GLOBAL REGULARITY ON 3-DIMENSIONAL SOLVMANIFOLDS

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ABSTRACT. Let $M$ be any 3-dimensional (nonabelian) compact solvmanifold. We apply the methods of representation theory to study the convergence of Fourier series of smooth global solutions to first order invariant partial differential equations $Df = g$ in $C^\infty(M)$. We show that smooth infinite-dimensional irreducible solutions, when they exist, satisfy estimates strong enough to guarantee uniform convergence of the irreducible (or primary) Fourier series to a smooth global solution.

1. Introduction

Let $S$ be a solvable Lie group, and $\Gamma$ a discrete subgroup of $S$ with compact quotient $\Gamma\backslash S$. There is then a unique probability measure $\nu$ on $\Gamma\backslash S$ that is invariant under translation on the right by elements of $S$. The regular representation of $S$ on $L^2(\Gamma\backslash S, \nu)$ decomposes into a direct sum of a countable number of irreducible unitary representations $\pi$ of $S$, each of finite multiplicity $m_\pi$ [G]. Let $D$ be a first order differential operator with complex coefficients, left-invariant on $S$ and viewed on $\Gamma\backslash S$. Let $(\Gamma\backslash S)^*$ denote the dual object of $\Gamma\backslash S$. If $g \in C^\infty(\Gamma\backslash S)$ and if $g_\pi$ is an orthogonal component of $g$ corresponding to some irreducible unitary representation $\pi$, then $g_\pi \in C^\infty(\Gamma\backslash S)$ too [A-B]. Modulo unitary equivalence, we may think of $g_\pi$ as being a $C^\infty$-vector in any concrete realization, or model, of $\pi$. We are interested in algebraically well-defined conditions on $D$ under which the global solvability of $Df = g$ in $C^\infty(\Gamma\backslash S)$ is equivalent to the solvability of $\pi(D)f_\pi = g_\pi$ in the $C^\infty$-vectors for each $\pi$ in the spectrum of $\Gamma\backslash S$. In a sense, we are looking for algebraic conditions on $D$ for the reduction of a global (geometrical) problem on $\Gamma\backslash S$ to a collection of purely group (representation) theoretic problems, none of which needs to be regarded as living on the manifold $\Gamma\backslash S$. Informally speaking, operators $D$ admitting such a reduction are called globally regular (Definition (1.1)).

In order to describe the results, we will recall the classical situation on a torus $T^2$ of two dimensions (the situation being similar for $T^n$ with $n > 2$). Let $D = \alpha \partial / \partial x + \beta \partial / \partial y$ and suppose for simplicity that $\alpha$ and $\beta$ are real.
Then $D$ is globally regular if and only if $\beta/\alpha$ is not a (transcendental) Liouville number. The problem with Liouville numbers is that, in solving for the Fourier transform of the solution function, small divisors occur. Now, every solvmanifold $\Gamma\backslash S$ contains the structure of a torus $T = \Gamma[S, S]\backslash S$ of dimension $\geq 1$, although this torus does not reflect any of the nonabelian structure of $S$. The only representations in $(\Gamma\backslash S)^\ast$ which are not infinite dimensional are the one-dimensional characters of $\Gamma[S, S]\backslash S$. Since the presence of this torus is inescapable, we denote, for each $g \in C^\infty(\Gamma\backslash S)$, the sum of the one-dimensional components of $g$ by $g_0$. Then global regularity is defined as follows. Let $L^2(\Gamma\backslash S) = \bigoplus_{\pi \in (\Gamma\backslash S)^\ast} m_\pi H_{\pi, j}$ be any (noncanonical) irreducible decomposition of $L^2(\Gamma\backslash S)$.

(1.1) **Definition.** A left-invariant differential operator $D$ on $\Gamma\backslash S$ is called globally regular if the three conditions

1. $g \in C^\infty(\Gamma\backslash S)$,
2. For each $\pi \in (\Gamma\backslash S)^\ast$ and $j = 1, \ldots, m_\pi$ there is a solution in $C^\infty(\Gamma\backslash S)$ to $Df_{\pi, j} = g_{\pi, j}$ (where $g_{\pi, j}$ is the $j$-component of $g$), and
3. $\exists f_0 \in C^\infty(T)$ such that $Df_0 = g_0$,

imply that there exists a solution in $C^\infty(\Gamma\backslash S)$ to $Df = g$.

Note that the solutions in (2) could be found in any convenient realization of $\pi$.

In previous papers we have dealt with nilpotent $S$. On the simplest nilmanifolds, the 3-dimensional Heisenberg manifolds, every first order differential operator $D$ in the complexified Lie algebra is globally regular [R2]. On more complicated nilmanifolds the problem of small divisors arises in the representation spaces of the group as well as on the associated torus. Moreover, if $D = X + iY$ is regular, both $\text{ad}_X$ and $\text{ad}_Y$ must map each step of the lower central series of the Lie algebra of $S$ (nilpotent) onto a sufficiently large subset of the next step. The details are explained in [C-R1, p. 349]. The purpose of this paper is to investigate the global regularity of first order differential operators on 3-dimensional compact solvmanifolds. We show that, as in the case of the simplest nilmanifolds, every first order differential operator on a 3-dimensional compact solvmanifold is globally regular.

2. 3-DIMENSIONAL SOLVMANIFOLDS

All 3-dimensional compact solvmanifolds can be described (up to homeomorphism) as quotients of two groups $S_h$ and $S_r$ by their various cocompact discrete subgroups. The groups $S_h$ and $S_r$ can both be described as $R^2 \times R^1$, where $(x, t)(x', t') = (x + A'x', t + t')$. Here $A'$ is a 1-parameter subgroup of $SL(2, R)$ through a matrix $A \in SL(2, Z)$. $S_h$ arises when the eigenvalues of $A$ are $\lambda > 1$ and $\lambda^{-1}$, so that the orbits of $R^1$ in $R^2$ are hyperbolic. $S_r$ arises when $A'$ is a compact group of rotations of $R^2$. Let $N := R^2 \times \{0\}$, the (abelian) nilradical of $S$. The cocompact discrete subgroups $\Gamma$ are described in [A-G-H], based upon the facts (due to Mostow) that $\Gamma \cap N$ is a discrete lattice in $R^2$, and that the image of $\Gamma$ under the natural projection $S \to S/N$ is a discrete lattice in $R^1$. We remark that $\Gamma\backslash S_h$ is determined up to homeomorphism by the eigenvalue $\lambda > 1$ of $A$, and $\lambda$ must be such that $\lambda + \lambda^{-1} \in Z$. For this reason, we denote the ‘hyperbolic’ manifolds $\Gamma_\lambda\backslash S_h$. Note however that
Sh is independent of the value of $\lambda > 1$. We have good use for the following lemma ((3.4) in [C-R2]).

(2.1) **Lemma.** If $S_h = R^2 \times R^1$ with the diagonalized matrix $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda > 1$, and if $\Gamma_\lambda$ is a cocompact discrete subgroup of $S_h$, then $\Gamma_\lambda \cap N$ is an abelian lattice $L$ of points $(\alpha, \beta)$ having the property that the product $\alpha \beta$ is bounded away from zero, except of course at the identity.

We need also the following

(2.2) **Corollary.** In the setting of the lemma above, the dual lattice $L^* = \{ \chi_{(a,b)} : L \to 1 \}$ is also a lattice of points $(a, b)$ such that the product $ab$ is bounded away from 0 (except of course for $(a, b) = (0, 0)$).

This corollary will give useful information about $\left( \Gamma \backslash S \right)_{\infty}$, the infinite dimensional representations in the spectrum of $\Gamma \backslash S$, in the hyperbolic case. For all 3-dimensional compact solvmanifolds, $\left( \Gamma \backslash S \right)_{\infty}$ is constructed as follows. Let $\chi_{(a,b)} \in \hat{N}$, where $\chi_{(a,b)} : \Gamma \cap N \to 1$. Now let $M$ be the extension of $N$ by the stabilizer of $\chi_{(a,b)}$ in $S$, and extend $\chi$ to $M$, so $\pi_{\alpha \beta} := \text{Ind}_M^S(\chi_{(a,b)}) \in \hat{S}$. If $S = Sh$ then $M = N$ and $M \backslash S \simeq R$, whereas if $S = S_r$ then $M \cong N \times Z$ and $M \backslash S \cong R/Z = \text{the circle group}$. If $H_{(\chi,M)}$ is the standard Mackey induced representation space, then $H_{(\chi,M)} = \{ f : S \to C | f(ms) = \chi_{(a,b)}(m)f(s), |f| \in L^2(M \backslash S) \}$. Define $L : H_{(\chi,M)} \to L^2(\Gamma \backslash S)$ by $(Lf)(G) = \sum_{(\gamma \in M \backslash S)} \gamma^* f(\gamma s)$. Then $L$ is a right $S$-invariant injection. If $\text{Int}(\chi, M) = \{ t \in R | \chi^{exp} : \Gamma \cap M \to 1 \}$ where $\chi^a(b) = \chi(a^{-1}ba)$ then the multiplicity of $\pi_{\alpha \beta}$ in $L^2(\Gamma \backslash S)$ equals the number of distinct $\Gamma$-orbits in $\text{Int}(\chi, M)$.

In the case of $S = S_h$, it is easiest to describe $\Gamma_\lambda \backslash S_h$ if we take $\Gamma_\lambda \cap N = Z^2$ and $A \in SL(2, Z)$. However, the model for $\pi_{\alpha \beta}$ is simplest if $A$ is diagonalized ($A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ as in Lemma (2.1)) and then $\Gamma \cap N$ is more difficult to describe. Nevertheless, the Corollary (2.2) above shows that in this model $\{ \pi_{\alpha \beta} \in (\Gamma_\lambda \backslash S_h)_{\infty} \}$ has $\alpha \beta$ bounded away from zero. We remark upon the fact that $\alpha \beta \neq 0$ prevents $\pi_{\alpha \beta}$ from being a representation of the (less well-behaved) ‘$ax + b$’ group which is a quotient of $S_h$.

We need to say a few words about the rotational three dimensional solvmanifolds $\Gamma \backslash S$, as well. Unlike the hyperbolic case, there are only finitely many $\Gamma \backslash S_r$ up to homeomorphism. We take $A \in SL(2, Z)$, but this time with no eigenvalues $> 1$. Now $A$ turns out to be similar to a rotation by $2\pi/p$, where $p = 1, 2, 3, 4, \text{ or } 6$ (see [A-G-H]). Since $S_r$ is independent of $p$ (up to group isomorphism), we denote the distinct rotational three dimensional solvmanifolds by $\Gamma_p \backslash S_r$, $p = 1, 2, 3, 4, \text{ or } 6$.

Our main result is

(2.3) **Theorem.** Let $\Gamma \backslash S$ be any nonabelian 3-dimensional compact solvmanifold. If $D \in \mathcal{P}^C$, the complexified Lie algebra of $S$, then $D$ is globally regular on $\Gamma \backslash S$.

Since this result has been proved in an earlier paper when $S$ is nilpotent, we concern ourselves here only with the manifolds $\Gamma_\lambda \backslash S_h$ and $\Gamma_p \backslash S_r$. Note also that the associated torus $T$ is 1-dimensional, so that small divisors cannot
occur on $T$. Thus condition (3) of Definition (1.1) will be satisfied automatically whenever (2) holds.

We will divide our proof of the theorem into two sections, one dealing with the rotational group $S_r$, the other with the hyperbolic one, $S_h$.

3. Proof of the Theorem—case of $\Gamma_p \setminus S_r$

Let $\mathcal{S}_r$ be the Lie algebra of $S_r$ with a linear basis $T, X, Y$ and the commutation relations $[T,X] = -\frac{2\pi}{p} Y$ and $[T,Y] = \frac{2\pi}{p} X$. Let $\pi_{\alpha\beta}$ be a generic infinite-dimensional representation in $(\Gamma_p \setminus S_r)^\prime$, acting in $L^2(T)$, where $T = M \setminus S_r$, a 1-dimensional torus as described in §2. For $f \in L^2(T)$ the action of $\pi_{\alpha\beta}$ on $f$ is given by

$$\pi_{\alpha\beta}(x, y; t) f(t) = \exp(2\pi i((\alpha, \beta) \sigma(x, y))) f(t),$$

where $\sigma(s) = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}$. For the basis vector fields $X, Y, T$ this amounts to

$$d\pi_{\alpha\beta}(X) = 2\pi i \left( \alpha \cos \frac{2\pi}{p} t - \beta \sin \frac{2\pi}{p} t \right),$$

$$d\pi_{\alpha\beta}(Y) = 2\pi i \left( \alpha \sin \frac{2\pi}{p} t + \beta \cos \frac{2\pi}{p} t \right),$$

$$d\pi_{\alpha\beta}(T) = \frac{d}{dt}.$$

Since the constant $p$ plays a negligible role in the proof (even though it classifies the rotational 3-dimensional solvmanifolds), we will set $p = 1$ in what follows.

We will break the proof into two cases of $D \in \mathcal{S}_r^\mathbb{C}$.

Case 1. $D = X + \gamma Y, \gamma \in \mathbb{C}$. This essentially covers all $D \in \mathcal{N}^\mathbb{C}$ (up to a constant factor) since $D = Y$ and $D = X$ behave alike and the case of $D = 0$ is trivial.

Case 2. $D = T + i(aX + bY), a, b \in \mathbb{R}$. This covers all $D \in \mathcal{S}_r^\mathbb{C} \sim \mathcal{N}^\mathbb{C}$ (up to an isomorphism) because the real part of $D$ can be absorbed into $T$.

Proof of Case 1. Write $D = X + (a + ib)Y, a, b \in \mathbb{R}$. Then, in view of (3.1), the operator $d\pi_{\alpha\beta}(D)$ is a multiplication by the function

$$D_{\alpha\beta}(t) = 2\pi i[(\alpha + a\beta + ib\beta) \cos 2\pi t + (a\alpha - \beta + ib\alpha) \sin 2\pi t] = 2\pi i(w \cos 2\pi t + z \sin 2\pi t).$$

By hypothesis, the equation $d\pi_{\alpha\beta}(D)f_{\alpha\beta} = g_{\alpha\beta}$ has the solution

$$f_{\alpha\beta}(t) = g_{\alpha\beta}(t)/D_{\alpha\beta}(t).$$

To prove the theorem we need to show that $\sum_{(\alpha, \beta)} Lf_{\alpha\beta} \in C^\infty(\Gamma \setminus S)$. Here $(\alpha, \beta)$ varies over a cross section of $\Gamma$-orbits, so that each infinite-dimensional primary summand will be spanned. By the Auslander-Brezin version of the Sobolev inequality it suffices to show $\sum_{(\alpha, \beta)} ||ULf_{\alpha\beta}||^2 < \infty$ for all $U \in \mathcal{N}(\mathcal{S}_r^\mathbb{C})$. Since $L$ is an $\mathcal{S}$-invariant isometry from $H_{\alpha\beta}$ into $L^2(\Gamma \setminus S)$, this is the same as to show $\sum_{(\alpha, \beta)} ||Uf_{\alpha\beta}||^2 < \infty$. We begin by estimating the sum

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The problem, of course, is that $D_{\alpha\beta}(t)$ in (3.3) can have zeros, and that even when it has no zeros, we must know how close $|D_{\alpha\beta}(t)|$ can come to 0 as $(\alpha, \beta) \to \infty$. We can write

$$D_{\alpha\beta}(t) = \pi i((w - iz)e^{2\pi it} + (w + iz)e^{-2\pi it}) = \pi i(Ae^{2\pi it} + Be^{-2\pi it}),$$

with $A$ and $B$ defined by the last equation. The minimum of $|D_{\alpha\beta}(t)|$ occurs where $A$ and $B$ are rotated to opposite directions, and then

$$\text{Min} |D_{\alpha\beta}(t)| = \pm \pi(|A| - |B|)$$

(3.4)

$$= \pi((\alpha^2 + \beta^2)^{1/2}((a^2 + (1 + b)^2)^{1/2} - (a^2 + (1 - b)^2)^{1/2})$$

$$= (\alpha^2 + \beta^2)^{1/2} \cdot K \geq \sqrt{2}|\alpha\beta|^{1/2} \cdot K.$$

If $b \neq 0$, the constant $K$ is $\neq 0$. Since $|\alpha\beta| \gg 0$, $|f_{\alpha\beta}(t)| \leq C|g_{\alpha\beta}(t)|$ with the constant $C$ independent of $(\alpha, \beta)$. Consequently, $\sum_{(\alpha, \beta)} \|f_{\alpha\beta}\|^2 \leq C \sum_{(\alpha, \beta)} \|g_{\alpha\beta}\|^2 < \infty$ for $g \in C^\infty(T\setminus S)$. If $b = 0$, $D_{\alpha\beta}(t)$ does have one or more zeros and

$$D_{\alpha\beta}(t) = 2\pi i((a + a\beta)\cos 2\pi t + (a\alpha - \beta)\sin 2\pi t).$$

If $D_{\alpha\beta}(t_0) = 0$, then $g_{\alpha\beta}(t_0) = 0$ too, since $f_{\alpha\beta}$ given by (3.3) is in $C^\infty(T)$. The idea is to control $f_{\alpha\beta}$ inside specified intervals around each of the $t_0$'s by the mean value theorem, and to use the monotonicity of $D_{\alpha\beta}$ on large exterior intervals to control $|f_{\alpha\beta}|$ by keeping $|D_{\alpha\beta}|$ big. We have

$$D'_{\alpha\beta}(t_0) = -2\pi(\alpha^2 + \beta^2)^{1/2}(1 + a^2)^{1/2},$$

and by the mean value theorem

$$|D'_{\alpha\beta}(t) - D'_{\alpha\beta}(t_0)| \leq |t - t_0|4\pi^2(\alpha^2 + \beta^2)^{1/2}(1 + a^2)^{1/2}.$$}

Consequently, for $|t - t_0| < 1/4\pi$, $|D'_{\alpha\beta}(t)| \geq \pi(\alpha^2 + \beta^2)^{1/2}(1 + a^2)^{1/2}$ and, since $D_{\alpha\beta}(t_0) = 0$,

$$|D_{\alpha\beta}(t)| \geq \pi(\alpha^2 + \beta^2)^{1/2}(1 + a^2)^{1/2}|t - t_0|$$

again by the mean value theorem. For $g_{\alpha\beta}(t)$ we have the estimate

(3.5)

$$|g_{\alpha\beta}(t)| = |g_{\alpha\beta}(t) - g_{\alpha\beta}(t_0)| \leq \|Tg_{\alpha\beta}\|_{\infty}|t - t_0|,$$

$\| \cdot \|_{\infty}$ denoting the sup norm on the torus $M\setminus S$. So

(3.6)

$$|f_{\alpha\beta}(t)| \leq \|Tg_{\alpha\beta}\|_{\infty}1/\pi(\alpha^2 + \beta^2)^{-1/2}(1 + a^2)^{-1/2},$$

for $|t - t_0| < 1/4\pi$.

On the intervals complementary to $|t - t_0| < 1/4\pi$,

$$|D_{\alpha\beta}(t)| \geq |D_{\alpha\beta}(t_0 \pm 1/4\pi)| \geq \pi(\alpha^2 + \beta^2)^{1/2}(1 + a^2)^{1/2}/4\pi,$$

hence

(3.7)

$$|f_{\alpha\beta}(t)| \leq 4\|g_{\alpha\beta}\|_{\infty}(\alpha^2 + \beta^2)^{-1/2}(1 + a^2)^{-1/2} \text{ for } |t - t_0| \geq 1/4\pi$$

for each of the two values $t_0$ of $t$ where $D_{\alpha\beta}$ vanishes. By (3.6) and (3.7), for all $t$ on the torus $M\setminus S$ we have the following estimate:

(3.8)

$$|f_{\alpha\beta}(t)| \leq (\|Tg_{\alpha\beta}\|_{\infty}/\pi + 4\|g_{\alpha\beta}\|_{\infty})(\alpha^2 + \beta^2)^{-1/2}(1 + a^2)^{-1/2}.$$

By Sobolev's inequality we may replace the sup norms in (3.8) by $L^2$-norms of $g_{\alpha\beta}$, $Tg_{\alpha\beta}$, and $T^2g_{\alpha\beta}$. The sum $\sum_{(\alpha, \beta)} \|f_{\alpha\beta}\|^2$ is finite because each sum $\sum_{(\alpha, \beta)} \|T^k g_{\alpha\beta}\|^2$ for $k = 0, 1, 2$ is finite and $|\alpha\beta|$ is bounded away from zero.
Next, we must show $\sum_{(\alpha, \beta)} \|Uf_{\alpha \beta}\|_2^2 < \infty$ for every fixed $U \in \mathcal{U}(\mathcal{S})$. Since $[D, \mathcal{M}] = 0$, this is true for all $U \in \mathcal{U}(\mathcal{M})$. It remains to show $\sum_{(\alpha, \beta)} \|T^k f_{\alpha \beta}\|_2^2 < \infty$ for $k = 1, 2, \ldots$, because every $U \in \mathcal{U}(\mathcal{S})$ can be written as a linear combination of monomials $T^k V$ with $V$ in $\mathcal{U}(\mathcal{M})$.

(3.9) Proposition. For $D = X + \gamma Y$, $\gamma \in C$ and $k = 1, 2, 3, \ldots$ the $(k+1)$-fold bracket product $[D[D[\cdots[D, T^k]\cdots]]]$ is 0.

Proof of Proposition. By Leibnitz's rule for the derivation $[D, \cdot]$ of the algebra $\mathcal{U}(\mathcal{S})$ we have $[D, T^k] = \sum_{j=1}^{k} T \cdots T[D, T]T \cdots T$, with $[D, T]$ at $j$th place. Since $D$ commutes with $\mathcal{M}$ and $[D, T] = -X + \gamma Y \in \mathcal{M}$, the derivation repeated $k + 1$ times is zero.

(3.10) Proposition. For $f_{\alpha \beta}$ as in (3.3) we have

\begin{equation}
T^k f_{\alpha \beta} = h_k / D^k_{\alpha \beta}
\end{equation}

with

\begin{align*}
h_k &= [D[D[\cdots[D, T^k]\cdots]]] g_{\alpha \beta} + D[D[\cdots[D, T^k]\cdots]] g_{\alpha \beta} \\
& \quad + \cdots + D^{k-1}[D, T^k] + D^k T^k g_{\alpha \beta}.
\end{align*}

The first bracket involves $k$ $D$'s with the number of $D$'s inside the brackets decreasing by one in each successive summand.

Proof of Proposition. In view of Proposition (3.9) this is formula (1.8) on p. 353 of [C-R1].

The estimates on $T^k f_{\alpha \beta}$ given by (3.11) can now be done in a manner similar to that already presented. If $D_{\alpha \beta} \neq 0$, we use the inequality (3.4) raised to the power $k + 1$ to estimate the denominator $|D_{\alpha \beta}|^{k+1}$. If $D_{\alpha \beta}(t_0) = 0$, the numerator $h_k$ in (3.11) must have a $(k + 1)$th order zero at $t_0$ since $T^k f_{\alpha \beta}$ is $C^\infty$. Instead of the estimate (3.5) we use

\begin{equation}
|h_k(t)| \leq \|T^{k+1} h_k\|_\infty |t - t_0|^{k+1}/(k + 1)!
\end{equation}

which follows from Taylor's formula. In the denominator of (3.6) and (3.7) we use the $(k + 1)$th power of the previous estimate for $D_{\alpha \beta}$.

Proof of Case 2. Here $D = T + i(aX + bY)$, $a, b \in R$, and

\begin{equation}
d\pi_{\alpha \beta}(D) = \frac{d}{dt} - 2\pi((a\alpha + b\beta) \cos 2\pi t + (b\alpha - a\beta) \sin 2\pi t)
\end{equation}

\begin{equation}
= \frac{d}{dt} - 2\pi(A \cos 2\pi t + B \sin 2\pi t).
\end{equation}

Write $A \cos 2\pi t + B \sin 2\pi t = (A^2 + B^2)^{1/2} \sin 2\pi(t + \phi)$, with the constant $\phi$ depending upon $A$ and $B$. If $d\pi_{\alpha \beta}(D)f_{\alpha \beta} = g_{\alpha \beta}$, we have

\begin{align*}
f_{\alpha \beta}(t) &= \exp(-2\pi(A^2 + B^2)^{1/2} \cos 2\pi(t + \phi)) \\
& \quad \times \left( \int_{-\phi - \frac{1}{2}}^{\phi} g_{\alpha \beta}(\tau) \exp(2\pi(A^2 + B^2)^{1/2} \cos 2\pi(t + \phi)) d\tau + C \right).
\end{align*}

Here we identify $M\setminus S = R/Z$ with the interval $[-\phi = \frac{1}{2}, \frac{1}{2} - \phi]$, and we use the fact that $f_{\alpha \beta}, g_{\alpha \beta}$, and the exponentials all have period 1. Since $C$ is
arbitrary, choosing \( C = 0 \) and changing the variables \( \tau' = \tau + \phi \), \( t' = t + \phi \) we have

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} g(\tau - \phi) \exp(2\pi(A^2 + B^2)^{1/2}(\cos 2\pi \tau - \cos 2\pi t)) \, d\tau
\]

\[
= -\int_{t}^{t+1/2} \text{(same integrand as above)}.
\]

The last equality follows from the periodicity of \( f_{\alpha\beta} \), because \( 0 = f_{\alpha\beta}(-1/2-\phi) = f_{\alpha\beta}(1/2-\phi) \). For the estimates on \( f_{\alpha\beta}(t) \) or on \( f_{\alpha\beta}(t-\phi) \) we use the integral over \(-1/2 \leq \tau \leq t\) if \(-1/2 \leq t \leq 0\) and over \( t \leq \tau \leq 1/2\) for \( 0 < t < 1/2\). That way we always have \( \cos 2\pi \tau < \cos 2\pi t \), so that the exponent in the integral defining \( f_{\alpha\beta} \) is negative. Then

\[
\|f_{\alpha\beta}\|_{\infty} \leq C\|g_{\alpha\beta}\|_{\infty} \leq C_1(\|g_{\alpha\beta}\|_2 + \|Tg_{\alpha\beta}\|_2)
\]

Since the \( f_{\alpha\beta} \)'s all live on an interval of length 1, the same inequality holds for \( \|f_{\alpha\beta}\|_2 \) with a constant, say \( C_1 \), independent of \( (\alpha, \beta) \). Thus \( \sum_{(\alpha, \beta)}\|f_{\alpha\beta}\|_2^2 < \infty \) and \( f = \sum f_{\alpha\beta} \in L^2(\Gamma_p \backslash S_r) \). Next, to estimate \( \sum_{(\alpha, \beta)}\|Uf_{\alpha\beta}\|_2^2 \) for every \( U \in \mathcal{Z}(\mathcal{F}) \) it suffices to consider \( U = X^k Y^l T^m \). By (3.12) \( T \) just differentiates \( f_{\alpha\beta} \) yielding \( 2\pi(A^2 + B^2)^{1/2} \sin 2\pi(t + \phi) f_{\alpha\beta}(t) + g_{\alpha\beta}(t) \) with \( A, B \) depending linearly on \( \alpha, \beta \). Successive powers of \( T \) differentiate \( g_{\alpha\beta} \) and \( \sin 2\pi(t + \phi) \), or \( f_{\alpha\beta}(t) \). Operating by \( X \)'s or \( Y \)'s just multiplies by a polynomial in \( (\alpha, \beta) \) times a (bounded) combination of sines and cosines. However, \( X^2 + Y^2 \) acts on \( L^2(\mathcal{M} \backslash S) \) by multiplying by \(-4\pi^2(\alpha^2 + \beta^2)\), so that \( \sum_{(\alpha, \beta)}(\alpha^2 + \beta^2)^q\|g_{\alpha\beta}\|_2^2 < \infty \). Thus \( \sum_{(\alpha, \beta)}\|X^k Y^l T^m f_{\alpha\beta}\|_2^2 \) can be estimated by \( C \sum \|(X^2 + Y^2)^q g_{\alpha\beta}\|_2^2 \) for some \( q \).

Remark. Although we have pretended \( p = 1 \) in the arguments above, this was just a notational convenience, and the same arguments work just as well for general \( \Gamma \backslash S_r \). Actually, the use of \( \Gamma \backslash S_r \) has some independent interest however. Here, although \( S_r \) is not nilpotent, \( \Gamma \cong Z^3 \). Thus, by the general theory of Mostow [M], \( \Gamma \backslash S_r \) is homeomorphic to a torus \( T^3 \). Here we get a proof of global regularity for elements of \( \mathcal{S}_m^\infty \) acting on \( T^3 \), keeping in mind that we mean global regularity with respect to the representation theory of \( S_r \). Thus we have a natural class of operators on \( T^3 \) which are best analyzed with respect to \( S_r \).

4. Proof of the theorem—case of \( \Gamma \backslash \mathcal{S}_h \)

Let \( \mathcal{S}_h \) be the Lie algebra of \( S_h \) with a linear basis \( T, X, Y \) and the commutation relations \([T, X] = X \ln \lambda \) and \([T, Y] = -Y \ln \lambda \). A generic infinite dimensional representation \( \pi_{\alpha\beta} \) in \((\Gamma \backslash \mathcal{S}_h)^\dagger\) acts on \( L^2(R) \) by

\[
\pi_{\alpha\beta}(x, y, t)f(\tau) = \exp 2\pi i(\alpha x + \beta y - t)f(\tau + t).
\]

For the basis vector fields \( X, Y, T \) this amounts to

\[
d\pi_{\alpha\beta}(X) = 2\pi i\alpha \lambda^t, \quad d\pi_{\alpha\beta}(Y) = 2\pi i\beta \lambda^{-t}, \quad d\pi_{\alpha\beta}(T) = \frac{d}{dt}.
\]

The space \( H_{\alpha\beta}^\infty \) of \( C^\infty \)-vectors for \( \pi_{\alpha\beta} \) consists of \( C^\infty(R) \) functions \( \phi \) with

\[
\lim_{|t| \to \infty} \lambda^{|t|} \phi^{(n)}(t) = 0
\]
for every $m, n \in N$. These $\phi$ are called "super-Schwartz". We will denote this space by $\mathcal{S}_h^C$.

We will break the proof into four cases of $D \in \mathcal{S}_h^C$.

**Case 1. $D = T$.** Any nonzero complex multiple of $D = T + aX + bY \in \mathcal{S}_h$ can be reduced up to isomorphism to $D = T$, if $a, b \in R$.

**Case 2. $D = aX + bY \in \mathcal{M} \subset \mathcal{S}_h$, $a, b \in R$.**

**Case 3. $D = aX + bY \in \mathcal{M}^C \subset \mathcal{S}_h^C$, $a, b \in C$.**

**Case 4. $D = T + i(aX + bY)$, $a, b \in R$.**

**Proof of Case 1. $D = T$.** The equation $d\pi_{\alpha\beta}(T)f_{\alpha\beta} = g_{\alpha\beta}$ becomes $(d/dt)f_{\alpha\beta} = g_{\alpha\beta}$. Since both $f_{\alpha\beta}$ and $g_{\alpha\beta}$ are in $H_{\alpha\beta}^\infty$, we have

\[
\int_{-\infty}^t g_{\alpha\beta}(\tau) d\tau = -\int_t^\infty g_{\alpha\beta}(\tau) d\tau.
\]

**Subcase 1a.** If $t \geq 0$, we use the second integral for the estimates on $f_{\alpha\beta}$ and the fact that $g_{\alpha\beta}(\tau) = Xg_{\alpha\beta}(\tau)/2\pi i\alpha\lambda^2$, so that

\[
|f_{\alpha\beta}(t)| \leq \|Xg_{\alpha\beta}\|_\infty (2\pi\alpha \ln \lambda)^{-1}\lambda^{-t} \in L^2([0, \infty), dt).
\]

Squaring, integrating $\int_0^\infty \cdots dt$, and applying Sobolev's inequality to estimate the norm $\|Xg_{\alpha\beta}\|_\infty$ by a combination of $L^2$-norms of derivatives of $Xg_{\alpha\beta}$ we obtain

\[
\|f_{\alpha\beta}\|^2_{L^2([0, \infty)} \leq \frac{c}{\alpha}(\|Xg_{\alpha\beta}\|^2_2 + \|TXg_{\alpha\beta}\|^2_2),
\]

for some constant $c$. As in the proof of Lemma (2.1) (see [C-R2]), we pick the representative $(\alpha, \beta)$ from the orbit corresponding to $\pi$ so that $\lambda^{-2} \leq |\beta/\alpha| \leq 1$. Since $|\alpha\beta| \gg 0$, we must have $|\alpha| \gg 0$ too. Thus $\sum_{(\alpha, \beta)} \|f_{\alpha\beta}\|^2_{L^2([0, \infty)} < \infty$.

**Subcase 1b.** If $t < 0$, we use the first integral formula for $f_{\alpha\beta}$ in (4.2), together with the fact that $g_{\alpha\beta}(\tau) = Yg_{\alpha\beta}(\tau)/2\pi i\beta\lambda^{-t}$. Picking $(\alpha, \beta)$ as in Subcase 1a, we insure that $|\beta| \gg 0$. Now an argument almost identical to that in Subcase 1a shows that $f = \sum f_{\alpha\beta} \in L^2(R)$.

To complete Case 1, we need to prove that $\sum \|Uf_{\alpha\beta}\|^2_2 < \infty$, for each fixed $U \in \mathcal{V}(\mathcal{S}_h)$. If $U = T^p$ there is no problem, since $[D, U] = 0$. If $U = X^m Y^n T^p$, $Uf_{\alpha\beta}(t) = (2\pi i)^{m+n} \alpha^m \beta^n \lambda^{(m-n)t} T^{p-1} g_{\alpha\beta}(t)$. If $p \geq 1$, there is no problem. But if $p = 0$, we estimate $|g_{\alpha\beta}(\tau)|$ by a $\|Vg_{\alpha\beta}\|_{\lambda^ti}$, where $V \in \mathcal{V}(\mathcal{S}_h)$ and $k \in Z$ are chosen one way for $t < 0$ and another way for $t \geq 0$, so as to assure that $Uf_{\alpha\beta} \in L^2(-\infty, 0)$ and $Uf_{\alpha\beta} \in L^2(0, \infty)$ separately. Hence $\sum Uf_{\alpha\beta} \in L^2(R)$.

**Case 2.** Suppose $D = aX + bY$, with $a$ and $b$ real. Here, $f_{\alpha\beta}(t) = g_{\alpha\beta}(t)/2\pi i(a\alpha\lambda^t + b\beta\lambda^{-t})$.

We will need to consider the occurrence of zeros in the denominator. Observe that the denominator vanishes if and only if $\lambda^{2t} = -b\beta/a\alpha$, which means $t = t_0 = -\log_i(-b\beta/a\alpha)$.

**Subcase 2a.** Suppose $b\beta/a\alpha > 0$, so that $h(t) = (a\alpha\lambda^t + b\beta\lambda^{-t})^{-1}$ has a maximum (absolute) value on $R$. At the maximum point $h'(t) = 0$, so that
\[
\lambda^{2t} = b\beta /a\alpha. \text{ Thus } h(t_{\max}) = \lambda^{t_{\max}/2}b\beta = \pm 1/2|a\beta|^{1/2}. \text{ Thus } |\sigma_{\alpha\beta}(t)| \leq |\sigma_{\alpha\beta}(t)|/2|ab\alpha\beta|^{1/2}. \text{ Since } |\alpha\beta| \gg 0, \text{ we have } \sum_{\alpha\beta} f_{\alpha\beta} \in L^2, \text{ where the sum is over those } (\alpha, \beta) \text{ such that } b\beta/a\alpha > 0. \text{ The same is true for } \sum U f_{\alpha\beta}, \text{ for each fixed } U \in \mathcal{U}(\mathcal{H}) \text{ since}
\]

1. If \( U \in \mathcal{U}(\mathcal{H}), [U, D] = 0, \text{ and} \)
2. If \( U = X^m Y^n T^k, \text{ then} \)

\[
\frac{d}{dt}(g_{\alpha\beta}(t)h(t)) = g'_{\alpha\beta}h + g_{\alpha\beta}h' = (Tg_{\alpha\beta})h + g_{\alpha\beta}h',
\]
and \( g_{\alpha\beta}h' = (X - Y)g_{\alpha\beta} \cdot h^2_{\alpha\beta}, \) which is still \( L^2 \)-summable.

**Subcase 2b.** Either \( a\alpha = 0 \) or \( b\beta = 0. \text{ Since } |\alpha\beta| \gg 0 \) (except at \((0, 0)) \text{ this means } a = 0 \text{ or } b = 0 \text{ (but not both). If } b = 0, \text{ then } f_{\alpha\beta}(t) = -Yg_{\alpha\beta}(t)/4\pi^2 a\alpha\beta. \text{ Then } \sum f_{\alpha\beta} \in L^2, \text{ since } |\alpha\beta| \gg 0. \text{ A similar argument applies if } a = 0. \)

**Subcase 2c.** Here, we suppose there exists a zero, \( t_0 = \frac{1}{2}\log_\lambda(\pm b\beta/a\alpha). \text{ Then } g_{\alpha\beta}(t_0) = 0 \text{ too, and } f_{\alpha\beta}(t) = (g_{\alpha\beta}(t) - g_{\alpha\beta}(t_0))/2\pi i(a\alpha\lambda^t + b\beta\lambda^{-t}). \text{ Since it would suffice to prove separately the summability of the real and imaginary parts of } f_{\alpha\beta}, \text{ we can assume wlog that } g_{\alpha\beta} \text{ is real-valued. For some } \tau \text{ between } 0 \text{ and } t, \text{ we have } f_{\alpha\beta}(t) = g'_{\alpha\beta}(\tau)/2\pi i(a\alpha\lambda^t + b\beta\lambda^{-t}) \text{ ln } \lambda, \text{ by the general mean value theorem. Since } b\beta/a\alpha < 0 \text{ in this case, the new denominator cannot vanish. Hence, recalling that } \tau \text{ depends on } t, \)

\[
4\pi^2 \int_\mathcal{R} |f_{\alpha\beta}(t)|^2\, dt
\]

\[
= \int_{|t - t_0| \leq 1} |g'_{\alpha\beta}(\tau)|^2 |a\alpha\lambda^t - b\beta\lambda^{-t}|^{-2} |\text{ln } \lambda|^{-2}\, dt
\]

\[
+ \int_{|t - t_0| \geq 1} |g_{\alpha\beta}(t)|^2 |a\alpha\lambda^t + b\beta\lambda^{-t}|^{-2}\, dt
\]

\[
\leq \frac{||g'_{\alpha\beta}||_\infty^2}{-4a\alpha b\beta \text{ ln } \lambda} + \int_{|t - t_0| \geq 1} |g_{\alpha\beta}(t)|^2 \cdot \text{Max } |a\alpha\lambda^t + b\beta\lambda^{-t}|^{-2}\, dt.
\]

The first summand is summable over \((\alpha, \beta), \text{ by Sobolev's inequality. However, by the mean value theorem, } |a\alpha\lambda^t + b\beta\lambda^{-t}| \geq |(a\alpha\lambda^t - b\beta\lambda^{-t})(\text{ln } \lambda)(t - t_0)| \geq 2(-a\alpha b\beta)^{1/2} |\text{ln } \lambda|, \text{ if } |t - t_0| \geq 1. \text{ Thus} \)

\[
\pi^2 \int_\mathcal{R} |f_{\alpha\beta}(t)|^2\, dt \leq \frac{||g'_{\alpha\beta}||_\infty^2}{|a\alpha b\beta| |\text{ln } \lambda|^2} + \int_\mathcal{R} |g_{\alpha\beta}(t)|^2\, dt,
\]
and so \( \sum f_{\alpha\beta} \in L^2. \)

If \( U \in \mathcal{U}(\mathcal{M}), \text{ then } \sum U f_{\alpha\beta} \in L^2 \text{ too, since } [D, U] = 0. \text{ So it suffice to consider } \sum T^k f_{\alpha\beta}, \text{ for each fixed } k \in \mathbb{Z}^+. \text{ First, consider } k = 1. \)

\[
2\pi i T f_{\alpha\beta}(t) = h(t)(a\alpha\lambda^t + b\beta\lambda^{-t})^{-2}, \text{ where } h(t) = (a\alpha\lambda^t + b\beta\lambda^{-t})g_{\alpha\beta}(t) - g_{\alpha\beta}(t)(a\alpha\lambda^t - b\beta\lambda^{-t}) \text{ ln } \lambda. \text{ Here } h(t) \text{ must have a zero of order at least two at } t = t_0. \text{ Therefore, using a Taylor remainder of degree two in } (t - t_0), \text{ there exist } \tau \text{ and } \tau' \text{ such that } T f_{\alpha\beta}(t) = h''(\tau')/D''(\tau)2\pi i \text{ where } D(t) = (a\alpha\lambda^t + b\beta\lambda^{-t}) \text{ ln } \lambda. \text{ But} \)

\[
|D''(t)| = 2[|(a\alpha\lambda^t - b\beta\lambda^{-t}) \text{ ln } \lambda|^2 + |(a\alpha\lambda^t + b\beta\lambda^{-t}) \text{ ln } \lambda|^2] \geq 8|a\alpha b\beta| |\text{ln } \lambda|^2,
\]
since the second square is positive. Thus \(|T f_{\alpha \beta}(t)| \leq |h''(\tau')|/16\pi |a\alpha b\beta|\). Hence
\[
4\pi^2 \int_R |T f_{\alpha \beta}|^2 \leq \int_{|t-t_0| \leq 1} \|h''\|_\infty/64(a\alpha b\beta)^2 \, dt \\
+ \int_R |h(t)|^2/\min\{|D(t)|^2 \mid |t-t_0| \geq 1\} \, dt.
\]
But \(\|h''\|_\infty\) is bounded by norms of derivatives of \(g_{\alpha \beta}\), providing good summability over \((\alpha, \beta)\) of the first term. Also, \(|D(t)|^2 = |D''(\tau)(t-t_0)^2/2!|^2 \geq 16(a\alpha b\beta)^2 \ln^4 \lambda \gg 0\). Thus \(\sum T f_{\alpha \beta} \in L^2\).

Next, we consider \(k > 1\). We make the following observations.

\[
[D, T^k] = \sum_{j=1}^k T \cdots [D, T]T \cdots T,
\]
where \([D, T]\) is in the \(j\)th position. Hence
\[
[D, [D, T^k]] = \sum_{i,j=1}^k T \cdots [D, T]T \cdots T
\]
\[
= 2 \sum_{1 \leq i < j \leq k} T \cdots [D, T]T \cdots [D, T]T \cdots T,
\]
where \([D, T]\) occupies the \(i\)th and \(j\)th positions. (This is a result of \(N\) being abelian and normal.) The \(k\)-fold bracket
\[
[D, [D, \ldots [D, T^k] \ldots]] = k![D, T]^k,
\]
while the \((k+1)\)-fold bracket vanishes. By (1.8) of [C-R1, p.353],
\[
2\pi i T^k f_{\alpha \beta}(t) = h_k(a\alpha \lambda^t + b\beta \lambda^{-t})^{-(k+1)}
\]
where
\[
h_k = [D[D \cdots [D, T^k] \cdots]]g_{\alpha \beta} + D[D \cdots [D, T^k] \cdots]g_{\alpha \beta}
\]
\[
+ \cdots + D^{k-1}[D, T^k]g_{\alpha \beta} + D^k T^k g_{\alpha \beta}.
\]
It follows that \(\|T^k f_{\alpha \beta}\|_2^2 = \int_{|t-t_0| \leq 1} + \int_{|t-t_0| > 1}\), with the integrand being determined by (4.8).

The second integral can be estimated as in the case of \(k = 1\), while the denominator in the integrand in the first integral has a Taylor expansion in \((t-t_0)\) using derivatives of order \(\leq k + 1\). For \(k > 1\) and odd, the \((k+1)\)th derivative of \((a\alpha \lambda^t + b\beta \lambda^{-t})^{k+1}\) is of the form \((d/dt)^{k+1}[(a\alpha \lambda^t + b\beta \lambda^{-t})^{k+1}] = \sum_{j=0}^{(k+1)/2} c_j(a\alpha \lambda^t + b\beta \lambda^{-t})^{k+1-2j}(a\alpha \lambda^t - b\beta \lambda^{-t})^{2j} \ln^k \lambda\) with all \(c_j \geq 0\) and \(c_{(k+1)/2} > 0\). Thus, if \(k+1\) is even,
\[
\frac{d^{k+1}}{dt^{k+1}} (a\alpha \lambda^t + b\beta \lambda^{-t})^{k+1} \geq C(a\alpha \lambda^t - b\beta \lambda^{-t})^{k+1}, \quad \text{where } C > 0.
\]
If \(k+1\) is odd, then
\[
\frac{d^{k+1}}{dt^{k+1}} (a\alpha \lambda^t + b\beta \lambda^{-t})^{k+1} \geq C'|a\alpha \lambda^t - b\beta \lambda^{-t}|^{k+1}, \quad \text{for } C' > 0.
\]
by a similar calculation. Hence the minima over \( |t - t_0| \leq 1 \) in the resulting estimates proceed as in the case of \( k = 1 \).

**Case 3.** \( D \in \mathcal{N}^C \). If \( D \in C \times X \) or \( D \in C \cdot Y \), then \( D \in C \cdot \mathcal{N} \) and is covered by Case 2. Otherwise, dividing by a constant, we can assume that \( D = X + (a + ib)Y \), where \( a, b \in \mathbb{R} \). Thus

\[
(4.9) \quad f_{\alpha\beta}(t) = g_{\alpha\beta}(t)/2\pi(i(\alpha\lambda^t + a\beta\lambda^{-t}) - b\beta\lambda^{-t}).
\]

Since it suffices to prove \( L^2 \)-summability for the parts of \( f_{\alpha\beta} \) corresponding to the real and imaginary parts of \( g_{\alpha\beta} \) separately, we can assume \( g_{\alpha\beta} \) is real in (4.9). Suppose also \( b \neq 0 \), since Case 2 would apply if \( b \) were 0. We note that \( |f_{\alpha\beta}(t)| \leq |g_{\alpha\beta}(t)|/2\pi b\beta\lambda^{-t} \). The methods of Case 2 can be applied to prove \( \sum f_{\alpha\beta} \in L^2 \).

If \( U \in \mathcal{W}(\mathcal{N}) \), then \([D, U] = 0\) so that \( \sum U f_{\alpha\beta} \in L^2 \). We need to prove \( \sum T^k f_{\alpha\beta} \in L^2 \) for each fixed \( k \in \mathbb{N} \).

We begin with \( k = 1 \). Then \( T f_{\alpha\beta}(t) \) is the derivative of the right side of (4.9). In the numerator, we get various derivatives of \( g_{\alpha\beta} \), while the modulus of the denominator exceeds \( 4\pi^2 b^2 \beta^2 \lambda^{-2t} \). The \( \lambda^{-2t} \) can be moved to the numerator as a derivation, and \( \sum T f_{\alpha\beta} \in L^2 \). For \( k > 1 \), similar reasoning applies.

**Case 4.** \( D = T + i(aX + bY) \), \( a, b \in \mathbb{R} \). Up to isomorphism, we could assume that \( a = b = 1 \), except for the case in which either \( a = 0 \) or \( b = 0 \). Since \( T + iX \) and \( T + iY \) are very similar, we need to treat only the cases \( T + iX \) and \( T + i(X + Y) \). Since \( T + i(X + Y) \) is more complicated, we will treat this case in detail, providing brief remarks to cover the simpler case of \( T + iX \).

Actually, to simplify the constants we suppose \( D = T + i(X + Y)/2\pi \), so that

\[
f_{\alpha\beta}(t) = \left( \int_0^t g_{\alpha\beta}(x) \exp(-((\alpha\lambda^x + \beta\lambda^{-x})/\ln \lambda) \, dx + C \right) \exp((\alpha\lambda^t + \beta\lambda^{-t})/\ln \lambda).
\]

**Subcase 4a.** \( \alpha > 0 \) and \( \beta > 0 \). In this case \( \exp((\alpha\lambda^t + \beta\lambda^{-t})/\ln \lambda) \to \infty \) as \( t \to \pm \infty \) (or in case \( b = 0 \), \( \exp(\alpha\lambda^t/\ln \lambda) \to \infty \) as \( t \to \infty \) and \( -1 \) as \( t \to -\infty \)). Thus

\[
(4.10) \quad f_{\alpha\beta}(t) = -\int_{-\infty}^t g_{\alpha\beta}(x) \exp(-((\alpha\lambda^x + \beta\lambda^{-x}) + (\alpha\lambda^t + \beta\lambda^{-t})/\ln \lambda) \, dx
\]

If \( b = 0 \), the \( \beta\lambda^{-x} \) and \( \beta\lambda^{-t} \) terms do not appear in (4.10) and we simply use \( t_{\alpha\beta} = 0 \) in what follows. Also, the restriction \( \beta > 0 \) is not necessary when we deal with the \( b = 0 \) case.

Note that \( \frac{d}{dx}(\alpha\lambda^x + \beta\lambda^{-x}) = (\alpha\lambda^x - \beta\lambda^{-x}) \ln \lambda = 0 \) iff \( x = \frac{1}{2} \log_4(\beta/\alpha) \), which we denote henceforth by \( -t_{\alpha\beta} \leq 0 \). Since \( \alpha \) and \( \beta > 0 \), \( \alpha\lambda^x + \beta\lambda^{-x} \) is a decreasing function on \( (-\infty, t_{\alpha\beta}) \) and an increasing function on \( (t_{\alpha\beta}, \infty) \). By using the first integral in (4.10) whenever \( t \geq t_{\alpha\beta} \) and the second integral whenever \( t < t_{\alpha\beta} \), we can assure that the exponential function in the integrand remains bounded by \( 1 \). In either case, \( |g_{\alpha\beta}| \) can be bounded by one of its \( (X \) or \( Y \) \) derivatives times an exponential function, with the result that \( \sum f_{\alpha\beta} \in L^2 \)
of \((-\infty, t_{\alpha \beta}) \cup [t_{\alpha \beta}, \infty) = (-\infty, \infty)\). (If \(b = 0\), use \((X^2 + Y^2)g\) instead.) Restriction to \(\lambda^2 \leq \beta/\alpha \leq 1\) will keep \(\alpha^{-1}\) bounded in absolute value.

Next, let \(U = (2\pi)^{-m-n}X^mY^n \in \mathcal{H}(\mathcal{A}')\). If \(m - n > 0\) and \(t > t_{\alpha \beta}\), then

\[
|Uf_{\alpha \beta}(t)| \leq |\alpha^m \beta^n \lambda^{(m-n)t} \int_{t}^{\infty} |g_{\alpha \beta}(x)| \, dx| \leq \int_{t}^{\infty} |Ug_{\alpha \beta}(x)| \, dx,
\]

which provides the necessary estimate. If \(m - n < 0\) and \(t > t_{\alpha \beta}\),

\[
|Uf_{\alpha \beta}(t)| \leq |\alpha^m \beta^n \lambda^{(m-n)t} \int_{t}^{\infty} |g_{\alpha \beta}(x)| \, dx|.
\]

But \(\lambda^2 \leq \beta/\alpha \leq 1\) implies \(t_{\alpha \beta} \geq -1\), so

\[
|Uf_{\alpha \beta}(t)| \leq \lambda^{n-m} \int_{t}^{\infty} |\alpha^m \beta^n g_{\alpha \beta}(x)| \, dx \leq \lambda^{n-m} \int_{t}^{\infty} |\alpha^m \beta^n g_{\alpha \beta}(x)| \, dx
\]

\[
= \lambda^{n-m} \int_{t}^{\infty} X^{m+n} g_{\alpha \beta}(x) \lambda^{-(m+n)x} (2\pi)^{-m-n} \, dx
\]

\[
\leq \lambda^{2n} (2\pi)^{-m-n} \int_{t}^{\infty} |X^{m+n} g_{\alpha \beta}(x)| \, dx.
\]

From here the \(L^2\)-estimates proceed as earlier.

Next, suppose \(m - n < 0\) and \(t < t_{\alpha \beta}\). Then \(\lambda^{(m-n)t} > \lambda^{(m-n)t_{\alpha \beta}}\). Thus

\[
|Uf_{\alpha \beta}(t)| \leq |\alpha^m \beta^n \lambda^{(m-n)t} \int_{-\infty}^{t} |g_{\alpha \beta}(x)| \, dx| \leq \int_{-\infty}^{t} |\alpha^m \beta^n \lambda^{(m-n)x} g_{\alpha \beta}(x)| \, dx
\]

\[
= \int_{-\infty}^{t} |Ug_{\alpha \beta}(x)| \, dx.
\]

The rest is as before.

Finally, for \(m - n \geq 0\) and \(t < t_{\alpha \beta}\), we write

\[
|Uf_{\alpha \beta}(t)| \leq |\alpha^m \beta^n \lambda^{(m-n)t} \int_{-\infty}^{t} |g_{\alpha \beta}(x)| \, dx|
\]

\[
\leq |\lambda^{2m} \beta^{m} \lambda^{(m-n)t_{\alpha \beta}} \int_{-\infty}^{t} |g_{\alpha \beta}(x)| \, dx|
\]

\[
\leq \lambda^{2m} \int_{-\infty}^{t} |\beta^{m+n} g_{\alpha \beta}(x)| \, dx
\]

\[
= \int_{-\infty}^{t} |Y^{m+n} g_{\alpha \beta}(x)| \lambda^{(m+n)x} / (2\pi)^{m+n} \, dx
\]

\[
\leq (2\pi)^{-m-n} \int_{-\infty}^{t} |Y^{m+n} g_{\alpha \beta}(x)| \, dx.
\]

The \(L^2\)-estimates can be completed as before.

Next, we show that \(\sum_{(\alpha, \beta)} T^k f_{\alpha \beta} \in L^2\), for each \(k \in \mathbb{N}\). This follows from the next lemma.

\((4.11)\)  Lemma. Let \(f\) be a solution of the equation

\[ (T + i(X + Y)) f = g. \]
Then $T^k f$, $k = 1, 2, 3, \ldots$, is a linear combination of monomials $X^j Y^l f$ with $j + l \leq k$ plus a linear combination of $X, Y, T$-derivatives of $g$.

Proof. We proceed by induction. For $k = 1$ we have $T f = g - iX f - iY f$. Next, $T^{k+1} f = T(T^k f) = T(X^j Y^l f)$ with $j + l \leq k$, where wlog we may assume $T^k f$ is a monomial $X^j Y^l f$.

\[ T(X^j Y^l f) = \left( \sum_{p=1}^j X \cdots X[T, X]X \cdots XY^l \right) f + \sum_{q=1}^l X^j Y^l Y^q f - iX^j Y^l f - iX^j Y^l Y^{q+1} f = (j \ln \lambda X^j Y^l - l \ln \lambda X^j Y^{l+1}) f + \ldots \]

which is the desired expression for $T^k f$.

Remark. Similarly, $T^k f$ is a linear combination of monomials $X^j f$ with $j \leq k$ plus a linear combination of $X, T$-derivatives of $g$ if $f$ is a solution of $(T + iX) f = g$.

Subcase 4b. $\alpha < 0$ and $\beta > 0$. (The case $\alpha > 0$ and $\beta < 0$ can be treated similarly.) Once again, we have

\[ f_{\alpha \beta}(t) = \left( \int_0^t g_{\alpha \beta}(x) \exp \left( - (\alpha \lambda^x + \beta \lambda^{-x}) / \ln \lambda \right) dx + C \right) \exp \left( (\alpha \lambda^t + \beta \lambda^{-t}) / \ln \lambda \right) \]

where the terms $\beta \lambda^{-x}$ and $\beta \lambda^{-t}$ are not present if $b = 0$. Moreover, the restriction $\beta > 0$ is not needed if $b = 0$. We observe $\exp((\alpha \lambda^t + \beta \lambda^{-t}) / \ln \lambda) \to 0$ as $t \to +\infty$ and $\to +\infty$ as $t \to -\infty$ (or $\to 1$ as $t \to -\infty$ in case $b = 0$). In either case, since $\lim_{t \to -\infty} f_{\alpha \beta}(t) = 0$,

\[ C = \int_{-\infty}^0 g_{\alpha \beta}(x) \exp \left( - (\alpha \lambda^x + \beta \lambda^{-x}) / \ln \lambda \right) dx \]

and

\[ f_{\alpha \beta}(t) = \int_{-\infty}^t g_{\alpha \beta}(x) e^{\psi(x, t)} dx, \]

where

\[ \psi(x, t) = (\alpha (\lambda^t - \lambda^x) + \beta (\lambda^{-t} - \lambda^{-x})) / \ln \lambda, \]

again with no $\beta (\lambda^{-t} - \lambda^{-x})$ term in case $b = 0$. We notice that $\psi(x, t) < 0$ for $x < t$. We have the estimates

\[ |f_{\alpha \beta}(t)| \leq \int_{-\infty}^t |g_{\alpha \beta}(x)| dx \leq \int_{-\infty}^t |Y g_{\alpha \beta}(x)| / |2\pi \beta \lambda^{-x}| dx \]

\[ \leq C \sum_{k=0}^{1} \|T^k Y g_{\alpha \beta}\|_2 |\beta|^{-1} \int_{-\infty}^t \lambda^x dx \quad \text{(by Sobolev)} \]

\[ \leq M \sum_{k=0}^{1} \|T^k Y g_{\alpha \beta}\|_2 \lambda^t / \ln \lambda \in L^2(-\infty, 0) \]

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since $|\beta|^{-1}$ is bounded. In fact, (4.13) implies that $\sum_{(\alpha, \beta)} f_{\alpha \beta} \in L^2(-\infty, 0)$. Next, we must consider convergence in $L^2(0, \infty)$. Thus for $t > 0$ we write

$$f_{\alpha \beta}(t) = \int_{-\infty}^{0} \cdots + \int_{0}^{t/2} \cdots + \int_{t/2}^{t} \cdots = I_{\alpha \beta}(t) + \Pi_{\alpha \beta}(t) + \PiI_{\alpha \beta}(t)$$

where the integrands are as in (4.12).

To estimate $I_{\alpha \beta}$ we notice that for $x < 0 < t$ we have $\beta(\lambda^{-t} - \lambda^{-x}) \leq 0$ and $\alpha(\lambda^t - \lambda^x) < \alpha(\lambda^t - 1) < 0$. Hence

$$(4.14) \quad |I_{\alpha \beta}(t)| \leq e^{\alpha(\lambda^t - 1)/\ln \lambda} \int_{-\infty}^{0} |g_{\alpha \beta}(x)| \, dx$$

and $\sum_{(\alpha, \beta)} I_{\alpha \beta} \in L^2(0, \infty)$. This is because $\alpha \gg 0$ makes the functions

$$t \mapsto \exp(\alpha(\lambda^t - \lambda^x)/\ln \lambda), \quad (\alpha, \beta) \in (\Gamma \backslash S_h)^2,$$

uniformly $L^2(0, \infty)$, while the integral $\int_{-\infty}^{0} \cdots$ in (4.14) can be estimated as in (4.13) making the sum finite.

For $\Pi_{\alpha \beta}$ we have the estimate

$$(4.15) \quad |\Pi_{\alpha \beta}(t)| \leq \int_{0}^{t/2} |g_{\alpha \beta}(x)| e^{\alpha(\lambda^t - \lambda^x)/\ln \lambda} \, dx \leq \|g_{\alpha \beta}\|_\infty \frac{1}{t} e^{\alpha(\lambda^t - \lambda^{t/2})/\ln \lambda}.$$ 

The right-hand side again is $\alpha$-uniformly in $L^2(0, \infty)$ with $\|g_{\alpha \beta}\|_\infty$ being $(\alpha, \beta)$-summable.

Finally,

$$(4.16) \quad |\PiI_{\alpha \beta}(t)| \leq \int_{t/2}^{t} |g_{\alpha \beta}(x)| \, dx \leq \int_{t/2}^{t} |X^m g_{\alpha \beta}(x)|/|2\pi \alpha \lambda^x|^m \, dx \leq \|X^m g_{\alpha \beta}\|_\infty \frac{1}{t} M^m \lambda^{-mt},$$

where $M$ is an upper bound on $|\alpha|^{-1}$, $t \lambda^{-mt} \in L^2(0, \infty)$, and $\|X^m\|_\infty$ is $(\alpha, \beta)$-summable.

Next, we must show $\sum_{(\alpha, \beta)} U f_{\alpha \beta} \in L^2(R)$ for every fixed $U \in \mathcal{U}(S_h)$. If $U = Y^k$ we have the estimate

$$(4.17) \quad |Y^k f_{\alpha \beta}(t)| \leq \int_{-\infty}^{t} |Y^k g_{\alpha \beta}(x)| e^{\psi(x,t)} \, dx.$$ 

As in the beginning of Subcase 4b we can show that $\sum Y^k f_{\alpha \beta} \in L^2(R)$.

For $U = X^k$, $\sum X^k f_{\alpha \beta} \in L^2(-\infty, 0)$ because for $t \leq 0$,

$$(4.18) \quad |X^k f_{\alpha \beta}(t)| = 2^{kt} |\alpha/\beta|^k |Y^k f_{\alpha \beta}(t)| \leq 2^t |Y^k f_{\alpha \beta}(t)|$$

if we choose $(\alpha, \beta)$ such that $\lambda^{-2} \leq |\beta/\alpha| \leq 1$. If $t > 0$, we consider $X^k I_{\alpha \beta}$, $X^k \Pi_{\alpha \beta}$, and $X^k \PiI_{\alpha \beta}$ and we get the estimates (4.14), (4.15), and (4.16), each multiplied by $\lambda^{kt}$ and with $g_{\alpha \beta}$ replaced by $Y^k g_{\alpha \beta}$, as it was done in (4.18).
Finally, let $U = \mathcal{X}^p Y^q T^r$. Case of $r \geq 1$ reduces to $r = 0$ by the Lemma (4.11). If $r = 0$, we apply $\mathcal{X}^p$ to $\int_{-\infty}^{t} |Y^q g_{\alpha \beta}(x)| e^{\psi(x, t)} \, dx$ as we applied $U = \mathcal{X}^k$ to $\int_{-\infty}^{t} g_{\alpha \beta}(x) e^{\psi(x, t)} \, dx$.

**Subcase 4c.** $\alpha < 0$ and $\beta < 0$. We have

(4.19) \[ f_{\alpha \beta}(t) = \left( \int_{0}^{t} g_{\alpha \beta}(x) e^{-\frac{(\alpha x^2 + \beta x^{-2})}{\ln \lambda}} \, dx + C \right) e^{\frac{(\alpha x^2 + \beta x^{-2})}{\ln \lambda}}. \]

The function $C e^{-\frac{(\alpha x^2 + \beta x^{-2})}{\ln \lambda}}$ is in $\mathcal{S}(R)$ if $\alpha < 0$ and $\beta < 0$. Hence if there is a constant $C$ such that $f_{\alpha \beta}$ in (4.19) is super-Schwartz, then $f_{\alpha \beta}$ in $\mathcal{S}(R)$ for any fixed $C$. We will pick $C = C_{\alpha \beta} = \int_{0}^{t} g_{\alpha \beta}(x) e^{-\frac{(\alpha x^2 + \beta x^{-2})}{\ln \lambda}} \, dx$, where $t_{\alpha \beta} := \frac{1}{2} \log_2(\beta/\alpha)$, and $\alpha, \beta$ are chosen so that $\lambda^{-2} < \beta/\alpha \leq 1$. Thus we will work with

(4.19') \[ f_{\alpha \beta}(t) = \left( \int_{0}^{t} g_{\alpha \beta}(x) e^{-\frac{(\alpha x^2 + \beta x^{-2})}{\ln \lambda}} \, dx \right) e^{\frac{(\alpha x^2 + \beta x^{-2})}{\ln \lambda}}. \]

We have the estimate

\[
\begin{align*}
|f_{\alpha \beta}(t)| &\leq e^{\frac{(\alpha x^2 + \beta x^{-2})}{\ln \lambda}} \int_{0}^{t} (2\pi)^{-m} |(X + Y)^m g_{\alpha \beta}(x)| \times e^{-\frac{(\alpha x^2 + \beta x^{-2})}{\ln \lambda}} (-\alpha x^2 - \beta x^{-2})^{-m} \, dx \\
&\leq e^{-\frac{(\alpha x^2 + \beta x^{-2})}{\ln \lambda}} \|X + Y\|_{\infty} \|e^{-\frac{2\pi \ldots}{\ln \lambda}}(\alpha x^2 - \beta x^{-2})^{-m} |t - t_{\alpha \beta}| \\
&\leq c \sum_{l=0}^{1} \|T'(X + Y)^m g_{\alpha \beta}\|_{2} (\alpha x^2 - \beta x^{-2})^{-m} |t - t_{\alpha \beta}|
\end{align*}
\]

where $\ldots$ stands for $\alpha x^2 + \beta x^{-2}$. We write the inequality (4.20) for $(\alpha, \beta)$ such that

(4.21) \[ 2(\alpha \beta)^{1/2} / \ln \lambda > m, \]

because then the function $u \mapsto e^{-\frac{(\alpha u + \beta u^{-1})}{\ln \lambda}} (-\alpha u - \beta u^{-1})^{-m}$ with $u = \lambda^t$ is increasing for $t \geq t_{\alpha \beta}$ and decreasing for $t \leq t_{\alpha \beta}$. (4.21) is valid for all but a finite number of $(\alpha, \beta) \in (\Gamma \setminus S)_{\infty}$. Similarly, for $Y^k X f_{\alpha \beta}$ we have the estimate

(4.22) \[ |Y^k X f_{\alpha \beta}(t)| \leq c_1 \sum_{p=0}^{1} \|T^p(X + Y)^m g_{\alpha \beta}\|_{2} |(\alpha \beta)^{1/2} (-\alpha x^2 - \beta x^{-2})^{-m} |t - t_{\alpha \beta}|. \]

But for $t \geq t_{\alpha \beta}$, if $m > k + l$

\[
|\beta x^{-l}| |(\alpha \beta)^{k} (-\alpha x^2 - \beta x^{-2})^{-m} |t - t_{\alpha \beta}| \leq (\beta / \alpha)^{k} (-\alpha)^{k+l-m} \lambda^{l(k-l-m)} |t - t_{\alpha \beta}| \\
\leq M^{m-k-l} |t - t_{\alpha \beta}| \lambda^{l(k-l-m)} \in L^2(0, \infty)
\]

since $\beta / \alpha \leq 1$ and $\alpha \gg 0$.

Similarly, for $t \leq t_{\alpha \beta}$

\[
|\beta x^{-l}| |(\alpha \beta)^{k} (-\alpha x^2 - \beta x^{-2})^{-m} |t - t_{\alpha \beta}| \leq (\alpha / \beta)^{k} (-\beta)^{k+l-m} \lambda^{l(m-l-k)} |t_{\alpha \beta} - t| \\
\leq \lambda^{2k} M^{m-k-l} |t - t_{\alpha \beta}| \lambda^{l(m-l-k)} \in L^2(-\infty, 0)
\]
since \( \lambda^{-2} \leq \beta/\alpha \) and \( \beta \gg 0 \). Thus \( \sum Y^l X^k f_{\alpha \beta} \in L^2(R) \). Finally,
\[
\sum Y^l X^k T^m f_{\alpha \beta} \in L^2(R)
\]
by Lemma (4.11).

**Bibliography**


