SYMOMETRIC LOCAL ALGEBRAS WITH 5-DIMENSIONAL CENTER

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Dedicated to Hiroyuki Tachikawa on the occasion of his sixtieth birthday

Abstract. We prove that a symmetric split local algebra whose center is 5-dimensional has dimension 5 or 8. This implies that the defect groups of a block of a finite group containing exactly five irreducible Frobenius characters and exactly one irreducible Brauer character have order 5 or are nonabelian of order 8.

Let $F$ be a field, and let $A$ be a finite-dimensional associative unitary $F$-algebra with center $Z$ and radical $J$. Then $A$ is called split local if $\dim A/J = 1$, and $A$ is called symmetric if there is a linear map $\lambda : A \to F$ whose kernel contains all Lie commutators $[x, y] := xy - yx$ ($x, y \in A$) but no nonzero ideal of $A$. Suppose now that $A$ is symmetric and split local. In [6] the second author proved that $A$ is necessarily commutative if $\dim Z \leq 4$. This incorporated earlier results by R. Brauer and J. Brandt [1]. In this paper we are dealing with the next case.

Theorem. Let $F$ be a field, and let $A$ be a symmetric split local $F$-algebra with center $Z$. If $\dim Z = 5$ then $\dim A \in \{5, 8\}$.

The group algebra of a group of order 5 over a field of characteristic 5 is an example for the case $\dim A = 5$, and the group algebra of a nonabelian group of order 8 over a field of characteristic 2 is an example for the case $\dim A = 8$.

Corollary. Let $F$ be an algebraically closed field, let $G$ be a finite group, let $P$ be an indecomposable projective $FG$-module, and set $A := \text{End}_{FG}(P)$. If the center of $A$ has dimension 5 then $\dim A \in \{5, 8\}$.

Proof. We choose a primitive idempotent $i$ in $FG$ such that $P$ is isomorphic to $iFGi$. Then $A$ is isomorphic to $iFGi$. Since $FG$ is a symmetric $F$-algebra, so are $iFGi$ and $A$. Since $P$ is indecomposable and $F$ is algebraically closed, $A$ is split local. Hence the corollary follows from the theorem.

We have the following application to block theory.

Proposition. Let $F$ be an algebraically closed field, let $G$ be a finite group, and let $B$ be a block of $FG$ containing exactly 5 irreducible complex characters.

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and exactly one irreducible Brauer character. Then the defect groups of $B$ have order 5 or are nonabelian of order 8.

Proof. Let $P$ denote the only indecomposable projective $FG$-module in $B$, and set $A := \text{End}_{FG}(P)$. By Lemma B in [4], $B$ is isomorphic to a complete matrix algebra over $A$; in particular, $A$ and $B$ have isomorphic centers. By (2G) in [2], the dimension of the center of $B$ coincides with the number of irreducible complex characters in $B$, so the center of $A$ has dimension 5. By the corollary, $A$ has dimension 5 or 8. On the other hand, Lemma B in [4] shows that the dimension of $A$ coincides with the order of a defect group $D$ of $B$. Hence $D$ has order 5 or 8. Assume now that $D$ is abelian of order 8. Then $B$ cannot be nilpotent in the sense of [3]; for otherwise $B$ would contain 8 irreducible complex characters by the main result of [3]. Thus $D$ must be elementary abelian. But in this case we obtain a contradiction using the results in [7].

The remainder of this paper consists of a proof of the theorem. Let $A$ be a symmetric split local algebra over a field $F$ and denote by $Z$ the center and by $J$ the radical of $A$. We may and do assume that $F$ is algebraically closed. For a subset $X$ of $A$, we denote by $FX$ the linear subspace of $A$ spanned by $X$. The subspace $K := F\{[x, y] : x, y \in A\}$ will be particularly important for us. Since $A = F1 + J$ we have $K = [J, J] \subset J^2$. We fix a linear map $\lambda : A \to F$ the kernel of which contains $K$ but no nonzero ideal of $A$. Then 0 is the only ideal of $A$ contained in $K$. For any linear subspace $U$ of $A$, $U^\perp := \{a \in A : \lambda(aU) = 0\}$ is a linear subspace of $A$ such that $\dim A = \dim U + \dim U^\perp$ and $(U^\perp)^\perp = U$. We have $Z^\perp = K$ (see [5]); moreover, $I^\perp = \{a \in A : aI = 0\} = \{a \in A : Ia = 0\}$ for any ideal $I$ of $A$; in particular, $I^\perp$ is an ideal of $A$. Furthermore, $\dim J^\perp = \dim A/J = 1$. Hence, if $J^n = 0$ for some positive integer $n$ then $J^{n-1} \subset J^\perp$; in particular, $\dim J^{n-1} \leq \dim J^\perp = 1$. We will often use this fact without special reference.

1. Preliminary results

From now on we suppose that $\dim Z = 5$. We may and will assume that $\dim A \geq 6$; for otherwise we are done.

(1.1) Lemma. We have $\dim A \geq 8$.

Proof. Assume that $\dim A \leq 7$. Then there are elements $a, b \in A$ such that $A = Z + Fa + Fb$. Therefore $K = F[a, b]$; in particular, $\dim K \cap Z \leq \dim K \leq 1$. Now Lemma D in [6] implies that $A$ is commutative, so $\dim A = \dim Z = 5$, a contradiction.

If $\dim A = 8$, then the theorem is proved, so we may and will assume that $\dim A \geq 9$. We are then looking for a contradiction.

(1.2) Lemma. We have $\dim A/K + J^3 = 4$, and one of the following occurs:

(1.3) $\dim J/J^2 = 2$, $\dim J^2/J^3 = 2$, $\dim J^3/J^4 \geq 2$, $\dim J^4/J^5 \geq 1$, $K + J^3 = K + J^4$. 

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(1.4) \( \text{dim} J/J^2 = 3, \quad \text{dim} J^2/J^3 = 2, \quad \text{dim} J^3/J^4 \geq 2, \quad \text{dim} J^4/J^5 \geq 1, \)
(1.5) \( J^2 = K + J^3 = K + J^4; \)
\( \text{dim} J/J^2 = 3, \quad \text{dim} J^2/J^3 = 3, \quad \text{dim} J^3/J^4 \geq 2, \quad \text{dim} J^4/J^5 \geq 1, \)
\( J^2 = K + J^3 = K + J^4. \)

Proof. Since \( \text{dim} J \geq 8 \) we have \( J^2 \neq 0 \). Thus Nakayama’s Lemma implies that \( J^2 \neq J^3 \). Furthermore, \( J \nleq Z \), so \( \text{dim} J^2/J^3 \geq 2 \) by Lemma G in [6]; in particular, \( \text{dim} J/J^2 \geq 2 \) by Lemma E in [6], and \( J^3 \neq 0 \). Hence \( J^3 \neq J^4 \) by Nakayama’s Lemma, and \( J^3 \nleq K \). Thus

\[
\text{dim} A/J^2 \leq \text{dim} A/K + J^3 < \text{dim} A/K = \text{dim} Z = 5;
\]
in particular, \( \text{dim} J/J^2 \in \{2, 3\} \), so \( \text{dim} J^2 \geq 5 \). This means that \( J^2 \nleq Z \) which implies by Lemma G in [6] that \( \text{dim} J^3/J^4 \geq 2 \). Hence \( J^4 \neq 0 \), and \( J^4 \neq J^5 \) by Nakayama’s Lemma again. Moreover, \( J^4 \nleq K \), so \( \text{dim} A/K + J^3 \leq \text{dim} A/K + J^4 < \text{dim} A/K = 5 \).

Suppose first that \( \text{dim} J/J^2 = 2 \) and write \( J = Fa + Fb + J^2 \) with elements \( a, b \in J \). Then \( A = F\{1, a, b\} + J^2 \) and \( K \subset F[a, b] + J^3 \); in particular, \( \text{dim} K + J^3/J^3 \leq 1 \), so \( \text{dim} A/J^3 \leq 5 \) and \( \text{dim} J^2/J^3 = 2 \). Thus \( \text{dim} A/J^3 = 5 \) and \( \text{dim} A/K + J^3 = 4 \). Hence also \( \text{dim} A/K + J^4 = 4 \).

Finally, suppose that \( \text{dim} J/J^2 = 3 \) and write \( J = Fa + Fb + J^2 \) with elements \( a, b, c \in J \). Then \( K \subset F[[a, b], [a, c], [b, c]] + J^3 \); in particular, \( \text{dim} K + J^3/J^3 \leq 3 \). Thus \( \text{dim} A/J^3 \leq 7 \) and \( \text{dim} J^2/J^3 \in \{2, 3\} \). Since \( 4 = \text{dim} A/J^2 \leq \text{dim} A/K + J^3 \leq \text{dim} A/K + J^4 \leq 4 \) the result follows.

We will deal with these cases in §§2, 3 and 4, respectively. The following results will be useful later on.

(1.6) Lemma. There is an element \( x \in J \) such that \( x^2 \notin J^3 \).

Proof. By (1.2) we have \( \text{dim} J/J^2 \leq 3 \). We write \( J = F\{a, b, c\} + J^2 \) with elements \( a, b, c \in J \). If \( x^2 \in J^3 \) for \( x \in J \) then \( ab + ba = (a+b)^2 - a^2 - b^2 \in J^3 \). Thus \( ba \equiv -ab \pmod{J^3} \). Similarly, \( ca \equiv -ac \pmod{J^3} \) and \( cb \equiv -bc \pmod{J^3} \).

Therefore

\[
J^2 = F\{a^2, ab, ac, ba, b^2, bc, ca, cb, c^2\} + J^3 = F\{ab, ac, bc\} + J^3;
\]
in particular, \( \text{dim} J^2/J^3 \leq 3 \). Now we apply Lemma E in [6] to obtain

\[
J^3 = F\{a^2b, a^2c, abc, bab, bac, b^2c\} + J^4 = Fabc + J^4
\]
and \( J^4 = Fa^2bc + J^5 = J^5 \) contradicting (1.2).

(1.7) Lemma. There are elements \( a, b \in J \) such that \( a^2 + J^3, ab + J^3 \) or \( a^2 + J^3, ba + J^3 \) are linearly independent in \( J^2/J^3 \).

Proof. By (1.6), there is an element \( a \in J \) such that \( a^2 \notin J^3 \); in particular, \( a \notin J^2 \). By (1.2) there are therefore elements \( b, c \in J \) such that \( J = F\{a, b, c\} + J^2 \). We may assume that \( ab, ba, ac, ca \in Fa^2 + J^3 \); for otherwise the result is proved. Then \( K + J^3 = F[[a, b], [a, c], [b, c]] + J^3 \subset F[a^2, [b, c]] + J^3 \); in particular, \( \text{dim} K + J^3/J^3 \leq 2 \). Hence, by (1.2), \( \text{dim} J^2/J^3 = 2 \).

Now consider the case where \( b^2 \notin Fa^2 + J^3 \); in particular, \( b^2 \notin J^3 \). Then we can interchange the roles of \( a \) and \( b \) and therefore assume that
ab, ba, bc, cb \in Fb^2 + J^3. Since a^2 + J^3 and b^2 + J^3 form a basis of 
J^2/J^3 this implies that ab, ba \in J^3. Thus (a + b)^2 + J^3 = a^2 + b^2 + J^3 and 
(a + b)b + J^3 = b^2 + J^3 are linearly independent, and the result follows in this 
case.

Therefore we may also assume that b^2 \in Fa^2 + J^3 and, similarly, c^2 \in 
Fa^2 + J^3. Then

\[ J^2 = F\{a^2, ab, ac, ba, b^2, bc, ca, cb, c^2\} + J^3 = F\{a^2, bc, cb\} + J^3. \]

Thus J^2 = F\{a^2, bc\} + J^3 or J^2 = F\{a^2, cb\} + J^3; we may assume that 
J^2 = F\{a^2, bc\} + J^3. Then Lemma E in [6] implies that

\[ J^3 = F\{a^3, abc, ba^2, b^2c\} + J^4 = F\{a^3, a^2c\} + J^4 = Fa^3 + J^4; \]
in particular, dim J^3/J^4 \leq 1 contradicting (1.2).

We now choose elements a, b \in J as in (1.7). By symmetry we may assume 
that ab \notin Fa^2 + J^3; in particular, a \notin J^2 and b \notin Fa + J^2. Thus a + 
J^2, b + J^2 are linearly independent in J/J^2. By (1.2), we can find an element 
c \in J such that J = F\{a, b, c\} + J^2.

2. THE CASE (1.3)

In this section we use the same hypothesis and notation as before, but we 
assume in addition that (1.3) holds. Then J = Fa + Fb + J^2 and J^2 = 
Fa^2 + Fab + J^3. Thus Lemma E in [6] implies that J^3 = Fa^3 + Fa^2b + J^4, 
J^4 = Fa^4 + Fa^3b + J^5 and J^5 = Fa^5 + Fa^4b + J^6; in particular, dim J^3/J^4 = 2.
Thus a^3 + J^4 and a^2b + J^4 form a basis of J^3/J^4. Moreover, dim J^4 \geq 2; 
in particular, J^5 \neq 0. Thus J^5 \not\subset K and 4 = dim A/K + J^4 \leq dim A/K + 
J^5 < dim A/K = dim Z = 5 by (1.2). We conclude that dim A/K + J^5 = 4.
Furthermore, A = F\{1, a, b, a^2, ab, a^3, a^2b\} + J^4, so

\[ K \subset F\{[a, b], [a, ab], [a, a^2b], [b, a^2], 
[b, ab], [b, a^3], [b, a^2b], [a^2, ab]\} + J^5. \]

Since J^2 = F\{a^2, ab, a^3, a^2b\} + J^4 there are elements \( \alpha_i, \beta_i, \gamma_i, \delta_i \in F \)
(i = 1, 2) such that

\[ ba \equiv \alpha_1a^2 + \beta_1ab + \gamma_1a^3 + \delta_1a^2b \pmod{J^4}, \]

\[ b^2 \equiv \alpha_2a^2 + \beta_2ab + \gamma_2a^3 + \delta_2a^2b \pmod{J^4}. \]

We have to distinguish between two cases.

Case 1. \( \beta_1 \neq 1 \). In this case we set \( \xi := \alpha_1/(1 - \beta_1) \) and \( b' := b - \xi a \). Then 
J = Fa + Fb' + J^2, J^2 = Fa^2 + Fab' + J^3 and

\[ b'a \equiv ba - \xi a^2 \equiv (\alpha_1 - \xi)a^2 + \beta_1ab \]
\[ \equiv (\alpha_1 - \xi + \beta_1 \xi)a^2 + \beta_1ab' \equiv \beta_1ab' \pmod{J^3}. \]

Thus we may replace b by b' and therefore assume that \( \alpha_1 = 0 \). Then

\[ 0 \equiv (b^2)a - b(ba) \equiv \alpha_2a^3 + \beta_2aba - \beta_1bab \equiv \alpha_2a^3 + \beta_1\beta_2a^2b - \beta_1^2ab^2 \]
\[ \equiv (\alpha_2 - \alpha_2\beta_1^2)a^3 + (\beta_1\beta_2 - \beta_1^2\beta_2)a^2b \pmod{J^4}. \]
and, similarly,

\[ 0 \equiv (b^2)b - b(b^2) \equiv (\alpha_2\beta_2 - \alpha_2\beta_1\beta_2)a^3 + (\alpha_2 + \beta_2^2 - \alpha_2\beta_1^2 - \beta_1\beta_2^2)a^2b \pmod{J^4}. \]

Since \( a^3 + J^4 \) and \( a^2b + J^4 \) form a basis of \( J^3/J^4 \) we conclude that

\begin{align*}
(2.1) \quad 0 &= \alpha_2 - \alpha_2\beta_1^2, \\
(2.2) \quad 0 &= \beta_1\beta_2 - \beta_2^3\beta_2, \\
(2.3) \quad 0 &= \alpha_2\beta_2 - \alpha_2\beta_1\beta_2, \\
(2.4) \quad 0 &= \alpha_2 + \beta_2^2 - \alpha_2\beta_1^2 - \beta_1\beta_2^2.
\end{align*}

Subtracting (2.1) from (2.4) we obtain \( \beta_2^2 = \beta_1\beta_2^2 \). Since \( \beta_1 \neq 1 \) this implies \( \beta_2 = 0 \). From (2.1) we also conclude that \( \alpha_2 = 0 \) or \( \beta_1^2 = 1 \). We assume first that \( \alpha_2 = 0 \). Then

\begin{align*}
[a, ab] &= a^2b - aba \equiv (1 - \beta_1)a^2b \pmod{J^4}, \\
[b, a^2] &= ba^2 - a^2b \equiv \beta_1aba - a^2b \equiv (\beta_1^2 - 1)a^2b \pmod{J^4}, \\
[b, ab] &= bab - ab^2 \equiv \beta_1ab^2 \equiv 0 \pmod{J^4}.
\end{align*}

This shows that \( K \subset F[a, b] + Fa^2b + J^4 \); in particular, \( \dim K + J^4/J^4 \leq 2 \). Thus \( \dim A/J^4 \leq 6 \) by (1.2), a contradiction.

Hence we must have \( \alpha_2 \neq 0 \) and \( \beta_1^2 = 1 \). Since \( \beta_1 \neq 1 \) this implies \( \beta_1 = -1 \) and \( \text{char} \ F \neq 2 \). It is now easy to check that

\begin{align*}
[a, a^2b] &= 2a^3b \pmod{J^5}, \\
[b, a^2a] &= -2\delta_1a^3b \pmod{J^5}, \\
[b, a^3] &= -2a^3b \pmod{J^5}, \\
[b, a^2b] &= [a^2, ab] \equiv 0 \pmod{J^5}.
\end{align*}

Thus \( K \subset F\{[a, b], [a, ab], [b, ab], a^3b\} + J^5 \); in particular, \( \dim K + J^5/J^5 \leq 4 \). Hence \( \dim A/J^5 \leq 8 \) and \( \dim J^4/J^5 = 1 \). By Lemma G in [6], this implies that \( J^3 \subset Z \); in particular, \( a^2b \in Z \). Thus \( a^3b \equiv a^2ba \equiv -a^3b \pmod{J^5} \). Since \( \text{char} \ F \neq 2 \) this implies \( a^3b \in J^5 \). Therefore

\[ K \subset F\{[a, b], [a, ab], [b, ab]\} + J^5; \]

in particular, \( \dim K + J^5/J^5 \leq 3 \). Hence \( \dim A/J^5 \leq 7 \), a contradiction.

Case 2. \( \beta_1 = 1 \). Assume first that \( \alpha_1 = 0 \). Then \( \{a, b\} \in J^3 \) and \( K \subset J^3 \), so \( \dim A/K + J^3 = \dim A/J^3 = 5 \) contradicting (1.2). Thus we must have \( \alpha_1 \neq 0 \). Now we set \( a' := \alpha_1a \). Then \( J = F(a') + Fb + J^2 \), \( J^2 = F(a')^2 + Fb^2 + J^3 \) and

\[ ba' \equiv \alpha_1ba \equiv \alpha_1^2a^2 + \alpha_1ab \equiv (a')^2 + a'b \pmod{J^3}. \]

Hence we may replace \( a \) by \( a' \) and therefore assume that \( \alpha_1 = 1 \). As in Case 1 we compute

\[ 0 \equiv (b^2)a - b(ba) \equiv (\beta_2 - 2)a^3 - 2a^2b \pmod{J^4}. \]

Since \( a^3 + J^4 \) and \( a^2b + J^4 \) form a basis of \( J^3/J^4 \) this implies that \( \text{char} \ F = 2 \) and \( \beta_2 = 0 \). Hence

\begin{align*}
[a, a^2b] &= a^4 \pmod{J^5}, \\
[b, a^2] &= \delta_1a^4 \pmod{J^5}, \\
[b, a^3] &= a^4 \pmod{J^5}, \\
[b, a^2b] &= [a^2, ab] \equiv 0 \pmod{J^5}.
\end{align*}

Therefore \( K \subset F\{[a, b], [a, ab], [b, ab], a^4\} + J^5 \); in particular, \( \dim K + J^5/J^5 \leq 4 \). Hence \( \dim A/J^5 \leq 8 \) and \( \dim J^4/J^5 = 1 \). By Lemma G in [6], this implies that \( J^3 \subset Z \); in particular, \( a^2b \in Z \). Thus \( a^3b = a^2ba \equiv a^4 + a^2b \)
Therefore \(a^4 \in J^5\) and \(J^5 = F a^5 + F a^4 b + J^6 = J^6\). Hence \(J^5 = 0\) by Nakayama's Lemma, a contradiction.

3. The case (1.4)

In this section we assume hypothesis and notation from §1. In addition, we assume that (1.4) holds. Then \(J^2 = F a^2 + F a b + J^3\) and \(J^3 = F a^3 + F a^2 b + J^4\) by Lemma E in [6]; in particular, \(\dim J^3 / J^4 = 2\). Hence \(a^3 + J^4, a^2 b + J^4\) form a basis of \(J^3 / J^4\). There are elements \(\alpha, \beta \in F\) such that \(ac \equiv \alpha a^2 + \beta ab \pmod{Z^3}\). Setting \(c' := c - \alpha a - \beta b\) we then have \(J = F \{a, b, c'\} + J^2\) and \(ac' \equiv ac - \alpha a^2 - \beta ab \equiv 0 \pmod{J^3}\). Hence we may replace \(c\) by \(c'\) and therefore assume that \(ac \in J^3\). We choose elements \(\alpha_i, \beta_i \in F\) \((i = 1, 2, 3, 4)\) such that

\[
bc \equiv \alpha_1 a^2 + \beta_1 ab \pmod{J^3}, \quad ca \equiv \alpha_2 a^2 + \beta_2 ab \pmod{J^3},
\]

\[
\text{and } cb \equiv \alpha_3 a^2 + \beta_3 ab \pmod{J^3}, \quad c^2 \equiv \alpha_4 a^2 + \beta_4 ab \pmod{J^3}.
\]

Then

\[
0 \equiv (ac)a = a(ca) = a^2 a^3 + \beta_2 a^2 b \pmod{J^4},
\]

\[
0 \equiv (ac)b = a(cb) = a^3 a^3 + \beta_3 a^2 b \pmod{J^4},
\]

\[
0 \equiv (ac)c = a(c^2) = a^4 a^3 + \beta_4 a^2 b \pmod{J^4}.
\]

Hence \(a^2 = \beta_2 = a_3 = \beta_3 = a_4 = \beta_4 = 0\); in particular, \(ca, cb, c^2 \in J^3\). Thus

\[
0 \equiv b(c^2) \equiv (bc)c \equiv \alpha_1 a^2 c + \beta_1 abc \equiv \alpha_1 \beta_1 a^3 + \beta_1^2 a^2 b \pmod{J^4},
\]

and we obtain \(\beta_1 = 0\). Thus \(0 \equiv b(cb) \equiv (bc)b \equiv \alpha_1 a^2 b \pmod{J^4}\). Therefore \(\alpha_1 = 0\); in particular, \(bc \in J^3\). Thus \([a, c], [b, c] \in J^3\) and \(K \subset F[[a, b], [a, c], [b, c]] + J^3 \subset F[a, b] + J^3\); in particular, \(\dim K + J^3 / J^3 \leq 1\). Thus \(\dim A / J^3 \leq 5\) by (1.2), a contradiction.

4. The case (1.5)

In this section we assume hypothesis and notation from §1. In addition, we assume that (1.5) holds. Since \(J = F \{a, b, c\}\) we have \(J^2 = F \{a^2, ab, ac, ba, b^2, bc, ca, cb, c^2\} + J^3\). Since \(J^2 / J^3 = 3\) we must have \(J^2 = F \{a^2, ab, d\} + J^3\) for some element \(d \in \{ac, ba, b^2, bc, ca, cb, c^2\}\). Since \(J^2 = K + J^4\) we obtain

\[
J^2 = F \{a, b, [a, c], [a, ab], [a, d], [b, c],
\]

\[
[b, a^2], [b, ab], [b, d], [c, a^2], [c, ab], [c, d]\} + J^4.
\]

We choose elements \(\alpha_i, \beta_i, \gamma_i \in F\) \((i = 1, 2, \ldots, 7)\) such that

\[
ac \equiv \alpha_1 a^2 + \beta_1 ab + \gamma_1 d \pmod{J^3}, \quad ba \equiv \alpha_2 a^2 + \beta_2 ab + \gamma_2 d \pmod{J^3},
\]

\[
b^2 \equiv \alpha_3 a^2 + \beta_3 ab + \gamma_3 d \pmod{J^3}, \quad bc \equiv \alpha_4 a^2 + \beta_4 ab + \gamma_4 d \pmod{J^3},
\]

\[
ca \equiv \alpha_5 a^2 + \beta_5 ab + \gamma_5 d \pmod{J^3}, \quad cb \equiv \alpha_6 a^2 + \beta_6 ab + \gamma_6 d \pmod{J^3},
\]

\[
c^2 \equiv \alpha_7 a^2 + \beta_7 ab + \gamma_7 d \pmod{J^3}.
\]

(4.1) Lemma. We may assume that \(d = ac\) or \(d = ba\).

Proof. Case 1. \(d = ac\). In this case there is nothing to prove.
Case 2. \( d = ba \). In this case there is nothing to prove either.

Case 3. \( d = b^2 \). In this case we may assume that \( ba \in Fa^2 + Fab + J^3 \); for otherwise we are in Case 2. Similarly, we may assume that \( ba \in Fb^2 + Fab + J^3 \); for otherwise we interchange \( a \) and \( b \) and are in Case 2 again. Hence \( ba \in Fab + J^3 \), and we may write \( ba \equiv \alpha ab \mod J^3 \) for some element \( \alpha \in F \).

Now we set \( b' := a + b \). Then we have

\[
ab' = a^2 + ab, \quad (b')^2 \equiv a^2 + (1 + \alpha)ab + b^2 \mod J^3 ;
\]

in particular, \( J = F\{a, b', c\} + J^2 \) and \( J^2 = F\{a^2, ab', (b')^2\} + J^3 \). Hence we may similarly assume that \( b'a \in Fab' + J^3 \). We write \( b'a \equiv \beta ab' \mod J^3 \) with some element \( \beta \in F \). Then

\[
\beta a^2 + \beta ab \equiv \beta ab' \equiv b'a \equiv (a + b)a \equiv a^2 + ba \equiv a^2 + \alpha ab \mod J^3 .
\]

Since \( a^2 + J^3 \) and \( ab + J^3 \) are linearly independent this means that \( \alpha = \beta = 1 \); in particular, \( [a, b] \in J^3 \), and \( J^2 = F[a, c] + F[b, c] + J^3 \) contradicting the fact that \( \dim J^2/J^3 = 3 \).

Case 4. \( d = bc \). In this case we may assume that \( ac, ba, b^2 \in Fa^2 + Fab + J^3 \); for otherwise we are in Cases 1, 2 or 3 again. Then we replace \( c \) by \( c - \alpha a - \beta b \) and may therefore assume that \( \alpha = \beta = 1 \). Moreover, we may assume that

\[
J^2 \neq (a + \xi b)J + J^3 = F\{(a + \xi b)a, (a + \xi b)b, (a + \xi b)c\} + J^3
\]

\[
= F\{(1 + \alpha_2 \xi)a^2 + \beta_2 \xi ab, \alpha_3 \xi^2 a^2 + (1 + \beta_3 \xi)ab, \xi bc\} + J^3
\]

for \( \xi \in F \); for otherwise we replace \( a \) by \( a + \xi b \) and are in Case 1. Since \( a^2 + J^3, ab + J^3, bc + J^3 \) form a basis of \( J^2/J^3 \) this implies that

\[
0 = \begin{vmatrix}
1 + \alpha_2 \xi & \beta_2 \xi & 0 \\
\alpha_3 \xi & 1 + \beta_3 \xi & 0 \\
0 & 0 & \xi
\end{vmatrix} = \xi + (\alpha_2 + \beta_3)\xi^2 + (\alpha_2 \beta_3 - \alpha_3 \beta_2)\xi^3
\]

for \( \xi \in F \). Since \( F \) is infinite this is impossible.

Case 5. \( d = ca \), i.e., \( \alpha_5 = \beta_5 = 0, \gamma_5 = 1 \). We may assume that \( \gamma_i = 0 \) for \( i = 1, 2, 3, 4 \); for otherwise we are in Cases 1, 2, 3, 4, respectively. Then we replace \( c \) by \( c - \alpha a - \beta b \) and may therefore assume that \( \alpha = \beta = 1 \). Moreover, \( \beta_2 = 0 \); for otherwise we are in Case 1. Similarly, we may assume that \( \alpha_3 = 0 \); for otherwise we interchange \( a \) and \( b \) and are then in Case 4 for the opposite algebra of \( A \). Now we replace \( b \) by \( b - \gamma_6 a \) and may then assume that \( \gamma_6 = 0 \). Furthermore, we may assume that \( \alpha_7 = 0 \); for otherwise we replace \( (a, b, c) \) by \( (b, c, a) \) and are then in Case 4 again. Finally, we may assume that \( \beta_7 = 0 \); for otherwise we interchange \( b \) and \( c \) and are then in Case 3 for the opposite algebra of \( A \). As in Case 4, we may assume

\[
J^2 \neq F\{((\xi a + \eta b + c)a, (\xi a + \eta b + c)b, (\xi a + \eta b + c)c) + J^3
\]

\[
= F\{(\xi + \alpha_2 \eta)a^2 + ca, \alpha_6 a^2 + (\xi + \beta_3 \eta + \beta_6)ab, \alpha_4 \eta a^2 + \beta_4 \eta ab + \gamma_7 ca\} + J^3
\]

for \( \xi, \eta \in F \). Since \( a^2 + J^3, ab + J^3, ca + J^3 \) form a basis of \( J^2/J^3 \) we may compute the corresponding determinant and obtain

\[
0 = \gamma_7 \xi^2 + (\beta_3 \gamma_7 + \alpha_2 \gamma_7 - \alpha_4)\xi \eta + \beta_6 \gamma_7 \xi + (\alpha_2 \beta_3 \gamma_7 - \alpha_4 \beta_3)\eta^2 + (\alpha_2 \beta_6 \gamma_7 + \alpha_6 \beta_4 - \alpha_4 \beta_6)\eta
\]
for $\xi, \eta \in F$. Since $F$ is infinite this implies that all coefficients on the right-hand side vanish; in particular, $0 = \gamma_7 = \alpha_4$. Then, similarly, we may assume that

$$J^2 \neq F \{a(a + \eta b + c), b(a + \eta b + c), c(a + \eta b + c)\} + J^3$$

$$= F \{a^2 + \eta ab, \alpha_2 a^2 + (\beta_3 \eta + \beta_4)ab, \alpha_6 \eta a^2 + \beta_6 \eta ab + ca\} + J^3$$

for $\eta \in F$. Computing the corresponding determinant we obtain $0 = (\beta_3 - \alpha_2)\eta + \beta_4$ for $\eta \in F$. As before this implies that $\beta_3 = \alpha_2$ and $\beta_4 = 0$. Finally, we may assume that

$$J^2 \neq F \{a(a + \eta b + c), b(a + \eta b + c), c(a + \eta b + c)\} + J^3$$

$$= F \{(\xi + \alpha_2 \eta + \alpha_5) a^2 + (\xi + \beta_3 \eta)ab + cb, \alpha_4 \eta + \alpha_7 a^2 + (\beta_4 \eta + \beta_7)ab + \gamma_7 cb\} + J^3$$

for $\xi, \eta \in F$. We work out the corresponding determinant and obtain

$$0 = \gamma_7 \xi^2 + (\beta_3 \gamma_7 + \alpha_2 \gamma_7 - \beta_4)\xi \eta + (\alpha_5 \gamma_7 - \beta_7)\xi + (\alpha_2 \beta_3 \gamma_7 - \alpha_2 \beta_4) \eta^2$$

$$+ (\alpha_5 \beta_3 \gamma_7 - \alpha_2 \beta_7 - \alpha_5 \beta_4 + \alpha_4 \beta_5) \eta + (\alpha_7 \beta_5 - \alpha_5 \beta_7)$$

for $\xi, \eta \in F$. Therefore all coefficients on the right-hand side vanish; in particular, $0 = \gamma_7 = \beta_4 = \beta_7$. Similarly, we have

$$J^2 \neq F \{a(a + b + c), b(a + b + c), c(a + b + c)\} + J^3$$

$$= F \{a^2 + ab, (\alpha_2 \xi + \alpha_4) a^2 + \beta_3 ab, (\alpha_5 \xi + \alpha_7) a^2 + \beta_5 \xi ab + cb\} + J^3$$

for $\xi \in F$. Computing the corresponding determinant we obtain $0 = (\beta_3 - \alpha_2)\xi - \alpha_4$ for $\xi \in F$ which again implies that $\beta_3 = \alpha_2$ and $\alpha_4 = 0$. We may also assume that

$$J^2 \neq F \{(\xi a + \eta b + c)^2, (\xi a + \eta b + c) a, a(\xi a + \eta b + c)\} + J^3$$

$$= F \{(\xi^2 + \alpha_2 \xi \eta + \alpha_5 \xi + \alpha_7) a^2 + (\xi \eta + \beta_5 \xi + \alpha_2 \eta^2) ab + \eta cb, \xi + \alpha_2 \eta + \alpha_5 a^2 + \beta_5 ab, \xi^2 + \eta ab\} + J^3$$

for $\xi, \eta \in F$; for otherwise we replace $(a, b, c)$ by $(\xi a + \eta b + c, a, b)$ and are then in Case 2 again. Working out the corresponding determinant we obtain $0 = \xi \eta^2 + \alpha_2 \eta^3 - \beta_5 \xi \eta + \alpha_5 \eta^2$ for $\xi, \eta \in F$ which is impossible.
Case 7. \( d = c^2 \). In this case we may assume that \( ac, ba, b^2, bc, ca, cb \in Fa^2 + Fab + J^3 \); for otherwise we are in Cases 1, 2, \ldots, 6, respectively. Then \( J^2 = F\{(a, b), [a, c], [b, c]\} + J^3 \subset Fa^2 + Fab + J^3 \); in particular, \( \dim J^2/J^3 \leq 2 \) contradicting (1.5).

(4.2) **Lemma.** We may assume that \( d = ac \).

**Proof.** We assume the contrary. Then we may assume that \( d = ba \), by (4.1); in particular, \( \alpha_2 = \beta_2 = 0, \gamma_2 = 1 \). We have \( \gamma_1 = 0 \). After replacing \( c \) by \( c - \alpha_1a - \beta_1b \) we may even assume \( 0 = \alpha_1 = \beta_1 \). Similarly, we may assume \( \beta_5 = 0 \). Moreover, after replacing \( b \) by \( b - \gamma_3a \) we may also assume that \( \gamma_3 = 0 \). We then have

\[
J^2 \neq (\xi a + \eta b + \zeta c)J + J^3
\]

\[
= F\{(\xi a + \eta b + \zeta c)a, (\xi a + \eta b + \zeta c)b, (\xi a + \eta b + \zeta c)c\} + J^3
\]

\[
= F\{(\xi + \alpha_5\zeta)a^2 + (\eta + \gamma_5\zeta)ba, (\alpha_3\eta + \alpha_6\zeta)a^2 + (\xi + \beta_3\eta + \beta_6\zeta)ab
\]

\[+ \gamma_6\zeta ba, (\alpha_4\eta + \alpha_7\zeta)a^2 + (\beta_4\eta + \beta_7\zeta)ab + (\gamma_4\eta + \gamma_7\zeta)ba\} + J^3
\]

for \( \xi, \eta, \zeta \in F \). Since \( a^2 + J^3, ab + J^3, ba + J^3 \) form a basis of \( J^2/J^3 \) this implies that

\[
0 = \begin{vmatrix}
\xi + \alpha_5\zeta & 0 & \eta + \gamma_5\zeta \\
\alpha_3\eta + \alpha_6\zeta & \xi + \beta_3\eta + \beta_6\zeta & \gamma_6\zeta \\
\alpha_4\eta + \alpha_7\zeta & \beta_4\eta + \beta_7\zeta & \gamma_4\eta + \gamma_7\zeta
\end{vmatrix}
\]

\[
= \gamma_4\zeta^2\eta + \gamma_7\zeta^2\xi + (\beta_3\gamma_4 - \alpha_4)\zeta\eta^2
\]

\[
+ (\beta_3\gamma_5 + \beta_6\gamma_4 - \alpha_5\gamma_4 - \beta_4\gamma_6 - \alpha_7 - \alpha_4\gamma_5)\xi\eta
\]

\[
+ (\beta_6\gamma_7 + \alpha_5\gamma_7 - \beta_7\gamma_6 - \alpha_7\gamma_5)\zeta^2 + (\alpha_3\beta_4 - \alpha_4\beta_3)\eta^3
\]

\[
+ (\alpha_5\beta_3\gamma_4 + \alpha_3\beta_7 + \alpha_6\beta_4 + \alpha_3\beta_4\gamma_5 - \alpha_6\beta_6 - \alpha_7\beta_3 - \alpha_4\beta_3\gamma_5)\eta^2\zeta
\]

\[
+ (\alpha_5\beta_3\gamma_7 + \alpha_5\beta_6\gamma_4 - \alpha_5\beta_4\gamma_6 + \alpha_5\beta_7 + \alpha_3\beta_7\gamma_5 + \alpha_6\beta_4\gamma_5
\]

\[
- \alpha_7\beta_6 - \alpha_4\beta_6\gamma_5 - \alpha_7\beta_3\gamma_5)\eta\zeta^2
\]

\[
+ (\alpha_5\beta_6\gamma_7 - \alpha_5\beta_7\gamma_6 + \alpha_6\beta_7\gamma_5 - \alpha_7\beta_6\gamma_5)\zeta^3
\]

for \( \xi, \eta, \zeta \in F \). Since \( F \) is infinite this implies that all coefficients on the right-hand side have to vanish; in particular, \( 0 = \gamma_4 = \gamma_7 = \alpha_4 = \alpha_3\beta_4 \) and \( \alpha_7 = -\beta_4\gamma_6 \). Then, similarly, we have

\[
J^2 \neq F\{a(\xi a + \eta b + \zeta c), b(\xi a + \eta b + \zeta c), c(\xi a + \eta b + \zeta c)\} + J^3
\]

\[
= F\{\xi a^2 + \eta ab, \alpha_3\eta a^2 + (\beta_3\eta + \beta_4\zeta)ab + \xi ba,
\]

\[
(\alpha_5\xi + \alpha_6\eta + \alpha_7\zeta)a^2 + (\beta_6\eta + \beta_7\zeta)ab + (\gamma_5\xi + \gamma_6\eta)ba\} + J^3
\]

for \( \xi, \eta, \zeta \in F \). As before, we work out the corresponding determinant and obtain

\[
0 = (\beta_3\gamma_5 - \beta_6 + \alpha_5)\zeta^2\eta + (\beta_4\gamma_5 - \beta_7)\xi^2\zeta + (\beta_3\gamma_6 - \alpha_3\gamma_5 + \alpha_6)\eta^2 - \alpha_3\gamma_6\eta^3
\]

for \( \xi, \eta, \zeta \in F \). Again, this implies that \( \beta_6 = \beta_3\gamma_5 + \alpha_5, \beta_7 = \beta_4\gamma_5, \alpha_6 = \alpha_3\gamma_5 - \beta_3\gamma_6, 0 = \alpha_3\gamma_6 \). On the other hand,

\[
J^2 = F\{(a, b), [a, c], [b, c]\} + J^3
\]

\[
= F\{ab - ba, \alpha_5a^2 + \gamma_5ba, (\beta_3\gamma_6 - \alpha_3\gamma_5)a^2
\]

\[+ (\beta_4 - \alpha_5 - \beta_3\gamma_5)ab - \gamma_6ba\} + J^3.
\]
Since \( a^2 + J^3, \ ab + J^3 \) and \( ba + J^3 \) form a basis of \( J^2/J^3 \) a computation of the corresponding determinant yields

\[
0 \neq \alpha_5\gamma_6 + \beta_3\gamma_5\gamma_6 - \alpha_3\gamma_5^2 - \alpha_5\beta_4 + \alpha_2^2 + \alpha_5\beta_3\gamma_5.
\]

Moreover, since \( J^2 = F\{a^2, \ ab, \ ba\} + J^3 \), Lemma E in [6] implies that

\[
J^3 = F\{a^3, \ a^2b, \ aba, \ ba^2, \ bab, \ b^2a\} + J^4 = F\{a^3, \ a^2b, \ aba, \ ba^2, \ bab\} + J^4.
\]

Now we distinguish two cases.

Case 1. \( \alpha_5 \neq 0 \). In this case we replace \( a \) by \( \alpha_5a \) and may then assume that \( \alpha_5 = 1 \). Thus

\[
0 \equiv a(ca) - (ac)a \equiv a^3 + \gamma_5aba \pmod{J^4},
\]

\[
0 \equiv b(ca) - (bc)a \equiv (\beta_3\gamma_5 - \alpha_3\gamma_5^2 - \beta_4)aba + ba^2 \pmod{J^4}.
\]

Now we distinguish two more cases.

Case 1.1. \( \beta_4 \neq 0 \). In this case we have \( \alpha_3 = 0 \) since \( 0 = \alpha_3\beta_4 \). Moreover,

\[
0 \equiv (b^2)c - b(bc) \equiv \beta_3\beta_4a^2b - \beta_4bab \pmod{J^4},
\]

\[
0 \equiv a(c^2) - (ac)c \equiv \beta_4\gamma_5a^2b + \beta_4\gamma_5\gamma_6aba \pmod{J^4};
\]

in particular, \( J^3 = F\{a^2b, \ Faba + J^4 \} \). Hence \( a^2b + J^4 \) and \( aba + J^4 \) are linearly independent. Then \( \gamma_5 = 0 \), and we obtain the contradiction

\[
0 \equiv a(cb) - (ac)b \equiv a^2b + \gamma_6aba \pmod{J^4}.
\]

Case 1.2. \( \beta_4 = 0 \). Here we have to distinguish two more cases.

Case 1.2.1. \( \beta_3 \neq 0 \). In this case we replace \( b \) by \( \beta_3^{-1}b \) and may then assume that \( \beta_3 = 1 \). Then

\[
0 \equiv (b^2)b - b(b^2) \equiv (1 + \alpha_3)a^2b - \alpha_3^2\gamma_5aba - bab \pmod{J^4},
\]

\[
0 \equiv a(cb) - (ac)b \equiv (1 + \gamma_5)a^2b + (\gamma_5\gamma_6 - \alpha_3\gamma_5^2 + \gamma_6)aba \pmod{J^4};
\]

in particular, \( J^3 = F\{a^2b + Faba + J^4 \} \). Thus \( a^2b + J^4 \) and \( aba + J^4 \) are linearly independent. Then \( \gamma_5 = -1 \) and \( \alpha_3 = 0 \). But now we obtain the contradiction

\[
\alpha_5\gamma_6 + \beta_3\gamma_5\gamma_6 - \alpha_3\gamma_5^2 - \alpha_5\beta_4 + \alpha_2^2 + \alpha_5\beta_3\gamma_5 = 0.
\]

Case 1.2.2. \( \beta_3 = 0 \). Here we have

\[
0 \equiv a(cb) - (ac)b \equiv a^2b + (\gamma_6 - \alpha_3\gamma_5^2)aba \pmod{J^4},
\]

\[
0 \equiv b(cb) - (bc)b \equiv \alpha_3^2\gamma_5aba + bab \pmod{J^4};
\]

in particular, \( J^3 = F\{aba + J^4 \}, \) a contradiction.

Case 2. \( \alpha_5 = 0 \). Then

\[
0 \neq \alpha_5\gamma_6 + \beta_3\gamma_5\gamma_6 - \alpha_3\gamma_5^2 - \alpha_5\beta_4 + \alpha_2^2 + \alpha_5\beta_3\gamma_5 = \beta_3\gamma_5\gamma_6 - \alpha_3\gamma_5^2;
\]

in particular, \( \gamma_5 \neq 0 \). Now we replace \( b \) by \( \gamma_5b \) and may therefore assume that \( \gamma_5 = 1 \). Hence

\[
0 \equiv a(ca) - (ac)a \equiv aba \pmod{J^4}.
\]

We distinguish two more cases.
Case 2.1. $\alpha_3 \neq 0$. In this case $\beta_4 = \gamma_6 = 0$ since $0 = \alpha_3 \beta_4 = \alpha_3 \gamma_6$. We now replace $a$ by $\sqrt{\alpha_3}a$ and may therefore assume that $\alpha_3 = 1$. Then

\begin{align*}
0 & \equiv b(ca) - (bc)a \equiv a^3 \quad (\text{mod } J^4), \\
0 & \equiv b(cb) - (bc)b \equiv ba^2 + \beta_3 bab \quad (\text{mod } J^4), \\
0 & \equiv a(cb) - (ac)b \equiv \beta_3 a^2 b \quad (\text{mod } J^4), \\
0 & \equiv (b^2)b - b(b^2) \equiv a^2 b \quad (\text{mod } J^4);
\end{align*}

in particular, $J^3 = F bab + J^4$, a contradiction.

Case 2.2. $\alpha_3 = 0$. In this case we have $0 \neq \beta_3 \gamma_6 - \alpha_3 = \beta_3 \gamma_6$, i.e. $\beta_3 \neq 0 \neq \gamma_6$. We now replace $a$ by $\beta_3 a$ and may then assume that $\beta_3 = 1$. We compute

\begin{align*}
0 & \equiv a(cb) - (ac)b \equiv a^2 b - \gamma_6 a^3 \quad (\text{mod } J^4), \\
0 & \equiv (b^2)b - b(b^2) \equiv \gamma_6 a^3 - bab \quad (\text{mod } J^4), \\
0 & \equiv (bc)c - b(c^2) \equiv (\beta_4^2 \gamma_6 - \beta_4 \gamma_6)a^3 + \beta_4 \gamma_6 bab^2 \quad (\text{mod } J^4);
\end{align*}

in particular, $J^3 = F a^3 + F ba^2 + J^4$. Thus $a^3 + J^4$ and $ba^2 + J^4$ are linearly independent. Then $\beta_4 = 0$ since $\gamma_6 \neq 0$. But now we obtain the contradiction

\[0 \equiv b(cb) - (bc)b \equiv \gamma_6 a^3 - \gamma_6 ba^2 \quad (\text{mod } J^4).\]

In the remainder of this paper we may and will assume that $J^2 = F \{a^2, ab, ac\} + J^3$. Then $J^3 = F \{a^3, a^2 b, a^2 c\} + J^4$ and $J^4 = F \{a^4, a^3 b, a^3 c\} + J^5$ by Lemma E in [6]; in particular, $\dim J^3/J^4 \in \{2, 3\}$. Since $J^4 \neq J^5$ we have $a^3 \notin J^4$.

(4.3) Lemma. The elements $a$, $b$, $c$ can be chosen such that one of the following holds:

\begin{align*}
(4.4) & \quad 0 = \alpha_2 = \beta_2 = \alpha_5, \ \gamma_2 = 1, \ \alpha_6 = \alpha_4 - 1, \ \beta_5 + \gamma_5 \neq 1; \\
(4.5) & \quad 0 = \alpha_2 = \beta_2, \ \gamma_2 = \alpha_5 = 1, \ \gamma_5 = 1 - \beta_5, \ \beta_6 - \beta_4 + \gamma_6 - \gamma_4 \neq 0; \\
(4.6) & \quad 0 = \alpha_2 = \gamma_2 = \alpha_5 = \beta_5, \ \gamma_5 = \beta_2 \neq 1, \ \alpha_4 = 1 \neq \alpha_6.
\end{align*}

Proof. We distinguish between two cases.

Case 1. $\gamma_2 \neq 0$. In this case we replace $c$ by $\alpha_2 a + \beta_2 b + \gamma_2 c$ and may therefore assume that $0 = \alpha_2 = \beta_2$ and $\gamma_2 = 1$. Now we distinguish two more cases.

Case 1.1. $\beta_5 + \gamma_5 \neq 1$. In this case we set $\xi := \alpha_5/(\beta_5 + \gamma_5 - 1)$ and replace $b$ by $b + \xi a$ and $c$ by $c + \xi a$. Then we have $\alpha_5 = 0$. Hence

\[J^2 = F \{[a, b], [a, c], [b, c]\} + J^3 = F \{ab - ac, \beta_5 ab + (\gamma_5 - 1)ac, (\alpha_4 - \alpha_6)a^2 + (\beta_4 - \beta_6)ab + (\gamma_4 - \gamma_6)ac\} + J^3;\]

in particular, $\alpha_4 \neq \alpha_6$. Now we replace $a$ by $(\alpha_4 - \alpha_6)^{1/2}a$ and may then assume that $\alpha_6 = \alpha_4 - 1$.

Case 1.2. $\beta_5 + \gamma_5 = 1$. In this case we have

\[J^2 = F \{[a, b], [a, c], [b, c]\} + J^3 = F \{ab - ac, \alpha_5 a^2 + \beta_5 ab - \beta_5 ac, (\alpha_4 - \alpha_6)a^2 + (\beta_4 - \beta_6)ab + (\gamma_4 - \gamma_6)ac\} + J^3.\]
Since \( a^2 + J^3, ab + J^3, ac + J^3 \) form a basis of \( J^2/J^3 \) we work out the corresponding determinant and obtain \( 0 \neq \alpha_5(\beta_6 - \beta_4 + \gamma_6 - \gamma_4) \), so \( \beta_6 - \beta_4 + \gamma_6 - \gamma_4 \neq 0 \neq \alpha_5 \). Then we replace \( a \) by \( \alpha_5a \) and may therefore assume that \( \alpha_5 = 1 \).

**Case 2.** \( \gamma_2 = 0 \). In this case we may assume that \( \beta_5 = 0 \); for otherwise we interchanged \( b \) and \( c \) and are then in Case 1 again. Similarly, we may assume that \( \gamma_5 = \beta_2 \); otherwise we replace \( b \) by \( b + c \) and are then in Case 1 again. Hence

\[
J^2 = F\{[a, b], [a, c], [b, c]\} + J^3
\]

\[
= F\{\alpha_2a^2 + (\beta_2 - 1)ab, \alpha_5a^2 + (\beta_2 - 1)ac, (\alpha_4 - \alpha_6)a^2 + (\gamma_4 - \gamma_6)ab + (\gamma_4 - \gamma_6)ac\} + J^3.
\]

Since \( \dim J^2/J^3 = 3 \) this implies that \( \beta_2 \neq 1 \). Now we replace \( b \) by \( b + \alpha_2(\beta_2 - 1)^{-1}a \) and \( c \) by \( c + \alpha_5(\beta_2 - 1)^{-1}a \) and may then assume that \( 0 = \alpha_2 = \alpha_5 \). In this situation we have \( \alpha_4 \neq 0 \) or \( \alpha_6 \neq 0 \). If necessary, we interchange \( b \) and \( c \) and may then assume that \( \alpha_4 \neq 0 \). Finally we replace \( b \) by \( \alpha_4^{-1}b \) and may therefore assume that \( \alpha_4 = 1 \).

Now we treat the cases above separately.

**Lemma.** The case (4.4) does not occur.

**Proof.** We assume the contrary and distinguish two cases.

**Case 1.** \( \dim J^3/J^4 = 3 \). In this case the elements \( a^3 + J^4, a^2b + J^4, a^2c + J^4 \) form a basis of \( J^3/J^4 \). Since

\[
0 \equiv (b^2)a - b(ba) \equiv (\alpha_3 - \alpha_7)a^3 + (\beta_5\gamma_3 - \beta_7)a^2b + (\beta_3 + \gamma_3\gamma_5 - \gamma_7)a^2c \quad (\text{mod } J^4)
\]

we conclude that \( \alpha_7 = \alpha_3, \beta_7 = \beta_5\gamma_3 \) and \( \gamma_7 = \beta_3 + \gamma_3\gamma_5 \). Similarly, using the fact that \( 0 = (bc)a - b(ca) + c(ba) - (cb)a + J^4 \) we obtain \( \beta_5 = -1 \), so \( \gamma_5 \neq 2 \). This also shows that \( \gamma_6 = \beta_4 - \beta_6 + \gamma_4 \) and \( 0 = (2 - \gamma_5)(\beta_4 - \beta_6) \).

Since \( \gamma_5 \neq 2 \) this implies that \( \beta_6 = \beta_4 \) and \( \gamma_6 = \gamma_4 \). Then, using the fact that \( 0 = (b^2)b - b(b^2) + J^4 \) and \( 0 = (bc)b - b(cb) + J^4 \) we see that \( 0 = (\alpha_3 - \alpha_4 + 1)(\beta_3 - \gamma_3) = (\alpha_3 - \alpha_4 + 1)(\beta_4 - \gamma_4) \). Now we distinguish two cases.

**Case 1.1.** \( \alpha_4 \neq \alpha_3 + 1 \). Then \( \gamma_3 = \beta_3 \) and \( \gamma_4 = \beta_4 \). Moreover, the fact that \( 0 = (bc)a - b(ca) + J^4 \) implies that \( 0 = \beta_3\gamma_5 \). We distinguish two more cases.

**Case 1.1.1.** \( \gamma_5 \neq 0 \), so \( \beta_3 = 0 \). In this case we use the fact that \( 0 = (bc)b - b(cb) + J^4 \) to obtain \( 0 = \gamma_5(1 - \alpha_4) \), so \( \alpha_4 = 1 \). But this leads to a contradiction using the fact that \( 0 = (bc)b - b(cb) + J^4 \) again.

**Case 1.1.2.** \( \gamma_5 = 0 \). In this case we use the fact that \( 0 = (bc)b - b(cb) + J^4 \) to obtain \( 2\alpha_4 = 1 \); in particular, \( \text{char } F \neq 2 \). Then we use the fact that \( 0 = (bc)a - b(ca) + J^4 \) to conclude that \( \beta_4 = 0 \), we use the fact that \( 0 = (c^2)a - c(ca) + J^4 \) to see that \( \beta_3 = 0 \), and we use the fact that \( 0 = (bc)c - b(c^2) + J^4 \) to show that \( \alpha_4 = 0 \). But this contradicts the fact that \( 0 = (bc)b - b(cb) + J^4 \).

**Case 1.2.** \( \alpha_4 = \alpha_3 + 1 \). In this case the fact that \( 0 = (bc)a - b(ca) + J^4 \) implies
that $0 = 2\alpha_3 + 1 - \alpha_3\gamma_5$. Thus

$$J^2 = F\{[a, b], [a, c], [b, c], [a, ab], [a, ac], [b, a^2], [b, ab], [b, ac], [c, a^2], [c, ab], [c, ac]\} + J^4$$

$$\subset F\{[a, b], [a, c], [b, c], a^2b, a^2c\} + J^4$$

as is easily checked. But this is a contradiction since $\dim J^2/J^4 = 6$.

Case 2. $\dim J^3/J^4 = 2$. Here we distinguish two more cases.

Case 2.1. $a^2b \in F a^3 + J^4$. In this case we have $J^3 = F\{a^3, a^2b, a^2c\} + J^4 = F\{a^3, a^2c\} + J^4$ and write $a^2b \equiv \delta a^3 \pmod{J^4}$ with some element $\delta \in F$. Then $a^3c \equiv a^2ba \equiv \delta a^4 \pmod{J^5}$, so $J^4 = F\{a^4, a^3c\} + J^5 = F a^4 + J^5$. Since $J^4 \neq J^5$ this implies that $\dim J^4/J^5 = 1$. By Lemma G in [6], $J^3 \subset Z$; in particular, $a^2c \in Z$. Hence

$$0 \equiv (a^2c)a - a(a^2c) \equiv a^2(ca) - a^3c \equiv (\beta_5 + \gamma_5 - 1)\delta a^3 \pmod{J^4}.$$ 

Since $\beta_5 + \gamma_5 \neq 1$ and $a^3 \notin J^4$ we conclude that $\delta = 0$. But now

$$0 \equiv a^2(bc) - (a^2b)c \equiv a^2ca - a^3b = \beta_5 + \gamma_5 - 1 \delta a^3 \pmod{J^5},$$

$$0 \equiv (a^2c)a - a(a^2c) \equiv a^2ca - a^3c \equiv (\beta_5 + \gamma_5 - 1)\delta a^3 + (\beta_5 + \gamma_5 - 1)\delta a^3 \pmod{J^4},$$

This leads to the contradiction $0 = \alpha_4 = \alpha_3 = \alpha_7 = -1$.

Case 2.2. $a^2b \notin F a^3 + J^4$. Since $a^3 \notin J^4$ and $\dim J^3/J^4 = 2$ the elements $a^3 + J^4$ and $a^2b + J^4$ form a basis of $J^3/J^4$ in this case. We write $a^2c \equiv \delta a^3 + \epsilon a^2b \pmod{J^4}$ with elements $\delta, \epsilon \in F$. Since $J^4 = F a^4 + F a^3b + J^5$ and $J^4 \neq J^5$ we have $\dim J^4/J^5 \in \{1, 2\}$. Let us distinguish the corresponding cases.

Case 2.2.1. $\dim J^4/J^5 = 1$. In this case $J^3 \subset Z$ by Lemma G in [6]; in particular, $a^2b, a^2c \in Z$. Hence

$$0 \equiv (a^2b)a - a(a^2b) \equiv a^2(ba) - a^3b$$

$$\equiv a^2c - a^3b \equiv \delta a^4 + (\epsilon - 1)a^3b \pmod{J^5},$$

$$0 \equiv (a^2c)a - a(a^2c) \equiv a^2(ca) - a^3c \equiv (\gamma_5 - 1)\delta a^4 + (\beta_5 + \gamma_5 \epsilon - \epsilon) a^3b$$

$$\equiv (\beta_5 + \gamma_5 - 1)\delta a^3 \pmod{J^5}.$$ 

Since $\beta_5 + \gamma_5 \neq 1$ this implies that $a^3b \in J^5$. Hence $J^4 = F a^4 + J^5$ and $\delta a^4 \in J^5$. Since $\dim J^4/J^5 = 1$ we must have $\delta = 0$. Therefore

$$0 \equiv (b^2)a - b(ba) \equiv (\alpha_3 - \alpha_7)a^3$$

$$+ (\beta_5 \gamma_5 - \beta_7 + \beta_5 \epsilon + \gamma_5 \gamma_5 \epsilon - \gamma_7 \epsilon) a^2b \pmod{J^4};$$

in particular, $\alpha_7 = \alpha_3$. Similarly, using the fact that $0 = (bc)a - b(ca) + c(ba) - (cb)a + J^4$ we see that $\beta_5 = -1$; in particular, $\gamma_5 \neq 2$. Hence

$$0 \equiv a^2(cb) - (a^2c)b \equiv (\alpha_4 - 1 - \alpha_3 \epsilon) a^4 \pmod{J^5},$$

$$0 \equiv a^2(c^2) - (a^2c)c \equiv (\alpha_3 - \alpha_4 \epsilon) a^4 \pmod{J^5}.$$
Since \( a^4 \notin J^5 \) this implies that \( \alpha_3 = \alpha_4 \delta \) and \( \alpha_4 - \alpha_4 \delta^2 = 1 \); in particular, \( \alpha_4 \neq 0 \) and \( \epsilon^2 \neq 1 \). But since \( a^2 c \in Z \) we have

\[
0 \equiv (a^2 c)b - b(a^2 c) \equiv (a^2 c)b - (ba)(ac) \pmod{J^5},
\]

\[
0 \equiv (a^2 c)b - b(a^2 c) \equiv (2 - \gamma_5 \epsilon)\alpha_4 \epsilon^2 a^4 \pmod{J^5}.
\]

Hence \( \gamma_5 \epsilon = 2 \) and \( \epsilon^2 = 1 \), a contradiction.

**Case 2.2.2.** \( \dim J^4/J^5 = 2 \). In this case

\[
0 \equiv (a^2 c)a - a^2 (ca) \equiv (\delta + \delta \epsilon - \gamma_5 \delta) a^4 + (\epsilon^2 - \beta_5 - \gamma_5 \epsilon) a^3 b \pmod{J^5},
\]

\[
0 \equiv (a^2 c)b - a^2 (cb) \equiv (\alpha_3 \epsilon + \gamma_3 \delta \epsilon - \alpha_4 + 1 - \gamma_6 \delta) a^4
\]

\+

\[
(\delta + \beta_3 \epsilon + \gamma_3 \epsilon^2 - \beta_6 - \gamma_6 \epsilon) a^3 b \pmod{J^5},
\]

\[
0 \equiv (a^2 c)c - a^2 c^2 \equiv (\alpha_4 \epsilon + \gamma_4 \delta \epsilon - \alpha_7 - \gamma_7 \delta + \delta^2) a^4
\]

\+

\[
(\delta + \beta_4 \epsilon + \gamma_4 \epsilon^2 - \beta_7 - \gamma_7 \epsilon) a^3 b \pmod{J^5}.
\]

Since \( a^4 + J^5 \) and \( a^3 b + J^5 \) form a basis of \( J^4/J^5 \) this implies that all coefficients on the right-hand side vanish; in particular, \( 0 = \delta + \delta \epsilon - \gamma_5 \delta \).

Assume that \( \delta \neq 0 \). Then \( \epsilon = \gamma_5 - 1 \) and we obtain the contradiction \( 0 = \epsilon^2 - \beta_5 - \gamma_5 \epsilon = 1 - \beta_5 - \gamma_5 \). Hence we must have \( \delta = 0 \). Therefore

\[
0 \equiv (b^2)a - b(ba) \equiv (\alpha_3 - \alpha_7) a^3
\]

\+

\[
(\beta_5 \gamma_3 - \beta_7 + \beta_3 \epsilon + \gamma_3 \gamma_5 \epsilon - \gamma_7 \epsilon) a^2 b \pmod{J^4}.
\]

in particular, \( \alpha_7 = \alpha_3 \). Similarly, using the fact that \( 0 = (bc)a - b(ca) + c(ba) - (cb)a + J^4 \) we see that \( \beta_5 = -1 \). Hence \( \epsilon^2 - \gamma_5 \epsilon = -1 \); in particular, \( \epsilon \neq 0 \).

Therefore \( 0 = \alpha_3 \epsilon + \gamma_3 \delta \epsilon - \alpha_4 + 1 - \gamma_6 \delta = \alpha_3 \epsilon - \alpha_4 + 1 \), and \( \alpha_4 = \alpha_3 \epsilon + 1 \).

Hence \( 0 = \alpha_4 \epsilon + \gamma_4 \delta \epsilon - \alpha_7 - \gamma_7 \delta + \delta^2 = \alpha_3 \epsilon^2 + \epsilon - \alpha_3 \); in particular, \( \alpha_3 \neq 0 \) and \( \epsilon^2 \neq 1 \). But this leads to the contradiction

\[
0 \equiv b(a^2 c) - (ba)ac 
\]

\[
\equiv -\epsilon^2 a^4 + (\beta_6 \gamma_5 \epsilon - \beta_3 \epsilon + \gamma_5 \gamma_6 \epsilon^2 - \gamma_3 \epsilon^2 - \beta_7 \gamma_5 + \beta_4 - \gamma_5 \gamma_7 \epsilon + \gamma_4 \epsilon) a^3 b
\]

\pmod{J^5}.

(4.8) **Lemma.** The case (4.5) does not occur.

**Proof.** We assume the contrary and distinguish two cases.

**Case 1.** \( \dim J^3/J^4 = 3 \). In this case we have

\[
0 \equiv (bc)a - b(ca) \equiv (\alpha_4 + \gamma_4 - 1 - \alpha_6 \beta_5 - \alpha_7 + \alpha_7 \beta_5) a^3 + (\beta_5 \gamma_4 - \beta_5 - \beta_5 \beta_6 - \beta_7 + \beta_5 \beta_7) a^2 b
\]

\+

\[
(\beta_4 + \gamma_4 - \beta_5 \gamma_4 - 1 + \beta_5 - \beta_5 \gamma_6 - \gamma_7 + \beta_5 \gamma_7) a^2 c \pmod{J^4}.
\]

Since \( a^3 + J^4 \), \( a^2 b + J^4 \), \( a^2 c + J^4 \) form a basis of \( J^3/J^4 \) we obtain

(4.9) \( 0 = \alpha_4 + \gamma_4 - 1 - \alpha_6 \beta_5 - \alpha_7 + \alpha_7 \beta_5 \),

(4.10) \( 0 = \beta_5 \gamma_4 - \beta_5 - \beta_5 \beta_6 - \beta_7 + \beta_5 \beta_7 \),

(4.11) \( 0 = \beta_4 + \gamma_4 - \beta_5 \gamma_4 - 1 + \beta_5 - \beta_5 \gamma_6 - \gamma_7 + \beta_5 \gamma_7 \).
Similarly, using the fact that \( 0 \equiv (ca)b - c(ab) \pmod{J^4} \) we obtain the following equations:

\[
\begin{align*}
(4.12) & \quad 0 = \alpha_6 + \gamma_6 - \alpha_4 \beta_5 - \alpha_7 + \alpha_7 \beta_5 , \\
(4.13) & \quad 0 = \beta_5 \gamma_6 - \beta_4 \beta_5 - \beta_7 + \beta_5 \beta_7 , \\
(4.14) & \quad 0 = \beta_6 + \gamma_6 - \beta_5 \gamma_6 - 1 - \beta_5 \gamma_4 - \gamma_7 + \beta_5 \gamma_7 .
\end{align*}
\]

Now we add (4.10) and (4.11) and subtract (4.13) and (4.14) from the result to obtain \( 0 = (\beta_5 + 1)(\beta_4 - \beta_6 + \gamma_4 - \gamma_6) \). Hence \( \beta_5 = -1 \). Then we subtract (4.12) from (4.9) and obtain \( 0 = \gamma_4 - \gamma_6 - 1 \). Hence \( \gamma_6 = \gamma_4 - 1 \). Next we subtract (4.14) from (4.11) and obtain \( 0 = \beta_4 - \beta_6 \). Hence \( \beta_6 = \beta_4 \). Then we use the fact that \( b(ba) \equiv (b^2)a \pmod{J^4} \) to obtain that \( \alpha_7 = \alpha_3 + \gamma_3, \beta_7 = -\gamma_3 \) and \( \gamma_7 = \beta_3 + 2 \gamma_3 \). Now (4.10) implies that \( \beta_4 = \gamma_4 - 1 - 2 \gamma_3 \). Using the fact that \( 0 \equiv (c^2)a - c(ca) \pmod{J^4} \) we obtain the following equations:

\[
\begin{align*}
(4.15) & \quad 0 = \beta_3 - \gamma_3 - 3 - 4 \alpha_3 + 2 \alpha_4 + 2 \alpha_6 , \\
(4.16) & \quad 0 = 4 \gamma_4 - 2 \beta_3 - 1 - 6 \gamma_3 ;
\end{align*}
\]

in particular, \( \text{char } F \neq 2 \). Now (4.11) forces \( 0 = 4 + 2 \beta_3 + 6 \gamma_3 - 4 \gamma_4 , \) so \( \beta_3 = 2 \gamma_4 - 3 \gamma_3 - 2 \). Next we multiply (4.9) by 2 and subtract (4.15) to obtain \( 0 = 3 \), so \( \text{char } F = 3 \). Thus

\[
J^2 = F \{[a, b], [a, c], [b, c], [a, ab], [a, ac], [b, a^2], [b, ab], [b, ac], [c, a^2], [c, ab], [c, ac] \} + J^4
\]

\[
\subset F \{[a, b], [a, c], [b, c], a^3, a^2 b - a^2 c \} + J^4
\]

as is easily checked. But this contradicts the fact that \( \text{dim } J^2/J^4 = 6 \).

**Case 2.** \( \text{dim } J^3/J^4 = 2 \). We distinguish two more cases.

**Case 2.1.** \( a^2 b \in Fa^3 + J^4 \). In this case \( J^3 = F \{a^3, a^2 b, a^2 c\} + J^4 = F \{a^3, a^2 c\} + J^4 \), and \( a^2 b \equiv \delta a^4 \pmod{J^4} \) for some element \( \delta \in F \). Since \( a^3 c \equiv a^2 b a \equiv \delta a^4 \pmod{J^5} \) we see that \( J^4 = Fa^4 + Fa^3 c + J^5 = Fa^4 + J^5 \). Since \( J^4 \neq J^5 \) this implies that \( \text{dim } J^4/J^5 = 1 \). Now Lemma G in [6] shows that \( J^3 \subset Z \); in particular, \( a^2 c \in Z \). But this leads to the contradiction \( 0 \equiv (a^2 c)a - a(a^2 c) \equiv a^2 (ca) - a^3 c \equiv a^4 \pmod{J^5} \).

**Case 2.2.** \( a^2 b \not\in Fa^3 + J^4 \). Since \( a^3 \not\in J^4 \) and \( \text{dim } J^3/J^4 = 2 \) the elements \( a^3 + J^4 \) and \( a^2 b + J^4 \) form a basis of \( J^3/J^4 \) in this case. We write \( a^2 c \equiv \delta a^3 + \varepsilon a^2 b \pmod{J^4} \) with elements \( \delta, \varepsilon \in F \). Since \( J^4 = Fa^4 + Fa^3 b + J^5 \) and \( J^4 \neq J^5 \) we have \( \text{dim } J^4/J^5 \in \{1, 2\} \). Let us distinguish the corresponding cases.

**Case 2.2.1.** \( \text{dim } J^4/J^5 = 2 \). In this case the elements \( a^4 + J^5 \) and \( a^3 b + J^5 \) form a basis of \( J^4/J^5 \). Since

\[
0 \equiv (a^2 c)a - a^2 (ca) \equiv (\delta e + \beta_3 \delta - 1)a^4 + (e - 1)(e + \beta_3)a^3 b \pmod{J^5}
\]

this implies that \( \delta e + \beta_3 \delta - 1 = 0 \) and \( (e - 1)(e + \beta_3) = 0 \). The first equation forces \( e \neq -\beta_3 \), so \( e = 1 \) by the second equation. Then, using the fact that \( 0 \equiv (bc)a - b(ca) + c(ba) - (cb)a \pmod{J^4} \) we obtain the contradiction \( 0 = (1 + \beta_3)(\beta_4 - \beta_6 + \gamma_4 - \gamma_6) \).
Case 2.2.2. \( \dim J^4/J^5 = 1 \). In this case \( J^3 \subset Z \) by Lemma G in [6]; in particular, \( a^2b, a^2c \in Z \). Hence \( a^2b \equiv a^2ba \equiv a^2c \pmod{J^5} \) and \( a^2b \equiv a^2c \equiv a^2ca \equiv a^4 + a^3b \pmod{J^5} \), so \( a^4 \in J^3 \) and \( J^4 = Fa^3b + J^5 \). Furthermore, since \( a^3b \equiv a^3c \equiv ea^3b \pmod{J^5} \) we must have \( \varepsilon = 1 \). Using the fact that \( 0 \equiv (bc)a - b(c a) + c(b a) - (cb)a \pmod{J^4} \) we obtain \( 0 = (1 + \beta_5)(\beta_4 - \beta_6 + \gamma_4 - \gamma_6) \), so \( \beta_5 = -1 \). Similarly, using the fact that \( 0 \equiv (b^2)a - b(ba) \pmod{J^4} \) we obtain \( \gamma_7 = \beta_3 + \gamma_3 - \beta_4 - \gamma_4 \). Then \( 0 \equiv (a^2c)b - (b^2)c \equiv (a^2c)b - (ba)c \pmod{J^5} \) implies that \( \delta = 1 + \beta_3 + \gamma_3 - \beta_4 - \gamma_4 \). But now the fact that \( 0 \equiv a^2(c^2) - (a^2c)c \pmod{J^5} \) leads to a contradiction.

(4.17) **Lemma.** The case (4.6) does not occur.

**Proof.** We assume the contrary and distinguish two cases.

Case 1. \( \dim J^3/J^4 = 3 \). In this case the elements \( a^3 + J^4, a^2b + J^4, a^2c + J^4 \) form a basis of \( J^3/J^4 \). Since \( \beta_2 \neq 1 \) and

\[
0 \equiv (bc)a - b(ca) \equiv (1 - \beta_2^2)a^3 + \beta_4(\beta_2 - \beta_2^2)a^2b + \gamma_4(\beta_2 - \beta_2^2)a^2c \pmod{J^4}
\]

this implies that \( \beta_2 = -1 \); in particular, \( \text{char } F \neq 2 \). Hence \( 0 = 2\beta_4 = 2\gamma_4 \), so \( 0 = \beta_4 = \gamma_4 \). Then, using similarly the fact that \( 0 \equiv c(ba) - (cb)a \pmod{J^4} \) we obtain \( 0 = \beta_6 = \gamma_6 \). But now the fact that \( 0 \equiv (bc)b - b(cb) \pmod{J^4} \) leads to a contradiction.

Case 2. \( \dim J^3/J^4 = 2 \). We distinguish two more cases.

Case 2.1. \( a^2b \in J^3 + J^4 \). In this case we have \( J^3 = F\{a^3, a^2b, a^2c\} + J^4 = Fa^3 + Fa^2c + J^4 \) and \( J^4 = Fa^4 + Fa^3c + J^5 \). Assume that \( a^4 \in J^5 \). Then \( J^4 = Fa^3c + J^5 \); in particular, \( \dim J^4/J^5 = 1 \) since \( J^4 \neq J^5 \). Hence Lemma G in [6] implies that \( J^3 \subset Z \); in particular, \( a^2c \in Z \). But this leads to the contradiction \( a^3c \equiv a^2ca \equiv \beta_2a^3c \pmod{J^5} \).

We write \( a^2b \equiv \delta a^3 \pmod{J^4} \) with some element \( \delta \in F \). Then \( \delta a^4 \equiv a^2ba \equiv \beta_2a^3b \equiv \beta_2\delta a^4 \pmod{J^5} \). Since \( a^4 \notin J^5 \) and \( \beta_2 \neq 1 \) this implies that \( \delta = 0 \). As in Case 1, now we use the fact that \( 0 \equiv (bc)a - b(ca) \pmod{J^4} \) to obtain that \( \beta_2 = -1 \), char \( F \neq 2 \) and \( \gamma_4 = 0 \). Similarly, using the fact that \( 0 \equiv a^2b - a(c a) \equiv (b^2)a - b(ba) \pmod{J^4} \) we obtain \( 0 = 2\gamma_6 = 2\gamma_3 \), so \( 0 = \gamma_6 = \gamma_3 \). But this yields a contradiction using the fact that \( 0 \equiv (bc)c - (bc)^2 \pmod{J^4} \).

Case 2.2. \( a^2b \notin J^3 + J^4 \). Since \( a^3 \notin J^4 \) and \( \dim J^3/J^4 = 2 \) the elements \( a^3 + J^4 \) and \( a^2b + J^4 \) form a basis of \( J^3/J^4 \) in this case, and \( J^4 = Fa^4 + Fa^3b + J^5 \). Assume that \( a^4 \in J^5 \). Then \( J^4 = Fa^3b + J^5 \); in particular, \( \dim J^4/J^5 = 1 \) since \( J^4 \neq J^5 \). Hence Lemma G in [6] implies that \( J^3 \subset Z \); in particular, \( a^2b \in Z \). But now we obtain the contradiction \( a^3b \equiv a^2ba \equiv \beta_2a^3b \pmod{J^5} \).

Hence \( a^4 \notin J^5 \), and we write \( a^2c \equiv \delta a^3 + ea^2b \pmod{J^4} \) with elements \( \delta, e \in F \). Then \( 0 \equiv (a^2c)a - a^2(ca) \equiv (1 - \beta_2)\delta a^4 \pmod{J^5} \), so \( \delta = 0 \) since \( \beta_2 \neq 1 \) and \( a^4 \notin J^5 \). As in Case 1, now we use the fact that \( 0 \equiv (bc)a - b(ca) \pmod{J^4} \) to obtain \( \beta_2 = -1 \) and char \( F \neq 2 \). Then we distinguish two more cases.

Case 2.2.1. \( \dim J^4/J^5 = 2 \). In this case the elements \( a^4 + J^5 \) and \( a^3b + J^5 \) form a basis of \( J^4/J^5 \). Using the fact that \( 0 \equiv b(a^2c) - (ba)ac \pmod{J^5} \)
we obtain $\alpha_3 e = 1$. But this leads to a contradiction using the fact that $0 \equiv (a^2c)b - a^2(cb) \pmod{J^4}$.

Case 2.2.2. $\dim J^4/J^5 = 1$. In this case we have $J^4 = Fa^4 + J^5$ since $a^4 \not\in J^3$. Moreover, Lemma G in [6] implies that $J^3 \subset Z$; in particular, $a^2b \in Z$. Thus $a^3b \equiv a^2ba \equiv -a^3b \pmod{J^5}$, so $a^3b \in J^5$ since $\text{char} F \neq 2$. This, however, leads to a contradiction using the fact that $0 \equiv (a^2c)b - b(a^2c) \equiv a^2(cb) - (ba)ac \pmod{J^5}$.

REFERENCES


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