid (d_W/d_{P_\lambda})$ as desired. □

1.2 **Corollary.** *Let W_λ, W_μ, W_θ be parabolic subgroups of W such that*

$$W_\theta \subseteq W_\mu^x, \qquad W_\mu \subseteq W_\lambda^y$$

where W_μ^x, W_λ^y are parabolic and $x, y \in W$. Assume that $d_{P_\theta} \neq d_{P_\lambda}$. Then $\Phi_l | (d_{W_\lambda}/d_{W_\theta})$.

Proof. By Lemma 1.1(a) we have

$$d_{W_\lambda} = (d_{W_\lambda}/d_{W_\theta})(d_{W_\theta}/d_{P_\theta}) d_{P_\theta}.$$

Let $d_{P_\theta} = (d_l)^m$, $d_{P_\lambda} = (d_l)^n$. Then $n > m$ and, by 1.1(b), $\Phi_l \nmid (d_{W_\theta}/d_{P_\theta})$ and $\Phi_l \nmid (d_{P_\theta}/\Phi_l^m)$, but $\Phi_l | (d_{W_\lambda}/\Phi_l^m)$. Since Φ_l is irreducible, it follows that $\Phi_l | (d_{W_\lambda}/d_{W_\theta})$. Hence the result. □

The following fact is worth noting.

(1.b) Suppose that P, Q are l-parabolic. Then $d_P | d_Q$ if and only if $P \subseteq_W Q$.

The notation $P \subseteq_W Q$ means that there is $w \in W$ such that $P^w \subseteq Q$ and P^w is parabolic.

1.3 **Lemma.** *Let $f(u)$ be a polynomial. If l is an odd number then*

$$\Phi_l(u) | f(u) \quad \textit{if and only if} \quad \Phi_l(u) | f(u^2).$$

Proof. We claim that

$$\Phi_l(u^2) = \Phi_l(u)\Phi_{2l}(u).$$

Indeed, if ω is a primitive lth root of unity then so is ω^2 and $-\omega$ is a $2l$th primitive root of unity since l is an odd number. Thus we have

$$\Phi_l(u) | \Phi_l(u^2) \quad \text{and} \quad \Phi_{2l}(u) | \Phi_l(u^2).$$

Hence

$$\Phi_l(u)\Phi_{2l}(u)|\Phi_l(u^2).$$

It is clear that $\Phi_{2l}(u)$ can be obtained by changing the sign of the coefficients of the terms in $\Phi_l(u)$ with odd degree. Therefore $\deg\Phi_l(u) = \deg\Phi_{2l}(u)$. Since $\deg\Phi_l(u^2) = 2\deg\Phi_l(u)$ the claim follows.

By the claim it is clear that $\Phi_l(u)|f(u)$ implies that $\Phi_l(u)|f(u^2)$. Conversely, it is easy to see that $\Phi_l(u)|f(u^2)$ implies that $\Phi_{2l}(u)|f(u^2)$, hence $\Phi_l(u^2)|f(u^2)$ by the claim. Therefore $\Phi_l(u)|f(u)$. □

1.4 **Definition.** Let λ, μ be compositions of r such that $W_\lambda \subseteq W_\mu$. Let M be an $\mathscr{H}_\mu$-$\mathscr{H}_\mu$ bimodule and $b \in M$. Define the *relative norm*

$$N_{W_\mu, W_\lambda}(b) = \sum_{w \in \mathscr{D}_\lambda \cap W_\mu} \widetilde{T}_{w^{-1}} b \widetilde{T}_w .$$

Note that if $W_\lambda = \{1\}$ we will write $N_{W_\mu, 1}(b)$.

Let M be an $\mathscr{H}_\lambda$-$\mathscr{H}_\lambda$ bimodule. We define

$$Z_M(\mathscr{H}_\lambda) = \{m \in M | hm = mh \text{ for all } h \in \mathscr{H}_\lambda\}.$$

It is easy to see that $Z_{\mathscr{H}_\lambda}(\mathscr{H}_\lambda)$ is the center of $\mathscr{H}_\lambda$ and, for $M = \mathrm{Hom}_R(N, N)$ where N is a right $\mathscr{H}_\lambda$-module,

$$Z_M(\mathscr{H}_\lambda) = \mathrm{Hom}_{\mathscr{H}_\lambda}(N, N).$$

The following material is from the unpublished work of P. Hoefsmit and L. Scott in 1977. The proof can be found in [Jo].

1.5 **Theorem.** *Let M be an $\mathscr{H}_R$-$\mathscr{H}_R$ bimodule and let W_λ and W_μ be parabolic subgroups of W.*

(a) (*Transitivity*) *If $W_\lambda \subseteq W_\mu$ and $b \in M$ then*

$$N_{W, W_\mu}(N_{W_\mu, W_\lambda}(b)) = N_{W, W_\lambda}(b).$$

(b) $N_{W, W_\lambda}(Z_M(\mathscr{H}_\lambda)) \subseteq Z_M(\mathscr{H}_R)$.

(c) *If N is an $\mathscr{H}_R$-$\mathscr{H}_\lambda$ bisubmodule of M such that $M \cong N \otimes_{\mathscr{H}_\lambda} \mathscr{H}_R$. Then*

$$Z_M(\mathscr{H}_R) = N_{W, W_\lambda}(Z_N(\mathscr{H}_\lambda)).$$

Moreover, if $W_\mu = W_\lambda^d$ for some $d \in \mathscr{D}_{\lambda\mu}$ then there exists an $\mathscr{H}_R$-$\mathscr{H}_\mu$ bisubmodule $N' \cong N \otimes_{\mathscr{H}_\lambda} \widetilde{T}_d$ of M such that

$$N_{W, W_\lambda}(Z_N(\mathscr{H}_\lambda)) = N_{W, W_\mu}(Z_{N'}(\mathscr{H}_\mu)).$$

(d) (*Mackey decomposition*) *If N is an $\mathscr{H}_R$-$\mathscr{H}_\lambda$ bimodule then*

$$N \otimes_{\mathscr{H}_\lambda} \mathscr{H}_R \cong \bigoplus_{d \in \mathscr{D}_{\lambda\mu}} [(N \otimes_{\mathscr{H}_\lambda} \widetilde{T}_d) \otimes_{\mathscr{H}_{\nu(d)}} \mathscr{H}_\mu],$$

where $\nu(d)$ is defined by $W_{\nu(d)} = W_\lambda^d \cap W_\mu$ for all $d \in \mathscr{D}_{\lambda\mu}$.

(e) *Let $b \in Z_M(\mathscr{H}_\lambda)$. Then*

$$N_{W, W_\lambda}(b) = \sum_{d \in \mathscr{D}_{\lambda\mu}} N_{W_\mu, W_{\nu(d)}}(\widetilde{T}_{d^{-1}} b \widetilde{T}_d). \quad \square$$

1.6 **Definition.** A right $\mathscr{H}_R$-module M is *projective relative* to $\mathscr{H}_\lambda$ or simply *$\mathscr{H}_\lambda$-projective* if for every pair of right $\mathscr{H}_R$-modules M', M'' the exact sequence

$$0 \to M' \to M'' \to M \to 0$$

splits provided it is a split exact sequence as $\mathscr{H}_\lambda$-modules.

The following result is also due to P. Hoefsmit and L. Scott. The notation $X|Y$ means that X is isomorphic to a direct summand of Y.

1.7 **Theorem** (Higman's criterion). *Let M be a right $\mathscr{H}_R$-module. Then the following are equivalent*:

(a) *M is $\mathscr{H}_\lambda$-projective*;
(b) $M|M \otimes_{\mathscr{H}_\lambda} \mathscr{H}_R$;
(c) *$M|U \otimes_{\mathscr{H}_\lambda} \mathscr{H}_R$ for some right $\mathscr{H}_\lambda$-module U*;
(d) $N_{W,W_\lambda}(\mathrm{Hom}_{\mathscr{H}_\lambda}(M, M)) = \mathrm{Hom}_{\mathscr{H}_R}(M, M)$. □

Let M be a finitely generated indecomposable right $\mathscr{H}_R$-module. Then, by [Jo, 3.35], there exists a parabolic subgroup W_λ of W unique up to conjugation such that M is $\mathscr{H}_\lambda$-projective and such that W_λ is W-conjugate to a parabolic subgroup of any parabolic subgroup W_μ of W for which M is $\mathscr{H}_\mu$-projective. We call W_λ the *vertex* of M.

2. q-Schur algebras

Let $\Lambda(n, r)$ be the set of all compositions $\lambda = (\lambda_1, \dots, \lambda_n)$ of r into n parts (each part λ_i being nonnegative). For any $\lambda \in \Lambda(n, r)$, we define

$$x_\lambda = \sum_{w \in W_\lambda} T_w \quad \text{and} \quad y_\lambda = \sum_{w \in W_\lambda} (-u)^{-l(w)} T_w .$$

Thus the external direct sum

$$\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathscr{H}_R \quad \left(\text{resp.} \bigoplus_{\lambda \in \Lambda(n,r)} y_\lambda \mathscr{H}_R \right)$$

of the right ideals $x_\lambda \mathscr{H}_R$ (resp. $y_\lambda \mathscr{H}_R$) is a right $\mathscr{H}_R$-module. Let

$$S_R(n, r) = \mathrm{End}_{\mathscr{H}_R} \left(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathscr{H}_R \right). \tag{2.a}$$

$S_R(n, r)$ is called the *q-Schur algebra* (see [DJ4]).

Let

$$T_s^\# = (u-1)T_1 - T_s \qquad (s \in S).$$

$\#$ induces an automorphism of $\mathscr{H}_R$ such that $x_\lambda^\# = r y_\lambda$ for some unit r of R, hence an isomorphism between $S_R(n, r)$ and $\mathrm{End}_{\mathscr{H}_R}(\bigoplus_{\lambda \in \Lambda(n,r)} y_\lambda \mathscr{H}_R)$ (see [DJ3, 2.1, 2.9]). Thus we have the following identification

$$S_R(n, r) = \mathrm{End}_{\mathscr{H}_R} \left(\bigoplus_{\lambda \in \Lambda(n,r)} y_\lambda \mathscr{H}_R \right). \tag{2.b}$$

The classical Schur algebra, as defined in Green's book [G1], is described as the centralizer ring of certain permutation representations. There is an analogue of such a description for the q-Schur algebra.

Let V be a free R-module of rank n with basis $X_1, X_2, \dots, X_n$. Then the tensor product $V^{\otimes r}$ of n copies of V is also a free R-module with basis $\{X_I | I \in I(n, r)\}$ where

$$I(n, r) = \{I = (i_1, i_2, \dots, i_r) | i_t \in [1, n], 1 \le t \le r\}$$

and, for $I = (i_1, i_2, \dots, i_r) \in I(n, r)$, X_I denotes the tensor $X_{i_1} \otimes X_{i_2} \otimes \cdots \otimes X_{i_r}$, or briefly, $X_I = X_{i_1} X_{i_2} \cdots X_{i_r}$.

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda(n, r)$ we let

$$I_\lambda = (\underbrace{1, \dots, 1}_{\lambda_1}, \underbrace{2, \dots, 2}_{\lambda_2}, \dots, \underbrace{n, \dots, n}_{\lambda_n}),$$

$$X_\lambda = X_{I_\lambda}, \quad X_{\lambda d} = X_{I_\lambda d} \quad \text{if } d \in \mathscr{D}_\lambda.$$

where Iw is the natural action of W on $I(n, r)$, called place permutation,

$$Iw = (i_{(1)w}, i_{(2)w}, \dots, i_{(r)w})$$

for $I = (i_1, i_2, \dots, i_r) \in I(n, r)$, $w \in W$. We define for $s = (a, a+1) \in S$,

$$\text{(2.c)} \qquad X_I \widetilde{T}_s = \begin{cases} u^{-1/2} X_I, & \text{if } i_a = i_{a+1}; \\ X_{I_s}, & \text{if } i_a < i_{a+1}; \\ (u^{-1/2} - u^{1/2}) X_I + X_{Is}, & \text{if } i_a > i_{a+1}. \end{cases}$$

It is easy to see that this action gives rise to an $\mathscr{H}_R$-module structure on $V^{\otimes r}$ (see [J, §4]).

2.1 **Lemma.** *$V^{\otimes r}$ is isomorphic to $\bigoplus_{\lambda \in \Lambda(n, r)} y_\lambda \mathscr{H}_R$ as $\mathscr{H}_R$-modules.*

Proof. There is a bijection between $\Lambda(n, r)$ and the set of all W-orbits on $I(n, r)$. Let $\mathscr{O}_\lambda$ be the orbit corresponding to $\lambda \in \Lambda(n, r)$. Then $|\mathscr{O}_\lambda| = |\mathscr{D}_\lambda|$. Therefore, the map

$$\varphi : \bigoplus_\lambda y_\lambda \mathscr{H}_R \to V^{\otimes r}$$

defined by $\varphi(y_\lambda \widetilde{T}_d) = X_{\lambda d} = X_\lambda \widetilde{T}_d$ for $d \in \mathscr{D}_\lambda$ is a linear isomorphism.

By [DJ2, 3.1] we have for $d \in \mathscr{D}_\lambda$, $s \in S$,

$$(y_\lambda \widetilde{T}_d) \widetilde{T}_s = \begin{cases} u^{-1/2} y_\lambda \widetilde{T}_d, & \text{if } ds > d,\ ds \notin \mathscr{D}_\lambda; \\ y_\lambda \widetilde{T}_{ds}, & \text{if } ds > d,\ ds \in \mathscr{D}_\lambda; \\ (u^{-1/2} - u^{1/2}) y_\lambda \widetilde{T}_d + y_\lambda \widetilde{T}_{ds}, & \text{otherwise}. \end{cases}$$

Now, one can check easily by (2.c) that $\varphi(y_\lambda \widetilde{T}_d \widetilde{T}_s) = \varphi(y_\lambda \widetilde{T}_d) \widetilde{T}_s$ for any $s \in S$. Hence φ is an $\mathscr{H}_R$-isomorphism. □

Combining this lemma with (2.b) we get

$$\text{(2.d)} \qquad S_R(n, r) \cong \operatorname{End}_{\mathscr{H}_R}(V^{\otimes r}) = \bigoplus_{\lambda, \mu \in \Lambda(n, r)} \operatorname{Hom}_{\mathscr{H}_R}(X_\mu \mathscr{H}_R, X_\lambda \mathscr{H}_R).$$

We are going to describe a basis of the q-Schur algebra in terms of relative norm which is defined in previous section. We make W act also on the set $I(n, r) \times I(n, r)$ by $(I, J)w = (Iw, Jw)$. Obviously, the orbits are indexed by the triples (μ, λ, d) with $\lambda, \mu \in \Lambda(n, r)$, $d \in \mathscr{D}_{\lambda\mu}$.

For $I, J \in I(n, r)$ we define $e_{IJ} \in \mathrm{Hom}_R(V^{\otimes r}, V^{\otimes r})$ to be the linear map:

$$(X_{I'})e_{IJ} = \begin{cases} X_J, & \text{if } I = I'; \\ 0, & \text{otherwise}. \end{cases}$$

If $(I, J) = (I_\mu, I_\lambda d)$ with $d \in \mathscr{D}_{\lambda\mu}$, we use the abbreviation $e_{\mu\lambda d}$ instead of $e_{I_\mu I_\lambda d}$.

2.2 **Lemma.** *Let $I, J \in I(n, r)$. Assume that W_τ is a parabolic subgroup of* $\mathrm{Stab}_W(I, J)$. *Then*

$$e_{IJ} \in \mathrm{End}_{\mathscr{H}_\tau}(V^{\otimes r}).$$

Proof. Let $s \in W_\tau \cap S$, $s = (a, a+1)$. Since $Is = I$ and $Js = J$ where $I = (i_1, i_2, \dots, i_r)$ and $J = (j_1, j_2, \dots, j_r)$, we have $i_a = i_{a+1}$, $j_a = j_{a+1}$. For any $I' \in I(n, r)$ we have, by (2.c), $X_{I'}(\widetilde{T}_s e_{IJ}) \neq 0$ if and only if $I' = I$ or $I's = I$. But $Is = I$, hence $I' = I$. Thus if $I' \neq I$ then

$$(X_{I'})\widetilde{T}_s e_{IJ} = 0 = (X_{I'})e_{IJ}\widetilde{T}_s$$

and

$$(X_I)\widetilde{T}_s e_{IJ} = u^{-1/2}X_I e_{IJ} = u^{-1/2}X_J = (X_I)e_{IJ}\widetilde{T}_s.$$

Hence $\widetilde{T}_s e_{IJ} = e_{IJ}\widetilde{T}_s$ for all $s \in W_\tau \cap S$. This proves the lemma. □

2.3 **Corollary.** *Let $\lambda, \mu \in \Lambda(n, r)$, $d \in \mathscr{D}_{\lambda\mu}$ and $W_\nu = W_\lambda^d \cap W_\mu$. Then* $e_{\mu\lambda d} \in \mathrm{End}_{\mathscr{H}_\nu}(V^{\otimes r})$. □

2.4 **Lemma.** *Let λ, μ, ν be as in* 2.3 *and $f \in \mathrm{Hom}_{\mathscr{H}_\nu}(X_\mu\mathscr{H}_R, X_\lambda\mathscr{H}_R)$. Then*
(a) $N_{W, W_\nu}(f) \in \mathrm{Hom}_{\mathscr{H}_R}(X_\mu\mathscr{H}_R, X_\lambda\mathscr{H}_R)$;
(b) *If $b \in V^{\otimes r}$ then $(b)N_{W, W_\nu}(f) = 0$ if the projection of b on $X_\mu\mathscr{H}_R$ is* 0. *In particular, $N_{W, W_\mu}(e_{\mu\mu})$ is identity on $X_\mu\mathscr{H}_R$, and* 0 *elsewhere.*

Proof. By (2.c) the space spanned by all X_I where I goes through one orbit is an $\mathscr{H}_R$-submodule of $V^{\otimes r}$. The first assertion follows from 1.5(b). Note that if $x \in \mathscr{D}_\mu$, $x \neq 1$, then $x^{-1} \notin \mathscr{D}_\mu$, hence $(X_\mu)N_{W, W_\mu}(e_{\mu\mu}) = X_\mu$. This implies the last assertion. □

It is well known that the induction is left adjoint to the restriction. The next lemma relates Frobenius reciprocity to the relative norm.

2.5 **Lemma.** *The natural isomorphism*

$$\varphi : \mathrm{Hom}_{\mathscr{H}_\mu}(RX_\mu, X_\lambda\mathscr{H}_R) \to \mathrm{Hom}_{\mathscr{H}_R}(X_\mu\mathscr{H}_R, X_\lambda\mathscr{H}_R)$$

is the restriction of $N_{W, W_\mu}(-)$.

Proof. Let $f \in \mathrm{Hom}_{\mathscr{H}_\mu}(RX_\mu, X_\lambda\mathscr{H}_R)$. We view f as an element of $\mathrm{End}_R(V^{\otimes r})$. Then $\varphi(f)$ is determined by the image $(X_\mu)\varphi(f) = (X_\mu)f$. Since

$$N_{W, W_\mu}(f) \in \mathrm{Hom}_{\mathscr{H}_R}(X_\mu\mathscr{H}_R, X_\lambda\mathscr{H}_R)$$

by 2.4, $N_{W, W_\mu}(f)$ is also determined by the image $(X_\mu)N_{W, W_\mu}(f)$. Since

$$(X_\mu)N_{W, W_\mu}(f) = (X_\mu)\sum_{x \in \mathscr{D}_\mu} \widetilde{T}_{x^{-1}} f \widetilde{T}_x = (X_\mu)f = (X_\mu)\varphi(f),$$

we have $N_{W, W_\mu}(f) = \varphi(f)$, hence the lemma. □

Let $A^d_{\mu\lambda} = N_{W, W_{\nu(d)}}(e_{\mu\lambda d})$ where $\nu(d)$ is defined by $W_{\nu(d)} = W^d_\lambda \cap W_\mu$ and let

$$B = \{A^d_{\mu\lambda} | \lambda, \mu \in \Lambda(n, r), d \in \mathscr{D}_{\lambda\mu}\}. \tag{2.e}$$

By 2.3 and 1.5(b) we have $B \subset S_R(n, r)$. When $u^{1/2} = 1$, $B|_{u^{1/2}=1}$ is the basis of the classical Schur algebra as described in [G1]. Now we have

2.6 **Theorem.** *B is a basis of the q-Schur algebra $S_R(n, r)$.*

Proof. Consider the $\mathscr{H}_\mu$-$\mathscr{H}_\mu$ bimodule

$$M = \mathrm{Hom}_R(RX_\mu, X_{\lambda d} \otimes_{\mathscr{H}_{\nu(d)}} \mathscr{H}_\mu) \quad \text{where } d \in \mathscr{D}_{\lambda\mu}.$$

M contains an $\mathscr{H}_\mu$-$\mathscr{H}_{\nu(d)}$ bimodule $N = \mathrm{Hom}_R(RX_\mu, RX_{\lambda d})$ and we have $M \cong N \otimes_{\mathscr{H}_{\nu(d)}} \mathscr{H}_\mu$. Thus, by 1.5(c), we have

$$N_{W_\mu, W_{\nu(d)}}(\mathrm{Hom}_{\mathscr{H}_{\nu(d)}}(RX_\mu, RX_{\lambda d})) = \mathrm{Hom}_{\mathscr{H}_\mu}(RX_\mu, X_{\lambda d} \otimes_{\mathscr{H}_{\nu(d)}} \mathscr{H}_\mu),$$

and by applying the Mackey decomposition theorem,

$$\begin{aligned}\mathrm{Hom}_{\mathscr{H}_\mu}(RX_\mu, X_\lambda \mathscr{H}_R) &= \bigoplus_{d \in \mathscr{D}_{\lambda\mu}} \mathrm{Hom}_{\mathscr{H}_\mu}(RX_\mu, X_{\lambda d} \otimes_{\mathscr{H}_{\nu(d)}} \mathscr{H}_\mu) \\ &= \bigoplus_{d \in \mathscr{D}_{\lambda\mu}} N_{W_\mu, W_{\nu(d)}}(\mathrm{Hom}_{\mathscr{H}_{\nu(d)}}(RX_\mu, RX_{\lambda d})).\end{aligned}$$

By 2.5 and the transitivity of relative norm we have

$$\begin{aligned}\mathrm{Hom}_{\mathscr{H}_R}(X_\mu \mathscr{H}_R, X_\lambda \mathscr{H}_R) &= N_{W, W_\mu}(\mathrm{Hom}_{\mathscr{H}_\mu}(RX_\mu, X_\lambda \mathscr{H}_R)) \\ &= \bigoplus_{d \in \mathscr{D}_{\lambda\mu}} N_{W, W_{\nu(d)}}(\mathrm{Hom}_{\mathscr{H}_{\nu(d)}}(RX_\mu, RX_{\lambda d})).\end{aligned}$$

Therefore

$$\{N_{W, W_{\nu(d)}}(e_{\mu\lambda d}) | d \in \mathscr{D}_{\lambda\mu}\}$$

is a basis of $\mathrm{Hom}_{\mathscr{H}_R}(X_\mu \mathscr{H}_R, X_\lambda \mathscr{H}_R)$. Hence B is a basis of $S_R(n, r)$ by (2.d). □

Note that there is another basis of $S_R(n, r)$ described in terms of x_λ's in [DJ4]. It is easy to check by using 2.1 that the basis B is not the image of the basis in [DJ4] under the isomorphism (2.d). For example, assume that $r = 3$, $\lambda = \mu = (1, 2)$, and $d = (12)$. Then the basis element $\varphi^d_{\mu\lambda}$ in the sense of [DJ4] maps x_μ to $x_\mu T_d x_\mu$ and $A^d_{\mu\lambda}$ maps y_μ to $y_\mu \widetilde{T}_d y_\mu$. But $(x_\mu T_d)^\# y_\mu \neq y_\mu \widetilde{T}_d y_\mu$, hence $(\varphi^d_{\mu\lambda})_\# \neq A^d_{\mu\lambda}$.

We may describe a basis for $\mathrm{End}_{\mathscr{H}_\theta}(V^{\otimes r})$, $\theta \vDash r$, in a similar way.

Let $d \in \mathscr{D}_{\mu\theta}$, $d' \in \mathscr{D}_{\lambda\theta}$ and $W_\alpha = W^d_\mu \cap W_\theta$, $W_\beta = W^{d'}_\lambda \cap W_\theta$. By the proof of 2.6 we have immediately

2.7 **Corollary.** $\mathrm{Hom}_{\mathscr{H}_\theta}(X_{\mu d} \otimes_{\mathscr{H}_\alpha} \mathscr{H}_\theta, X_{\lambda d'} \otimes_{\mathscr{H}_\beta} \mathscr{H}_\theta)$ *has a basis of the form*

$$B(\mu, \lambda, d, d') = \{N_{W_\theta, W^y_\beta \cap W_\alpha}(e_{\mu d \lambda d' y}) | y \in \mathscr{D}_{\beta\alpha} \cap W_\theta\}. \quad \square$$

Let

$$B(\theta) = \bigcup_{\substack{\lambda, \mu \in \Lambda(n, r) \\ d \in \mathscr{D}_{\mu\theta}, d' \in \mathscr{D}_{\lambda\theta}}} B(\mu, \lambda, d, d').$$

2.8 **Theorem.** *$B(\theta)$ is a basis of* $\mathrm{End}_{\mathscr{H}_\theta}(V^{\otimes r})$.

Proof. By (2.1) and [Jo, (3.22)] we have

$$\begin{aligned}\mathrm{End}_{\mathscr{H}_\theta}(V^{\otimes r}) &= \bigoplus_{\lambda,\mu\in\Lambda(n,r)} \mathrm{Hom}_{\mathscr{H}_\theta}(X_\mu\mathscr{H}_R, X_\lambda\mathscr{H}_R)\\ &= \bigoplus_{\substack{\lambda,\mu\in\Lambda(n,r)\\ d\in\mathscr{D}_{\mu\theta},\, d'\in\mathscr{D}_{\lambda\theta}}} \mathrm{Hom}_{\mathscr{H}_\theta}(X_{\mu d}\otimes_{\mathscr{H}_{\alpha(d)}}\mathscr{H}_\theta, X_{\lambda d'}\otimes_{\mathscr{H}_{\beta(d')}}\mathscr{H}_\theta).\end{aligned}$$

The theorem follows from 2.7. □

3. Defect groups

From now on we assume that l is an odd number. Let R_l be the completion of the polynomial ring in the indeterminate $u^{1/2}$ over $\mathbf{Q}$ localized at the maximal ideal generated by $\Phi_l(u^{1/2})$. Let K be the quotient field of R_l and F is the residue class field $R_l/\pi R_l$ where π is the generator of the maximal ideal of R_l. Thus (K, R_l, F) is a characteristic 0 modular system. Let $R\in\{K, R_l, F\}$.

Let $A^d_{\mu\lambda} = N_{W,W_\nu}(e_{\mu\lambda d})$ be a standard basis element as described in (2.e). We define the *defect group* $D^d_{\mu\lambda}$ of $A^d_{\mu\lambda}$ to be the maximal l-parabolic subgroup of W_ν. We are going to study the coefficients of the product of two standard basis elements. First of all we need some lemmas.

3.1 **Lemma.** *Let $\lambda, \mu\in\Lambda(n,r)$ and $d\in\mathscr{D}_{\lambda\mu}$. Then*

$$N_{W,W^d_\lambda\cap W_\mu}(e_{\mu\lambda d}) = N_{W,W_\lambda\cap W^{d^{-1}}_\mu}(e_{\mu d^{-1}\lambda}).$$

Proof. Let

$$W_\nu = W^d_\lambda\cap W_\mu, \qquad W_{\nu'} = W_\lambda\cap W^{d^{-1}}_\mu.$$

We claim that $\mathscr{D}_{\nu'}\cap dW_\mu = d(\mathscr{D}_\nu\cap W_\mu)$. Indeed, let $y = dw\in\mathscr{D}_{\nu'}\cap dW_\mu$ for $w\in W_\mu$. Then $l(v'y) = l(v')+l(y)$ for any $v'\in W_{\nu'}$. For any $v\in W_\nu$, $v = d^{-1}v'd$ for some $v'\in W_{\nu'}$, and we have $l(v) = l(v')$. Thus

$$l(d)+l(vw) = l(dvw) = l(v'dw) = l(v')+l(d)+l(w).$$

Hence, $l(vw) = l(v')+l(w) = l(v)+l(w)$. This implies $w\in\mathscr{D}_\nu$, and so $y\in d(\mathscr{D}_\nu\cap W_\mu)$.

Conversely, let $w\in\mathscr{D}_\nu\cap W_\mu$. For any $v'\in W_{\nu'}$, $v' = dvd^{-1}$ for some $v\in W_\nu$, we have $l(v') = l(v)$, and

$$l(v'dw) = l(dvw) = l(d)+l(v)+l(w) = l(v')+l(dw).$$

Hence $dw\in\mathscr{D}_{\nu'}\cap dW_\mu$. So our claim is proved.

Obviously, by 2.4, it suffices to check that

$$(X_\mu)N_{W,W_\nu}(e_{\mu\lambda d}) = (X_\mu)N_{W,W_{\nu'}}(e_{\mu d^{-1}\lambda}).$$

Since

$$\begin{aligned}\text{L.H.S.} &= \sum_{x\in\mathscr{D}_\nu}(X_\mu)\widetilde{T}_{x^{-1}}e_{\mu\lambda d}\widetilde{T}_x\\ &= \sum_{x\in\mathscr{D}_\nu\cap W_\mu} u^{-l(x)/2}X_{\lambda dx}\end{aligned}$$

and

$$\begin{aligned}\text{R.H.S.} &= \sum_{y\in\mathscr{D}_{\nu'}\cap dW_\mu} (X_\mu)\widetilde{T}_{y^{-1}}e_{\mu d^{-1}}\widetilde{T}_y \\ &= \sum_{y\in d(\mathscr{D}_\nu\cap W_\mu)} (X_\mu)\widetilde{T}_{y^{-1}}e_{\mu d^{-1}\lambda}\widetilde{T}_y\,, \quad \text{by the claim}, \\ &= \sum_{\dot{x}\in\mathscr{D}_\nu\cap W_\mu} u^{-l(x)/2}X_{\lambda dx}\end{aligned}$$

so the lemma follows. □

We now fix the following notation: Let λ, μ, ρ be compositions of r,

$$d \in \mathscr{D}_{\lambda\mu}\,, \qquad d' \in \mathscr{D}_{\rho\lambda}\,,$$

$$W_\nu = W_\lambda^d \cap W_\mu\,, \qquad W_{\nu'} = W_\rho \cap W_\lambda^{d'^{-1}}\,, \qquad W_\tau = W_{\tau(y)} = W_\rho^y \cap W_\mu\,,$$

where $y \in \mathscr{D}_{\rho\mu}$.

3.2 **Lemma.** *Let* $y \in \mathscr{D}_{\rho\mu}$. *Then*

$$W_\rho y W_\mu \cap \mathscr{D}_{\nu'\nu} = \{h_i y k_j | 1 \le i \le n_y\,,\ 1 \le j \le m_y\}$$

where $yk_j \in \mathscr{D}_{\rho\nu}$, $k_j \in \mathscr{D}_{\tau\nu} \cap W_\mu$ *and* $h_i \in \mathscr{D}_{\nu'} \cap W_\rho$ *for all* i, j.

Proof. By [DJ1, 1.6] we have

$$W_\rho y W_\mu = W_\rho y(\mathscr{D}_\tau \cap W_\mu) \quad \text{and} \quad W_\mu = (\mathscr{D}_\nu \cap W_\mu)^{-1} W_\nu\,.$$

We may write the element of $\mathscr{D}_\tau \cap W_\mu$ in the form: kw where $k \in \mathscr{D}_\nu^{-1} \cap W_\mu$, $w \in W_\nu$. It is clear that $k \in \mathscr{D}_\tau$ (see [DJ1, 1.4]), hence $k \in \mathscr{D}_{\tau\nu} \cap W_\mu$ and $yk \in \mathscr{D}_\rho$ since

$$\mathscr{D}_\rho = \bigcup_{y\in\mathscr{D}_{\rho\mu}} y(\mathscr{D}_{\tau(y)} \cap W_\mu)\,.$$

Since $y \in \mathscr{D}_\mu^{-1}$, $k \in \mathscr{D}_\nu^{-1} \cap W_\mu$ we have $yk \in \mathscr{D}_\nu^{-1}$, hence $yk \in \mathscr{D}_{\rho\nu}$. Thus we may express $W_\rho y W_\mu$ as a disjoint union of the double cosets of the form:

(3.a) $\qquad W_\rho y k W_\nu$ where $yk \in \mathscr{D}_{\rho\nu}$ and $k \in \mathscr{D}_{\tau\nu} \cap W_\mu$.

Also, each $W_\rho y k W_\nu$ is a disjoint union of the double cosets of the form:

(3.b) $\qquad W_{\nu'} h y k W_\nu$ where $h \in \mathscr{D}_{\nu'} \cap W_\rho$.

Hence the lemma follows. □

3.3 **Lemma.** *Let* y, k *be as in* (3.a) *and let* $z = yk$. *Then*

$$N_{W,W_\theta}(e_{\mu\rho z}) = N_{W,W_{\theta'}}(c e_{\mu\rho y})$$

where $W_\theta = W_\rho^z \cap W_\nu$ *and* $W_{\theta'} = W_\rho^y \cap W_\nu^{k^{-1}}$ *and* $c \in R$.

Proof. Since $W_\rho^z \cap W_\nu = (W_\rho^y \cap W_\mu)^k \cap W_\nu$ and $k \in \mathscr{D}_{\tau\nu} \cap W_\mu$ by 3.2, we have both W_θ and $W_{\theta'}$ parabolic subgroups of W_μ. By transitivity of relative norm (Theorem 1.5) it is enough to show that

(3.c) $\qquad N_{W_\mu,W_\theta}(e_{\mu\rho z}) = N_{W_\mu,W_{\theta'}}(c e_{\mu\rho y})\,.$

To do this we consider the $\mathscr{H}_\mu$-$\mathscr{H}_\mu$ bimodule

$$M = \mathrm{Hom}_R(RX_\mu, X_{\rho y} \otimes_{\mathscr{H}_{\theta'}} \mathscr{H}_\mu).$$

It is clear that there exist an $\mathscr{H}_\mu$-$\mathscr{H}_{\theta'}$ bisubmodule $N = \mathrm{Hom}_R(RX_\mu, RX_{\rho y})$ and an $\mathscr{H}_\mu$-$\mathscr{H}_\theta$ bisubmodule $N' = \mathrm{Hom}_R(RX_\mu, RX_{\rho z})$ such that

$$M \cong N \otimes_{\mathscr{H}_{\theta'}} \mathscr{H}_\mu \quad \text{and} \quad M \cong N' \otimes_{\mathscr{H}_\theta} \mathscr{H}_\mu.$$

By 1.5(c) we get

$$N_{W_\mu, W_{\theta'}}(\mathrm{Hom}_{\mathscr{H}_{\theta'}}(RX_\mu, RX_{\rho y})) = N_{W_\mu, W_\theta}(\mathrm{Hom}_{\mathscr{H}_\theta}(RX_\mu, RX_{\rho z})).$$

Therefore there exists $c \in R$ such that (3.c) holds. □

In 3.4 and 3.5 we assume that $R = R_l$.

3.4 **Lemma.** *Let $hyk \in \mathscr{D}_{\nu'\nu}$ be as in* (3.b) *and let*

$$g(u) = \frac{d_{W_\rho^{yk} \cap W_\nu}}{d_{W_{\nu'}^{hyk} \cap W_\nu}}, \qquad f(u) = \frac{d_{W_\rho^y \cap W_\mu}}{d_{W_\rho^y \cap W_\nu^{k-1}}}.$$

Then

$$N_{W, W_{\nu'}^{hyk} \cap W_\nu}(e_{\mu\rho yk}) = cf(u)g(u)N_{W, W_\rho^y \cap W_\mu}(e_{\mu\rho y})$$

for some $c \in R$.

Moreover, if $P_{\tau(y)} \not\subseteq_W D^d_{\mu\lambda}$ or $P_{\tau(y)} \not\subseteq_W D^{d'}_{\lambda\rho}$ then $\Phi_l(u) | f(u)g(u)$.

Proof. By 1.5, 2.2, 2.4(a) and 3.3 we have

$$\begin{aligned}
N_{W, W_{\nu'}^{hyk} \cap W_\nu}(e_{\mu\rho yk}) &= N_{W, W_\rho^{yk} \cap W_\nu}(N_{W_\rho^{yk} \cap W_\nu, W_{\nu'}^{hyk} \cap W_\nu}(e_{\mu\rho yk})) \\
&= N_{W, W_\rho^{yk} \cap W_\nu}(N_{W_\rho^{yk} \cap W_\nu, W_{\nu'}^{hyk} \cap W_\nu}(\widetilde{T}_1)e_{\mu\rho yk}) \\
&= N_{W, W_\rho^{yk} \cap W_\nu}(g(u^{-1})e_{\mu\rho yk}) \\
&= N_{W, W_\rho^y \cap W_\nu^{k-1}}(g(u^{-1})ce_{\mu\rho y}) \\
&= cg(u^{-1})N_{W, W_\rho^y \cap W_\mu}(N_{W_\rho^y \cap W_\mu, W_\rho^y \cap W_\nu^{k-1}}(e_{\mu\rho y})) \\
&= cf(u^{-1})g(u^{-1})N_{W, W_\rho^y \cap W_\mu}(e_{\mu\rho y})
\end{aligned}$$

as desired.

We now assume that $d_{P_\nu} | d_{P_{\tau(y)}}$ but $d_{P_\nu} \neq d_{P_{\tau(y)}}$. Let P be the maximal l-parabolic subgroup of $W_{\nu'}^{hyk} \cap W_\nu$, then $d_P | d_{P_{\tau(y)}}$ and $d_P \neq d_{P_{\tau(y)}}$. Applying 1.2 we obtain $\Phi_l(u) | f(u)g(u)$. The lemma is proved. □

We now prove the following theorem which is a natural generalization of the classical situation (see [F, S]).

3.5 **Theorem.** *Let $A^d_{\mu\lambda}, A^{d'}_{\lambda\rho} \in B$. Assume that*

$$A^d_{\mu\lambda} A^{d'}_{\lambda\rho} = \sum_{y \in \mathscr{D}_{\mu\rho}} a_y A^y_{\mu\rho}$$

where $a_y \in R$. If $a_y \not\equiv 0 \bmod(\Phi_l(u))$, then

$$P_{\tau(y)} \subseteq_W D^d_{\mu\lambda} \quad \text{and} \quad P_{\tau(y)} \subseteq_W D^{d'}_{\lambda\rho}.$$

Proof. By 3.1 and 1.5(e) we have

$$\begin{aligned} A^d_{\mu\lambda}A^{d'}_{\lambda\rho} &= N_{W,W_\nu}(e_{\mu\lambda d})N_{W,W_{\nu'}}(e_{\lambda d'^{-1}\rho}) \\ &= N_{W,W_\nu}(e_{\mu\lambda d}N_{W,W_{\nu'}}(e_{\lambda d'^{-1}\rho})) \\ &= N_{W,W_\nu}\left(e_{\mu\lambda d}\sum_{x\in\mathscr{D}_{\nu'\nu}} N_{W_\nu,W^x_{\nu'}\cap W_\nu}(\widetilde{T}_{x^{-1}}e_{\lambda d'^{-1}\rho}\widetilde{T}_x)\right) \\ &= \sum_{x\in\mathscr{D}_{\nu'\nu}} N_{W,W^x_{\nu'}\cap W_\nu}(e_{\mu\lambda d}\widetilde{T}_{x^{-1}}e_{\lambda d'^{-1}\rho}\widetilde{T}_x). \end{aligned}$$

For $x \in \mathscr{D}_{\nu'\nu}$ there exist $y \in \mathscr{D}_{\rho\mu}$ and h_i, k_j as in 3.2 such that $x = h_i y k_j$ and $yk_j \in \mathscr{D}_{\rho\nu}$, $h_i \in \mathscr{D}_{\nu'} \cap W_\rho$. By a direct computation we have

$$e_{\mu\lambda d}\widetilde{T}_{x^{-1}}e_{\lambda d'^{-1}\rho}\widetilde{T}_x = b_{ijy}e_{\mu\rho yk_j}$$

for some $b_{ijy} \in R$. Thus by 3.4,

$$\begin{aligned} A^d_{\mu\lambda}A^{d'}_{\lambda\rho} &= \sum_{y\in\mathscr{D}_{\mu\rho}}\sum_{i,j} N_{W,W^{h_iyk_j}_{\nu'}\cap W_\nu}(b_{ijy}e_{\mu\rho yk_j}) \\ &= \sum_{y\in\mathscr{D}_{\mu\rho}}\left(\sum_{i,j} b'_{ijy}f_{iy}(u)g_{jy}(u)\right)N_{W,W^y_\rho\cap W_\mu}(e_{\mu\rho y}). \end{aligned}$$

Therefore, we get

$$a_y = \sum_{i=1}^{n_y}\sum_{j=1}^{m_y} b'_{ijy}f_{iy}(u)g_{jy}(u).$$

If $P_{\tau(y)} \not\subseteq_W D^d_{\mu\lambda}$ or $P_{\tau(y)} \not\subseteq_W D^{d'}_{\lambda\rho}$ then $\Phi_l(u)|f_{iy}(u)g_{jy}(u)$ for all i, j by 3.4. Hence $a_y \equiv 0 \bmod(\Phi_l(u))$. This completes the proof of the theorem. □

Let P be an l-parabolic subgroup of W. We define $I_F(P)$ to be the subspace of $S_F(n, r)$ spanned by all $A^d_{\mu\lambda}$ satisfying $D^d_{\mu\lambda} \subseteq_W P$. By the above theorem and 1.3 we immediately have

3.6 **Corollary.** *$I_F(P)$ is an ideal of $S_F(n, r)$.* □

Let H be a parabolic subgroup of W. Define $\nu_l(d_H)$ to be the largest power of $\Phi_l(u)$ which divides d_H. That is, if $\Phi_l^m|d_H$ but $\Phi_l^{m+1}\nmid d_H$ then $\nu_l(d_H) = m$. Let $r = kl + s$, $s < l$. For each m, $0 \le m \le k$, we choose an l-parabolic subgroup P_m of W such that $\nu_l(d_{P_m}) = m$. Thus, by (1.b), we have a chain of ideals in $S_F(n, r)$

$$0 \subseteq I_F(P_0) \subseteq I_F(P_1) \subseteq \cdots \subseteq I_F(P_k) = S_F(n, r). \tag{3.d}$$

Given a primitive idempotent $e \in S_F(n, r)$, there is a number $n(e)$ such that $e \in I_F(P_{n(e)})$, $e \notin I_F(P_{n(e)-1})$. We set $D(e) = P_{n(e)}$ and call $D(e)$ the *defect group* of e. If F is a splitting field for $S_F(n, r)$, then there is a unique irreducible modular character ξ of $S_F(n, r)$ satisfying $\xi(e) = 1$.

The following characterization of defect groups for the q-Schur algebra holds (see [S]).

3.7 **Proposition.** *Assume that F is a splitting field of $S_F(n, r)$. Let $e, e' \in S_F(n, r)$ be primitive idempotents, and let ξ, ξ' be the associated irreducible modular characters. Then*

(a) $\xi(hA^d_{\mu\lambda}) = 0$ *for all* $h \in S_F(n, r)$ *unless* $D(e) \subseteq_W D^d_{\mu\lambda}$;

(b) $\xi(A^d_{\mu\lambda}) \neq 0$ *for some* μ, λ, d *with* $D(e) =_W D^d_{\mu\lambda}$;

(c) e *is equivalent to* e' *if and only if* $D(e) =_W D(e')$ *and* $\xi(a^d_{\mu\lambda}) = \xi'(A^d_{\mu\lambda})$ *for all* μ, λ, d *with* $D^d_{\mu\lambda} =_W D(e)$.

Proof. By Corollary 3.6 the arguments in [S, p. 106] are valid. □

4. Vertex

Let $R = R_l$. If M is a right $\mathscr{H}_R$-module then we denote by $\overline{M}$ the right $\mathscr{H}_F$-module $M/\pi M \cong M \otimes_R F$.

Recall from §1 the definition of the vertex of an indecomposable module. It is natural to expect that the vertex is an l-parabolic subgroup. Indeed, we have (see [Du, 3.1]).

4.1 **Theorem.** *Let M be a finitely generated indecomposable $\mathscr{H}_R$-module. Then the vertex of M is an l-parabolic subgroup of W.* □

4.2 **Proposition.** *Let M, N be indecomposable direct summands of $V^{\otimes r}$. Then*

(a) $\overline{\mathrm{Hom}_{\mathscr{H}_R}(M, N)} \cong \mathrm{Hom}_{\mathscr{H}_F}(\overline{M}, \overline{N})$;

(b) $\overline{M}$ *is indecomposable and has the same vertex as* M.

Proof. (a) Since R is a p.i.d. and M, N are the direct summands of $V^{\otimes r}$, M and N are free R-modules. Thus the natural map

$$f : \overline{\mathrm{Hom}_{\mathscr{H}_R}(M, N)} \to \mathrm{Hom}_{\mathscr{H}_F}(\overline{M}, \overline{N})$$

is a monomorphism. Since the Hom is additive and by Theorem 2.6, we have $\overline{S_R(n, r)} = S_F(n, r)$. Hence f must be an isomorphism.

(b) By (a) it is easy to see that $\overline{M}$ is indecomposable since $\mathrm{Hom}_{\mathscr{H}_F}(\overline{M}, \overline{M})$ is a local ring.

Let W_λ be vertex of M and let

$$L = N_{W, W_\lambda}(\mathrm{Hom}_{\mathscr{H}_\lambda}(M, M)), \qquad N = \mathrm{Hom}_{\mathscr{H}_R}(M, M).$$

If $L = N$ then by noting 2.8 we have

$$\begin{aligned}\mathrm{Hom}_{\mathscr{H}_F}(\overline{M}, \overline{M}) &= \overline{\mathrm{Hom}_{\mathscr{H}_R}(M, M)} \\ &= \overline{N_{W, W_\lambda}(\mathrm{Hom}_{\mathscr{H}_\lambda}(M, M))} \\ &= N_{W, W_\lambda}(\overline{\mathrm{Hom}_{\mathscr{H}_\lambda}(M, M)}) \\ &= N_{W, W_\lambda}(\mathrm{Hom}_{\mathscr{H}_\lambda}(\overline{M}, \overline{M})).\end{aligned}$$

Hence W_λ is the vertex of $\overline{M}$.

Conversely, assume that $\overline{L} = \overline{N}$. Then $N = L + \pi N$ and L is an ideal of N. Since N is a local ring and is a free R-module, by [G, 3.3a], we have $L = N$. Hence W_λ is the vertex of M. □

There is a close relation between the vertex of an indecomposable direct summand M of $V^{\otimes r}$ and the defect group of the corresponding primitive idempotent of $S_F(n, r)$ in classical situation (see [S]). We are going to generalize such a relation in the case of q-Schur algebra.

Let $\theta \vDash r$ be such that W_θ is an l-parabolic subgroup of W. Let $\lambda, \mu \in \Lambda(n, r)$, $d \in \mathscr{D}_{\mu\theta}$ and $d' \in \mathscr{D}_{\lambda\theta}$.

4.3 **Lemma.** *Let $\rho \vDash r$ be such that $W_\theta \subseteq W_\rho$. Assume that $W_\theta \subseteq W_\mu^d \cap W_\lambda^{d'}$. Then*

$$N_{W_\rho, W_\theta}(e_{\mu d\lambda d'}) = N_{W_\rho, W_\theta}(e_{\mu d\theta}) N_{W_\rho, W_\theta}(e_{\theta\lambda d'}).$$

Proof. By 1.5(a), (b) and (e) we have

$$\begin{aligned} N_{W_\rho, W_\theta}(e_{\mu d\theta}) N_{W_\rho, W_\theta}(e_{\theta\lambda d'}) &= N_{W_\rho, W_\theta}(e_{\mu d\theta} N_{W_\rho, W_\theta}(e_{\theta\lambda d'})) \\ &= N_{W_\rho, W_\theta}\left(e_{\mu d\theta} \sum_{w \in \mathscr{D}_{\theta\theta} \cap W_\rho} N_{W_\theta, W_\theta^w \cap W_\theta}(\widetilde{T}_{w^{-1}} e_{\theta\lambda d'} \widetilde{T}_w)\right) \\ &= \sum_{w \in \mathscr{D}_{\theta\theta} \cap W_\rho} N_{W_\rho, W_\theta^w \cap W_\theta}(e_{\mu d\theta} \widetilde{T}_{w^{-1}} e_{\theta\lambda d'} \widetilde{T}_w) \\ &= N_{W_\rho, W_\theta}(e_{\mu d\lambda d'}) \end{aligned}$$

since $e_{\nu d\theta} \widetilde{T}_{w^{-1}} e_{\theta\lambda d'} \widetilde{T}_w \neq 0$ if and only if $w \in \mathscr{D}_{\theta\theta} \cap W_\theta = \{1\}$. □

In particular, we have for $W_\rho = W$,

(4.a) $$N_{W, W_\theta}(e_{\mu d\lambda d'}) = N_{W, W_\theta}(e_{\mu d\theta}) N_{W, W_\theta}(e_{\theta\lambda d'}).$$

4.4 **Lemma.** *Let $z \in \mathscr{D}_{\lambda\theta}$ and $k \in \mathscr{D}_{W_\lambda^z \cap W_\theta} \cap \mathscr{D}^{-1}_{W_\mu \cap W_\theta} \cap W_\theta$. Then there are $z' \in \mathscr{D}_{\lambda\mu}$ and $k' \in \mathscr{D}_{W_\lambda^{z'} \cap W_\mu} \cap \mathscr{D}^{-1}_{W_\mu \cap W_\theta} \cap W_\mu$ such that $zk = z'k'$ and $z'k' \in \mathscr{D}_\lambda \cap \mathscr{D}^{-1}_{W_\mu \cap W_\theta}$.*

Moreover, we have

$$(W_\lambda^{z'} \cap W_\mu)^{k'} \cap (W_\theta \cap W_\mu) = (W_\lambda^z \cap W_\theta)^k \cap (W_\mu \cap W_\theta).$$

Proof. Consider

$$W_\lambda z W_\theta = W_\lambda z(\mathscr{D}_{W_\lambda^z \cap W_\theta} \cap W_\theta).$$

The elements in $\mathscr{D}_{W_\lambda^z \cap W_\theta} \cap W_\theta$ can be written in the form kw where $k \in \mathscr{D}^{-1}_{W_\theta \cap W_\mu} \cap W_\theta$, $w \in W_\theta \cap W_\mu$. Since $l(kw) = l(k) + l(w)$, it follows that

(4.b) $$k \in \mathscr{D}_{W_\lambda^z \cap W_\theta} \cap \mathscr{D}^{-1}_{W_\mu \cap W_\theta} \cap W_\theta.$$

Also, we have $zk \in \mathscr{D}_\lambda$. On the other hand, since $z \in \mathscr{D}_\theta^{-1}$ and $k \in \mathscr{D}^{-1}_{W_\mu \cap W_\theta} \cap W_\theta$, we have $zk \in \mathscr{D}^{-1}_{W_\mu \cap W_\theta}$. Hence $zk \in \mathscr{D}_\lambda \cap \mathscr{D}^{-1}_{W_\mu \cap W_\theta}$.

Thus the elements in $\mathscr{D}_\lambda \cap \mathscr{D}^{-1}_{W_\mu \cap W_\theta} \cap W_\lambda z W_\theta$ have the form zk where k as described in (4.b). Therefore

$$\mathscr{D}_\lambda \cap \mathscr{D}^{-1}_{W_\mu \cap W_\theta} = \{zk \mid z \in \mathscr{D}_{\lambda\theta},\ k \in \mathscr{D}_{W_\lambda^z \cap W_\theta} \cap \mathscr{D}^{-1}_{W_\mu \cap W_\theta} \cap W_\theta\}.$$

By a similar argument we have

$$\mathscr{D}_\lambda \cap \mathscr{D}^{-1}_{W_\mu \cap W_\theta} = \{z'k' \mid z' \in \mathscr{D}_{\lambda\mu},\ k' \in \mathscr{D}_{W_\lambda^{z'} \cap W_\mu} \cap \mathscr{D}^{-1}_{W_\mu \cap W_\theta} \cap W_\mu\}.$$

So the first assertion follows.

Now,

$$\begin{aligned} (W_\lambda^{z'} \cap W_\mu)^{k'} \cap (W_\theta \cap W_\mu) &= W_\lambda^{z'k'} \cap W_\theta \cap W_\mu = W_\lambda^{zk} \cap W_\mu \cap W_\theta \\ &= (W_\lambda^z \cap W_\theta)^k \cap (W_\mu \cap W_\theta) \end{aligned}$$

as desired. □

4.5 **Lemma.** *Let* z, k *be as in* 4.4. *Then*

$$N_{W,(W_\lambda^z\cap W_\theta)^k\cap(W_\mu\cap W_\theta)}(e_{\mu\lambda zk})\in I_F(W_\theta).$$

Proof. By 4.4 there exist $z'\in\mathscr{D}_{\lambda\mu}$ and $k'\in\mathscr{D}_{W_\lambda^{z'}\cap W_\mu}\cap\mathscr{D}^{-1}_{W_\mu\cap W_\theta}\cap W_\mu$ such that $zk=z'k'$ and

$$\begin{aligned}
&N_{W,(W_\lambda^z\cap W_\theta)^k\cap(W_\mu\cap W_\theta)}(e_{\mu\lambda zk})\\
&\quad=N_{W,(W_\lambda^{z'}\cap W_\mu)^{k'}\cap(W_\theta\cap W_\mu)}(e_{\mu\lambda z'k'})\\
&\quad=cN_{W,(W_\lambda^{z'}\cap W_\mu)\cap(W_\theta\cap W_\mu)^{k'-1}}(e_{\mu\lambda z'}),\qquad\text{for some } c\in F \text{ by } 3.3,\\
&\quad=cN_{W,W_\lambda^{z'}\cap W_\mu}(N_{W_\lambda^{z'}\cap W_\mu,(W_\lambda^{z'}\cap W_\mu)\cap(W_\theta\cap W_\mu)^{k'-1}}(e_{\mu\lambda z'}))\\
&\quad=c\frac{d_1(u^{-1})}{d_2(u^{-1})}N_{W,W_\lambda^{z'}\cap W_\mu}(e_{\mu\lambda z'})
\end{aligned}$$

where $d_1(u)=d_{W_\lambda^{z'}\cap W_\mu}$ and $d_2(u)=d_{(W_\lambda^{z'}\cap W_\mu)\cap(W_\theta\cap W_\mu)^{k'-1}}$.

If the maximal l-parabolic subgroup P of $W_\lambda^{z'}\cap W_\mu$ satisfies $d_P\neq d_{W_\theta}$ and $d_{W_\theta}|d_P$, then

$$\frac{d_1(u)}{d_2(u)}\equiv 0\quad \mathrm{mod}(\Phi_l(u))$$

since $d_2(u)|d_{W_\theta}$. Hence our result follows. □

4.6 **Lemma.** *We have*

$$I_F(\{1\})=N_{W,1}(\mathrm{End}_F(V^{\otimes r})).$$

Proof. Clearly,

$$\{e_{\mu d\lambda d'}|\lambda,\mu\in\Lambda(n,r),\ d\in\mathscr{D}_\mu,\ d'\in\mathscr{D}_\lambda\}$$

is a basis of $\mathrm{End}_F(V^{\otimes r})$. By 1.5 and 2.4 we have

$$\begin{aligned}
N_{W,1}(e_{\mu d\lambda d'})&=N_{W,W_\mu}(e_{\mu\mu})N_{W,1}(e_{\mu d\lambda d'})\\
&=N_{W,W_\mu}(e_{\mu\mu}N_{W,1}(e_{\mu d\lambda d'}))\\
&=N_{W,W_\mu}\left(e_{\mu\mu}\sum_{x\in\mathscr{D}_\mu^{-1}}N_{W_\mu,1}(\widetilde{T}_{x^{-1}}e_{\mu d\lambda d'}\widetilde{T}_x)\right)\\
&=\sum_{x\in\mathscr{D}_\mu^{-1}\cap d^{-1}W_\mu}N_{W,1}(e_{\mu\mu}\widetilde{T}_{x^{-1}}e_{\mu d\lambda d'}\widetilde{T}_x)\\
&=N_{W,1}(e_{\mu\mu}\widetilde{T}_d e_{\mu d\lambda d'}\widetilde{T}_{d^{-1}}).
\end{aligned}$$

Let

$$e_{\mu\mu}\widetilde{T}_d e_{\mu d\lambda d'}\widetilde{T}_{d^{-1}}=\sum_{y\in\mathscr{D}_\lambda}a_y e_{\mu\lambda y}$$

where $a_y\in F$. Then

$$N_{W,1}(e_{\mu d\lambda d'})=\sum_{y\in\mathscr{D}_\lambda}a_y N_{W,1}(e_{\mu\lambda y}).$$

By Lemma 4.5 we have

$$N_{W,1}(e_{\mu\lambda y}) \in I_F(\{1\}).$$

Hence,

$$N_{W,1}(e_{\mu d\lambda d'}) \in I_F(\{1\}).$$

Thus we have proved that

$$N_{W,1}(\mathrm{End}_F(V^{\otimes r})) \subseteq I_F(\{1\}).$$

The proof of the other direction is obvious (see the proof of [Du, 3.1]). □

4.7 **Proposition.** *Let W_θ be an l-parabolic subgroup of W. Then*

$$I_F(W_\theta) = N_{W,W_\theta}(\mathrm{End}_{\mathscr{H}_\theta}(V^{\otimes r})).$$

Proof. We first show that

$$N_{W,W_\theta}(\mathrm{End}_{\mathscr{H}_\theta}(V^{\otimes r})) \subseteq I_F(W_\theta).$$

By 2.8 it is enough to prove that

$$N_{W,W_\beta^y \cap W_\alpha}(e_{\mu d\lambda d' y}) \in I_F(W_\theta) \tag{4.c}$$

for all μ, λ, d, d', y as described in 2.7.

To do this we proceed induction on $\nu_l(d_{W_\theta})$ (see the definition after 3.6). If $\nu_l(d_{W_\theta}) = 0$ i.e. $W_\theta = \{1\}$, (4.c) follows from 4.6. Assume that $W_\theta \neq \{1\}$. Let $W_\gamma = W_\beta^y \cap W_\alpha$. If $\nu_l(d_{P_\gamma}) < \nu_l(d_{W_\theta})$ then

$$N_{W,W_\gamma}(e_{\mu d\lambda d' y}) = (d_{W_\gamma}/d_{P_\gamma})^{-1} N_{W,P_\gamma}(e_{\mu d\lambda d' y}) \in I_F(P_\gamma)$$

by induction. Hence, by (3.d),

$$N_{W,W_\gamma}(e_{\mu d\lambda d' y}) \in I_F(W_\theta).$$

Thus, we may assume that $W_\gamma = W_\theta$, that is

$$W_\theta = (W_\lambda^{d'} \cap W_\theta)^y \cap (W_\mu^d \cap W_\theta) = W_\lambda^{d'y} \cap W_\mu^d \cap W_\theta.$$

Since $y \in \mathscr{D}_{\beta\alpha} \cap W_\theta$ and $\alpha = \beta = \theta$ we have $y = 1$ and hence $W_\theta \subseteq W_\lambda^{d'} \cap W_\mu^d$. Since $d \in \mathscr{D}_{\mu\theta}$, $d' \in \mathscr{D}_{\lambda\theta}$, by 3.1 and 4.5, we have

$$N_{W,W_\theta}(e_{\mu d\theta}) = N_{W,W_\theta^{d^{-1}}}(e_{\mu\theta d^{-1}}) \in I_F(W_\theta)$$

and

$$N_{W,W_\theta}(e_{\theta\lambda d'}) \in I_F(W_\theta).$$

By (4.a) and 3.6 we obtain

$$N_{W,W_\theta}(e_{\mu d\lambda d'}) = N_{W,W_\theta}(e_{\mu d\theta}) N_{W,W_\theta}(e_{\theta\lambda d'}) \in I_F(W_\theta).$$

Hence (4.c) holds.

We next prove that

$$I_F(W_\theta) \subseteq N_{W,W_\theta}(\mathrm{End}_{\mathscr{H}_\theta}(V^{\otimes r})).$$

Let

$$N_{W,W_\nu}(e_{\mu\lambda d}) \in I_F(W_\theta) \cap B$$

where $W_\nu = W_\lambda^d \cap W_\mu$, $d \in \mathscr{D}_{\lambda\mu}$. Then $d_{P_\nu} | d_{W_\theta}$.

Let z be a distinguished representative of the $P_\nu - W_\theta$ double coset such that $P_\nu^z \subseteq W_\theta$ and P_ν^z is also parabolic (of course l-parabolic, see (1.b)). z is also a distinguished representative of the double coset $P_\nu - P_\nu^z$. Let τ, τ' be the compositions of r such that $W_\tau = P_\nu$, $W_{\tau'} = P_\nu^z$.

We consider $\mathscr{H}_F$-$\mathscr{H}_F$ bimodule

$$M = \mathrm{Hom}_F(X_\mu \mathscr{H}_F, X_{\lambda d} \otimes_{\mathscr{H}_\tau} \mathscr{H}_F).$$

M has an $\mathscr{H}_F$-$\mathscr{H}_\tau$ bisubmodule

$$N = \mathrm{Hom}_F(X_\mu \mathscr{H}_F, X_{\lambda d})$$

and an $\mathscr{H}_F$-$\mathscr{H}_{\tau'}$ bisubmodule

$$N' = \mathrm{Hom}_F(X_\mu \mathscr{H}_F, X_{\lambda d} \otimes_{\mathscr{H}_\tau} \widetilde{T}_z).$$

Since

$$\begin{aligned} M &\cong \bigoplus_{w \in \mathscr{D}_\tau} \mathrm{Hom}_F(X_\mu \mathscr{H}_F, X_{\lambda d} \otimes \widetilde{T}_w) \\ &\cong \bigoplus_{w \in \mathscr{D}_\tau} \mathrm{Hom}_F(X_\mu \mathscr{H}_F, X_{\lambda d}) \otimes \widetilde{T}_w \\ &\cong N \otimes_{\mathscr{H}_\tau} \mathscr{H}_F \end{aligned}$$

as $\mathscr{H}_F$-$\mathscr{H}_F$ bimodules and similarly,

$$M \cong N' \otimes_{\mathscr{H}_{\tau'}} \mathscr{H}_F,$$

it follows from 1.5 that

$$N_{W, W_\tau}(\mathrm{Hom}_{\mathscr{H}_\tau}(X_\mu \mathscr{H}_F, X_{\lambda d})) = N_{W, W_{\tau'}}(\mathrm{Hom}_{\mathscr{H}_{\tau'}}(X_\mu \mathscr{H}_F, X_{\lambda d} \otimes_{\mathscr{H}_\tau} \widetilde{T}_z)).$$

Thus, there exists $h \in \mathrm{Hom}_{\mathscr{H}_{\tau'}}(X_\mu \mathscr{H}_F, X_{\lambda d} \otimes_{\mathscr{H}_\tau} \widetilde{T}_z)$ such that

$$N_{W, P_\nu}(e_{\mu\lambda d}) = N_{W, P_\nu^z}(h).$$

Hence,

$$\begin{aligned} N_{W, W_\nu}(e_{\mu\lambda d}) &= \left(\frac{d_{W_\nu}}{d_{P_\nu}}\right)^{-1} N_{W, P_\nu}(e_{\mu\lambda d}) = \left(\frac{d_{W_\nu}}{d_{P_\nu}}\right)^{-1} N_{W, P_\nu^z}(h) \\ &= N_{W, W_\theta}\left(N_{W_\theta, P_\nu^z}\left(\left(\frac{d_{W_\nu}}{d_{P_\nu}}\right)^{-1} h\right)\right), \end{aligned}$$

we have

$$N_{W, W_\nu}(e_{\mu\lambda d}) \in N_{W, W_\theta}(\mathrm{End}_{\mathscr{H}_\theta}(V^{\otimes r})).$$

The proposition is proved. □

Let A be a ring and M a right A-module. Then M is also a right $\mathrm{End}_A(M)$-module in a natural way. If e is an idempotent of $\mathrm{End}_A(M)$ we have obviously,

$$\mathrm{End}_A(Me) = e\,\mathrm{End}_A(M)e. \tag{4.d}$$

Let e be a primitive idempotent of $S_F(n, r)$. Then $V^{\otimes r}e$ is an indecomposable $\mathscr{H}_F$-submodule of $V^{\otimes r}$. Let $D(e)$ denote the defect group of e. We can now prove the following result (see [S, Proposition 2]).

4.8 **Theorem.** *$D(e)$ is the vertex of $V^{\otimes r}e$.*

Proof. Let W_λ be the vertex of $M = V^{\otimes r}e$. Then by 4.1, W_λ is an l-parabolic subgroup of W. By Higman's criterion we have

$$\mathrm{End}_{\mathscr{H}_F}(M) = N_{W,W_\lambda}(\mathrm{End}_{\mathscr{H}_\lambda}(M)).$$

Thus, by 4.7 and (4.d),

$$\begin{aligned} eS_F(n,r)e &= \mathrm{End}_{\mathscr{H}_F}(M) \\ &= N_{W,W_\lambda}(\mathrm{End}_{\mathscr{H}_\lambda}(M)) \\ &= N_{W,W_\lambda}(e\,\mathrm{End}_{\mathscr{H}_\lambda}(V^{\otimes r})e) \\ &= eN_{W,W_\lambda}(\mathrm{End}_{\mathscr{H}_\lambda}(V^{\otimes r}))e \\ &= eI_F(W_\lambda)e. \end{aligned}$$

Therefore, $e \in I_F(W_\lambda)$ and hence, $d_{D(e)} | d_{W_\lambda}$.

On the other hand, we have

$$eS_F(n,r)e = eI_F(D(e))e$$

since $e \in I_F(D(e))$ and $I_F(D(e))$ is an ideal. Similar reasoning as above shows that

$$\mathrm{End}_{\mathscr{H}_F}(M) = N_{W,D(e)}(\mathrm{End}_{\mathscr{H}_{D(e)}}(M)).$$

By Higman's criterion again, we see that M is $\mathscr{H}_{D(e)}$-projective. Therefore, $d_{W_\lambda} | d_{D(e)}$ and hence $d_{W_\lambda} = d_{D(e)}$, as desired. □

4.9 **Corollary.** *Let e_0 be a primitive idempotent of $S_{R_l}(n,r)$ such that $\overline{e_0} = e$. Then $D(e)$ is also a vertex of $V^{\otimes r}e_0$.*

Proof. This follows from the above theorem and 4.2.

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