GLOBAL REGULARITY ON 3-DIMENSIONAL SOLVMANIFOLDS

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Abstract. Let $M$ be any 3-dimensional (nonabelian) compact solvmanifold. We apply the methods of representation theory to study the convergence of Fourier series of smooth global solutions to first order invariant partial differential equations $Df = g$ in $C^\infty(M)$. We show that smooth infinite-dimensional irreducible solutions, when they exist, satisfy estimates strong enough to guarantee uniform convergence of the irreducible (or primary) Fourier series to a smooth global solution.

1. Introduction

Let $S$ be a solvable Lie group, and $\Gamma$ a discrete subgroup of $S$ with compact quotient $\Gamma\backslash S$. There is then a unique probability measure $\nu$ on $\Gamma\backslash S$ that is invariant under translation on the right by elements of $S$. The regular representation of $S$ on $L^2(\Gamma\backslash S, \nu)$ decomposes into a direct sum of a countable number of irreducible unitary representations $\pi$ of $S$, each of finite multiplicity $m_\pi$ [G]. Let $D$ be a first order differential operator with complex coefficients, left-invariant on $S$ and viewed on $\Gamma\backslash S$. Let $(\Gamma\backslash S)^*$ denote the dual object of $\Gamma\backslash S$. If $g \in C^\infty(\Gamma\backslash S)$ and if $g_\pi$ is an orthogonal component of $g$ corresponding to some irreducible unitary representation $\pi$, then $g_\pi \in C^\infty(\Gamma\backslash S)$ too [A-B]. Modulo unitary equivalence, we may think of $g_\pi$ as being a $C^\infty$-vector in any concrete realization, or model, of $\pi$. We are interested in algebraically well-defined conditions on $D$ under which the global solvability of $Df = g$ in $C^\infty(\Gamma\backslash S)$ is equivalent to the solvability of $\pi(D)f_\pi = g_\pi$ in the $C^\infty$-vectors for each $\pi$ in the spectrum of $\Gamma\backslash S$. In a sense, we are looking for algebraic conditions on $D$ for the reduction of a global (geometrical) problem on $\Gamma\backslash S$ to a collection of purely group (representation) theoretic problems, none of which needs to be regarded as living on the manifold $\Gamma\backslash S$. Informally speaking, operators $D$ admitting such a reduction are called globally regular (Definition (1.1)).

In order to describe the results, we will recall the classical situation on a torus $T^2$ of two dimensions (the situation being similar for $T^n$ with $n > 2$). Let $D = \alpha \partial/\partial x + \beta \partial/\partial y$ and suppose for simplicity that $\alpha$ and $\beta$ are real.
Then \( D \) is globally regular if and only if \( \beta/\alpha \) is not a (transcendental) Liouville number. The problem with Liouville numbers is that, in solving for the Fourier transform of the solution function, small divisors occur. Now, every solvmanifold \( \Gamma\backslash S \) contains the structure of a torus \( T = \Gamma[S, S]\backslash S \) of dimension \( \geq 1 \), although this torus does not reflect any of the nonabelian structure of \( S \). The only representations in \( (\Gamma\backslash S)^\sim \) which are not infinite dimensional are the one-dimensional characters of \( \Gamma[S, S]\backslash S \). Since the presence of this torus is inescapable, we denote, for each \( g \in C^\infty(\Gamma\backslash S) \), the sum of the one-dimensional components of \( g \) by \( g_0 \). Then global regularity is defined as follows. Let \( L^2(\Gamma\backslash S) = \bigoplus_{\pi \in (\Gamma\backslash S)^\sim} \bigoplus_{j=1}^{m_\pi} H_{\pi, j} \) be any (noncanonical) irreducible decomposition of \( L^2(\Gamma\backslash S) \).

(1.1) **Definition.** A left-invariant differential operator \( D \) on \( \Gamma\backslash S \) is called globally regular if the three conditions

1. \( g \in C^\infty(\Gamma\backslash S) \),
2. For each \( \pi \in (\Gamma\backslash S)^\sim \) and \( \forall j = 1, \ldots, m_\pi \exists \) a solution in \( C^\infty(\Gamma\backslash S) \) to \( Df_{\pi, j} = g_{\pi, j} \) (where \( g_{\pi, j} = (\pi, j)\)-component of \( g \)), and
3. \( \exists f_0 \in C^\infty(\Gamma) \text{ such that } Df_0 = g_0 \),

imply that there exists a solution in \( C^\infty(\Gamma\backslash S) \) to \( Df = g \).

Note that the solutions in (2) could be found in any convenient realization of \( \pi \).

In previous papers we have dealt with nilpotent \( S \). On the simplest nilmanifolds, the 3-dimensional Heisenberg manifolds, every first order differential operator \( D \) in the complexified Lie algebra is globally regular [R2]. On more complicated nilmanifolds the problem of small divisors arises in the representation spaces of the group as well as on the associated torus. Moreover, if \( D = X + iY \) is regular, both \( ad_X \) and \( ad_Y \) must map each step of the lower central series of the Lie algebra of \( S \) (nilpotent) onto a sufficiently large subset of the next step. The details are explained in [C-R1, p. 349]. The purpose of this paper is to investigate the global regularity of first order differential operators on 3-dimensional compact solvmanifolds. We show that, as in the case of the simplest nilmanifolds, every first order differential operator on a 3-dimensional compact solvmanifold is globally regular.

2. 3-DIMENSIONAL SOLVMANIFOLDS

All 3-dimensional compact solvmanifolds can be described (up to homeomorphism) as quotients of two groups \( S_h \) and \( S_r \) by their various cocompact discrete subgroups. The groups \( S_h \) and \( S_r \) can both be described as \( R^2 \times R^1 \), where \( (x, t)(x', t') = (x + A'x', t + t') \). Here \( A' \) is a 1-parameter subgroup of \( SL(2, R) \) through a matrix \( A \in SL(2, Z) \). \( S_h \) arises when the eigenvalues of \( A \) are \( \lambda > 1 \) and \( \lambda^{-1} \), so that the orbits of \( R^1 \) in \( R^2 \) are hyperbolic. \( S_r \) arises when \( A' \) is a compact group of rotations of \( R^2 \). Let \( N := R^2 \times \{0\} \), the (abelian) nilradical of \( S \). The cocompact discrete subgroups \( \Gamma \) are described in [A-G-H], based upon the facts (due to Mostow) that \( \Gamma \cap N \) is a discrete lattice in \( R^2 \), and that the image of \( \Gamma \) under the natural projection \( S \to S/N \) is a discrete lattice in \( R^1 \). We remark that \( \Gamma\backslash S_h \) is determined up to homeomorphism by the eigenvalue \( \lambda > 1 \) of \( A \), and \( \lambda \) must be such that \( \lambda + \lambda^{-1} \in Z \). For this reason, we denote the ‘hyperbolic’ manifolds \( \Gamma\backslash S_h \). Note however that
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$S_h$ is independent of the value of $\lambda > 1$. We have good use for the following lemma (3.4 in [C-R2]).

(2.1) **Lemma.** If $S_h = R^2 \times R^1$ with the diagonalized matrix $A = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$, $\lambda > 1$, and if $\Gamma_\lambda$ is a cocompact discrete subgroup of $S_h$, then $\Gamma_\lambda \cap N$ is an abelian lattice $L$ of points $(\alpha, \beta)$ having the property that the product $\alpha \beta$ is bounded away from zero, except of course at the identity.

We need also the following

(2.2) **Corollary.** In the setting of the lemma above, the dual lattice $L^* = \{x(a,b) : L \to 1\}$ is also a lattice of points $(a, b)$ such that the product $ab$ is bounded away from 0 (except of course for $(a, b) = (0, 0)$).

This corollary will give useful information about $(\Gamma \setminus S)_\infty$, the infinite dimensional representations in the spectrum of $\Gamma \setminus S$, in the hyperbolic case. For all 3-dimensional compact solvmanifolds, $(\Gamma \setminus S)_\infty$ is constructed as follows. Let $x(a, b) \in \hat{N}$, where $x(a, b) : \Gamma \cap N \to 1$. Now let $M$ be the extension of $N$ by the stabilizer of $x(a, b)$ in $S$, and extend $x$ to $M$, so $\pi_{a,b} := \text{Ind}^M_N(x(a, b)) \in \hat{S}$. If $S = S_h$ then $M = N$ and $M \setminus S \cong R$, whereas if $S = S_r$ then $M \cong N \times Z$ and $M \setminus S \cong R/Z = \text{the circle group}$. If $H(x, M)$ is the standard Mackey induced representation space, then $H(x, M) = \{f : S \to C|f| = \chi_{x, M}(m)f(s), |f| \in L^2(M \setminus S)\}$. Define $L : H(x, M) \to L^2(\Gamma \setminus S)$ by $(Lf)(s) = \sum_{y \in M} f(y)$. Then $L$ is a right $S$-invariant injection. If $\text{Int}(x, M) = \{t \in R|\chi^{exp}t : \Gamma \cap M \to 1\}$ where $\chi^t(b) = \chi(a^{-1}ba)$ then the multiplicity of $\pi_{a,b}$ in $L^2(\Gamma \setminus S)$ equals the number of distinct $\Gamma$-orbits in $\text{Int}(x, M)$.

In the case of $S = S_h$, it is easiest to describe $\Gamma \setminus S_h$ if we take $\Gamma \cap N = Z^2$ and $A \in SL(2, Z)$. However, the model for $\pi_{a,b}$ is simplest if $A$ is diagonalized $A = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$ as in Lemma (2.1)) and then $\Gamma \cap N$ is more difficult to describe. Nevertheless, the Corollary (2.2) above shows that in this model $\{\pi_{a,b} \in (\Gamma \setminus S)_\infty\}$ has $\alpha \beta$ bounded away from zero. We remark upon the fact that $\alpha \beta \neq 0$ prevents $\pi_{a,b}$ from being a representation of the (less well-behaved) ‘$ax + b$’ group which is a quotient of $S_h$.

We need to say a few words about the rotational three dimensional solvmanifolds $\Gamma \setminus S_r$, as well. Unlike the hyperbolic case, there are only finitely many $\Gamma \setminus S_r$ up to homeomorphism. We take $A \in SL(2, Z)$, but this time with no eigenvalues $> 1$. Now $A$ turns out to be similar to a rotation by $2\pi/p$, where $p = 1, 2, 3, 4, \text{ or } 6$ (see [A-G-H]). Since $S_r$ is independent of $p$ (up to group isomorphism), we denote the distinct rotational three dimensional solvmanifolds by $\Gamma_p \setminus S_r$, $p = 1, 2, 3, 4, \text{ or } 6$.

Our main result is

(2.3) **Theorem.** Let $\Gamma \setminus S$ be any nonabelian 3-dimensional compact solvmanifold. If $D \in \mathcal{F}^C$, the complexified Lie algebra of $S$, then $D$ is globally regular on $\Gamma \setminus S$.

Since this result has been proved in an earlier paper when $S$ is nilpotent, we concern ourselves here only with the manifolds $\Gamma_\lambda \setminus S_h$ and $\Gamma_p \setminus S_r$. Note also that the associated torus $T$ is 1-dimensional, so that small divisors cannot
occur on $T$. Thus condition (3) of Definition (1.1) will be satisfied automatically whenever (2) holds.

We will divide our proof of the theorem into two sections, one dealing with the rotational group $S_r$, the other with the hyperbolic one, $S_h$.

3. Proof of the Theorem—case of $\Gamma_p \setminus S_r$

Let $\mathfrak{g}$ be the Lie algebra of $S_r$ with a linear basis $T, X, Y$ and the commutation relations $[T, X] = -\frac{2\pi}{p} Y$ and $[T, Y] = \frac{2\pi}{p} X$. Let $\pi_{\alpha\beta}$ be a generic infinite-dimensional representation in $(\Gamma_p \setminus S_r)^\ast$, acting in $L^2(T)$, where $T = M \setminus S_r$, a 1-dimensional torus as described in §2. For $f \in L^2(T)$ the action of $\pi_{\alpha\beta}$ on $f$ is given by

$$\pi_{\alpha\beta}(x, y; t) f(\tau) = \exp(2\pi i ((\alpha, \beta) \sigma(\frac{2\pi}{p} \tau), (x, y))) f(\tau + t),$$

where $\sigma(s) = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}$. For the basis vector fields $X, Y, T$ this amounts to

$$d\pi_{\alpha\beta}(X) = 2\pi i \begin{pmatrix} \cos \frac{2\pi}{p} t - \beta \sin \frac{2\pi}{p} t \\ \alpha \cos \frac{2\pi}{p} t - \beta \sin \frac{2\pi}{p} t \end{pmatrix},$$

(3.1)

$$d\pi_{\alpha\beta}(Y) = 2\pi i \begin{pmatrix} \alpha \sin \frac{2\pi}{p} t + \beta \cos \frac{2\pi}{p} t \\ -\alpha \sin \frac{2\pi}{p} t - \beta \cos \frac{2\pi}{p} t \end{pmatrix},$$

$$d\pi_{\alpha\beta}(T) = \frac{d}{dt}.$$

Since the constant $p$ plays a negligible role in the proof (even though it classifies the rotational 3-dimensional solvmanifolds), we will set $p = 1$ in what follows.

We will break the proof into two cases of $D \in \mathfrak{g}^C$.

Case 1. $D = X + \gamma Y$, $\gamma \in C$. This essentially covers all $D \in \mathfrak{g}^C$ (up to a constant factor) since $D = Y$ and $D = X$ behave alike and the case of $D = 0$ is trivial.

Case 2. $D = T + i(aX + bY)$, $a, b \in R$. This covers all $D \in \mathfrak{g}^C \sim \mathfrak{g}^C$ (up to an isomorphism) because the real part of $D$ can be absorbed into $T$.

Proof of Case 1. Write $D = X + (a + ib) Y$, $a, b \in R$. Then, in view of (3.1), the operator $d\pi_{\alpha\beta}(D)$ is a multiplication by the function

$$D_{\alpha\beta}(t) = 2\pi i[(\alpha + a\beta + ib\beta) \cos 2\pi t + (a\alpha - \beta + ib\alpha) \sin 2\pi t]$$

(3.2)

$$= 2\pi i(w \cos 2\pi t + z \sin 2\pi t).$$

By hypothesis, the equation $d\pi_{\alpha\beta}(D)f_{\alpha\beta} = g_{\alpha\beta}$ has the solution

$$f_{\alpha\beta}(t) = g_{\alpha\beta}(t)/D_{\alpha\beta}(t).$$

(3.3)

To prove the theorem we need to show that $\sum_{(\alpha, \beta)} Lf_{\alpha\beta} \in C^\infty(\Gamma \setminus S)$. Here $(\alpha, \beta)$ varies over a cross section of $\Gamma$-orbits, so that each infinite-dimensional primary summand will be spanned. By the Auslander-Brezin version of the Sobolev inequality it suffices to show $\sum_{(\alpha, \beta)} \|UF_{\alpha\beta}\|^2 < \infty$ for all $U \in \mathcal{Z}(\mathfrak{g}^C)$. Since $L$ is an $S$-invariant isometry from $H_{\alpha\beta}$ into $L^2(\Gamma \setminus S)$, this is the same as to show $\sum_{(\alpha, \beta)} \|UF_{\alpha\beta}\|^2 < \infty$. We begin by estimating the sum
The problem, of course, is that $D_{\alpha \beta}(t)$ in (3.3) can have zeros, and that even when it has no zeros, we must know how close $|D_{\alpha \beta}(t)|$ can come to 0 as $(\alpha, \beta) \to \infty$. We can write

$$D_{\alpha \beta}(t) = \pi i ((w - iz) e^{2\pi it} + (w + iz) e^{-2\pi it}) = \pi i (A e^{2\pi it} + B e^{-2\pi it}),$$

with $A$ and $B$ defined by the last equation. The minimum of $|D_{\alpha \beta}(t)|$ occurs where $A$ and $B$ are rotated to opposite directions, and then

$$\text{Min} |D_{\alpha \beta}(t)| = \pm \pi (|A| - |B|)$$

(3.4)

$$= \pi (\alpha^2 + \beta^2)^{1/2}((\alpha^2 + (1 + b)^2)^{1/2} - (\alpha^2 + (1 - b)^2)^{1/2})$$

$$= (\alpha^2 + \beta^2)^{1/2} \cdot K \geq \sqrt{2} |\alpha \beta|^{1/2} \cdot K.$$

If $b \neq 0$, the constant $K$ is $\neq 0$. Since $|\alpha \beta| \gg 0$, $|f_{\alpha \beta}(t)| \leq C |g_{\alpha \beta}(t)|$ with the constant $C$ independent of $(\alpha, \beta)$. Consequently, $\sum_{(\alpha, \beta)} \|f_{\alpha \beta}\|_2 \leq C \sum_{(\alpha, \beta)} \|g_{\alpha \beta}\|_2^2 < \infty$ for $g \in C^\infty(\Gamma \setminus S)$.

If $b = 0$, $D_{\alpha \beta}(t)$ does have one or more zeros and

$$D_{\alpha \beta}(t) = 2\pi i ((a + \beta \alpha \cos 2\pi t + (a \alpha - \beta \sin 2\pi t)).$$

If $D_{\alpha \beta}(t_0) = 0$, then $g_{\alpha \beta}(t_0) = 0$ too, since $f_{\alpha \beta}$ given by (3.3) is in $C^\infty(T)$. The idea is to control $f_{\alpha \beta}$ inside specified intervals around each of the $t_0$'s by the mean value theorem, and to use the monotonicity of $D_{\alpha \beta}$ on large exterior intervals to control $|f_{\alpha \beta}|$ by keeping $|D_{\alpha \beta}|$ big. We have

$$D_{\alpha \beta}'(t_0) = -2\pi (\alpha^2 + \beta^2)^{1/2}(1 + a^2)^{1/2},$$

and by the mean value theorem

$$|D_{\alpha \beta}'(t) - D_{\alpha \beta}'(t_0)| \leq |t - t_0| 4\pi^2 (\alpha^2 + \beta^2)^{1/2}(1 + a^2)^{1/2}.$$

Consequently, for $|t - t_0| < 1/4\pi$, $|D_{\alpha \beta}'(t)| \geq \pi (\alpha^2 + \beta^2)^{1/2}(1 + a^2)^{1/2}$ and, since $D_{\alpha \beta}(t_0) = 0$,

$$|D_{\alpha \beta}(t)| \geq \pi (\alpha^2 + \beta^2)^{1/2}(1 + a^2)^{1/2}|t - t_0|$$

again by the mean value theorem. For $g_{\alpha \beta}(t)$ we have the estimate

(3.5)

$$|g_{\alpha \beta}(t)| = |g_{\alpha \beta}(t) - g_{\alpha \beta}(t_0)| \leq \|T g_{\alpha \beta}\|_\infty |t - t_0|,$$

$\| \|_\infty$ denoting the sup norm on the torus $M \setminus S$. So

(3.6) $|f_{\alpha \beta}(t)| \leq \|T g_{\alpha \beta}\|_\infty \frac{1}{\pi} (\alpha^2 + \beta^2)^{-1/2}(1 + a^2)^{-1/2} \quad \text{for} \quad |t - t_0| < 1/4\pi.$

On the intervals complementary to $|t - t_0| < 1/4\pi$,

$$|D_{\alpha \beta}(t)| \geq |D_{\alpha \beta}(t_0 \pm 1/4\pi)| \geq \pi (\alpha^2 + \beta^2)^{1/2}(1 + a^2)^{1/2}/4\pi,$$

hence

(3.7) $|f_{\alpha \beta}(t)| \leq 4\|g_{\alpha \beta}\|_\infty (\alpha^2 + \beta^2)^{-1/2}(1 + a^2)^{-1/2}$ for $|t - t_0| \geq 1/4\pi$

for each of the two values $t_0$ of $t$ where $D_{\alpha \beta}$ vanishes. By (3.6) and (3.7), for all $t$ on the torus $M \setminus S$ we have the following estimate:

(3.8) $|f_{\alpha \beta}(t)| \leq (\|T g_{\alpha \beta}\|_\infty /\pi + 4\|g_{\alpha \beta}\|_\infty)(\alpha^2 + \beta^2)^{-1/2}(1 + a^2)^{-1/2}.$

By Sobolev's inequality we may replace the sup norms in (3.8) by $L^2$-norms of $g_{\alpha \beta}$, $T g_{\alpha \beta}$, and $T^2 g_{\alpha \beta}$. The sum $\sum_{(\alpha, \beta)} \|f_{\alpha \beta}\|_2^2$ is finite because each sum $\sum_{(\alpha, \beta)} \|T^k g_{\alpha \beta}\|_2^2$ for $k = 0, 1, 2$ is finite and $|\alpha \beta|$ is bounded away from zero.
Next, we must show \( \sum_{(\alpha, \beta)} \|Uf_{\alpha\beta}\|_{L^2}^2 < \infty \) for every fixed \( U \in \mathcal{U}(\mathcal{S}) \). Since \( [D, \mathcal{N}] = 0 \), this is true for all \( U \in \mathcal{U}(\mathcal{N}) \). It remains to show \( \sum_{(\alpha, \beta)} \|Tk f_{\alpha\beta}\|_{L^2}^2 < \infty \) for \( k = 1, 2, \ldots \), because every \( U \in \mathcal{U}(\mathcal{S}) \) can be written as a linear combination of monomials \( T^k V \) with \( V \) in \( \mathcal{U}(\mathcal{N}) \).

(3.9) **Proposition.** For \( D = X + \gamma Y, \gamma \in C \) and \( k = 1, 2, 3, \ldots \) the \((k+1)\)-fold bracket product \([D[\ldots [D, T^k] \ldots]]\) is 0.

**Proof of Proposition.** By Leibnitz’s rule for the derivation \([D, \cdot]\) of the algebra \( \mathcal{U}(\mathcal{S}) \) we have \( [D, T^k] = \sum_{j=1}^{k} T \cdots T[D, T] T \cdots T \), with \([D, T]\) at \( j \)th place. Since \( D \) commutes with \( \mathcal{N} \) and \([D, T] = -X + \gamma Y \in \mathcal{N}\), the derivation repeated \( k + 1 \) times is zero.

(3.10) **Proposition.** For \( f_{\alpha\beta} \) as in (3.3) we have

\[
(3.11) \quad T^k f_{\alpha\beta} = h_k/D_{\alpha\beta}^{k+1}
\]

with

\[
(3.12) \quad h_k = [D[D[\ldots [D, T^k] \ldots]]]f_{\alpha\beta} + D[D[\ldots [D, T^k] \ldots]]f_{\alpha\beta} + \cdots + D^{k-1}[D, T^k] + D^k T^k f_{\alpha\beta}.
\]

The first bracket involves \( k \) \( D \)'s with the number of \( D \)'s inside the brackets decreasing by one in each successive summand.

**Proof of Proposition.** In view of Proposition (3.9) this is formula (1.8) on p. 353 of [C-R1].

The estimates on \( T^k f_{\alpha\beta} \) given by (3.11) can now be done in a manner similar to that already presented. If \( D_{\alpha\beta} \neq 0 \), we use the inequality (3.4) raised to the power \( k + 1 \) to estimate the denominator \( |D_{\alpha\beta}|^{k+1} \). If \( D_{\alpha\beta}(t_0) = 0 \), the numerator \( h_k \) in (3.11) must have a \((k+1)\)th order zero at \( t_0 \) since \( T^k f_{\alpha\beta} \) is \( C^\infty \). Instead of the estimate (3.5) we use

\[
|h_k(t)| \leq \|T^{k+1}h_k\|_{L^\infty}|t - t_0|^{k+1}/(k + 1)!
\]

which follows from Taylor’s formula. In the denominator of (3.6) and (3.7) we use the \((k+1)\)th power of the previous estimate for \( D_{\alpha\beta} \).

**Proof of Case 2.** Here \( D = T + i(aX + bY), a, b \in R \), and

\[
(3.12) \quad d\pi_{\alpha\beta}(D) = \frac{d}{dt} - 2\pi((a\alpha + b\beta) \cos 2\pi t + (b\alpha - a\beta) \sin 2\pi t)
\]

Write \( A \cos 2\pi t + B \sin 2\pi t = (A^2 + B^2)^{1/2} \sin 2\pi(t + \phi) \), with the constant \( \phi \) depending upon \( A \) and \( B \). If \( d\pi_{\alpha\beta}(D)f_{\alpha\beta} = g_{\alpha\beta} \), we have

\[
f_{\alpha\beta}(t) = \exp(-2\pi(A^2 + B^2)^{1/2} \cos 2\pi(t + \phi)) \times \left( \int_{\phi-\frac{1}{2}}^{t} g_{\alpha\beta}(\tau) \exp(2\pi(A^2 + B^2)^{1/2} \cos 2\pi(\tau + \phi)) d\tau + C \right).
\]

Here we identify \( M \backslash S = R/Z \) with the interval \([-\phi - \frac{1}{2}, \frac{1}{2} - \phi]\), and we use the fact that \( f_{\alpha\beta}, g_{\alpha\beta} \), and the exponentials all have period 1. Since \( C \) is
arbitrary, choosing \( C = 0 \) and changing the variables \( t' = \tau + \phi \), \( t' = t + \phi \) we have

\[
\int_{-1/2}^{t} g(t - \phi) \exp(2\pi(A^2 + B^2)^{1/2}(\cos 2\pi \tau - \cos 2\pi t)) \, d\tau
\]

\[
= - \int_{t}^{1/2} \quad \text{(same integrand as above)}.
\]

The last equality follows from the periodicity of \( f_{\alpha\beta} \), because \( 0 = f_{\alpha\beta}(-1/2 - \phi) = f_{\alpha\beta}(1/2 - \phi) \). For the estimates on \( f_{\alpha\beta}(t) \) or on \( f_{\alpha\beta}(t - \phi) \) we use the integral over \(-1/2 \leq \tau \leq t\) if \(-1/2 \leq t \leq 0\) and over \( t \leq \tau \leq 1/2\) for \( 0 < t < 1/2\). That way we always have \( \cos 2\pi \tau < \cos 2\pi t \), so that the exponent in the integral defining \( f_{\alpha\beta} \) is negative. Then

\[
\|f_{\alpha\beta}\|_{\infty} \leq C \|g_{\alpha\beta}\|_{\infty} \leq C_1 (\|g_{\alpha\beta}\|_2 + \|Tg_{\alpha\beta}\|_2)
\]

Since the \( f_{\alpha\beta} \)'s all live on an interval of length 1, the same inequality holds for \( \|f_{\alpha\beta}\|_2 \) with a constant, say \( C'_1 \), independent of \( (\alpha, \beta) \). Thus \( \sum_{(\alpha, \beta)} \|f_{\alpha\beta}\|_2^2 < \infty \) and \( f = \sum f_{\alpha\beta} \in L^2(\Gamma_p \backslash S_r) \). Next, to estimate \( \sum_{(\alpha, \beta)} \|Uf_{\alpha\beta}\|_2^2 \) for every \( U \in \mathcal{Z}(\mathcal{F}) \) it suffices to consider \( U = X^k Y^l T^l \). By (3.12) \( T \) just differentiates \( f_{\alpha\beta} \) yielding \( 2\pi(A^2 + B^2)^{1/2} \sin 2\pi(t + \phi)f_{\alpha\beta}(t) + g_{\alpha\beta}(t) \) with \( A, B \) depending linearly on \( \alpha, \beta \). Successive powers of \( T \) differentiate \( g_{\alpha\beta} \) and \( \sin 2\pi(t + \phi) \), or \( f_{\alpha\beta}(t) \). Operating by \( X \)'s or \( Y \)'s just multiplies by a polynomial in \( (\alpha, \beta) \) times a (bounded) combination of sines and cosines. However, \( X^2 + Y^2 \) acts on \( L^2(M \backslash S) \) by multiplying by \(-4\pi^2(\alpha^2 + \beta^2)\), so that \( \sum_{(\alpha, \beta)} (\alpha^2 + \beta^2)^q \|g_{\alpha\beta}\|^2 < \infty \). Thus \( \sum_{(\alpha, \beta)} \|X^k Y^l T^l f_{\alpha\beta}\|_2^2 \) can be estimated by \( C \sum \|(X^2 + Y^2)^q g_{\alpha\beta}\|^2 \) for some \( q \).

Remark. Although we have pretended \( p = 1 \) in the arguments above, this was just a notational convenience, and the same arguments work just as well for general \( \Gamma \backslash S_r \). Actually, the use of \( \Gamma \backslash S_r \) has some independent interest however. Here, although \( S_r \) is not nilpotent, \( \Gamma \cong Z^3 \). Thus, by the general theory of Mostow [M], \( \Gamma \backslash S_r \) is homeomorphic to a torus \( T^3 \). Here we get a proof of global regularity for elements of \( \mathcal{S}^C \) acting on \( T^3 \), keeping in mind that we mean global regularity with respect to the representation theory of \( S_r \). Thus we have a natural class of operators on \( T^3 \) which are best analyzed with respect to \( S_r \).

### 4. Proof of the theorem—case of \( \Gamma \backslash S_h \)

Let \( \mathcal{S}_h \) be the Lie algebra of \( S_h \) with a linear basis \( T, X, Y \) and the commutation relations \( [T, X] = X \ln \lambda \) and \( [T, Y] = -Y \ln \lambda \). A generic infinite dimensional representation \( \pi_{\alpha\beta} \) in \( (\Gamma \backslash S_h) \) acts on \( L^2(R) \) by

\[
\pi_{\alpha\beta}(x, y, t) f(\tau) = \exp 2\pi i(\alpha \tau^t x + \beta \lambda^{-t} y) f(\tau + t).
\]

For the basis vector fields \( X, Y, T \) this amounts to

\[
d \pi_{\alpha\beta}(X) = 2\pi i \lambda^t, \quad d \pi_{\alpha\beta}(Y) = 2\pi i \lambda^{-t}, \quad d \pi_{\alpha\beta}(T) = \frac{d}{dt}.
\]

The space \( H^\infty_{\alpha\beta} \) of \( C^\infty \)-vectors for \( \pi_{\alpha\beta} \) consists of \( C^\infty(R) \) functions \( \phi \) with

\[
\lim_{|t| \to \infty} \lambda^{|t|} \phi(t) = 0.
\]
for every \( m, n \in N \). These \( \phi \) are called “super-Schwartz”. We will denote this space by \( \mathcal{S}^p(R) \).

We will break the proof into four cases of \( D \in \mathcal{S}^c_h \).

**Case 1.** \( D = T \). Any nonzero complex multiple of \( D = T + aX + bY \in \mathcal{S}_h \) can be reduced up to isomorphism to \( D = T \), if \( a, b \in R \).

**Case 2.** \( D = aX + bY \in \mathcal{N} \subset \mathcal{S}_h \), \( a, b \in R \).

**Case 3.** \( D = aX + bY \in \mathcal{N}^c \subset \mathcal{S}_h^c \), \( a, b \in C \).

**Case 4.** \( D = T + i(aX + bY) \), \( a, b \in R \).

**Proof of Case 1.** \( D = T \). The equation \( d\pi_{\alpha\beta}(T)f_{\alpha\beta} = g_{\alpha\beta} \) becomes \( (d/dt)f_{\alpha\beta} = g_{\alpha\beta} \). Since both \( f_{\alpha\beta} \) and \( g_{\alpha\beta} \) are in \( H^\infty_{\alpha\beta} \), we have

\[
(4.2) \quad f_{\alpha\beta}(t) = \int_{-\infty}^{t} g_{\alpha\beta}(\tau) d\tau = -\int_{t}^{\infty} g_{\alpha\beta}(\tau) d\tau.
\]

**Subcase 1a.** If \( t \geq 0 \), we use the second integral for the estimates on \( f_{\alpha\beta} \) and the fact that \( g_{\alpha\beta}(\tau) = Xg_{\alpha\beta}(\tau)/2\pi i\alpha \lambda^2 \), so that

\[
(4.3) \quad |f_{\alpha\beta}(t)| \leq \|Xg_{\alpha\beta}\|_\infty (2\pi \alpha \ln \lambda)^{-1} \lambda^{-t} \in L^2([0, \infty), dt).
\]

Squaring, integrating \( \int_0^\infty \cdots dt \), and applying Sobolev’s inequality to estimate the norm \( \|Xg_{\alpha\beta}\|_\infty \) by a combination of \( L^2 \)-norms of derivatives of \( Xg_{\alpha\beta} \) we obtain

\[
(4.4) \quad \|f_{\alpha\beta}\|_{L^2([0, \infty), t)}^2 \leq \frac{c}{\alpha} \left( \|Xg_{\alpha\beta}\|_2^2 + \|TXg_{\alpha\beta}\|_2^2 \right),
\]

for some constant \( c \). As in the proof of Lemma (2.1) (see [C-R2]), we pick the representative \( (\alpha, \beta) \) from the orbit corresponding to \( \pi \) so that \( \lambda^{-2} \leq |\beta/\alpha| \leq 1 \). Since \( |\alpha\beta| \gg 0 \), we must have \( |\alpha| \gg 0 \) too. Thus \( \sum_{(\alpha, \beta)} \|f_{\alpha\beta}\|_{L^2([0, \infty), t)}^2 \ll \infty \).

**Subcase 1b.** If \( t < 0 \), we use the first integral formula for \( f_{\alpha\beta} \) in (4.2), together with the fact that \( g_{\alpha\beta}(\tau) = Yg_{\alpha\beta}(\tau)/2\pi i\beta \lambda^{-t} \). Picking \( (\alpha, \beta) \) as in Subcase 1a, we insure that \( |\beta| \gg 0 \). Now an argument almost identical to that in Subcase 1a shows that \( f = \sum f_{\alpha\beta} \in L^2(R) \).

To complete Case 1, we need to prove that \( \sum \|Uf_{\alpha\beta}\|_2^2 \ll \infty \), for each fixed \( U \in \mathcal{Z}(\mathcal{S}_h) \). If \( U = T^p \) there is no problem, since \( [D, U] = 0 \). If \( U = X^m Y^n T^p \), \( Uf_{\alpha\beta}(t) = (2\pi i)^m+n\alpha^m\beta^n\lambda^{(m-n)t}T^{p-1}g_{\alpha\beta}(t) \). If \( p \geq 1 \), there is no problem. But if \( p = 0 \), we estimate \( |g_{\alpha\beta}(\tau)| \) by a \( \|Vg_{\alpha\beta}\|_{\lambda^{kt}} \), where \( V \in \mathcal{Z}(\mathcal{S}_h) \) and \( k \in Z \) are chosen one way for \( t < 0 \) and another way for \( t \geq 0 \), so as to assure that \( Uf_{\alpha\beta} \in L^2(-\infty, 0) \) and \( Uf_{\alpha\beta} \in L^2(0, \infty) \) separately. Hence \( \sum Uf_{\alpha\beta} \in L^2(R) \).

**Case 2.** Suppose \( D = aX + bY \), with \( a \) and \( b \) real. Here,

\[
f_{\alpha\beta}(t) = g_{\alpha\beta}(t)/2\pi i(a\alpha \lambda^t + b\beta \lambda^{-t}).
\]

We will need to consider the occurrence of zeros in the denominator. Observe that the denominator vanishes if and only if \( \lambda^{2t} = -b\beta/a\alpha \), which means \( t = t_0 = \frac{1}{2} \log_i(-b\beta/a\alpha) \).

**Subcase 2a.** Suppose \( b\beta/a\alpha > 0 \), so that \( h(t) = (a\alpha \lambda^t + b\beta \lambda^{-t})^{-1} \) has a maximum (absolute) value on \( R \). At the maximum point \( h'(t) = 0 \), so that...
\[ \lambda^{2t} = b \beta / a \alpha. \] Thus \( h(t_{\text{max}}) = \lambda^{t_{\text{max}}}/2b \beta = \pm 1/2|a \alpha \beta|^{1/2}. \) Thus \( |f_{\alpha \beta}(t)| \leq |g_{\alpha \beta}(t)|/2|a \alpha \beta|^{1/2}. \) Since \( |\alpha \beta| \gg 0, \) we have \( \sum f_{\alpha \beta} \in L^2 \), where the sum is over those \((\alpha, \beta)\) such that \( b \beta/a \alpha > 0. \) The same is true for \( \sum U f_{\alpha \beta} \), for each fixed \( U \in \mathcal{U}(\mathcal{F}_h) \) since

1. If \( U \in \mathcal{U}(\mathcal{N}), [U, D] = 0, \) and
2. If \( U = X^m Y^n T^k \), then

\[ \frac{d}{dt}(g_{\alpha \beta}(t) h(t)) = g'_{\alpha \beta} h + g_{\alpha \beta} h' = (T g_{\alpha \beta}) h + g_{\alpha \beta} h', \]

and \( g_{\alpha \beta} h' = (X - Y) g_{\alpha \beta} \cdot h_{\alpha \beta}^2 \), which is still \( L^2 \)-summable.

Subcase 2b. Either \( a \alpha = 0 \) or \( b \beta = 0. \) Since \( |\alpha \beta| \gg 0 \) (except at \((0, 0)\)) this means \( a = 0 \) or \( b = 0 \) (but not both). If \( b = 0, \) then \( f_{\alpha \beta}(t) = -Y g_{\alpha \beta}(t)/4\pi^2 a \alpha \beta. \) Then \( \sum f_{\alpha \beta} \in L^2, \) since \( |\alpha \beta| \gg 0. \) A similar argument applies if \( a = 0. \)

Subcase 2c. Here, we suppose there exists a zero, \( t_0 = \frac{1}{2} \log_2(-b \beta/a \alpha). \) Then \( g_{\alpha \beta}(t_0) = 0 \) too, and \( f_{\alpha \beta}(t) = (g_{\alpha \beta}(t) - g_{\alpha \beta}(t_0))/2\pi i(a \alpha \lambda^t + b \beta \lambda^{-t}). \) Since it would suffice to prove separately the summability of the real and imaginary parts of \( f_{\alpha \beta}, \) we can assume wlog that \( g_{\alpha \beta} \) is real-valued. For some \( \tau \) between 0 and \( t, \) we have \( f_{\alpha \beta}(t) = g'_{\alpha \beta}(\tau)/2\pi i(a \alpha \lambda^\tau + b \beta \lambda^{-\tau}) \ln \lambda, \) by the general mean value theorem. Since \( b \beta/a \alpha < 0 \) in this case, the new denominator cannot vanish. Hence, recalling that \( \tau \) depends on \( t, \)

\[ 4\pi^2 \int_R |f_{\alpha \beta}(t)|^2 dt = \int_{|t - t_0| \leq 1} |g'_{\alpha \beta}(\tau)|^2 |a \alpha \lambda^\tau - b \beta \lambda^{-\tau}|^{-2} |\ln \lambda|^{-2} dt + \int_{|t - t_0| \geq 1} |g_{\alpha \beta}(t)|^2 |a \alpha \lambda^t + b \beta \lambda^{-t}|^{-2} dt \]

\[ \leq \frac{||g_{\alpha \beta}'||_\infty^2}{4a \alpha \beta b \beta \ln \lambda^2} + \int_{|t - t_0| \geq 1} |g_{\alpha \beta}(t)|^2 \cdot \text{Max} |a \alpha \lambda^t + b \beta \lambda^{-t}|^{-2} dt. \]

The first summand is summable over \((\alpha, \beta), \) by Sobolev's inequality. However, by the mean value theorem, \( |a \alpha \lambda^t + b \beta \lambda^{-t}| = |(a \alpha \lambda^t - b \beta \lambda^{-t})(\ln \lambda)(t - t_0)| \geq 2(-a \alpha b \beta)^{1/2} |\ln \lambda|, \) if \( |t - t_0| \geq 1. \) Thus

\[ \pi^2 \int_R |f_{\alpha \beta}(t)|^2 dt \leq \frac{||g_{\alpha \beta}'||_\infty^2}{|a \alpha b \beta| \ln \lambda^2} + \int_R \frac{|g_{\alpha \beta}(t)|^2}{|a \alpha b \beta| \ln \lambda^2} dt, \]

and so \( \sum f_{\alpha \beta} \in L^2. \)

If \( U \in \mathcal{U}(\mathcal{N}), \) then \( \sum U f_{\alpha \beta} \in L^2 \) too, since \([D, U] = 0. \) So it suffices to consider \( \sum T^k f_{\alpha \beta}, \) for each fixed \( k \in \mathbb{Z}^+. \) First, consider \( k = 1. \)

\[ 2\pi i T f_{\alpha \beta}(t) = h(t)(a \alpha \lambda^t + b \beta \lambda^{-t})^{-2}, \] where \( h(t) = (a \alpha \lambda^t + b \beta \lambda^{-t}) g_{\alpha \beta}(t) - g_{\alpha \beta}(t)(a \alpha \lambda^t - b \beta \lambda^{-t}) \ln \lambda. \) Here \( h(t) \) must have a zero of order at least two at \( t = t_0. \) Therefore, using a Taylor remainder of degree two in \((t - t_0), \) there exist \( \tau \) and \( \tau' \) such that \( T f_{\alpha \beta}(t) = h''(\tau)/D''(\tau)2\pi i \) where \( D(t) = (a \alpha \lambda^t + b \beta \lambda^{-t}) \ln \lambda. \) But

\[ |D''(\tau)| = 2[(a \alpha \lambda^t - b \beta \lambda^{-t}) \ln \lambda]^2 + [(a \alpha \lambda^t + b \beta \lambda^{-t}) \ln \lambda]^2 \geq 8|a \alpha b \beta| \ln \lambda^2, \]
since the second square is positive. Thus \(|Tf_{\alpha\beta}(t)| \leq |h''(\tau')|/16\pi|a\alpha b\beta|. Hence

\[4\pi^2 \int_R |Tf_{\alpha\beta}|^2 \leq \int_{|t-t_0|\leq 1} \|h''\|_\infty/64(a\alpha b\beta)^2 dt + \int_R |h(t)|^2/\text{Min}\{|D(t)^2| \ |t-t_0| \geq 1\} dt.\]

But \(\|h''\|_\infty\) is bounded by norms of derivatives of \(g_{\alpha\beta}\), providing good summability over \((\alpha, \beta)\) of the first term. Also, \(|D(t)|^2 = |D''(\tau)(t-t_0)/2!|^2 \geq 16(a\alpha b\beta)^2 \ln^4 \lambda \gg 0\). Thus \(\sum Tf_{\alpha\beta} \in L^2\).

Next, we consider \(k > 1\). We make the following observations.

(4.5) \([D, T^k] = \sum_{j=1}^k T \cdots T[D, T]T \cdots T\),

where \([D, T]\) is in the \(j\)th position. Hence

(4.6) \([D, [D, T^k]] = \sum_{i,j=1}^k T \cdots T[D, T]T \cdots T\)

\(= 2 \sum_{1 \leq i < j \leq k} T \cdots T[D, T]T \cdots T[D, T]T \cdots T\),

where \([D, T]\) occupies the \(i\)th and \(j\)th positions. (This is a result of \(\mathcal{N}\) being abelian and normal.) The \(k\)-fold bracket

(4.7) \([D, [D, \ldots [D, T^k], \ldots]] = k![D, T]^k\),

while the \((k + 1)\)-fold bracket vanishes. By (1.8) of [C-R1, p.353],

(4.8) \(2\pi i T^k f_{\alpha\beta}(t) = h_k(a\alpha \lambda^i + b\beta \lambda^{-i})^{-(k+1)}\)

where

\[h_k = [D[D \cdots [D, T^k], \ldots]g_{\alpha\beta} + D[D \cdots [D, T^k], \ldots]g_{\alpha\beta} + \cdots + D^{k-1}[D, T^k]g_{\alpha\beta} + D^k T^k g_{\alpha\beta}].\]

It follows that \(||T^k f_{\alpha\beta}||_2^2 = \int_{|t-t_0|\leq 1} + \int_{|t-t_0|>1} , with the integrand being determined by (4.8).

The second integral can be estimated as in the case of \(k = 1\), while the denominator in the integrand in the first integral has a Taylor expansion in \((t-t_0)\) using derivatives of order \(\leq k + 1\). For \(k > 1\) and odd, the \((k + 1)\)th derivative of \((a\alpha \lambda^i + b\beta \lambda^{-i})^{k+1}\) is of the form \((d/dt)^{k+1}[(a\alpha \lambda^i + b\beta \lambda^{-i})^{k+1}] = \sum_{j=0}^{(k+1)/2} c_j(a\alpha \lambda^i + b\beta \lambda^{-i})^{k+1-j}(a\alpha \lambda^i - b\beta \lambda^{-i})^{j} \ln^k \lambda\) with all \(c_j \geq 0\) and \(c_{(k+1)/2} \geq 0\). Thus, if \(k + 1\) is even,

\[
\frac{d^{k+1}}{dt^{k+1}}(a\alpha \lambda^i + b\beta \lambda^{-i})^{k+1} \geq C(a\alpha \lambda^i - b\beta \lambda^{-i})^{k+1}, \quad \text{where } C > 0.
\]

If \(k + 1\) is odd, then

\[
\frac{d^{k+1}}{dt^{k+1}}(a\alpha \lambda^i + b\beta \lambda^{-i})^{k+1} \geq C'|a\alpha \lambda^i - b\beta \lambda^{-i}|^{k+1}, \quad \text{for } C' > 0,
\]
by a similar calculation. Hence the minima over $|t - t_0| \leq 1$ in the resulting estimates proceed as in the case of $k = 1$.

**Case 3.** $D \in \mathcal{N}^C$. If $D \in C \cdot X$ or $D \in C \cdot Y$, then $D \in C \cdot \mathcal{N}$ and is covered by Case 2. Otherwise, dividing by a constant, we can assume that $D = X + (a + ib)Y$, where $a, b \in \mathbb{R}$. Thus

$$f_{\alpha \beta}(t) = g_{\alpha \beta}(t)/2\pi(i(\alpha \lambda^t + a \beta \lambda^{-t}) - b \beta \lambda^{-t}).$$

Since it suffices to prove $L^2$-summability for the parts of $f_{\alpha \beta}$ corresponding to the real and imaginary parts of $g_{\alpha \beta}$ separately, we can assume $g_{\alpha \beta}$ is real in (4.9). Suppose also $b \neq 0$, since Case 2 would apply if $b$ were 0. We note that $|f_{\alpha \beta}(t)| \leq |g_{\alpha \beta}(t)|/2\pi b \beta \lambda^{-t}$. The methods of Case 2 can be applied to prove $\sum f_{\alpha \beta} \in L^2$.

If $U \in \mathcal{U}(\mathcal{N})$, then $[D, U] = 0$, so that $\sum U f_{\alpha \beta} \in L^2$. We need to prove $\sum T^k f_{\alpha \beta} \in L^2$ for each fixed $k \in \mathbb{N}$.

We begin with $k = 1$. Then $T^1 f_{\alpha \beta}(t)$ is the derivative of the right side of (4.9). In the numerator, we get various derivatives of $g_{\alpha \beta}$, while the modulus of the denominator exceeds $4\pi^2 b^2 \beta^2 \lambda^{-2t}$. The $\lambda^{-2t}$ can be moved to the numerator as a derivation, and $\sum T^k f_{\alpha \beta} \in L^2$. For $k > 1$, similar reasoning applies.

**Case 4.** $D = T+i(aX+bY), \ a, b \in \mathbb{R}$. Up to isomorphism, we could assume that $a = b = 1$, except for the case in which either $a = 0$ or $b = 0$. Since $T + iX$ and $T + iY$ are very similar, we need to treat only the cases $T + iX$ and $T + i(X + Y)$. Since $T + i(X + Y)$ is more complicated, we will treat this case in detail, providing brief remarks to cover the simpler case of $T + iX$.

Actually, to simplify the constants we suppose $D = T + i(X + Y)/2\pi$, so that

$$f_{\alpha \beta}(t) = \left(\int_0^t g_{\alpha \beta}(x)\exp(-(\alpha \lambda^x + \beta \lambda^{-x})/\ln \lambda) \, dx + C\right)\exp((\alpha \lambda^t + \beta \lambda^{-t})/\ln \lambda).$$

**Subcase 4a.** $\alpha > 0$ and $\beta > 0$. In this case $\exp((\alpha \lambda^t + \beta \lambda^{-t})/\ln \lambda) \to \infty$ as $t \to \pm \infty$ (or in case $b = 0$, $\exp(\alpha \lambda^t/\ln \lambda) \to \infty$ as $t \to \infty$ and $\to 1$ as $t \to -\infty$). Thus

$$f_{\alpha \beta}(t) = -\int_0^\infty g_{\alpha \beta}(x)\exp(-(\alpha \lambda^x + \beta \lambda^{-x}) + (\alpha \lambda^t + \beta \lambda^{-t})/\ln \lambda) \, dx,$$

$$= \int_{-\infty}^t \cdots \, dx.$$
of \((-\infty, t_{\alpha\beta}) \cup [t_{\alpha\beta}, \infty) = (-\infty, \infty)\). (If \(b = 0\), use \((X^2 + Y^2)^g\) instead.)

Restriction to \(\lambda^{-2} \leq \beta / \alpha \leq 1\) will keep \(\alpha^{-1}\) bounded in absolute value.

Next, let \(U = (2\pi)^{-m-n} X^m Y^n \in \mathcal{H}(\mathcal{A})\). If \(m - n > 0\) and \(t > t_{\alpha\beta}\), then
\[
|U f_{\alpha\beta}(t)| \leq \left| \alpha^m \beta^n \lambda^{(m-n)t} \int_t^\infty |g_{\alpha\beta}(x)| \, dx \right| \leq \int_t^\infty |U g_{\alpha\beta}(x)| \, dx,
\]
which provides the necessary estimate. If \(m - n < 0\) and \(t > t_{\alpha\beta}\),
\[
|U f_{\alpha\beta}(t)| \leq |\alpha^m \beta^n \lambda^{-(m-n)t} \int_t^\infty |g_{\alpha\beta}(x)| \, dx .
\]

But \(\lambda^{-2} \leq \beta / \alpha \leq 1\) implies \(t_{\alpha\beta} \geq -1\), so
\[
|U f_{\alpha\beta}(t)| \leq \lambda^{n-m} \int_t^\infty |\alpha^m \beta^n g_{\alpha\beta}(x)| \, dx \leq \lambda^{n-m} \int_t^\infty |\alpha^{m+n} g_{\alpha\beta}(x)| \, dx
\]
\[
= \lambda^{n-m} \int_t^\infty |X^{m+n} g_{\alpha\beta}(x)| \lambda^{-(m+n)x} (2\pi)^{-m-n} \, dx
\]
\[
\leq \lambda^{2n} (2\pi)^{-m-n} \int_t^\infty |X^{m+n} g_{\alpha\beta}(x)| \, dx .
\]

From here the \(L^2\)-estimates proceed as earlier.

Next, suppose \(m - n < 0\) and \(t < t_{\alpha\beta}\). Then \(\lambda^{(m-n)t} > \lambda^{(m-n)t_{\alpha\beta}}\). Thus
\[
|U f_{\alpha\beta}(t)| \leq |\alpha^m \beta^n \lambda^{(m-n)t} \int_t^{-\infty} |g_{\alpha\beta}(x)| \, dx \leq \int_t^{-\infty} |\alpha^m \beta^n \lambda^{(m-n)x} g_{\alpha\beta}(x)| \, dx
\]
\[
= \int_t^{-\infty} |U g_{\alpha\beta}(x)| \, dx .
\]

The rest is as before.

Finally, for \(m - n > 0\) and \(t < t_{\alpha\beta}\), we write
\[
|U f_{\alpha\beta}(t)| \leq |\alpha^m \beta^n \lambda^{(m-n)t} \int_{-\infty}^t |g_{\alpha\beta}(x)| \, dx
\]
\[
\leq |\lambda^{2m} \beta^m \lambda^{(m-n)t_{\alpha\beta}} \int_{-\infty}^t |g_{\alpha\beta}(x)| \, dx
\]
\[
\leq \lambda^{2m} \int_{-\infty}^t |\beta^{m+n} g_{\alpha\beta}(x)| \, dx
\]
\[
= \int_{-\infty}^t |Y^{m+n} g_{\alpha\beta}(x)| \lambda^{(m+n)x} / (2\pi)^{m+n} \, dx
\]
\[
\leq (2\pi)^{-m-n} \int_{-\infty}^t |Y^{m+n} g_{\alpha\beta}(x)| \, dx .
\]

The \(L^2\)-estimates can be completed as before.

Next, we show that \(\sum_{(\alpha, \beta)} T^k f_{\alpha\beta} \in L^2\), for each \(k \in \mathbb{N}\). This follows from the next lemma.

(4.11) Lemma. Let \(f\) be a solution of the equation
\[
(T + i(X + Y)) f = g .
\]
Then $T^k f$, $k = 1, 2, 3, \ldots$, is a linear combination of monomials $X^j Y^l f$ with $j + l \leq k$ plus a linear combination of $X, Y, T$-derivatives of $g$.

**Proof.** We proceed by induction. For $k = 1$ we have $T f = g - iX f - iY f$. Next, $T^{k+1} f = T(T^k f) = T(X^j Y^l f)$ with $j + l \leq k$, where wlog we may assume $T^k f$ is a monomial $X^j Y^l f$.

$$T(X^j Y^l f) = \left( \sum_{p=1}^{j} X \cdots X[T, X]X \cdots XY^l + \sum_{q=1}^{l} X^j Y \cdots Y[T, Y]Y \cdots Y \right) f$$

$$+ Y^l f - iX^{j+1} Y^l f - iX^j Y^{l+1} f$$

$$= (j \ln \lambda X^j Y^l f + l \ln \lambda X^j Y^{l+1} f) + \ldots$$

which is the desired expression for $T^k f$.

**Remark.** Similarly, $T^k f$ is a linear combination of monomials $X^j f$ with $j \leq k$ plus a linear combination of $X, T$-derivatives of $g$ if $f$ is a solution of $(T + iX) f = g$.

**Subcase 4b.** $\alpha < 0$ and $\beta > 0$. (The case $\alpha > 0$ and $\beta < 0$ can be treated similarly.) Once again, we have

$$f_{a\beta}(t) = \left( \int_{0}^{t} g_{a\beta}(x) \exp(-\alpha x + \beta x^{-1}) \ln \lambda \, dx + C \right) \exp(\alpha x + \beta x^{-1}) \ln \lambda$$

where the terms $\beta x^{-1}$ and $\beta x^{-1}$ are not present if $b = 0$. Moreover, the restriction $\beta > 0$ is not needed if $b = 0$. We observe $\exp(\alpha x + \beta x^{-1}) \ln \lambda \to 0$ as $t \to +\infty$ and $-\infty$ as $t \to -\infty$ (or $-1$ as $t \to -\infty$ in case $b = 0$). In either case, since $\lim_{t \to -\infty} f_{a\beta}(t) = 0$,

$$C = \int_{-\infty}^{0} g_{a\beta}(x) \exp(-\alpha x + \beta x^{-1}) \ln \lambda \, dx$$

and

$$f_{a\beta}(t) = \int_{-\infty}^{t} g_{a\beta}(x) e^{\psi(x, t)} \, dx,$$

where

$$\psi(x, t) = (\alpha (x - x^{-1}) + \beta (x^{-1} - x^{-1})) \ln \lambda,$$

again with no $\beta (x^{-1} - x^{-1})$ term in case $b = 0$. We notice that $\psi(x, t) < 0$ for $x < t$. We have the estimates

$$|f_{a\beta}(t)| \leq \int_{-\infty}^{t} |g_{a\beta}(x)| \, dx \leq \int_{-\infty}^{t} |Y g_{a\beta}(x)| / 2\pi \beta x^{-1} \, dx$$

$$\leq C \sum_{k=0}^{1} \|T^k Y g_{a\beta}\|_2 |\beta|^{-1} \int_{-\infty}^{t} \lambda x \, dx \quad \text{(by Sobolev)}$$

$$\leq M \sum_{k=0}^{1} \|T^k Y g_{a\beta}\|_2 \lambda^k / \ln \lambda \in L^2(-\infty, 0).$$

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since $|\beta|^{-1}$ is bounded. In fact, (4.13) implies that $\sum_{(\alpha, \beta)} f_{\alpha \beta} \in L^2(-\infty, 0)$. Next, we must consider convergence in $L^2(0, \infty)$. Thus for $t > 0$ we write

$$f_{\alpha \beta}(t) = \int_{-\infty}^{0} \cdots + \int_{0}^{t/2} \cdots + \int_{t/2}^{t} \cdots$$

$$= I_{\alpha \beta}(t) + II_{\alpha \beta}(t) + III_{\alpha \beta}(t)$$

where the integrands are as in (4.12).

To estimate $I_{\alpha \beta}$ we notice that for $x < 0 < t$ we have $\beta(\lambda^{-t} - \lambda^{-x}) \leq 0$ and $\alpha(\lambda^{t} - \lambda^{x}) < \alpha(\lambda^{t} - 1) < 0$. Hence

(4.14) $$|I_{\alpha \beta}(t)| \leq e^{a(\lambda^{t} - \lambda^{x})/\ln \lambda} \int_{-\infty}^{0} |g_{\alpha \beta}(x)| \, dx$$

and $\sum_{(\alpha, \beta)} I_{\alpha \beta} \subset L^2(0, \infty)$. This is because $\alpha \gg 0$ makes the functions

$$t \mapsto \exp(\alpha(\lambda^{t} - \lambda^{x})/\ln \lambda), \quad (\alpha, \beta) \in (\Gamma \setminus S_h),$$

uniformly $L^2(0, \infty)$, while the integral $\int_{-\infty}^{0} \cdots$ in (4.14) can be estimated as in (4.13) making the sum finite.

For $II_{\alpha \beta}$ we have the estimate

(4.15) $$|II_{\alpha \beta}(t)| \leq \int_{0}^{t/2} |g_{\alpha \beta}(x)| e^{a(\lambda^{t} - \lambda^{x})/\ln \lambda} \, dx$$

$$\leq \|g_{\alpha \beta}\|_{\infty} \frac{1}{2} e^{a(\lambda^{t} - \lambda^{x})/\ln \lambda}.$$ 

The right-hand side again is $\alpha$-uniformly in $L^2(0, \infty)$ with $\|g_{\alpha \beta}\|_{\infty}$ being $(\alpha, \beta)$-summable.

Finally,

(4.16) $$|III_{\alpha \beta}(t)| \leq \int_{t/2}^{t} |g_{\alpha \beta}(x)| \, dx$$

$$\leq \int_{t/2}^{t} \left| X^m g_{\alpha \beta}(x) \right| / |2\pi \alpha \lambda|^m \, dx$$

$$\leq \|X^m g_{\alpha \beta}\|_{\infty} I_{\alpha \beta} \leq M^m \lambda^{-mt},$$

where $M$ is an upper bound on $|\alpha|^{-1}$, $t \lambda^{-mt} \in L^2(0, \infty)$, and $\|X^m\|_{\infty}^2$ is $(\alpha, \beta)$-summable.

Next, we must show $\sum_{(\alpha, \beta)} U f_{\alpha \beta} \subset L^2(R)$ for every fixed $U \in U(S_h)$.

If $U = Y^k$ we have the estimate

(4.17) $$|Y^k f_{\alpha \beta}(t)| \leq \int_{-\infty}^{t} |Y^k g_{\alpha \beta}(x)| e^{\psi(x, t)} \, dx.$$ 

As in the beginning of Subcase 4b we can show that $\sum Y^k f_{\alpha \beta} \subset L^2(R)$.

For $U = X^k$, $\sum X^k f_{\alpha \beta} \subset L^2(-\infty, 0)$ because for $t \leq 0$,

(4.18) $$|X^k f_{\alpha \beta}(t)| = 2k^t |\alpha|/|\beta|^{k|X^k f_{\alpha \beta}(t)|} \leq 2^t |Y^k f_{\alpha \beta}(t)|$$

if we choose $(\alpha, \beta)$ such that $\lambda^{-2} \leq |\beta|/|\alpha| \leq 1$. If $t > 0$, we consider $X^k I_{\alpha \beta}$, $X^k II_{\alpha \beta}$, and $X^k III_{\alpha \beta}$ and we get the estimates (4.14), (4.15), and (4.16), each multiplied by $\lambda^{k^t}$ and with $g_{\alpha \beta}$ replaced by $Y^k g_{\alpha \beta}$, as it was done in (4.18).
Finally, let $U = X^pY^qT^r$. Case of $r \geq 1$ reduces to $r = 0$ by the Lemma (4.11). If $r = 0$, we apply $X^p$ to $\int_{-\infty}^{t} |Y^qg_{a\beta}(x)| e^{\psi(x,t)} dx$ as we applied $U = X^k$ to $\int_{-\infty}^{t} g_{a\beta}(x) e^{\psi(x,t)} dx$.

Subcase 4c. $\alpha < 0$ and $\beta < 0$. We have

$$(4.19)\quad f_{a\beta}(t) = \left( \int_{0}^{t} g_{a\beta}(x) e^{-\frac{(\alpha\lambda^x + \beta\lambda^{-x})}{\ln \lambda}} dx + C \right) e^{\frac{(\alpha\lambda^t + \beta\lambda^{-t})}{\ln \lambda}}.$$ 

The function $Ce^{\frac{(\alpha\lambda^t + \beta\lambda^{-t})}{\ln \lambda}}$ is in $\mathcal{S}^f(R)$ if $\alpha < 0$ and $\beta < 0$. Hence if there is a constant $C$ such that $f_{a\beta}$ in (4.19) is super-Schwartz, then $f_{a\beta} \in \mathcal{S}^f(R)$ for any fixed $C$. We will pick $C = C_{a\beta} = \int_{a\beta}^{0} g_{a\beta}(x) e^{-\frac{(\alpha\lambda^x + \beta\lambda^{-x})}{\ln \lambda}} dx$, where $t_{a\beta} := \frac{1}{2} \log (\beta/\alpha)$, and $\alpha$, $\beta$ are chosen so that $\lambda^{-2} < \beta/\alpha \leq 1$. Thus we will work with

$$(4.19')\quad f_{a\beta}(t) = \left( \int_{a\beta}^{t} g_{a\beta}(x) e^{-\frac{(\alpha\lambda^x + \beta\lambda^{-x})}{\ln \lambda}} dx \right) e^{\frac{(\alpha\lambda^t + \beta\lambda^{-t})}{\ln \lambda}}.$$ 

We have the estimate

$$|f_{a\beta}(t)| \leq e^{\frac{(\alpha\lambda^t + \beta\lambda^{-t})}{\ln \lambda}} \int_{a\beta}^{t} (2\pi)^{-m}|(X + Y)^m g_{a\beta}(x)|$$

$$\times e^{-\frac{(\alpha\lambda^x + \beta\lambda^{-x})}{\ln \lambda}}(\alpha\lambda^x - \beta\lambda^{-x})^{-m} dx$$

$$\leq e^{-\frac{\alpha\lambda^t + \beta\lambda^{-t}}{\ln \lambda}}(\alpha\lambda^x - \beta\lambda^{-x})^{-m} \|g_{a\beta}\|_{\infty} e^{-\frac{\alpha\lambda^t + \beta\lambda^{-t}}{\ln \lambda}}|t - a\beta|$$

$$\leq c \sum_{l=0}^{1} \|T^l(X + Y)^m g_{a\beta}\|_2 (\alpha\lambda^t - \beta\lambda^{-t})^{-m}|t - a\beta|$$

where $\cdots$ stands for $\alpha\lambda^t + \beta\lambda^{-t}$. We write the inequality (4.20) for $(\alpha, \beta)$ such that

$$(4.21)\quad 2(\alpha\beta)^{1/2} / \ln \lambda > m,$$

because then the function $u \mapsto e^{-\frac{(\alpha u + \beta u^{-1})}{\ln \lambda}}(\alpha u - \beta u^{-1})^{-m}$ with $u = \lambda^t$ is increasing for $t \geq a\beta$ and decreasing for $t \leq a\beta$. (4.21) is valid for all but a finite number of $(\alpha, \beta) \in (\Gamma \setminus S_h)_{\infty}$. Similarly, for $y^l x^k f_{a\beta}$ we have the estimate

$$(4.22)\quad |y^l x^k f_{a\beta}(t)| \leq c_1 \sum_{p=0}^{1} \|T^p(X + Y)^m g_{a\beta}\|_2 |\beta\lambda^{-t}|^m |\alpha\lambda^t|^k(-\alpha\lambda^t - \beta\lambda^{-t})^{-m}|t - a\beta|.$$

But for $t \geq a\beta$, if $m > k + l$

$$|\beta\lambda^{-t}|^m |\alpha\lambda^t|^k(-\alpha\lambda^t - \beta\lambda^{-t})^{-m}|t - a\beta| \leq (\beta/\alpha)^l (-\alpha)^{k+l-m} \lambda^{(k-l-m)}|t - a\beta|$$

$$\leq M^{m-k-l}|t - a\beta| \lambda^{(k-l-m)} \in L^2(0, \infty)$$

since $\beta/\alpha \leq 1$ and $\alpha \gg 0$.

Similarly, for $t \leq a\beta$

$$|\beta\lambda^{-t}|^m |\alpha\lambda^t|^k(-\alpha\lambda^t - \beta\lambda^{-t})^{-m}|t - a\beta| \leq (\alpha/\beta)^k (-\beta)^{k+l-m} \lambda^{(m-l+k)}|a\beta - t|$$

$$\leq \lambda^{2k} M^{m-k-l}|t - a\beta| \lambda^{(m-l+k)} \in L^2(-\infty, 0).$$
since \( \lambda^{-2} \leq \beta/\alpha \) and \( \beta \gg 0 \). Thus \( \sum Y^l X^k f_{\alpha \beta} \in L^2(R) \). Finally,

\[
\sum Y^l X^k T^m f_{\alpha \beta} \in L^2(R)
\]

by Lemma (4.11).

BIBLIOGRAPHY


