PRINJECTIVE MODULES, REFLECTION FUNCTORS, QUADRATIC FORMS, AND AUSLANDER-REITEN SEQUENCES

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Abstract. Let \( A, B \) be artinian rings and let \( A^{}M^{}B^{} \) be an \((A - B)\)-bimodule which is a finitely generated left \( A \)-module and a finitely generated right \( B \)-module. A right \( A^{}M^{}B^{} \)-prinjective module is a finitely generated module \( X_R = (X'_A, X''_B, \varphi : X'_A \otimes_A M_B \to X''_B) \) over the triangular matrix ring

\[
R = \begin{pmatrix} A & A^{}M^{}B^{} \\ 0 & B \end{pmatrix}
\]

such that \( X'_A \) is a projective \( A \)-module, \( X''_B \) is an injective \( B \)-module, and \( \varphi \) is a \( B \)-homomorphism.

We study the category \( \text{prin}(R)^A_B \) of right \( A^{}M^{}B^{} \)-prinjective modules. It is an additive Krull-Schmidt subcategory of \( \text{mod}(R) \) closed under extensions. For every \( X, Y \) in \( \text{prin}(R)^A_B \), \( \text{Ext}^2_R(X, Y) = 0 \). When \( R \) is an Artin algebra, the category \( \text{prin}(R)^A_B \) has Auslander-Reiten sequences and they can be computed in terms of reflection functors. In the case that \( R \) is an algebra over an algebraically closed field we give conditions for \( \text{prin}(R)^A_B \) to be representation-finite or representation-tame in terms of a Tits form. In some cases we calculate the coordinates of the Auslander-Reiten translation of a module using a Coxeter linear transformation.

0. Introduction

Let \( A, B \) be artinian rings and let \( A^{}M^{}B^{} \) be an \((A - B)\)-bimodule which is a finitely generated left \( A \)-module and a finitely generated right \( B \)-module. By a right \( A^{}M^{}B^{} \)-prinjective module we shall mean a finitely generated module

\[ X_R = (X'_A, X''_B, \varphi : X'_A \otimes_A M_B \to X''_B) \]

over the triangular matrix ring

\[
R = \begin{pmatrix} A & A^{}M^{}B^{} \\ 0 & B \end{pmatrix}
\]

such that \( X'_A \) is a projective \( A \)-module, \( X''_B \) is an injective \( B \)-module, and \( \varphi \) is a \( B \)-homomorphism.

The aim of this paper is to study the category \( \text{prin}(R)^A_B \) of right \( A^{}M^{}B^{} \)-prinjective modules. It is an additive Krull-Schmidt subcategory of \( \text{mod}(R) \) closed under extensions. It has enough relative projective and relative injective modules and it is a hereditary subcategory of \( \text{mod}(R) \) in the sense that \( \text{Ext}^2_R(X, Y) = 0 \) for all \( X \) and \( Y \) in \( \text{prin}(R)^A_B \).

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We prove in this paper that if $R$ is an Artin algebra then $\text{prin}(R)_B$ has Auslander-Reiten sequences and we show how to compute them in terms of reflection functors. In the case that $R$ is an algebra over an algebraically closed field we give necessary conditions for $\text{prin}(R)_B$ to be a representation-finite or a representation-tame category in terms of a Tits form. Moreover, if $\text{prin}(R)_B$ has a preprojective component, this gives a criterion for the representation-finiteness of $\text{prin}(R)_B$. In the latter case, we calculate the coordinates of the Auslander-Reiten translation of a module by means of a Coxeter linear transformation.

The main motivation of our study is the fact observed in [10, 18, 19] (see also §1) that given finite Krull-Schmidt categories $K$, $L$ and a $(K-L)$-bimodule $N: K^\text{op} \times L \to \text{mod}(k)$ there is an algebra $R$ of the form (0.1) and an equivalence of categories

$$\text{Mat}(K^N_L) \sim \text{prin}(R)_B,$$

where $\text{Mat}(K^N_L)$ is the category of $K^N_L$-matrices in the sense of Drozd [7]. In particular, the subspace category $\mathbb{Z}((K_F)$ of a vector space category $K_F$ is equivalent to $\text{prin}(E^M_F)$ for some $E_M F$. Therefore, our results give us tools to study the representation type of $\text{Mat}(K^N_L)$, which turns out to be an important class of matrix problems with many useful applications [16, 19].

The notion of an $AM_B$-prjective module was introduced in [10] and independently in [19] under the name of $AM_B$-matrix module. The category $\text{prin}(R)_B$ is denoted by $\text{mod}_{\text{pr}}(R)$ in [19]. Our Coxeter scheme, studied in §4, generalizes the one defined in [17] for the case where $B$ is a division ring.

Throughout this paper $R$ is an Artin algebra of the form (0.1). We denote by $\text{mod}(R)$, $\text{pr}(A)$, and $\text{inj}(B)$ the categories of finitely generated right $R$-modules, projective right $A$-modules, and injective right $B$-modules, respectively. We assume that $A$ and $B$ are basic algebras and we fix complete sets $\{e_1, \ldots, e_n\}$ and $\{\eta_1, \ldots, \eta_m\}$ of orthogonal primitive idempotents in $A$ and $B$ respectively. Given $X_R = (X'_A, X'_B, \phi)$ in $\text{mod}(R)$ we put $X'_i = X' e_i = X e_i$ and $X''_j = X'' \eta_j = X \eta_j$.

We fix a duality $D: \text{mod}(R) \to \text{mod}(R^\text{op})$; that is, $D(-) = \text{Hom}_C(-, E(C/\mathcal{F}(C)))$,

where $C$ is a commutative artinian ring contained in the center of $R$ such that $R$ is a finitely generated $C$-module, $\mathcal{F}(C)$ is its Jacobson radical, and $E(-)$ denotes the injective envelope. We denote by

$$\mathfrak{M}: \text{pr}(A) \to \text{inj}(A)$$

the Nakayama equivalence defined by $\mathfrak{M}(P) = D\text{Hom}_A(P, A)$ (we will write $\mathfrak{M}_A$ whenever confusion may arise). We will frequently use the equivalence

$$P \otimes_A M_B \sim \text{Hom}_A(B \widetilde{M}_A, \mathfrak{M}(P)),$$

with $\widetilde{M} = DM$ and $P$ in $\text{pr}(A)$, defined as the composed map

$$P \otimes_A M_B \overset{\sigma}{\to} \text{Hom}_A(\text{Hom}_A(P, A), A M_B) \overset{D}{\to} \text{Hom}_A(B \widetilde{M}_A, \mathfrak{M}(P)),$$

with $\sigma(p \otimes m)(f) = f(p)m$.

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1. THE CATEGORY OF $AM_B$-MATRICES

Let us recall from [7] the definition of $\operatorname{Mat}^{A\oplus B}$. Suppose that $k$ is a commutative artinian ring, $K$, $L$ are finite Krull-Schmidt $k$-categories, and $N: K^{\text{op}} \times L \to \text{mod}(k)$ is a $(K \otimes L)$-bimodule, i.e., it is an additive $k$-bilinear functor. The objects of $\operatorname{Mat}^{A\oplus B}$ are triples $(x, y, f)$ with $x \in \mathcal{O}b K$, $y \in \mathcal{O}b L$, and $f \in N(x, y)$. A morphism from $(x, y, f)$ to $(x', y', f')$ is a pair $(\varphi, \psi)$, where $\varphi \in K(x, x')$, $\psi \in L(y, y')$ are such that $N(x, \psi)f = N(\varphi, y')f' \in N(x, y')$. It is known that $\operatorname{Mat}^{A\oplus B}$ is an additive Krull-Schmidt category.

Following [18, 19] we associate to $K\oplus L$ an Artin $k$-algebra $R = \begin{pmatrix} A & M_B \\ 0 & B \end{pmatrix}$ as follows. We take complete sets $K_1, \ldots, K_n$ and $L_1, \ldots, L_m$ of pairwise nonisomorphic indecomposable objects in $K$ and $L$ respectively and we set

$$K = K_1 \oplus \cdots \oplus K_n, \quad L = L_1 \oplus \cdots \oplus L_m,$$

$$A = K(K, K), \quad B = L(L, L), \quad \text{and} \quad AM_B = DN(K, L).$$

Proposition 1.1. With the notation above, there is an equivalence of categories $\mu^*: \operatorname{Mat}^{A\oplus B} \simeq \text{prin}(R)^d$.

Proof. We repeat the proof given in [19, §5; 10, §1]. Let $\omega: K \to \text{pr}(A)$ and $\omega': L \to \text{pr}(B)$ be the Yoneda equivalences given by $\omega(-) = K(K, -)$ and $\omega'(-) = L(L, -)$. For $x \in \mathcal{O}b K$ and $y \in \mathcal{O}b L$ the Yoneda lemma yields a natural isomorphism

(1.2) $\mu: N(x, y) \to \operatorname{Hom}_B(\omega(x) \otimes_A M_B, \mu\omega'(y))$

which is the composed map

$$N(x, y) \simeq \operatorname{Nat}(L(y, -), N(x, -)) \simeq \operatorname{Hom}_B(L(y, L), N(x, L))$$

$$\simeq \operatorname{Hom}_B(DN(x, L), DL(y, L))$$

$$\simeq \operatorname{Hom}_B(\mathbb{K}(-, x) \otimes_K DN(-, L), \mu\omega'(y))$$

$$\simeq \operatorname{Hom}_B(\omega(x) \otimes_A M_B, \mu\omega'(y)),$$

where $\operatorname{Nat}(-, -)$ denotes the set of $k$-linear natural transformations.

Now we associate the module $\mu^*(x, y, f) = (\omega(x), \mu\omega'(y), \mu(f))$ in $\text{prin}(R)^d$ to $(x, y, f)$ in $\operatorname{Mat}^{A\oplus B}$ with $f \in N(x, y)$. Applying (1.2) one can easily check that $\mu^*$ defines an equivalence of categories. □

2. THE CATEGORY OF PRINJECTIVE MODULES

Let $R = \begin{pmatrix} A & M_B \\ 0 & B \end{pmatrix}$ be an Artin algebra. A module $P$ (resp. $Q$) in $\text{prin}(R)^d$ will be called $\text{prin-projective}$ (resp. $\text{prin-injective}$) if $\operatorname{Hom}_R(P, -)$ (resp. $\operatorname{Hom}_R(-, Q)$) carries over short exact sequences in $\text{mod}(R)$ with prinjective terms into exact ones.
To describe the indecomposable prin-projective and prin-injective $R$-modules we need the following notation. Given $X = (X', X'', \phi)$ in $\text{mod}(R)$ we form two modules,

$$\widehat{X} = (X', E(X''), \phi) \quad \text{and} \quad \widetilde{X} = (P(X'), X'', \bar{\phi}),$$

where $E(X'')$ denotes the injective envelope of $X''$ in $\text{mod}(B)$ and $P(X')$ denotes the projective cover of $X'$ in $\text{mod}(A)$; the maps $\phi$ and $\bar{\phi}$ are defined in the obvious way. Now we form two families of indecomposable prin-injective $R$-modules,

$$\widehat{P}_1, \ldots, \widehat{P}_n, \widehat{0}_1, \ldots, \widehat{0}_m,$$

$$\widetilde{P}_1, \ldots, \widetilde{P}_n, \widetilde{0}_1, \ldots, \widetilde{0}_m,$$

where $\widehat{P}_j = e_jR$, $\widehat{0}_i = (0, E(\text{top} \eta_i B), 0)$, $\widehat{P}_j = (e_jA, 0, 0)$, and $\widetilde{Q}_i = \text{E}(\text{top} \eta_i B)$, $\text{top} \eta_i B = \eta_i B / \text{rad} \eta_i B$; $E_R(-)$ denotes the injective envelope in $\text{mod}(R)$.

**Proposition 2.4.** (a) The category $\text{prin}(R)_R^A$ is closed under extensions in $\text{mod}(R)$ and for any $X$ in $\text{prin}(R)_R^A$ there are exact sequences in $\text{mod}(R)$

$$0 \to H_1 \to H_0 \to X \to 0,$$

$$0 \to X \to U_0 \to U_1 \to 0,$$

where $H_0$ and $U_0$ are direct sums of modules in (2.2) and (2.3) respectively, whereas $H_1$ and $U_1$ are direct sums of modules $\widehat{0}_1, \ldots, \widehat{0}_m$ and $\widehat{P}_1, \ldots, \widehat{P}_n$ respectively.

(b) A module $P$ in $\text{prin}(R)_R^A$ is prin-projective if and only if $\text{Ext}(P, X) = 0$ for all $X$ in $\text{prin}(R)_R^A$. The modules (2.2) form a complete list of pairwise nonisomorphic indecomposable prin-projective modules.

(c) A module $Q$ in $\text{prin}(R)_R^A$ is prin-injective if and only if $\text{Ext}(X, Q) = 0$ for all $X$ in $\text{prin}(R)_R^A$. The modules (2.3) form a complete list of pairwise nonisomorphic indecomposable prin-injective modules.

(d) $\text{prin}(R)_R^A$ is a hereditary subcategory of $\text{mod}(R)$, i.e., $\text{Ext}(X, Y) = 0$ for all $X$ and $Y$ in $\text{prin}(R)_R^A$.

**Proof.** (a) Let $X = (X', X'', \phi)$ be in $\text{prin}(R)_R^A$. Since $X'$ is in $\text{pr}(A)$, then $L = (X', X' \otimes_A M_B, \text{id})$ is in $\text{pr}(R)$ and therefore $\widehat{L}$ is a direct sum of copies of $\widehat{P}_1, \ldots, \widehat{P}_n$. The maps $\text{id}_{X'}$ and $\text{id}_{X''}$ induce the exact sequence

$$0 \to (0, Y, 0) \to \widehat{L} \oplus (0, X'', 0) \to X \to 0.$$ 

Since obviously $Y$, $X''$ are in $\text{inj}(B)$, we then get the first sequence in (a). The second one arises dually.

(b) It is clear that $\text{Ext}(P, X) = 0$ for all $X$ in $\text{prin}(R)_R^A$ and $P$ of the form (2.2). Then statement (b) follows from (a).

Since (c) follows similarly from (a) and (d) is a consequence of (a)–(c), the proof is complete.

Let $X$ be indecomposable. A map $f: X \to Y$ in an additive Krull-Schmidt category $K (= \text{prin}(R)_R^A, \text{mod}(R), \ldots)$ is called a source map for $X$ if it satisfies:
(a) \( f \) is not split mono;
(b) given \( f': X \to Y' \) not split mono, there exists \( \eta: Y \to Y' \) with \( f' = \eta f \); and
(c) if \( \gamma \in \text{End}_K(Y) \) satisfies \( f\gamma = f \), then \( \gamma \) is an automorphism.

The dual concept is a sink map (see [16, (2.2)]).

We describe now the sink morphisms in \( \text{prin}(R)_B^A \) ending at the prin-projective indecomposable modules.

Consider the case \( \tilde{P}_j = \tilde{e}_jR = (e_jA, E(e_jM), \text{id}) \). Let \( \xi: X \to e_jA \) be a sink morphism in \( \text{pr}(A) \); that is, \( \xi: X = P(\text{rad} e_jA) \to (\text{rad} e_jA) \hookrightarrow e_jA \). Then

\[
(\xi, \text{id}): (X, e_jM, \xi \otimes 1_M) \rightrightarrows \tilde{P}_j
\]

is a sink morphism.

Consider the case \( 0I_1 = (0, E(\text{top} \eta_1 B), 0) \). Let \( \lambda: Y \to \eta_1^{-1} E(\text{top} \eta_1 B) \) be a sink morphism in \( \text{pr}(B) \). Then \( \nu = \eta_1 \lambda: J = \eta_1 Y \to E(\text{top} \eta_1 B) \) is a sink morphism in \( \text{inj}(B) \). Let \( j: \ker \nu \hookrightarrow J \) be the inclusion. Then

\[
(0, \nu): (\text{Hom}_B(M, \ker \nu), J, j) \rightrightarrows 0I_1,
\]

where \( \tilde{j} \) is the adjoint map to \( \text{Hom}_B(M, j) \), is a sink morphism.

The description of the source morphisms in \( \text{prin}(R)_B^A \) starting at \( 0P_j \) and \( \tilde{Q}_j \) is dual.

### 3. Auslander-Reiten sequences for prinjective modules

Let \( R \) be an Artin algebra of the form (0.1).

In this section we establish the existence of Auslander-Reiten sequences in \( \text{prin}(R)_B^A \) (even in two different ways!) and we give some useful relations of the category \( \text{prin}(R)_B^A \) with other module categories.

We start by recalling some definitions from [19]. By \( \text{mod}^{pg}(R)_A \) we denote the full subcategory of \( \text{mod}(R) \) consisting of modules of the form \( X = (X', X'', \varphi) \) such that \( X' \in \text{pr}(A) \) and \( \varphi: X' \otimes_A M \to X'' \) is onto. Dually, \( \text{mod}^{ic}(R)_B \) is the full subcategory of \( \text{mod}(R) \) consisting of modules \( X = (X', X'', \varphi) \) in \( \text{mod}(R) \) such that \( X'' \in \text{inj}(B) \) and the adjoint map to \( \varphi, \overline{\varphi}: X' \to \text{Hom}_B(M, X'') \) is mono. Observe that modules in \( \text{mod}^{pg}(R)_A \) have no top at \( B \) and modules in \( \text{mod}^{ic}(R)_B \) have no socle at \( A \). The category of adjusted modules \( \text{adj}(R)_B^A \) consists of finitely generated modules of the form \( X = (X', X'', \varphi) \) such that \( \varphi \) is onto and \( \overline{\varphi} \) is mono.

Let \( \Theta_B: \text{prin}(R)_B^A \to \text{mod}^{pg}(R)_A \) be the functor given by \( (X', X'', \varphi) \mapsto (X', \text{Im} \varphi, \text{res} \varphi) \). Dually, \( \Theta^A: \text{prin}(R)_B^A \to \text{mod}^{ic}(R)_B \) is given by

\[
(X', X'', \varphi) \mapsto (\text{Im} \overline{\varphi}, X'', j_{\varphi}),
\]

where \( j_{\varphi} \) is the adjoint map to the inclusion \( \text{Im} \overline{\varphi} \hookrightarrow \text{Hom}_A(M, X'') \).

We get the following commutative diagram:

\[
\begin{array}{ccc}
\text{prin}(R)_B^A & \xrightarrow{\Theta_B} & \text{mod}^{pg}(R)_A \\
\Theta_B & \downarrow \circ & \downarrow \Theta_A \\
\text{adj}(R)_B^A & \xrightarrow{\Theta_B} & \text{mod}^{ic}(R)_B
\end{array}
\]
Clearly, we get

$$
\Theta^A(\tilde{X}) \cong X, \quad \Theta^B(\tilde{Y}) \cong Y \quad \text{for } X \in \text{mod}_{ic}(R_B), \, Y \in \text{mod}^{pg}(R)^A.
$$

**Lemma 3.3.** (a) If $X \in \text{mod}^{pg}(R)^A$ (or $X \in \text{prin}(R)^A$), then there exists a natural epimorphism $\varepsilon_X: X \to \Theta^A(X)$ such that for any morphism $f: Z \to \Theta^A(X)$ with $Z \in \text{mod}^{pg}(R)^A$ (or $Z \in \text{prin}(R)^A$), there is a lifting $\tilde{f}: Z \to X$ with $\varepsilon_X \tilde{f} = f$.

(b) If $X \in \text{mod}_{ic}(R)_B$ (or $X \in \text{prin}(R)^A$), then there exists a natural monomorphism $\eta_X: \Theta^B(X) \to X$ such that for any morphism $f: \Theta^B(X) \to Z$ with $Z \in \text{mod}_{ic}(R)_B$ (or $Z \in \text{prin}(R)^A$), there is an extension $\tilde{f}: Z \to X$ with $\eta_X \tilde{f} = f$.

(c) If $e: 0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is an exact sequence in $\text{adj}(R)^A_B$ (resp. in $\text{mod}_{ic}(R)_B$), then there exists an exact sequence $\tilde{e}: 0 \to \tilde{X} \xrightarrow{\mu} \tilde{E} \xrightarrow{\nu} \tilde{Z} \to 0$ in $\text{mod}^{pg}(R)^A$ (resp. in $\text{prin}(R)^A$) such that $\Theta^A(\tilde{e})$ and $e$ are equivalent sequences.

(d) If $e: 0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is an exact sequence in $\text{adj}(R)^A_B$ (resp. in $\text{mod}_{ic}(R)_B$), then there exists an exact sequence $\tilde{e}: 0 \to \tilde{X} \xrightarrow{\mu} \tilde{L} \xrightarrow{\nu} \tilde{Z} \to 0$ in $\text{mod}_{ic}(R)_B$ (resp. in $\text{prin}(R)^A$) such that $\Theta^B(\tilde{e})$ and $e$ are equivalent sequences.

(e) Let $X$ be an indecomposable in $\text{mod}^{pg}(R)^A$ (or in $\text{prin}(R)^A$). Then $\Theta^A(X) = 0$ if and only if $X \cong 0P_j$ for some $j$. If $X \not\cong 0P_j$, then $\Theta^A(X)$ is indecomposable and $\Theta^A(X) \simeq X$. Moreover, $\ker \Theta^A = [0P_1, \ldots, 0P_n]$, i.e., $\Theta^A(f) = 0$ if and only if $f$ factorizes through a direct sum of $0P_1, \ldots, 0P_n$.

(f) Let $X$ be an indecomposable in $\text{mod}_{ic}(R)_B$ (or in $\text{prin}(R)^A$). Then $\Theta^B(X) = 0$ if and only if $X \cong 0I_t$ for some $t$. If $X \not\cong 0I_t$, then $\Theta^B(X)$ is indecomposable and $\Theta^B(X) \simeq X$. Moreover, $\ker \Theta^B = [0I_1, \ldots, 0I_m]$.

(g) Let $X \in \text{mod}^{pg}(R)^A$ (resp. $X \in \text{prin}(R)^A$) be an indecomposable such that $X \not\cong 0P_j$ ($1 \leq j \leq n$). Then $\Theta^A(X)$ is projective in $\text{adj}(R)^A_B$ (resp. in $\text{mod}_{ic}(R)_B$) if and only if $X$ is projective in $\text{mod}^{pg}(R)^A$ (resp. prin-projective).

(h) Let $X \in \text{mod}_{ic}(R)_B$ (resp. $X \in \text{prin}(R)^A$) be an indecomposable such that $X \not\cong 0I_t$ ($1 \leq t \leq m$). Then $\Theta^B(X)$ is injective in $\text{adj}(R)^A_B$ (resp. in $\text{mod}^{pg}(R)^A$) if and only if $X$ is injective in $\text{mod}_{ic}(R)_B$ (resp. prin-injective).

(i) The functors $\Theta^A, \Theta^B$ are full and dense with

$$
\ker \Theta^A \Theta^B = [0P_1, \ldots, 0P_n, 0I_1, \ldots, 0I_m],
$$

and

$$
\text{prin}(R)^A_B/[0P_1, \ldots, 0P_n, 0I_1, \ldots, 0I_m] \simeq \text{adj}(R)^A_B.
$$

**Proof.** We give some indications of the proofs of (a), (c), (e), (g), (i); the other claims are dual.

(a) Let $X = (X', \ X'', \ \varphi) \in \text{mod}^{pg}(R)^A$. Then there is an onto $A$-morphism $\nu: X' \to \text{Im} \varphi$ such that $\varphi = i_\varphi \circ \nu$, where $i_\varphi$ is the inclusion $\text{Im} \varphi \hookrightarrow \text{Hom}_B(M, \ X'')$. Then $e_X = (\nu, 1_{X''})$: $X \to \Theta^A(X)$ is a natural epimorphism. If $f = (f', \ f'')$: $Z \to \Theta^A(X)$ is a morphism with $Z \in \text{mod}^{pg}(R)^A$, then there is an $A$-morphism $\tilde{f}'': Z' \to X'$ such that $f' = \nu \tilde{f}'$. Therefore, $\tilde{f} = (\tilde{f}', \ f'')$: $Z \to X$ is a morphism such that $f = e_X \tilde{f}$. 

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(c) Consider the following exact and commutative diagram of $A$-modules:

\[
\begin{array}{ccccccccc}
0 & \to & P(X') & \xrightarrow{\mu'} & P & \xrightarrow{\nu'} & P(Z') & \to & 0 \\
\downarrow{p_{X'}} & & \downarrow{p} & & \downarrow{p_{Y'}} & & \downarrow{p_{Z'}} & & \\
0 & \to & X' & \xrightarrow{\nu'} & Y' & \xrightarrow{\nu'} & Z' & \to & 0 \\
\end{array}
\]

where $P = P(X') \oplus P(Z')$. We get an exact sequence in $\text{mod}^{Pg}(R)_A$,

\[\tilde{e} : 0 \to \bar{X} (\mu', \nu'') E = (P, Y'', P \otimes_A M \xrightarrow{p} Y' \otimes_A M \xrightarrow{\nu''} Z) \to 0.\]

Since $\bar{p}_Y(p \otimes 1_{M}) = \bar{p}_Y$, then $\Theta_A(E) = \Theta_A(\tilde{Y}) \simeq Y$. It follows easily that $\Theta_A(\tilde{e})$ and $e$ are equivalent sequences.

(e) This is left to the reader.

(g) Assume that $X \in \text{mod}^{Pg}(R)_A$ is an indecomposable such that $X \not\cong 0_{P_j}$. Then $\Theta_A(X) \neq 0$. Assume that $\Theta_A(X)$ is projective in $\text{adj}(R)_B$.

Let $e : 0 \to V \xrightarrow{\psi} E \xrightarrow{\varphi} X \to 0$ be an exact sequence in $\text{mod}^{Pg}(R)_A$. Consider the following induced exact and commutative diagram of $A$-modules:

\[
\begin{array}{ccccccccc}
0 & \to & K & \xrightarrow{j} & \text{Im} \varphi_E & \to & \text{Im} \varphi_X & \to & 0 \\
\downarrow{i} & & \downarrow{\varphi_E} & & \downarrow{\varphi_X} & & \downarrow{\varphi_X} & & \\
0 & \to & \text{Hom}_B(M, V'') & \to & \text{Hom}_B(M, E'') & \to & \text{Hom}_B(M, X'') & & \\
\end{array}
\]

We get an exact sequence in $\text{adj}(R)_B$

\[\tilde{e} : 0 \to \bar{X} (\mu', \nu'') E = (P, Y'', P \otimes_A M \xrightarrow{\nu''} Y' \otimes_A M \xrightarrow{\psi}) \bar{Z} \to 0.\]

\[\Theta_A(E) = \Theta_A(\bar{Y}) \simeq Y.\]

The endomorphism $\mu \tilde{h}$ satisfies $\Theta_A(\mu \tilde{h}) = \Theta_A(X) \to 0$, where $\bar{\varphi} = i$. Therefore, there exists $h : \Theta_A(X) \to \Theta_A(E)$, a morphism such that $\Theta_A(h) = \varphi_X$. By (a), there is a lifting $\tilde{h} : X \to E$ such that $\varphi_X = \tilde{h} \varphi_X$. Since $X$ is indecomposable, $\nu \tilde{h}$ is an isomorphism. Thus $\mu$ splits and $X$ is projective.

(i) Follows easily. \(\square\)

If $\eta : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is an exact sequence in $\text{mod}(R)$, where $X, Y, Z$ are in $K (= \text{adj}(R)_B, \text{prin}(R)_B, \ldots)$ with $X$ and $Z$ indecomposable, then $\eta$ is an Auslander-Reiten sequence if $f$ is a source map, or, equivalently, $g$ is a sink map in $K$. The category $K$ is said to have Auslander-Reiten sequences if for every indecomposable noninjective (resp. nonprojective) object $X$ in $K$ there exists an Auslander-Reiten sequence in $K$ starting (resp. ending) at $X$ (see [16]). Moreover, $K$ is said to have source maps (resp. sink maps) if for every indecomposable object $X$ in $K$ there exists a source map in $K$ starting (resp. sink map in $K$ ending) at $X$.

**Theorem 3.4.** The categories $\text{adj}(R)_B$, $\text{mod}^{Pg}(R)_A$, $\text{mod}_{ic}(R)_B$, and $\text{prin}(R)_A$ have Auslander-Reiten sequences, source maps, and sink maps.

**Proof.** As observed in [19], the existence of Auslander-Reiten sequences in $\text{adj}(R)_B$ is a direct application of [2].

Let $X$ be a nonprojective indecomposable in $\text{mod}_{ic}(R)_B$. By Lemma 3.3, $\Theta_B(X)$ is indecomposable and nonzero. If $\Theta_B(X)$ is nonprojective, then there
exists an Auslander-Reiten sequence \( e: 0 \rightarrow Z \xrightarrow{u} E \xrightarrow{v} \Theta_B(X) \rightarrow 0 \) in \( \text{adj}(R)^B \).

By (3.3)(d),(f), there is an exact commutative diagram in \( \text{mod}_{ic}(R)_B \) as follows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & Z & \xrightarrow{u} & E & \xrightarrow{v} & \Theta_B(X) & \rightarrow & 0 \\
& & \downarrow{t} & \swarrow{\eta_x} & & & & \downarrow{\eta_x} & \\
0 & \rightarrow & \tilde{Z} & \xrightarrow{\mu} & L & \xrightarrow{\mu} & X \cong \Theta_B(X) & \rightarrow & 0
\end{array}
\]

Moreover, \( \Theta_B(\tilde{e}) \) and \( e \) are equivalent sequences. We claim that \( \tilde{e} \) is an Auslander-Reiten sequence in \( \text{mod}_{ic}(R)_B \).

In fact, let \( f: Y \rightarrow X \) be a noninvertible map in \( \text{mod}_{ic}(R)_B \), where \( Y \) is indecomposable. If \( \Theta_B(f) = 0 \), then \( f \) has a factorization through a direct sum of modules \( 0I_1, \ldots, 0I_m \) (3.3)(f). Since every \( 0I_t \) is projective in \( \text{mod}_{ic}(R)_B \), then \( f = \mu g \) for some \( g: Y \rightarrow L \). Suppose now that \( \Theta_B(f) \neq 0 \). By (3.3)(f), \( \Theta_B(X) \) and \( \Theta_B(Y) \) are indecomposable and \( \Theta_B(f) \) is not invertible. Since \( e \) is an Auslander-Reiten sequence, there exists \( h: \Theta_B(Y) \rightarrow E \) such that \( \Theta_B(f) = vh \). By (3.3)(b), there exists \( g \in \text{Hom}_R(Y, L) \) such that \( th = g \eta_Y \). Hence, \( (\mu g - f)\eta_Y = \mu h - \eta_x \Theta_B(f) = \eta_x(vh - \Theta_B(f)) = 0 \); that is, \( \Theta_B(\mu g - f) = 0 \). As above, \( \mu g - f \) factorizes through \( \mu \). Therefore, \( f \) factorizes through \( \mu \).

Hence our claim follows. If \( \Theta_B(X) \) is projective, see the note added in proof.

For the proof that for every noninjective module \( X \) in \( \text{mod}_{ic}(R)_B \) there exists an Auslander-Reiten sequence starting at \( X \), we recall from [19, §2; 18, §7B] the following. There is a reflection duality

\[
(D^\bullet): \text{mod}_{ic}(R)_B \rightarrow \text{mod}_{ic}(R^\bullet)_B,
\]

where \( R^\bullet \) is the algebra opposite to \( R^\circ \) defined in (3.6) below and \( D^\bullet \) is the composed functor

\[
\text{mod}_{ic}(R)_B \xrightarrow{\nabla^-} \text{mod}^{\phi}(R^\circ)_B \xrightarrow{D} \text{mod}_{ic}(R^\bullet)_B
\]

with \( \nabla^- (X', X'', \varphi) = (\mathcal{M}_B^{-1}(X''), \text{coker } \varphi, \nu_X) \) and \( \nu_X: \mathcal{M}_B^{-1}(X'') \otimes_B \tilde{M} \cong \text{Hom}_A(M', X'') \rightarrow \text{coker } \varphi \).

In view of the duality \( D^\bullet \), the required property easily follows.

The existence of source and sink maps in \( \text{mod}_{ic}(R)_B \) can be shown similarly as in \( \text{prin}(R)^B \) as discussed in §2.

The proof for \( \text{prin}(R)^B \) can be done in a similar way as above, whereas for \( \text{mod}^{\phi}(R^\circ) \) the proof follows from the case above using the duality \( D \).

If \( X \) is an indecomposable nonprojective module in \( \text{prin}(R)^B \), we denote by \( \Delta X \) the starting term of the Auslander-Reiten sequence ending at \( X \). Dually, if \( X \) is indecomposable noninjective, \( \Delta^- X \) is the ending term of the Auslander-Reiten sequence in \( \text{prin}(R)^B \) starting at \( X \).

We introduce a Coxeter scheme similar to that given in [17]. For this purpose, we consider the reflection form

\[
R^\circ = \begin{bmatrix} B & B\tilde{M}_A \\ 0 & A \end{bmatrix}
\]

of \( R \), where \( \tilde{M} = DM \).

We introduce reflection functors in the following way:

\[
S^B: \text{prin}(R)^B \rightarrow \text{mod}_{ic}(R^\circ)_A
\]
defined on objects as
\[ X = (X', X'', \varphi) \mapsto (\ker \varphi, \mathfrak{M}_A(X'), j_X), \]
where \( j_X : \ker \varphi \otimes_B \overline{M} \to \mathfrak{M}_A(X') \) is the adjoint map to the composition \( \ker \varphi \hookrightarrow X' \otimes_A M \cong \Hom_A(M, \mathfrak{M}_A(X')); \)
\[ S_A : \text{prin}(R)^A_B \to \text{mod}^{pg}(R^\vee)^B \]
defined on objects as
\[ X = (X', X'', \varphi) \mapsto (\mathfrak{M}_B^{-1}(X''), \coker \varphi, \nu_X), \]
where \( \nu_X : X' \to \Hom_A(M, X'') \) is the adjoint map to \( \varphi \) and \( \nu_X \) is the composition \( \mathfrak{M}_B^{-1}(X'') \otimes_B \overline{M} \cong \Hom_A(M, X'') \to \coker \varphi. \)

We define dually the functors
\[ S_B : \text{prin}(R^\vee)^B_A \to \text{mod}^{pc}(R)^A, \quad S^A : \text{prin}(R^\vee)^A_B \to \text{mod}^{ic}(R)^B. \]

The partial Coxeter maps are given by
\[
\begin{align*}
\Delta^B : \text{prin}(R)^A_B &\to \text{prin}(R^\vee)^A_B, & \Delta^B(X) &= S^B(X), \\
\Delta^B : \text{prin}(R^\vee)^B_A &\to \text{prin}(R)^A_B, & \Delta^B(Y) &= S_B(Y), \\
\Delta^A : \text{prin}(R^\vee)^B_A &\to \text{prin}(R)^A_B, & \Delta^A(Y) &= S^A(Y), \\
\Delta_A : \text{prin}(R)^A_B &\to \text{prin}(R^\vee)^B_A, & \Delta_A(X) &= S_A(X);
\end{align*}
\]
therefore, by (3.2), we get the following commutative diagrams:
\[
\begin{array}{c}
\text{prin}(R)^A_B \quad \Delta^B \quad \text{mod}^{pg}(R)^A \\
\downarrow S_B \quad \downarrow \Theta_B \\
\text{mod}^{pc}(R)^A \\
\end{array}
\begin{array}{c}
\text{prin}(R^\vee)^B_A \quad \Delta^A \quad \text{mod}^{pc}(R^\vee)^A \\
\downarrow \Theta_A \quad \downarrow S^A \\
\text{mod}^{ic}(R)^B \\
\end{array}
\]

We will use this scheme to give formulas connecting the left-hand and right-hand terms of the Auslander-Reiten sequences in \( \text{prin}(R)^A_B \). We follow ideas in [19, Proposition 2.17, Theorem 3.28; 17, Corollary 3.7].

By \( \text{tr} : \text{mod}(R) \to \text{mod}(R^{op}) \) we denote the usual transpose construction.

**Lemma 3.9.** Let \( X \) be an indecomposable in \( \text{prin}(R)^A_B \). Then
\[ D \text{tr} \Theta_B(X) \cong S^A \Delta^B(X), \quad \text{tr} D \Theta^A(X) \cong S_B \Delta_A(X). \]

In particular, \( D \text{tr} \Theta_B(X) \) is in \( \text{mod}^{ic}(R)^B \) and \( \text{tr} D \Theta^A(X) \) is in \( \text{mod}^{pg}(R)^A \).

**Proof.** First we note that the following diagram
\[
\begin{array}{c}
\text{prin}(R^\vee)^B_A \quad S^A \quad \text{mod}^{ic}(R)^B \\
\downarrow D \quad \downarrow D \\
\text{prin}(R^\vee)^{op}(R^{op}) \quad S_{A^{op}} \quad \text{mod}^{pg}(R^{op})^{B^{op}} \\
\end{array}
\]
is commutative. This follows from the definitions of \( S^A \) and \( S_{A^{op}} \) and the fact that given \( Y_{R^\vee} = (P_B, Q_A, t) \) in \( \text{prin}(R^\vee)^B_A \) we have
\[ D(Y) = (D(Q_A), D(P_B), \psi), \]
where \( \psi \) is the image of \( t : P \otimes_B \tilde{M}_A \to Q \) under the composed isomorphism

\[
\text{Hom}_A(P \otimes_B \tilde{M}_A, Q_A) \cong \text{Hom}_A(D(Q_A), \text{Hom}(P \otimes_B \tilde{M}_A, Q_A)) \\
\cong \text{Hom}_A(D(Q_A), \text{Hom}_B(P_B, \tilde{M}_A)) \\
\cong \text{Hom}_A(D(Q_A), \text{Hom}_B(\tilde{M}_A, D(P_B))) \\
\cong \text{Hom}_B(\tilde{M}_A \otimes_A D(Q_A), D(P_B)).
\]

(3.11)

Now, let \( X_R = (X'_A, X'_B, \phi) \) be in prin \((R)^A_B\) and let \( P_B \) be the projective cover of \( \ker \phi \). We get an exact sequence \( P_B \xrightarrow{u} X'_A \otimes_A M_B \xrightarrow{\phi} X'_B \) and

\[
\Delta^B(X) = (P_B, \eta_A(X'_A), t),
\]

where \( t \) is the adjoint map to the composition

\[
P_B \xrightarrow{u} X'_A \otimes_A M_B \xrightarrow{\eta_A(X'_A)} \text{Hom}_A(D(QA), D(PB)).
\]

Let us consider the module \( Z = S_A \Delta^B(X) \) in \( \text{mod}^g(R)^{Bopp} \). Note that \( Z = (\rho Z', A Z'', h) \), where \( B Z' = \text{Hom}_B(P_B, B) \) and \( h : A M \otimes_B Z' \to A Z'' \) is the cokernel of the composed map

\[
e : \text{Hom}_A(X'_A, A) \xrightarrow{\rho} \text{Hom}_B(M_B, D(P_B)) \to A M \otimes_B Z',
\]

where \( \rho \) is the adjoint map to the image \( \psi \) of \( t \) under the isomorphism (3.11).

On the other hand, \( P_R(\Theta_B(X)) \cong (X'_A, X'_A \otimes_A M_B, id) \) and therefore the sequence

\[
(0, P_B, 0) \xrightarrow{(0, u^*)} P_R(\Theta_B(X)) \to \Theta_B(X) \to 0
\]

is exact. It is not hard to show the commutativity of the following diagram:

\[
\begin{array}{c}
\text{Hom}_R(\Theta_B(X), R) \xrightarrow{(0, u^*)} \text{Hom}_R((0, P_B, 0), R) \\
\downarrow \quad \downarrow \\
(0, \text{Hom}_A(X'_A, A), 0) \xrightarrow{(0, \rho^*)} (\text{Hom}_B(P_B, B), A M \otimes_B \text{Hom}_B(P_B, B), id)
\end{array}
\]

Therefore, \( \text{tr} \Theta_B(X) \cong \text{coker}(0, u^*) \cong \text{coker}(0, \rho^*) \cong Z \). Since the diagram (3.10) is commutative, we get \( S^A \Delta^B(X) \cong D(Z) \cong D \text{tr} \Theta_B(X) \), as desired.

The remaining part of the lemma follows in a similar way. \( \square \)

Lemma 3.12. Let \( 0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0 \) be an Auslander-Reiten sequence in \( \text{mod}(R) \) with \( X, Z \) indecomposable.

(a) If \( X \) is in \( \text{mod}_{ic}(R)_B \), then \( Z \) is in \( \text{mod}^p(R)^A \). If \( Z \) is not of the form \( 0 P_j \), then the induced sequence \( 0 \to X \to \Theta^4(Y) \to \Theta^4(Z) \to 0 \) is exact.

(b) If \( Z \) is in \( \text{mod}^p(R)^A \), then \( X \) is in \( \text{mod}_{ic}(R)_B \). If \( X \) is not of the form \( 0 I_i \), then the induced sequence \( 0 \to \Theta_B(X) \to \Theta_B(Y) \to Z \to 0 \) is exact.

Proof. (a) If \( X \) is in \( \text{mod}_{ic}(R)_B \) then \( X \cong \Theta^4(X) \) and by (3.9), \( Z \cong \text{tr} D(X) \cong \text{tr} D \Theta^4(X) \cong S_B \Delta_A(X) \) is in \( \text{mod}^p(R)^A \).

Let \( X = (X', X'', \phi), \ Y = (Y', Y'', \psi), \) and \( Z = (Z', Z'', \lambda) \). We have a commutative diagram

\[
\begin{array}{c}
\ker \varphi & \overset{\iota'}{\longrightarrow} & \ker \lambda \\
\downarrow & \downarrow & \downarrow \\
0 & \to & X' \xrightarrow{u'} Y' \xrightarrow{i'} Z' \to 0 \\
\downarrow \varphi & \downarrow \psi & \downarrow \iota \\
\text{Im} \varphi & \overset{\iota''}{\longrightarrow} & \text{Im} \psi & \overset{i''}{\longrightarrow} & \text{Im} \lambda \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & \text{Hom}_B(M, X'') & \to & \text{Hom}_B(M, Y'') & \to \text{Hom}_B(M, Z'') & \to 0
\end{array}
\]
with the second and the last row exact, where $\overline{\varphi}, \overline{\psi}, \overline{\lambda}$ are the adjoint maps to $\varphi, \psi, \lambda$ and $\overline{u'}, \overline{v'}, \overline{t'}_0$ are maps induced by $u, v, t$. It is clear that $\overline{t'}_0$ and $\overline{u'}$ are injective, $\overline{v'}$ is surjective, and $\overline{v'}\overline{u'} = 0$. We claim that $\overline{t'}_0$ is surjective. If $\ker \overline{\lambda} = 0$, this is clear. If $\ker \overline{\lambda} \neq 0$, the inclusion $(\ker \overline{\lambda}, 0, 0) \hookrightarrow \mathbb{Z}$ is nonsurjective (since otherwise $\mathbb{Z} \cong \mathfrak{P}_j$ for some $j$) and by the right almost split property, it has a factorization through $t$. It follows that $\overline{t'}_0$ is surjective and hence it is an isomorphism. The snake lemma implies that $\ker \overline{v'} = \text{Im} \overline{u'}$ and (a) follows. Statement (b) is dual to (a). \[ \square \]

**Theorem 3.13.** If $0 \to X \to Y \to Z \to 0$ is an Auslander-Reiten sequence in $\text{prin}(R)_A^\mathfrak{p}$, then

$$X \cong D\text{tr} \Theta_B(Z) \cong \Delta^A \Delta^B(Z) \text{ and } Z \cong \text{tr} \Theta^A(X) \wedge \cong \Delta_B \Delta_A(X).$$

**Proof.** Let $X$ be an indecomposable non-prin-injective module in $\text{prin}(R)_A^\mathfrak{p}$. By (3.3)(a),(e), $0 \cong \mathfrak{P}^A(X)$ is indecomposable and not injective in $\text{mod}(R)$ (compare [19, Theorem 3.28]). Let

$$e: 0 \to \Theta^A(X) \to W \overset{u}{\to} L \to 0$$

be an Auslander-Reiten sequence in $\text{mod}(R)$. Assume first that $L \cong \mathfrak{P}_j$. Then as in the proof of (3.4) we get an Auslander-Reiten sequence

$$\tilde{e}: 0 \to \Theta^A(X) \to E \overset{\overline{u}}{\to} \overline{L} \to 0.$$

Since $L = \overline{L}$ and $X \cong \Theta^A(X)$, it follows that

$$Z \cong L = \overline{L} = \text{tr} \Theta^A(X) \wedge \cong S_B \Delta_A(X) \wedge \cong \Delta_B \Delta_A(X). \quad (3.9)$$

Assume $L$ is not of the form $\mathfrak{P}_j$. By (3.12), we get an exact sequence

$$0 \to \Theta^A(X) \overset{u}{\to} \Theta^A(W) \to \Theta^A(L) \to 0$$

and also an exact nonsplit sequence in $\text{mod}_{ic}(R)_B$,

$$e': 0 \to \Theta^A(X) \overset{u'}{\to} \Theta^A(W) \to \Theta^A(L) \to 0,$$

where $u'$ is the composition of $u$ with the natural embedding $\Theta^A(W) \to \Theta^A(W)$. Since $L$ is in $\text{mod}^{\mathfrak{P}^A}(R)_A$ by (3.9), $\Theta^A(L)$ is an indecomposable module in $\text{adj}(R)_B^\mathfrak{p}$ and $\Theta^A(L)$ is indecomposable in $\text{mod}_{ic}(R)_B$. It is easy to check that $u'$ is a source map in $\text{mod}_{ic}(R)_B$ and, therefore, $e'$ is an Auslander-Reiten sequence in $\text{mod}_{ic}(R)_B$. Applying (3.3) (as in the proof of (3.4)), we get an Auslander-Reiten sequence

$$0 \to \Theta^A(X) \to E \to \Theta^A(L) \to 0$$

in $\text{prin}(R)_B^\mathfrak{p}$. Since $\Theta^A(X) \cong X$, it follows that

$$Z \cong \Theta^A(L) \cong \overline{L} \cong \text{tr} D\Theta^A(X) \wedge \cong \Delta_B \Delta_A(X).$$

The remaining part is dual. \[ \square \]

We remark that Theorem 3.13 gives an explicit construction of the Auslander-Reiten sequences in $\text{prin}(R)_B^\mathfrak{p}$ starting with those in $\text{mod}(R)$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Corollary 3.14. $\Delta = \Delta^A \Delta^B$ and $\Delta^- = \Delta_B \Delta_A$. \hfill $\square$

We denote by $\overline{\text{prin}(R)^4_B}$ (resp. $\overline{\text{prin}(R)^4_B}$) the factor category of $\text{prin}(R)^4_B$ modulo the ideal formed by all maps which admit a factorization through a prin-projective (resp. prin-injective) module. The corresponding Hom functor is denoted by $\overline{\text{Hom}}$ and $\overline{\text{Hom}}$, respectively. We shall prove the relative Auslander-Reiten formula.

Proposition 3.15. (a) For $X, Y \in \text{prin}(R)^4_B$ there are natural isomorphisms

$$D\text{Ext}_R^1(X, Y) \simeq \overline{\text{Hom}}_R(Y, \Delta X) \simeq \overline{\text{Hom}}_R(\Delta^- Y, X).$$

(b) $\Delta$ and $\Delta^-$ induce inverse equivalences

$$\overline{\text{prin}(R)^4_B} \xrightarrow{\Delta} \overline{\text{prin}(R)^4_B},$$

$$\text{adj}(R)^4_B / [e_1 R, \ldots, e_n R] \simeq \text{adj}(R)^4_B / [Q_1, \ldots, Q_m],$$

where $Q_i = E_R(\text{top} \eta_i B)$.

Proof. (a) Let $e: 0 \to \Delta X \xrightarrow{\mu} E \xrightarrow{\nu} X \to 0$ be an Auslander-Reiten sequence in $\text{prin}(R)^4_B$. Choose $\varphi \in D\text{Ext}_R^1(X, \Delta X)$ such that $\varphi(e) \neq 0$. We define a natural morphism

$$h: \text{Hom}_R(-, \Delta X) \to D\text{Ext}_R^1(X, -)$$

by $h_{\Delta X}(1_{\Delta X}) = \varphi$.

Let $Y \in \text{prin}(R)^4_B$. We show that $h_Y$ is epi by proving that $Dh_Y$ is mono. Let $0 \neq \eta \in \text{Ext}_R^1(X, Y)$. Then we obtain the following exact commutative diagram:

$$\begin{array}{cccccc}
\eta: 0 & \to & Y & \xrightarrow{f} & E' & \xrightarrow{f} & X & \to & 0 \\
& & & \downarrow \ | & \downarrow & & \downarrow \ | & & \\
0 & \to & \Delta X & \to & E & \to & X & \to & 0
\end{array}$$

Thus, $f \eta = e$ and $Dh_Y(\eta)(f) = \varphi(f \eta) = \varphi(e) \neq 0$.

Clearly, $h$ factorizes through $\overline{\text{Hom}}_R(-, \Delta X)$. Then we get a natural epimorphism $\overline{h}: \overline{\text{Hom}}_R(-, \Delta X) \to D\text{Ext}_R^1(X, -)$.

Let $f \in \text{Hom}_R(Y, \Delta X)$ and assume that $f$ does not admit a factorization through prin-injective modules. By (2.4), there is an exact sequence $\eta: 0 \to Y \xrightarrow{\eta} U_0 \to U_1 \to 0$, where $U_0$ and $U_1$ are prin-injective modules. Therefore, $0 \neq f \eta \in \text{Ext}_R^1(U_1, \Delta X)$ and we get a morphism $g \in \text{Hom}_R(X, U_1)$ such that $(f \eta)g = \varepsilon$. Thus we have constructed $\eta g \in \text{Ext}_R^1(X, Y)$ such that $h_Y(f)(\eta g) = \varphi(f(\eta g)) = \varphi(e) \neq 0$. This shows that $\overline{h}$ is a natural isomorphism.

The remaining part of the proof is simple. \hfill $\square$

In [10], the existence of Auslander-Reiten sequences in $\text{prin}(R)^4_B$ is shown by means of functorial methods. We give here a brief survey of the main ideas.

Let $X$ be any module in $\text{mod}(R)$. We denote the restriction

$$\text{Hom}_R(-, X)|\text{prin}(R)^4_B$$

by $\text{prin}_R(-, X)$.
Proposition 3.16. Let \( X \in \text{mod}(R) \). Then there exists a module \( \overline{X} \) in \( \text{prin}(R)_{B}^{A} \) such that \( \text{prin}_{R}(-, \overline{X}) \to \text{prin}_{R}(-, X) \) is a projective cover in the category of contravariant functors from \( \text{prin}(R)_{B}^{A} \) to \( Ab \).

Proof. Let 
\[
X = (X_{A}', X_{B}', \varphi) \in \text{prin}(R)_{B}^{A}.
\]

First we show that \( \text{Hom}_{B}(-, X'')|_{\text{inj}(B)} \) is finitely generated in the category of contravariant functors from \( \text{inj}(B) \) to \( Ab \). Indeed, let \( 0 \to X'' \to I_{0} \to I_{1} \) be an injective presentation of \( X'' \) in \( \text{mod}(B) \). Consider the exact sequence
\[
0 \to K \to \Omega_{B}^{-1}(I_{0}) \xrightarrow{\varphi_{i}^{-1}(i)} \Omega_{B}^{-1}(I_{1})
\]
and let \( P \xrightarrow{p} K \) be a projective cover. As \( MBU) = \Omega_{B}^{-1}(I_{1}) = 0 \), we get the following commutative diagram:
\[
\begin{array}{ccc}
\Omega_{B}(P) & \xrightarrow{\varphi_{i}} & \Omega_{B}^{-1}(I_{1}) \\
\varphi_{i}(p) \downarrow & & \downarrow \\
\Omega_{B}(K) & \xrightarrow{s} & \Omega_{B}(j) \\
0 \to X'' \xrightarrow{m} I_{0}
\end{array}
\]

Therefore, \( \nu = s\Omega_{B}(p) : I = \Omega_{B}(P) \to X'' \) yields the wanted morphism.

Consider the fibered product in \( \text{mod}(A) \):
\[
L \xrightarrow{h} \text{Hom}_{B}(M, I) \quad \text{Hom}_{B}(M, \nu) \\
\psi \downarrow \quad \downarrow \text{Hom}_{B}(M, X'')
\]
where \( \overline{\varphi} \) is the adjoint map to \( \varphi \). Let \( P(L) \xrightarrow{l} L \) be a projective cover in \( \text{mod}(A) \) and let \( \psi : P(L) \otimes_{A} M \to I \) be the adjoint map to \( hl : P(L) \to \text{Hom}_{B}(M, I) \). Then
\[
f = (tl, \nu) : \overline{X} = (P(L), I, \psi) \to X
\]
yields an onto transformation
\[
(-, f) : \text{prin}_{R}(-, \overline{X}) \to \text{prin}_{R}(-, X)
\]
in the category \( \text{prin}(R)_{B}^{A}, Ab \). The existence of the projective cover easily follows. \( \Box \)

The existence of Auslander-Reiten sequences can be proved as follows. By (3.16), the category of all finitely presented modules \( F : (\text{prin}(R)_{B}^{A})^{\text{op}} \to Ab \) is abelian. Let \( F : (\text{prin}(R)_{B}^{A})^{\text{op}} \to Ab \) be a finitely presented functor; then \( DF : \text{prin}(R)_{B}^{A} \to Ab \) is also finitely presented. If \( 0 \neq F \), a simple quotient of \( DF \) provides a simple subfunctor of \( F \). Let \( Z \) be an indecomposable non-prin-projective module in \( \text{prin}(R)_{B}^{A} \). Then the functor \( \text{Ext}_{R}^{1}(Z, -) : \text{prin}(R)_{B}^{A} \to Ab \) is finitely presented and nonzero. Let \( S_{X} \) be a simple subfunctor of \( \text{Ext}_{R}^{1}(Z, -) \)
with $X$ an indecomposable module in $\text{prin}(R)^A_B$ such that $S_X(X) \neq 0$. An exact sequence $\varepsilon : 0 \to X \to Y \to Z \to 0$ which generates $\text{Im}(S_X \hookrightarrow \text{Ext}^1_R(Z, -))$ is the wanted sequence.

In [10] the following formula is shown by applying the functorial arguments above.

**Proposition 3.17.** Let $Z \in \text{prin}(R)^A_B$ be an indecomposable non-prin-projective module. There exists a prin-injective module $I$ such that $D\text{tr}Z \cong \Delta Z \oplus I$. □

The Auslander-Reiten quiver $\Gamma(\text{prin}(R)^A_B)$ of the category $\text{prin}(R)^A_B$ is defined as usual.

4. **Algebras over Fields**

Let $R$ be a basic finite-dimensional algebra over a field $k$. In the case $R$ is schurian of upper triangular form we will study the representation type of $\text{prin}(R)^A_B$ in terms of a Cartan matrix and a corresponding quadratic form. The methods will be particularly successful when $R$ is of finite prinjective type (i.e., there are only finitely many indecomposable modules in $\text{prin}(R)^A_B$, up to isomorphism) and the Auslander-Reiten quiver $\Gamma(\text{prin}(R)^A_B)$ is a preprojective component.

We recall that we have fixed complete sets $e_1, \ldots, e_n$ and $\eta_1, \ldots, \eta_m$ of pairwise orthogonal primitive idempotents for $A$ and $B$ respectively. We define numbers

$$a_{ij} = \dim_k e_i A e_j, \quad b_{st} = \dim_k \eta_s B \eta_t, \quad c_{is} = \dim_k e_i M \eta_s,$$

for all $i, j \in \{1, \ldots, n\}$ and $s, t \in \{1, \ldots, m\}$. We consider the bilinear form

$$(4.1) \langle \cdot, \cdot \rangle_R: \mathbb{Z}^{n+m} \times \mathbb{Z}^{n+m} \to \mathbb{Z}$$

defined by the formula

$$\langle z, w \rangle_R = \sum_{i,j=1}^n a_{ij} z_i w_j - \sum_{i=1}^n \sum_{s=1}^m c_{is} z_i w_{n+s} + \sum_{s,t=1}^m b_{st} z_{n+s} w_{n+t}.$$

The associated quadratic form $\chi_R(z) = \langle z, z \rangle_R$ is called the Tits form of $\text{prin}(R)^A_B$.

Given $X = (X', X'', \varphi)$ in $\text{prin}(R)^A_B$, we consider decompositions

$$X' = \bigoplus_{i=1}^n (e_i A)^{x_i} \quad \text{and} \quad X'' = \bigoplus_{s=1}^m (DB \eta_s)^{x_{n+s}}.$$

Following [4, 17], we define the coordinate vector of $X$ as $\text{cdn} X = (x_i)_i \in \mathbb{N}^{n+m}$. The dimension vector of $X$ is $\text{dim} X = ((\dim_k X'e_i)_i, (\dim_k X''\eta_s)_s) \in \mathbb{N}^{n+m}$.

In the case $k$ is algebraically closed, we say that $R$ is of tame prinjective type if for each vector $w \in \mathbb{N}^{n+m}$, there is a finite family $M_i^{(w)}, \ldots, M_s^{(w)}$ of $(k[x] - R)$-bimodules which are free as $k[x]$-modules and such that every indecomposable $X \in \text{prin}(R)^A_B$ with $\text{cdn} X = w$ is isomorphic to $N \otimes_{k[x]} M_i^{(w)}$ for some $i$ and some simple $k[x]$-module $N$.

One of the motivations for considering the Tits form is the following
Proposition 4.2. (a) If \( R \) is of finite prinjective type, then \( \chi_R \) is weakly positive (i.e., \( \chi_R(z) > 0 \), for \( 0 \neq z \in \mathbb{N}^{n+m} \)).

(b) If \( k \) is algebraically closed and \( R \) is of tame prinjective type, then \( \chi_R \) is weakly nonnegative (i.e., \( \chi_R(z) \geq 0 \), for \( z \in \mathbb{N}^{n+m} \)).

Proof. We recall here briefly the arguments given in [5] for (a) and in [15] for (b).

(a) Let \( z \in \mathbb{N}^{n+m} \). By \( V_z \) we denote the variety of all \( X \in \text{prin}(R)^n \) with \( \text{cdn} \ X = z \). Therefore, \( \dim V_z = \sum_{i=1}^{n} \sum_{s=1}^{m} c_{is} z_i z_{n+s} \).

Let \( P_z = \bigoplus_{i=1}^{n} (e_i A)^{z_i} \) and \( I_z = \bigoplus_{s=1}^{m} (DB^s) z_{n+s} \). Isomorphy in \( V_z \) is given by the action of the group \( G(z) \) with elements of the form \( (\alpha \otimes \beta, \alpha) \) \( \in \text{Aut}_A P_z \times \text{Aut}_B I_z \). Hence,

\[
\dim G(z) \leq \sum_{i,j=1}^{n} a_{ij} z_i z_j + \sum_{s,t=1}^{m} b_{st} z_{n+s} z_{n+t}.
\]

As \( k^* \subset G(z) \) acts trivially and there are only finitely many orbits under the action of \( G(z) \) on \( V_z \), \( \dim G(z) - 1 \geq \dim V_z \). Thus, \( \chi_R(z) \geq 1 \).

(b) Let \( w \in \mathbb{N}^{n+m} \). There are modules \( M^{(w)}_1, \ldots, M^{(w)}_{s_w} \) as in the definition. Let \( V_1^{(w)}, \ldots, V_{s_w}^{(w)} \) be the varieties of dimension at most 1 in \( V_w \) obtained as the images of the functors \( M_i^{(w)} \otimes_k [x]^{-} \) evaluated in the \( k[x] \)-simples. For any family of vectors \( (w_i)_i \) with \( w_i \in \mathbb{N}^{n+m} \), \( \sum_i w_i = \bar{z} \), and numbers \( (j_i)_i \) with \( j_i \in \{1, \ldots, s_w\} \), we consider the (algebraic) map \( \prod_i V_{j_i}^{(w_i)} \times G(z) \rightarrow V_z \), \( ((M_i)_i, g) \mapsto (\bigoplus_i M_i)^g \). Clearly, any module in \( V_z \) is in the image of one of these maps. Since there are only finitely many of these maps and their images are constructible,

\[
\dim V_z \leq \max \left\{ \sum_i \dim V_{j_i}^{(w_i)} \right\} + \dim G(z).
\]

Therefore,

\[
\dim V_z \leq |z| + \dim G(z),
\]

where \( |z| = \sum_{i=1}^{n} z_i + \sum_{s=1}^{m} z_{n+s} \). Thus, \( \chi_R(z) \geq -|z| \). If \( \chi_R(z) = -s \), for some \( s > 0 \), then \(-l|z| \leq \chi_R(lz) = l^2 \chi_R(z) = -l^2 s \) and \( |z| \geq l s \), for every \( l \in \mathbb{N} \). A contradiction proving that \( \chi_R(z) \geq 0 \). □

The following remark in [16, 2.5] is useful: Let \( e : 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \) be an exact sequence in \( \text{prin}(R)_B^A \). Then there exists a morphism \( \delta \in \text{Hom}_A(Z', \text{Hom}_B(M, X'')) \) such that \( e \) is equivalent to the sequence:

\[
0 \rightarrow X' \xrightarrow{(\bar{\delta})} X' \oplus Z' \xrightarrow{(0,1)} Z' \rightarrow 0
\]

where \( X = (X', X'', \varphi_X) \) and \( \bar{\varphi}_X \) is the adjoint map to \( \varphi_X \); similarly for \( Y, Z \).

We write \( e = [\delta] \). We get an exact sequence

\[
0 \rightarrow \text{Hom}_R(Z, X) \xrightarrow{\iota} \text{Hom}_A(Z', X') \times \text{Hom}_B(Z'', X'')
\]

\[
\xrightarrow{\iota_2} \text{Hom}_A(Z', \text{Hom}_B(M, X'')) \xrightarrow{\iota} \text{Ext}_R(Z, X) \rightarrow 0
\]
with \( \nu_1(g) = (g', g'') \), \( \nu_2(\alpha, \beta) = \overline{\alpha} \overline{\alpha} - \text{Hom}_B(M, \beta) \overline{\beta} \), and \( \nu(\delta) = [\delta] \).

As a direct consequence we have

**Proposition 4.4.** Let \( X, Z \in \text{prin}(R)^A \). Then

\[
(\text{cdn } Z, \text{cdn } X)_R = \dim_k \text{Hom}_R(Z, X) - \dim_k \text{Ext}^1_R(Z, X). \quad \square
\]

We now develop some general arguments about reflections that will be useful for our considerations of algebras over fields.

Following [14, 20] by a bipartite Cartan matrix we shall mean a matrix \( C \) of the form

\[
C = \begin{bmatrix}
  1 & a_{12} & \cdots & a_{1n} & c_{11} & c_{12} & \cdots & c_{1m} \\
  a'_{12} & 1 & \cdots & a'_{1n} & c_{21} & c_{22} & \cdots & c_{2m} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  a'_{in} & a'_{in} & \cdots & 1 & c_{n1} & c_{n2} & \cdots & c_{nm} \\
  c'_{11} & c'_{12} & \cdots & c'_{1n} & 1 & b_{12} & \cdots & b_{1m} \\
  c'_{12} & c'_{22} & \cdots & c'_{n2} & 1 & b_{12} & \cdots & b_{2m} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  c'_{im} & c'_{rm} & \cdots & c'_{nm} & 1 & b'_{1m} & b'_{2m} & \cdots & b'_{nm}
\end{bmatrix}
\]

with integral nonnegative coefficients. We say that \( C \) is symmetrizable by positive natural numbers \( f_1, \ldots, f_n, g_1, \ldots, g_m \) if \( a_{ij} f_j = f_i a_{ij} \), \( b_{ij} g_j = g_i b_{ij} \), and \( c_{ij} g_j = f_i c'_{ij} \) for all \( i, j \). In this situation we construct the bilinear form

\[
\langle \cdot, \cdot \rangle_C : \mathbb{Z}^{n+m} \times \mathbb{Z}^{n+m} \to \mathbb{Z}
\]

given by the formula

\[
\langle x, y \rangle_C = \sum_{i,j=1}^n a_{ij} f_j x_i y_j - \sum_{i=1}^n \sum_{s=1}^m c_{is} g_s x_i y_{n+s} + \sum_{s,t=1}^m b_{st} g_t x_{n+s} y_{n+t},
\]

where \( a_{ii} = 1 = b_{ss} \). Let \( \chi_C(z) = \langle z, z \rangle_C \) be the quadratic form associated to \( \langle \cdot, \cdot \rangle_C \) and let

\[
\langle -,- \rangle_C : \mathbb{Z}^{n+m} \times \mathbb{Z}^{n+m} \to \mathbb{Z}
\]

be the associated symmetric bilinear form, i.e., \( \langle z, w \rangle_C = \frac{1}{2} \langle (z, w)_C + (w, z)_C \rangle \).

Let \( \xi_1, \ldots, \xi_{n+m} \in \mathbb{Z}^{n+m} = \mathbb{Z}^n \oplus \mathbb{Z}^m \) be the standard basis (i.e., \( \xi_i(j) = \delta_{ij} \) is the Kronecker delta). We define reflections

\[
\delta_j : \mathbb{Z}^{n+m} \to \mathbb{Z}^{n+m}, \quad j = 1, \ldots, n + m,
\]

by the formulas

\[
\delta_i(z) = z - \frac{2 \langle z, \xi_i \rangle_C}{f_i} \xi_i \quad \text{for } i \leq n
\]

and

\[
\delta_{n+t}(z) = z - \frac{2 \langle z, \xi_{n+t} \rangle_C}{g_t} \xi_{n+t} \quad \text{for } t = 1, \ldots, m.
\]

The composed map

\[
\delta = \delta_1 \cdots \delta_n \cdots \delta_{n+m}
\]

is called the Coxeter transformation associated to \( C \). Clearly, \( \chi_C(\delta(z)) = \chi_C(z) \) for any vector \( z \in \mathbb{Z}^{n+m} \).
Lemma 4.8. Let \( z \in \mathbb{Z}^{n+m} \). Then

(a) \( \delta(z) = \delta_j \delta_{j+1} \cdots \delta_{n+m}(z) \);
(b) \( \langle \delta_{j+1} \cdots \delta_{n+m}(z), \xi_j \rangle_C = \langle z, \xi_j \rangle_C \).

Proof. (a) is straightforward.
(b) is by downwards induction on \( j \). If \( j = n + m \), there is nothing to show. Let \( j < n + m \). We may assume that \( n \leq j \); then

\[
\langle \delta_{j+1} \cdots \delta_{n+m}(z), \xi_j \rangle_C = \langle \delta_{j+2} \cdots \delta_{n+m}(z) - \frac{2(z, \xi_{j+1})}{g_{j+1-n}} \xi_{j+1}, \xi_j \rangle_C.
\]

Since \( \langle \xi_{j+1}, \xi_j \rangle_C = 0 \), the result follows by induction. \( \square \)

The following result is similar to [16, 2.4].

Lemma 4.9. For any two vectors \( v, w \in \mathbb{Z}^{n+m} \) we have \( \langle w, \delta(v) \rangle_C = -\langle v, w \rangle_C \).

Proof. It is enough to show the formula for \( w = \xi_j, j = 1, \ldots, n + m \). Let \( n < j < n + m \). By applying (4.8) we get

\[
\langle \xi_j, \delta(v) \rangle_C = g_{j-n}\delta(v)_j + \sum_{i=j-n+1}^{m} b_{(j-n)i} g_i \delta(v)_{n+i} = g_{j-n} \delta_{j+1} \cdots \delta_{n+m}(v)_j - 2(\delta_{j+1} \cdots \delta_{n+m}(v), \xi_j)_C + \sum_{i=j-n+1}^{m} b_{(j-n)i} g_i \delta(v)_{n+i}.
\]

Since

\[
2(\delta_{j+1} \cdots \delta_{n+m}(v), \xi_j)_C = \langle v, \xi_j \rangle_C + \langle \xi_j, \delta_{j+1} \cdots \delta_{n+m}(v) \rangle_C = \langle v, \xi_j \rangle_C + \sum_{i=j-n}^{m} b_{(j-n)i} g_i \delta_{j+1} \cdots \delta_{n+m}(v)_{n+i},
\]

we get the desired result. The proof for \( 1 \leq j \leq n \) is similar. \( \square \)

We say that \( R \) has a schurian upper triangular form if \( R \) has the shape

\[
R = \begin{bmatrix}
A_1 & iA_j & iM_1 & \cdots & iM_m \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & A_n & iM_1 & \cdots & iM_m \\
0 & \cdots & 0 & B_1 & \cdots & B_m
\end{bmatrix},
\]

where \( A_1, \ldots, A_n \) and \( B_1, \ldots, B_m \) are division rings, and \( iA_j, iM_s \) and \( iB_t \) are bimodules. We assume that \( A_i = e_iAe_i, \ iA_j = e_iAe_j, \ B_s = \eta_sB\eta_s, \ iB_t = \eta_sB\eta_t, \) and \( iA_j = e_iR\eta_j \).

We associate to \( R \) the bipartite Cartan matrix \( C = C(R) \) with \( a_{ij} = \dim(iA_jA_i), \ a'_{ij} = \dim(A_iA_j), \ c_{it} = \dim(iM_tB_i), \ c'_{it} = \dim(A_i(iM_t), \ b_{st} = \dim(sB_t)B_i, \) and \( b'_{st} = \dim(B_s(iB_t)). \) Then \( C(R) \) is symmetrizable by the numbers \( f_j = \dim_k A_j \) and \( g_t = \dim_k B_t. \)
It is clear that
\[(4.11) \langle \cdot, \cdot \rangle_R = \langle \cdot, \cdot \rangle_{C(R)}.
\]

From here on, we assume that \( R \) is an indecomposable ring (or connected in the sense of [9]). In this case, by well-known arguments of Auslander, \( R \) is of finite prinjective type if and only if \( \Gamma(\text{prin}(R)^A_B) \) has a finite component.

**Lemma 4.12.** Assume that \( R \) is of finite prinjective type and \( \Gamma(\text{prin}(R)^A_B) \) has a preprojective component. Then:

(a) \( R \) has (up to isomorphism) a schurian upper triangular form \((4.10)\).
(b) \( C(R) \) has the property that \( c_{is}c'_{is} \leq 3 \) for all \( i, s \). If in addition \( R \) is an algebra over an algebraically closed field, then \( c_{is}c'_{is} \leq 1 \).

**Proof.** (a) Any cycle \( \eta_iB\eta_i \neq 0, \eta_iB\eta_i \neq 0, \ldots, \eta_iB\eta_i \neq 0 \) would produce a cycle \( \text{Hom}_R(P_0, P_1) \neq 0, \text{Hom}_R(P_1, P_0) \neq 0, \ldots, \text{Hom}_R(P_1, P_0) \neq 0 \) and thus is in \( \Gamma(\text{prin}(R)^A_B) \). Here, \( P_0 = (0, \eta_iB, 0) \in \text{prin}(R)^A_B \). If \( x \in \eta_iB\eta_i \) (resp. \( x \in e_iAe_j \)) is noninvertible, then we get a noninvertible endomorphism of \( P_0 \) (resp. of \( (e_iA, 0, 0) \)).

(b) Let \( 1 \leq i \leq n \) and \( 1 \leq s \leq m \). Let
\[ R' = \begin{bmatrix} A_i & M_s \\ 0 & B_s \end{bmatrix}. \]
There is a fully-faithful functor \( \phi : \text{prin}(R')^A_B \rightarrow \text{prin}(R)^A_B \) such that
\[ \text{res} \circ \phi(X) \cong X \]
(compare with [18, 19]). Therefore, \( R' \) is of finite prinjective type. The Tits form associated with \( \text{prin}(R')^A_B \) is
\[ f_ix^2 + g_sy^2 - c_{is}g_sp_{xy}. \]
By \((4.2)\), this form is weakly positive; then the discriminant
\[ c_{is}^2g_s^2 - 4f_ig_s = c_{is}c'_{is}f_ig_s - 4f_ig_s \]
is negative. Thus, \( c_{is}c'_{is} < 4 \).

If \( R \) is a \( k \)-algebra with \( k \) algebraically closed, then \( f_i = 1 = g_s \) for all \( i, s \). Therefore, \( c_{is} = c'_{is} \) and the condition above implies \( c_{is}c'_{is} \leq 1 \).

A large class of examples of rings \( R \) satisfying the assumptions of the above lemma are the \( sp \)-representation-finite right peak algebras studied in [14] and the piecewise peak algebras of finite prinjective type studied in [20]. In both cases there is a diagrammatic characterization of these algebras and sincere algebras are described.

**Proposition 4.13.** Let \( R \) be a ring of the form \((0.1)\) and assume that \( \Gamma(\text{prin}(R)^A_B) \) has a preprojective component. Then \( R \) is of finite prinjective type if and only if \( \chi_R \) is weakly positive. Moreover, in this case, if \( R \) is a \( k \)-algebra with \( k \) an algebraically closed field, then:

(a) \( X \leftrightarrow \text{cdn} X \) is a bijection between the indecomposable modules in \( \text{prin}(R)^A_B \) and the positive roots of \( \chi_R \).
(b) If \( X \) and \( Y \) are indecomposables in \( \text{prin}(R)^A_B \) and \( \dim X = \dim Y \), then \( X \cong Y \).
Proof. The first claim and (a) follow from the well-known Drozd’s arguments [8, 16].

(b) Consider the matrix

\[
H = \begin{pmatrix}
\dim_k e_i M e_j & \dim_k e_i M \eta_i \\
0 & \dim_k \eta_i B \eta_i
\end{pmatrix}.
\]

Then we have

\[
\left( \dim_k X'e_i, \left( \sum_{i=1}^n (\dim_k X'e_i) (\dim_k e_i M \eta_i) + \dim_k X'' \right) \right) = (\text{cdn } X)H.
\]

By (4.12), \( H \) is invertible and the result follows from (a).

Our aim is to use the Coxeter transformation \( \delta \) associated with \( C(R) \) to calculate the Auslander-Reiten translation in \( \text{prin}(R) \). We start by showing that in some cases \( \delta \) preserves positiveness of vectors.

**Lemma 4.14.** Let \( R \) be of finite prinjective type over an algebraically closed field and such that \( \Gamma(\text{prin}(R)^A) \) has a preprojective component. Let \( v \in \mathbb{N}^{n+m} \) be such that \( \chi_R(v) = 1 \). Then:

(a) \(-1 \leq 2(v, \xi_i)_R \leq 1 \) for \( 1 \leq i \leq n + m \) with \( v_i \neq 0 \).

(b) If \( v \) is a sincere vector (that is, \( v(i) > 0 \) for every \( 1 \leq i \leq n + m \)), then \( \delta(v) \in \mathbb{N}^{n+m} \).

**Proof.** (a) Since \( \chi_R(\xi_i) = 1 \) and \( v - \xi_i \) is a vector with nonnegative coordinates, \( 0 < \chi_R(v - \xi_i) = 2 - 2(v, \xi_i)_R \) and \( 2(v, \xi_i)_R \leq 1 \). Similarly, \( -1 \leq 2(v, \xi_i)_R \).

(b) By downwards induction on \( j \) \((0 < j < n + m)\) we show that \( \delta_j \cdots \delta_{n+m}(v) \in \mathbb{N}^{n+m} \).

If \( j = n + m \), there is nothing to show. Let \( j < n + m \) and \( 1 \leq i \leq n + m \). If \( i < j \), \( \delta_j \cdots \delta_{n+m}(v)_i = v_i > 0 \). If \( i \geq j \), by (4.8)

\[
\delta_j \cdots \delta_{n+m}(v)_i = \delta_i \cdots \delta_{n+m}(v)_i
\]

\[
= \delta_{i+1} \cdots \delta_{n+m}(v)_i - 2(\delta_{i+1} \cdots \delta_{n+m}(v)_i, \xi_i)_R
\]

\[
= v_i - 2(\delta_{i+1} \cdots \delta_{n+m}(v)_i, \xi_i)_R.
\]

By the induction hypothesis, \( \delta_{i+1} \cdots \delta_{n+m}(v)_i \in \mathbb{N}^{n+m} \). By (a),

\[
2(\delta_{i+1} \cdots \delta_{n+m}(v), \xi_i)_R \leq 1.
\]

Thus, \( \delta_j \cdots \delta_{n+m}(v)_i \geq 0 \).

**Theorem 4.15.** Let \( R \) be an algebra over an algebraically closed field \( k \). Assume that \( R \) is of finite prinjective type and that \( \Gamma(\text{prin}(R)^A) \) has a preprojective component. Let \( V \in \text{prin}(R)^A \) be an indecomposable module with \( v = \text{cdn } V \) such that \( \delta(v) \in \mathbb{N}^{n+m} \). Then \( \delta(v) = \text{cdn } \Delta V \).

**Proof.** Since \( \delta(v) \) is a positive root of \( \chi_R \), by (4.13) there exists an indecomposable module \( V' \in \text{prin}(R)^A \) with \( \delta(v) = \text{cdn } V' \).

We claim that \( \Delta V \) is the unique indecomposable in \( \text{prin}(R)^A \) satisfying:

(i) \( \langle v, \text{cdn } \Delta V \rangle_R < 0 \).

(ii) If \( W \) is an indecomposable with \( \langle v, \text{cdn } W \rangle_R < 0 \), then \( \langle \text{cdn } W, \text{cdn } \Delta V \rangle_R > 0 \).
For, since $\Gamma(\text{prin}(R)^d_B)$ is a preprojective component, then $\text{Hom}_R(V, \Delta V) = 0$ and by (4.4),

$$\langle v, \text{cdn} \Delta V \rangle_R = -\dim_k \text{Ext}^1_R(V, \Delta V) < 0.$$  

The property defining Auslander-Reiten sequences implies (ii). Moreover, if $Z$ is an indecomposable satisfying (i) and (ii), we should have $\langle \text{cdn} Z, \text{cdn} \Delta V \rangle_R > 0$ and $\langle \text{cdn} \Delta V, \text{cdn} Z \rangle_R > 0$. Hence, $\text{Hom}_R(Z, \Delta V) \neq 0 \neq \text{Hom}_R(\Delta V, Z)$, and therefore $Z \simeq \Delta V$.

Now, note that by (4.9) the vector $\delta(v) = \text{cdn} V'$ satisfies (i) and (ii). It follows that $V' \simeq \Delta V$ and $\delta(v) = \text{cdn} \Delta V$.

Corollary 4.16. Let $R$ and $V$ be as in (4.15). If $v = \text{cdn} V$ is sincere, then $\delta(v) = \text{cdn} \Delta V$.

Proof. The proof follows from (4.14) and (4.15). □

Remark 4.17. Let $R$ be a basic, indecomposable triangular $k$-algebra of the form (4.10) and assume that $k$ is algebraically closed.

(a) It is possible to give an algorithmic construction of the modules on a preprojective component of $\Gamma(\text{prin}(R)^d_B)$. This generalization of (4.15) makes use of a nonlinear map $\delta'$ instead of $\delta$. The construction follows that given in [12] for the case of algebras.

(b) We say that $R$ is sincere if its Tits form $\chi_R$ has a sincere root. The separation-criterion for $R$ can be defined as in the case of algebras. If $R$ satisfies the separation-criterion, then $\Gamma(\text{prin}(R)^d_B)$ has a preprojective component. There is an algorithmic construction of all the $k$-algebras $R$ which are sincere, of finite-prinjective type, and which satisfy the separation-criterion.

We will come back to these and other algorithmic problems in a forthcoming publication.

Note added in proof

In the proof of Theorem 3.4 the following situation remains to be considered: Suppose that $\Theta_B(X)$ is projective in $\text{adj}(R)^d_B$. Since $X$ is not projective in $\text{mod}_{ic}(R)$, then $X$ is not of the form $e_i R$ ($i = 1, \ldots, n$) and therefore we conclude from Lemma 3.3 that there exists $j \in \{1, \ldots, m\}$ such that $\Theta_B(X) \simeq \Theta^A \nabla_\sim(\eta_j R\nabla)$, where $R\nabla$ is defined in (3.6) and

$$\nabla_\sim : \text{mod}_{ic}(R\nabla)_A \rightarrow \text{mod}^P(R)^A$$

is an equivalence defined below (3.5). It follows from the definition of $\nabla_\sim$ that the inclusion $\text{rad} \eta_j R\nabla \hookrightarrow \eta_j R\nabla$ induces a commutative diagram

$$
eq:\begin{array}{ccc}
eq\end{array}$$

$$
\text{e}_0 : 0 \rightarrow \text{top} \eta_j R \rightarrow E_0 \xleftarrow{w} \Theta_B(X) \rightarrow 0 \quad \eta_X \quad 0 \quad 0
\text{e}_0 : 0 \rightarrow I_j \rightarrow L_0 \xleftarrow{w'} X \simeq \Theta_B(X) \rightarrow 0
$$

with exact rows, where $E_0 = \Theta^A \nabla_\sim(\text{rad} \eta_j R\nabla)$ and $L_0$ is such that $\Theta_B(L_0) \simeq E_0$. Since $\nabla_\sim$ is an equivalence, $w$ is a sink map in $\text{adj}(R)^d_B$ and one can easily prove as above that $\text{e}_0$ is an Auslander-Reiten sequence in $\text{mod}_{ic}(R)_B$.
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