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HOLOMORPHIC FLOWS IN $\mathbf{C}^3$, 0 WITH RESONANCES

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ABSTRACT. The topological classification, by conjugacy, of the germs of holomorphic diffeomorphisms $f\colon \mathbf{C}^2, 0 \to \mathbf{C}^2, 0$ with $df(0) = \operatorname{diag}(\lambda_1, \lambda_2)$, where λ_1 is a root of unity and $|\lambda_2| \neq 1$ is given.

This type of diffeomorphism appears as holonomies of singular foliations $\mathscr{F}_X$ induced by holomorphic vector fields $X\colon \mathbf{C}^3, 0 \to \mathbf{C}^3, 0$ normally hyperbolic and resonant. An explicit example of a such vector field without holomorphic invariant center manifold is presented.

We prove that there are no obstructions in the holonomies for $\mathscr{F}_X$ to be topologically equivalent to a product type foliation.

INTRODUCTION

In this paper we study germs of holomorphic vector fields $X\colon \mathbf{C}^3, 0 \to \mathbf{C}^3, 0$ such that:

(i) 0 is an isolated singularity,
(ii) $DX(0) = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_j \neq 0$, $p\lambda_1 + q\lambda_2 = 0$, p, q relatively prime and $\lambda_3/\lambda_1, \lambda_3/\lambda_2 \notin \mathbf{R}$.

Any such vector field is normally hyperbolic.

Consider the foliation $\mathscr{F}_X$ induced by the differential equation

$$\frac{dz}{dT} = X(z), \qquad z \in \mathbf{C}^3, 0, \ T \in \mathbf{C},$$

in a neighborhood of $0 \in \mathbf{C}^3$.

The problem is to describe and to classify these foliations in a neighborhood of the singularity. Here we consider the topological description of $\mathscr{F}_X$.

This note is divided in five sections:

(I) Formal classification
(II) Center manifold and holomorphic normal form
(III) Study of the holonomies
(IV) Normally hyperbolic diffeomorphisms in $\mathbf{C}^2, 0$ with resonance—topological classification
(V) The problem of the topological classification for the foliation $\mathscr{F}_X$.

Firstly in §I we present the formal classification. We prove that the formal

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normal form is

$$Y: \begin{cases} \dot{x} = x(\lambda_1 + a_1 \cdot x^p y^q + \cdots + a_k (x^p y^q)^k), \\ \dot{y} = y(\lambda_2 + b_1 \cdot x^p y^q + \cdots + b_k (x^p y^q)^k), \\ \dot{z} = \lambda_3 z. \end{cases}$$

The solutions of this equation are obtained by means of the singular transformation $u = x^p y^q$; $v = x^r y^s$, $ps - qr^* = 1$ which will transform it into a vector field of type saddle-node.

The formal normal form always has the invariant subspace $\{z = 0\} = \mathbf{C}^2_{xy}$.

Then all foliations $\mathcal{F}_X$ have a formal invariant surface corresponding to this.

In §II we give an example of a holomorphic vector field which does not have a holomorphic invariant surface corresponding to this formal one.

Then, we prove that this class of foliations has the following holomorphic normal form:

$$X: \begin{cases} \dot{x} = \lambda_1 x + x^p y^q \cdot A(x, y, z), \\ \dot{y} = \lambda_2 y + x^p y^q \cdot B(x, y, z), \\ \dot{z} = \lambda_3 z + x^p y^q \cdot C(x, y, z), \end{cases}$$

where $A, B, C\colon \mathbf{C}^3, 0 \to \mathbf{C}$ are holomorphic functions.

Definition 1. A foliation $\mathcal{F}_X$ will be called *product type* if it is analytically equivalent (near $0 \in \mathbf{C}^3$) to the foliation $\mathcal{F}_{X_0}$ defined by

$$X_0: \begin{cases} \dot{x} = \lambda_1 x + x^p y^q a(x, y), \\ \dot{y} = \lambda_2 y + x^p y^q b(x, y), \\ \dot{z} = \lambda_3 z, \end{cases}$$

where $a, b\colon \mathbf{C}^2 0 \to \mathbf{C}, 0$ are holomorphic functions.

The foliation $\mathcal{F}_{X_0}$ is the product of the saddle-resonant in $\mathbf{C}^2_{xy}$:

$$\begin{cases} \dot{x} = \lambda_1 x + x^p y^q a(x, y), \\ \dot{y} = \lambda_2 y + x^p y^q b(x, y) \end{cases}$$

by the linear equation $\dot{z} = \lambda_3 z$.

The saddle-resonant in $\mathbf{C}^2$ have a well-known structure; the topological classification was obtained by C. Camacho and P. Sad in [1]; the analytic classification (as the differentiable ones) by J. Martinet and J. P. Ramis in [3].

The holomorphic normal form X has the three coordinate axes invariant, and they are the only separatrices of the foliation.

We denote their holonomies by $H_{X,x}$; $H_{X,y}$ and $H_{X,z}$ (where $H_{X,\cdot}$ is the holonomy of $\mathcal{F}_X$ with respect to the axis).

The main objective of this note is to prove

Theorem B. *The holonomies give no obstruction for $\mathcal{F}_X$ to be topologically equivalent to a foliation of product type.*

The diffeomorphisms of holonomy $H_{X_0,x}$ and $H_{X_0,y}$ (X_0 is a vector field of product type) are diffeomorphisms of $\mathbf{C}^2, 0$ that satisfy:

(1) The linear part has eigenvalues λ_1, λ_2 such that $\lambda_1^n = 1$; $0 \neq |\lambda_2| \neq 1$ (for some n: natural number).

(2) They have the two coordinate axes invariant.

In the axis corresponding to the eigenvalue λ_1 we have a diffeomorphism of $\mathbf{C}, 0$ with linear part multiplication by a root of unity.

The dynamics of these diffeomorphisms is well known; see [2] where one can find their topological classification; in [3] one finds the analytical and differentiable classification.

The diffeomorphisms of $\mathbf{C}^2, 0$ with $\lambda_1^n = 1$ and $|\lambda_2| \neq 1$ are normally hyperbolic, then applying the results of J. Palis and F. Takens (see [5]) we obtain that the holonomies $H_{X_0, x}$ and $H_{X_0, y}$ are topologically conjugate to diffeomorphism of the form

$$(x, y) \mapsto (\lambda_1 x + x^{kn+1}, \lambda_2 y).$$

In §III we study the diffeomorphisms $f\colon \mathbf{C}^2, 0 \to \mathbf{C}^2, 0$ with eigenvalues λ_1, λ_2 such that $\lambda_1^n = 1$; $|\lambda_2| \neq 1$ (normally hyperbolic with resonance).

There exist diffeomorphisms of this class that have no holomorphic invariant curve tangent to the direction corresponding to λ_1 (see §III, (b)).

By the Center Manifold Theorem we can choose an f-invariant curve S of class C^m where m can be taken arbitrarily large (see [7, pp. 64–67]).

If we take m sufficiently big we can determine the dynamics of $f|S$ by the results of Dumortier, Rodrigues and Roussarie (see [6]).

In this way we obtain the topological classification of these diffeomorphisms.

Theorem A. *Let $f\colon \mathbf{C}^2, 0 \to \mathbf{C}^2, 0$ a holomorphic diffeomorphism.*
Suppose $df(0) = \operatorname{diag}(\lambda_1, \lambda_2)$ with $\lambda_1^n = 1$ and $|\lambda_2| \neq 1$.
Then f is topologically conjugate to

$$(x, y) \to (\lambda_1 x + x^{kn+1}, \lambda_2 y)$$

where k is the order of the first resonance of f.

Finally, although the holonomies give no topological obstructions for the foliation to be topologically a product, we cannot yet prove this.

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I. Formal classification

Consider an equation X, with the hypotheses given in the introduction.

It is well known (see (4)) that there exists a formal transformation conjugating X and Y where Y is defined by

$$Y\colon \begin{cases} \dot{x} = \lambda_1 x + \sum_{j=1}^{\infty} a_j x^{jp+1} y^{jq}, \\ \dot{y} = \lambda_2 y + \sum_{j=1}^{\infty} b_j x^{jp} y^{jq+1}, \\ \dot{z} = \lambda_3 z + z \sum_{j=1}^{\infty} c_j x^{jp} y^{jq}. \end{cases}$$

Y is the best formal normal form that we can obtain with transformations tangent to the identity of $\mathbf{C}^3$.

Consider now the transformation

$$(*) \qquad \begin{array}{l} u = x^p y^q \\ v = x^r y^s \end{array} \qquad \left(\begin{array}{l} x = u^2 v^{-q} \\ y = u^{-r} v^p \end{array} \right)$$

where $p, q, r, s \in \mathbf{N}$; $ps - qr = 1$.

Substitution into Y yields

$$Y_1: \begin{cases} \dot{u} = \sum_{j=1}^{\infty}(pa_j + qb_j)u^{j+1}, \\ \dot{v} = v(r\lambda_1 + s\lambda_2) + v\sum_{j=1}^{\infty}(ra_j + sb_j)u^j, \\ \dot{z} = z\left(\lambda_3 + \sum_{j=1}^{\infty} c_j u^j\right). \end{cases}$$

Note that Y_1 is a vector field of type saddle-node.

Suppose $pa_j + qb_j = 0$, $j = 1, \dots, k-1$, and $pa_k + qb_k \neq 0$. Then Y_1 is analytically equivalent to

$$Y_2: \begin{cases} \dot{u} = u^{k+1}, \\ \dot{\xi} = \xi(\alpha_0 + \alpha_1 u + \cdots + \alpha_k u^k), \\ \dot{\eta} = \eta(\beta_0 + \beta_1 u + \cdots + \beta_k u^k). \end{cases}$$

Now, using $(*)$ we obtain that X is formally equivalent to

$$Y_3: \begin{cases} \dot{x} = x(\lambda_1 + \alpha_1 x^p y^q + \cdots + \alpha_k (x^p y^q)^k), \\ \dot{y} = y(\lambda_2 + \beta_1 x^p y^q + \cdots + \beta_k (x^p y^q)^k), \\ \dot{z} = \lambda_3 z. \end{cases}$$

Then, Y_3 is the formal normal form for the foliation $\mathscr{F}_X$. So, $\mathscr{F}_X$ has only finitely many formal invariants.

Using the general solution of Y_2 we have that the leaves of $\mathscr{F}_{Y_3}$ are given by the level lines of

$$F(x, y, z) = (x^r y^s \varphi(u) u^{-\alpha_k} \exp(\Gamma_1(u)); z\psi(u)u^{-\beta_k}\exp(\Gamma_2(u)))$$

where

$$u = x^p y^q, \quad \varphi(u) = 1 + \varphi_1 u + O(u^2), \quad \psi(u) = 1 + \psi_1 u + O(u^2),$$
$$\Gamma_1(u) = \frac{\alpha_0}{ku^k} + \cdots + \frac{\alpha_{k-1}}{u}, \quad \Gamma_2(u) = \frac{\beta_0}{ku^k} + \cdots + \frac{\beta_{k-1}}{u}.$$

That is, for each $(c, d) \in \mathbf{C}^2$, the curve $F(x, y, z) = (c, d)$ is a leaf of the foliation $\mathscr{F}_{Y_3}$. This provides us with a good description of $\mathscr{F}_{Y_3}$.

Note that $\mathscr{F}_{Y_3}$ is the simplest product type foliation that we can have.

Remarks. (1) F has an essential singularity in $u = x^p y^q = 0$.

(2) In the plane $\{z = 0\}$ we have the first integral

$$x^r y^s \varphi(u) u^{-\alpha_k} \exp(\Gamma_1(u)) \quad \text{for } \mathscr{F}_{Y_3}|\mathbf{C}^2_{xy}.$$

(3) If in the formal normal form Y we have

$$pa_j + qb_j = 0 \qquad \forall j = 1, 2, \dots,$$

then X and Y are analytically conjugate (see [4]).

In this case $\mathscr{F}_X$ is given by the equation

$$\dot{x} = \lambda_1 x, \quad \dot{y} = \lambda_2 y,$$
$$\dot{z} = \lambda_3 z(1 + f(u)), \quad \text{where } u = x^p y^q,$$

and $f\colon \mathbf{C}, 0 \to \mathbf{C}, 0$ is holomorphic. Note that $G(x, y, z) = x^p y^q$ is a first integral for this equation.

II. Center manifold and holomorphic normal form

We have seen that the class of vector fields in study has the formal normal form Y.

Then $\mathscr{F}_Y$ is a product type foliation with $\{z = 0\}$ invariant. Thus all vector fields X have a formal invariant surface corresponding to $\{z = 0\}$.

We present now an example of a holomorphic vector field which has no invariant surface of the type

$$z = \varphi(x, y), \qquad \varphi(0, 0) = D\varphi(0, 0) = 0,$$

φ holomorphic in a neighborhood of $0 \in \mathbf{C}^2$.

Example. Consider the differential equation

$$Z\colon \begin{cases} \dot{x} = \lambda_1 x, \\ \dot{y} = \lambda_2 y + a \cdot x^p y^{q+1}, \\ \dot{z} = \lambda_3 z + \sum_{n=1}^{\infty} (x^p y^q)^n. \end{cases}$$

If $S = \{(x, y, z) \in \mathbf{C}^3 \mid z = \varphi(x, y)\}$ is invariant by Z, then φ is a solution of the partial differential equation

$$\lambda_1 x \cdot \frac{\partial \varphi}{\partial x} + (\lambda_2 y + a x^p y^{q+1}) \frac{\partial \varphi}{\partial y} = \lambda_3 \varphi + \sum_{n=1}^{\infty} (x^p y^q)^n. \tag{E.1}$$

Substitution of $\varphi = \sum_{j+k \geq 2} \varphi_{jk} x^j y^k$ into (E.1) yields

$$\sum_{j+k \geq 2} (j\lambda_1 + k\lambda_2 - \lambda_3) \varphi_{jk} x^j y^k + \sum_{j+k \geq 2} a k \varphi_{jk} x^{j+p} y^{k+q} = \sum_{n=1}^{\infty} (x^p y^q)^n. \tag{E.2}$$

Solving formally we have

- if $(j, k) \neq n(p, q)$ then $\varphi_{jk} = 0$,
- if $(j, k) = n(p, q)$ then $(-\lambda_3)\varphi_{n(p,q)} + a \cdot q(n-1)\varphi_{(n-1)(p,q)} = 1$ for $n \geq 1$.

For $n = 1$: $\varphi_{p,q} = -1/\lambda_3$.
For $n = 2$: $\varphi_{2(p,q)} = -1/\lambda_3 \cdot (1 + aq/\lambda_3)$.
For $n = 3$: $\varphi_{3(p,q)} = -(1/\lambda_3)(1 + (2aq/\lambda_3)(1 + aq/\lambda_3))$.

$\vdots$

For $n = k + 1$:

$$\varphi_{k+1(p,q)} = -\frac{1}{\lambda_3}\left(1 + \frac{kaq}{\lambda_3}(1 + (k-1)\frac{aq}{\lambda_3}\left(1 + \cdots + \left(1 + \frac{aq}{\lambda_3}\right)\right) \cdots\right).$$

Then, we can verify that

$$|\varphi_{n(p,q)}| > (n-1)|\varphi_{(n-1)(p,q)}|.$$

So, φ has a divergent power series, and is not a holomorphic germ.

Remarks. (1) The coefficient $j\lambda_1 + k\lambda_2 - \lambda_3$ of φ_{jk} in equation (E.2) is never zero but it has minimum module when $(j, k) = n(p, q)$ $n \geq 1$. The idea for obtaining divergence is to take the subseries of the coefficients φ_{jk} corresponding to the minimum of $|j\lambda_1 + k\lambda_2 - \lambda_3|$.

(2) The divergent series that we obtained in the example diverges as $\sum k!x^k$. Then it is Gevrey of order two, and we can prove that it belongs to $\mathbf{C}\{x\}[[y]] \cap \mathbf{C}\{y\}[[x]]$; see [3]. In the following we obtain the best holomorphic normal form for the class of foliations in study.

Proposition 1 (Holomorphic normal form). *There exists a holomorphic change of coordinates near $0 \in \mathbf{C}^3$ transforming a vector field X into the normal form*

$$\begin{aligned} \dot{x} &= \lambda_1 x + x^p y^q A(x, y, z), \\ \dot{y} &= \lambda_2 y + x^p y^q B(x, y, z), \\ \dot{z} &= \lambda_3 z + x^p y^q C(x, y, z), \end{aligned}$$

where A, B and C are holomorphic functions.

Proof. Consider the vectors fields

$$X: \begin{cases} \dot{x}_1 = \lambda_1 x_1 + f_1(x), \\ \dot{x}_2 = \lambda_2 x_2 + f_2(x), \\ \dot{x}_3 = \lambda_3 x_3 + f_3(x), \end{cases} \qquad f_j(x) = \textstyle\sum_{|Q|\geq 2} f_{jQ} x^Q,$$

and

$$Y: \begin{cases} \dot{y}_1 = \lambda_1 y_1 + g_1(y), \\ \dot{y}_2 = \lambda_2 y_2 + g_2(y), \\ \dot{y}_3 = \lambda_3 y_3 + g_3(y). \end{cases} \qquad g_j(y) = \textstyle\sum_{|Q|\geq 2} g_{jQ} y^Q,$$

Let $H(y) = y + h(y) = x$, $y = (y_1, y_2, y_3)$, $x = (x_1, x_2, x_3)$, $h(y) = (h_1(y), h_2(y), h_3(y))$, such that $dH(Y) = X(H)$; suppose X holomorphic. Then we have the equations

$$\text{(E.3)} \qquad \begin{aligned} &\sum_{|Q|\geq 2} [((Q, \Lambda) - \lambda_j)h_{jQ} + g_{jQ}]y^Q \\ &\qquad = f_j(y + h(y)) - \sum_{k=1}^{3} g_k(y) \cdot \frac{\partial h_j}{\partial y_k} \quad \text{for } j = 1, 2, 3 \end{aligned}$$

where $Q = (q_1, q_2, q_3)$, $q_j \geq 0$: natural numbers

$$\Lambda = (\lambda_1, \lambda_2, \lambda_3), \qquad h_j(y) = \sum_{|Q|\geq 2} h_{jQ} y^Q.$$

Define

- if $y^Q \notin \langle y_1^p y_2^q \rangle$ then $g_{jQ} = 0$ and $h_{jQ} = ((Q, \Lambda) - \lambda_j)^{-1}$ (coefficient of y^Q in the right member of (E_j)).
- if $y^Q \in \langle y_1^p y_2^q \rangle$ then $h_{jQ} = 0$ and $g_{jQ} =$ coefficient of y^Q in the right member of (E_j). Note that $y^Q \notin \langle y_1^p y_2^q \rangle$ if and only if $q_1 < p$ or $q_2 < q$.

Then there exists $\delta > 0$: cte. such that

$$|(Q, \Lambda) - \lambda_j| \geq \delta \cdot |Q| \quad \forall j = 1, 2, 3; \forall Q, |Q| \geq 2.$$

In the following we use only that

$$|(Q, \Lambda) - \lambda_j| \geq \delta > 0$$

for Q such that $y^Q \notin \langle y_1^p y_2^q \rangle$.

Observing that

$$\sum_{k=1}^{3} g_k(y) \frac{\partial h_j}{\partial y_k} \in \langle y_1^p y_2^q \rangle \quad \text{for } j = 1, 2, 3$$

we obtain the majorations in (E_j):

$$\delta \sum |h_{jQ}| y^Q < \sum |(Q, \Lambda) - \lambda_j| \, |h_{jQ}| y^Q < \overline{f}_j(y + \overline{h}(y))$$

(where if $f(y) = \sum f_Q y^Q$ then $\overline{f}(y) = \sum |f_Q| y^Q$ and $\overline{\overline{f}}(w) = \sum |f_Q| w^{|Q|}$ ($w = y_1 = y_2 = y_3$), and $<$ is the notation for majorations between series).

Then addition with respect to j yields

$$\delta \sum_{|Q| \geq 2} (|h_{1Q}| + |h_{2Q}| + |h_{3Q}|) y^Q < \sum_{j=1}^{3} \overline{f}_j(y + \overline{h}(y))$$

making $y_1 = y_2 = y_3 = w$, $\sum (|h_{1Q}| + |h_{2Q}| + |h_{3Q}|) w^{|Q|} = w \cdot u$, $u(w) = u_1 w + u_2 w^2 + \cdots$, we obtain

$$u \cdot w < \delta^{-1} \sum_{j=1}^{3} \overline{\overline{f}}_j(w + wu) < \frac{A_0 w^2 (1+u)^2}{1 - Aw(1+u)} \quad \text{where } A_0 > 0,$$

and $A > 0$ are constants, and

$$\sum_{j=1}^{3} \overline{\overline{f}}_j(w) < \frac{A_0 w^2}{1 - Aw}.$$

Then

$$u < \frac{A_0 w (1+u)^2}{1 - Aw(1+u)}.$$

Now we can prove that the holomorphic solution $v = A_0 w + \cdots$ of the equality

$$v = \frac{A_0 w (1+v)^2}{1 - Aw(1+v)}$$

is a majorant for u (see [4]).

So u is holomorphic and consequently $H(y)$. As $g_j(y) \in \langle y_1^p y_2^q \rangle$ we can write $g_j(y) = y_1^p y_2^q \cdot \overline{g}_j(y_1, y_2)$. Thus Y is in the form enunciated in the proposition.

III. Study of the holonomies

The holomorphic normal form given by Proposition 1 of §II has the coordinate axes invariant.

Now we compute their holonomies.

(a) *Holonomy of the z axis.* Let Σ be a transversal section to the $\mathbf{C}_z$-axis by the point $(0, 0, 1)$.

Take the loop $z = e^{i\theta}$, $0 \le \theta \le 2\pi$, in the $\mathbf{C}_z$ axis. Then $dz = ie^{i\theta}\,d\theta$, and substitution in

$$\frac{dx}{dz} = \frac{\lambda_1 x + x^p y^q A(x, y, z)}{\lambda_3 z + x^p y^q C(x, y, z)},$$
$$\frac{dy}{dz} = \frac{\lambda_2 y + x^p y^q B(x, y, z)}{\lambda_3 z + x^p y^q C(x, y, z)}$$

yields

$$\frac{dx}{d\theta} = \left(i \cdot \frac{\lambda_1}{\lambda_3}\right) x + x^p y^q \overline{A}(x, y, e^{i\theta}),$$
$$\frac{dy}{d\theta} = \left(i \cdot \frac{\lambda_2}{\lambda_3}\right) y + x^p y^q \overline{B}(x, y, e^{i\theta}).$$

Integrating for $0 \le \theta \le 2\pi$ we obtain the diffeomorphism $H_{X,z}$: $(\Sigma, 1) \approx (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ defined by

$$H_{X,z}(x, y) = (e^{2\pi i\lambda_1/\lambda_3} x + x^p y^q h(x, y), e^{2\pi i\lambda_2/\lambda_3} y + x^p y^q g(x, y))$$

(h, g are holomorphic functions).

Note that $H_{X,z}$ is a hyperbolic resonant diffeomorphism of $\mathbf{C}^2$, 0, that is, we have

$$(e^{2\pi i\lambda_1/\lambda_3})^p \cdot (e^{2\pi i\lambda_2/\lambda_3})^q = 1$$

and

$$|e^{2\pi i\lambda_j/\lambda_3}| \ne 1 \qquad (j = 1, 2).$$

So $H_{X,x}$ has a well-known dynamics, it is topologically linearizable (see (5)).

(b) *Holonomy of the x-axis.* Take a section Σ transversal to the $\mathbf{C}_x$ axis in the point $(1, 0, 0)$; and the loop $x = e^{i\theta}$. Then $dx = ie^{i\theta}\,d\theta$, and substitution in

$$\frac{dy}{dx} = \frac{\lambda_2 y + x^p y^q B(x, y, z)}{\lambda_1 x + x^p y^q A(x, y, z)},$$
$$\frac{dz}{dx} = \frac{\lambda_3 z + x^p y^q C(x, y, z)}{\lambda_1 x + x^p y^q A(x, y, z)}$$

yields

$$\frac{dy}{d\theta} = \left(i\frac{\lambda_2}{\lambda_1}\right) y + y^q \overline{A}(e^{i\theta}, y, z),$$
$$\frac{dz}{d\theta} = \left(i\frac{\lambda_3}{\lambda_1}\right) z + y^q \overline{B}(e^{i\theta}, y, z).$$

Integrating for $0 \le \theta \le 2\pi$ we have the diffeomorphism $H_{X,x}$: $\mathbf{C}^2, 0 \to \mathbf{C}^2, 0$ defined by

$$H_{X,x}(y, z) = (e^{2\pi i\lambda_2/\lambda_1} y + y^q h(y, z), e^{2\pi i\lambda_3/\lambda_1} z + y^q g(y, z)).$$

Note that

$$(e^{2\pi i\lambda_2/\lambda_1})^q = 1 \quad \text{and} \quad |e^{2\pi i\lambda_3/\lambda_1}| \ne 1.$$

Thus, $H_{X,x}$ is a resonant, normally hyperbolic diffeomorphism of $\mathbf{C}^2$, 0.

The axis $y = 0$ is invariant by $H_{X,x}$, and $H_{X,x}$ restricted to it is linear (a contraction or an expansion).

The axis $z = 0$ may or may not be invariant by $H_{X,x}$. We can construct examples where no holomorphic invariant curve (tangent to the y-axis) by $H_{X,x}$ exists, e.g. take the holonomy of the equation considered in the example of §II.

(c) *Holonomy of the y-axis.* Making $y = e^{i\theta}$ and proceeding analogously to case (b) we obtain the diffeomorphism

$$H_{X,y}(x, z) = (e^{2\pi i\lambda_1/\lambda_2}x + x^p h(x, z), e^{2\pi i\lambda_3/\lambda_1}z + x^p h(x, z)).$$

Note that

$$(e^{2\pi i\lambda_1/\lambda_2})^p = 1, \qquad |e^{2\pi i\lambda_3/\lambda_1}| \neq 1.$$

So $H_{X,y}$ has the same properties as $H_{X,x}$.

Remark. If the foliation $\mathscr{F}_X$ has a holomorphic center surface (tangent in 0 to the $\mathbf{C}^2_{xy}$-plane) then the holonomy $H_{X,x}$ (and $H_{X,y}$) has one holomorphic invariant curve (center manifold) tangent to the $\mathbf{C}_y$-axis. In this case $H_{X,x}$ has the form

$$(y, z) \mapsto (e^{2\pi i\lambda_2/\lambda_1}y + y^q h(y, z), e^{2\pi i\lambda_3/\lambda_1}z + zy^q g(y, z)).$$

In the invariant axis $\{z = 0\}$ we obtain the diffeomorphism

$$(y, 0) \mapsto (e^{2\pi i\lambda_2/\lambda_1}y + y^q h(y, 0), 0).$$

This is a diffeomorphism of $\mathbf{C}, 0$ with linear part a root of unity.

The dynamics of these diffeomorphisms is well known, they have a dynamic like a flower (see [2, 3]) (see Figure 1).

In this way, a foliation of product type like $\mathscr{F}_{X_0}$, has the following picture for their holonomies. (See Figure 2.)

In Σ_1 we have the illustration shown in Figure 3.

In Σ_2 we have the illustration shown in Figure 4.

In Σ_3 we have the illustration shown in Figure 5.

In Σ_1 and Σ_2 we are in presence of normal hyperbolicity, with known dynamics in the center manifold.

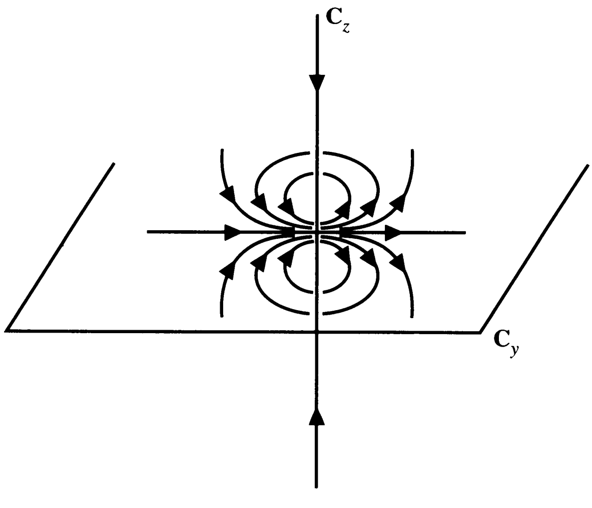
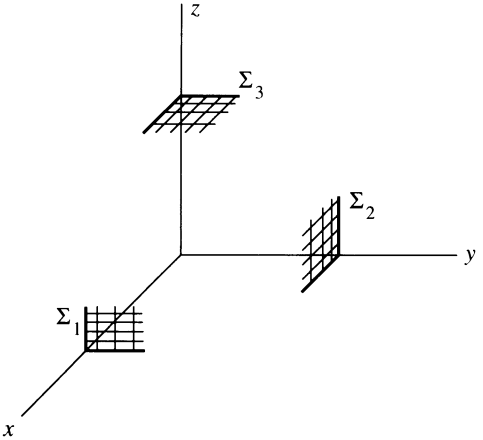


FIGURE 1. $(y, z) \mapsto (y + y^2, \lambda_3 z)$, $|\lambda_3| < 1$



$\Sigma_1, \Sigma_2, \Sigma_3$: transversal sections

FIGURE 2. $(\lambda_1 = 1, \lambda_2 = -1)$ (order of resonance two)

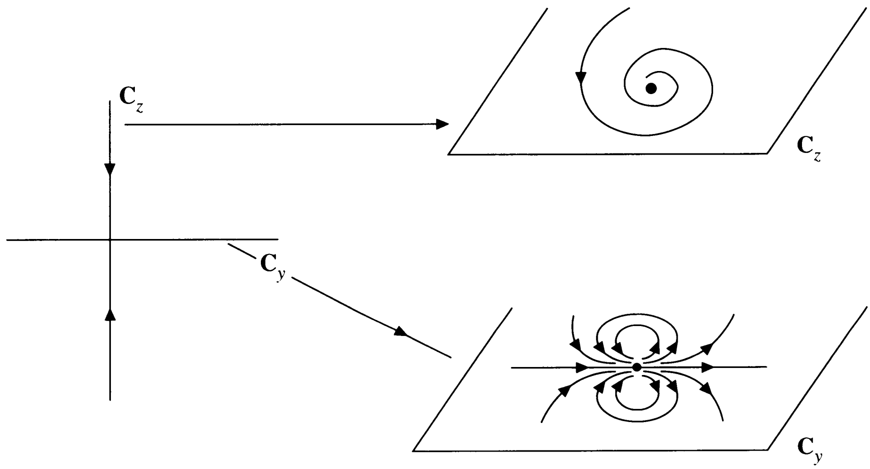


FIGURE 3

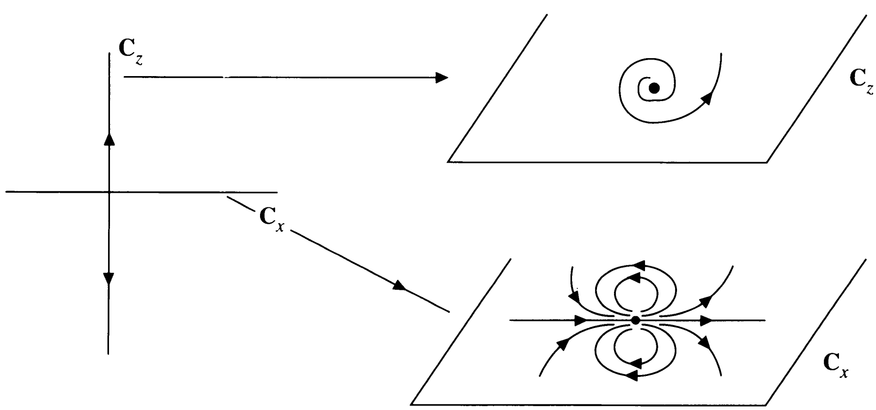


FIGURE 4

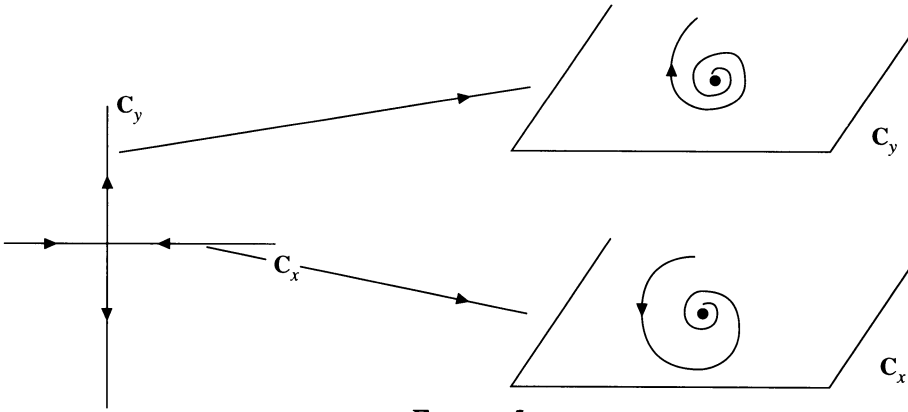


FIGURE 5

By the theorem of Palis and Takens (see [5]), we get that

$$H(y, z) = (e^{2\pi i\lambda_2/\lambda_1}y + y^q h(y, z), e^{2\pi i\lambda_3/\lambda_1}z + zy^q g(y, z))$$

is topologically conjugate to

$$G(y, z) = (e^{2\pi i\lambda_2/\lambda_1}y + y^q h(y, 0), e^{2\pi i\lambda_3/\lambda_1}z).$$

But, by the classification theorem of Camacho (see [2]) G is topologically conjugate to

$$(y, z) \mapsto (e^{2\pi i\lambda_2/\lambda_1}y + y^{kq+1}, e^{2\pi i\lambda_3/\lambda_1}z)$$

(here k is the order of the first resonance).

IV. NORMALLY HYPERBOLIC DIFFEOMORPHISMS OF $\mathbf{C}^2, 0$ WITH RESONANCE–TOPOLOGICAL CLASSIFICATION

Consider a germ of holomorphic diffeomorphism

$$f\colon \mathbf{C}^2, 0 \to \mathbf{C}^2, 0,$$

$$f(x, y) = (\lambda_1 x + \alpha(x, y), \lambda_2 y + \beta(x, y))$$

with $\lambda_1 = e^{2\pi i p/q}$, $0 \neq |\lambda_2| \neq 1$, $\alpha, \beta\colon \mathbf{C}^2, 0 \to \mathbf{C}$ holomorphic functions.

We can choose coordinates in $\mathbf{C}^2, 0$ relatively to which f is written as

$$f(x, y) = (\lambda_1 x + x^q\overline{\alpha}(x, y), \lambda_2 y + x^q\overline{\beta}(x, y)).$$

In this system of coordinates the $\mathbf{C}y$-axis is invariant. As we have already observed, in some cases there does not exist a holomorphic center manifold invariant by f.

In this case we ask about the existence of an invariant curve for f which is differentiable of class C^m, and how large we can take m.

By the Center Manifold Theorem (see [7, pp. 64–67]) we have always an invariant curve S for f, tangent to the x-axis through 0 in $\mathbf{C}^2$, and this curve can be chosen with class of differentiability m, m as large as we want (if m increases the neighborhood in which S is defined decreases).

This invariant curve is of the form

$$S = \{(x, y) | y = u(x); u\colon \mathbf{C}, 0 \to \mathbf{C}, 0 \text{ is of class } C^m\}.$$

Note that we cannot assert that S is unique. But we know that two invariant curves S and S' of class C^m are such that $f|S$ and $f|S'$ have the same dynamics.

Now, using the Normal Hyperbolicity Theory of Palis and Takens (see [5]) we have that the dynamics of f in a neighborhood of $0 \in \mathbf{C}^2$ depends only on the dynamics of $f|S$, from the topological point of view.

A natural question now is: What is the dynamics of $f|S$?

The answer is given by the following theorem.

Theorem (Dumortier, Rodrigues, and Roussarie [6]). *Let* $f\colon \mathbf{C}, 0 \to \mathbf{C}, 0$ *a germ of diffeomorphism of class* C^m *with*

$$f(z) = \lambda z + a \cdot z^k + O(|z|^k), \qquad \lambda^n = 1, a \neq 0, k \geq 2.$$

Suppose $m > k$. *Then* f *is topologically conjugate to* $z \mapsto \lambda z + z^k$.

If m is big enough we can take $k = ln + 1$ for some $l \geq 1$. (l is the order of the first resonance.)

With these results we can prove the following:

Theorem A (Topological classification of resonant normally hyperbolic diffeomorphisms). *Let* $f\colon \mathbf{C}^2, 0 \to \mathbf{C}^2, 0$ *holomorphic with*

$$df(0) = \operatorname{diag}(\lambda_1, \lambda_2), \qquad \lambda_1^q = 1, 0 \neq |\lambda_2| \neq 1.$$

Then f *is topologically conjugate to*

$$g(x, y) = (\lambda_1 x + x^{kq+1}, \lambda_2 y)$$

where k *is the order of the first resonance in the formal normal form of* f.

Proof. It is well known that f has the formal normal form

$$F(x, y) = \left(\lambda_1 x + \sum_{j=1}^{\infty} a_j x^{jq+1}, \lambda_2 y + y \sum_{j=1}^{\infty} b_j x^{jq}\right).$$

Let $k = \min\{l \in \mathbf{N} \mid a_l \neq 0\}$. (If $k = \infty$ f is analytically linearizable, see [4].) Suppose $k < \infty$.

By a holomorphic change of coordinates we can write f in the form

$$f(x, y) = (\lambda_1 x + a_k x^{kq+1} + R_1(x, y), \lambda_2 y + x^q R_2(x, y))$$

where $R_1 = o(|(x, y)|^{kq+1})$.

Now, we can choose an invariant curve for f of class C^m, through 0 in $\mathbf{C}^2$ and tangent to the $\mathbf{C}_x$-axis in 0, with $m > kq + 1$ (see [7, pp. 64–67]).

This invariant curve is defined by

$$S = \{(x, y) \mid y = u(x), u(x) = u_{kq} x^{kq} + r(x); r(x) = O(|x|^{kq}), r \text{ of class } C^m\}.$$

Then $f|S$ is given by

$$f(x, u(x)) = (\lambda_1 x + a_k x^{kq+1} + R_1(x, u(x)), \lambda_2 u(x) + x^q R_2(x, u(x))).$$

In this way, we obtain that $f|S$ is a diffeomorphism of $\mathbf{R}^2, 0$ of class C^m defined by the expression:

$$(C, 0) \approx (\mathbf{R}^2, 0) \to \mathbf{R}^2, 0,$$
$$x \to \lambda_1 x + a_k x^{kq+1} + R_1(x, u(x)).$$

By the theorem of Dumortier, Rodriques, and Roussarie we get that $f|S$ is topologically conjugate to $x \to \lambda_1 x + x^{kq+1}$.

Finally, by the normal hyperbolicity we have that f is topologically conjugate to

$$(x, y) \to (\lambda_1 x + x^{kq+1}, \lambda_2 y).$$

V. The problem of the topological classification for the foliation $\mathscr{F}_X$

We are interested in the description of the foliation $\mathscr{F}_X$, when X is any holomorphic normal form given by Proposition 1 of §II.

If two foliations $\mathscr{F}_X$ and $\mathscr{F}_Y$ are topologically equivalent, then their holonomies are topologically conjugate.

So, by Theorem A of §IV we see that all equations in study have holonomies of one of the two types:

- saddle-hyperoblic (holonomy of the $\mathbf{C}_z$-axis), or
- normally hyperbolic (holonomies of the $\mathbf{C}_x$ and $\mathbf{C}_y$ axes).

In the second case they have dynamics like a product of a "flower" in the center manifold with a linear contraction or expansion.

As the foliations type product (like $\mathscr{F}_{X_0}$) have these same types of holonomies we can resume these facts in the following.

Theorem B. *The holonomies of the separatrices of $\mathscr{F}_X$ give no obstructions for the foliation to be topologically equivalent to a product type foliation.*

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