RIGIDITY OF \( p \)-COMPLETED CLASSIFYING SPACES OF ALTERNATING GROUPS AND CLASSICAL GROUPS OVER A FINITE FIELD

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Abstract. A \( p \)-adic rigid structure of the classifying spaces of certain finite groups \( \pi \), including alternating groups \( A_n \) and finite classical groups, is shown in terms of the maps into the \( p \)-completed classifying spaces of compact Lie groups. The spaces \( (B\pi)^{\wedge}_p \) have no nontrivial retracts. As an application, it is shown that \( (B_{A_n})^{\wedge}_p \simeq (B_{\Sigma_n})^{\wedge}_p \) if and only if \( n \not\equiv 0, 1 \mod p \). It is also shown that \( (B_{\text{SL}(n, \mathbb{F}_q)})^{\wedge}_p \simeq (B_{\text{GL}(n, \mathbb{F}_q)})^{\wedge}_p \) where \( q \) is a power of \( p \) if and only if \( (n, q - 1) = 1 \).

If \( K \) and \( G \) are compact Lie groups, there are usually relatively few homotopy classes of maps \( BK \to BG \) or \( (BK)^{\wedge}_p \to (BG)^{\wedge}_p \). For instance, if \( K \) is connected and simple and \( G \) is connected with \( \text{rank}(K) > \text{rank}(G) \), the homotopy sets \( [BK, BG] \) and \( [(BK)^{\wedge}_p, (BG)^{\wedge}_p] \) are trivial \([1, 20]\) and the \( p \)-completion \( (BK)^{\wedge}_p \) has no nontrivial retracts at any prime \( p \) \([11]\). We will prove similar results with \( (BK)^{\wedge}_p \) replaced by \( (B\pi)^{\wedge}_p \), where \( \pi \) is an alternating group or a classical group over a finite field, and the notion of rank replaced by the notion of \( p \)-rank. (The \( p \)-rank of a group \( \pi \) is the maximal rank of an elementary abelian \( p \)-subgroup of \( \pi \).)

Let \( G \) be a compact Lie group. Recall that if \( \pi \) is a finite group with \( p \)-Sylow subgroup \( \pi_p \) and \( f \) is a map \( (B\pi)^{\wedge}_p \to (BG)^{\wedge}_p \), then the restriction \( f|B\pi_p \) must be of the form \( B\rho \) for some homomorphism \( \rho : \pi_p \to G \), \([7, 2, 15]\). The following theorem gives a sufficient condition that the homomorphism \( \rho \) be one-to-one in terms of weak closures of elements of the center of the \( p \)-Sylow subgroup \( \pi_p \). The weak closure of the one-element set \( \{z\} \) in \( \pi_p \) with respect to \( \pi \) is the subgroup of the \( p \)-Sylow subgroup generated by the set \( \{xzx^{-1} | x \in \pi \} \cap \pi_p \). We prove

Theorem 1. Let \( \pi \) be a finite group with \( p \)-Sylow subgroup \( \pi_p \). Suppose that, for any nonidentity element \( z \) of the center of \( \pi_p \), the weak closure of \( \{z\} \) in \( \pi_p \) with respect to \( \pi \) is equal to the \( p \)-Sylow subgroup \( \pi_p \). If \( G \) is a compact Lie group and \( f : (B\pi)^{\wedge}_p \to (BG)^{\wedge}_p \) is a nonzero map with \( f|B\pi_p \simeq B\rho \) for a homomorphism \( \rho \), then \( \rho : \pi_p \to G \) is injective.

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Corollary 2. Let $\pi$ be a finite group with $p$-Sylow subgroup $\pi_p$ and let $G$ be a compact Lie group. Assume that if $f: (B\pi_p)^\wedge \to (BG)^\wedge_p$ is a nonzero map with $f|B\pi_p \simeq B\rho$, then the homomorphism $\rho: \pi_p \to G$ is injective. Then each of the following holds:

(a) If $p\text{-rank}(\pi) > p\text{-rank}(G)$, then $[(B\pi_p)^\wedge, (BG)^\wedge_p] = 0$, and the evaluation map $\text{map}((B\pi_p)^\wedge, (BG)^\wedge_p) \to (BG)^\wedge_p$ is a weak equivalence.

(b) The $p$-complete classifying space $(B\pi_p)^\wedge$ has no nontrivial retracts.

We will show that the hypothesis of $\pi$ in Theorem 1 is satisfied by many finite (simple) groups at $p$. The list of such groups contains the alternating groups $A_n$ at any prime $p$, the finite classical groups $GL(n, \mathbb{F}_q)$, $O(n, \mathbb{F}_q)$ with $n \geq 5$ and $q$ odd, $Sp(2n, \mathbb{F}_q)$ with $(n, q) \neq (2, 2)$ and $U(2n, \mathbb{F}_{q^2})$ at $p$ which is the characteristic of the finite fields. All of the finite simple groups of types $A$, $B$, $C$, and $D$ associated with the above classical groups at the prime $p$ also satisfy the hypothesis in Theorem 1. Consequently Theorem 1 and Corollary 2 hold for these groups.

The proof of Theorem 1 makes use of the property of the images of conjugacy classes in a $p$-Sylow subgroup under the homomorphism $\rho$. This property is stated in Lemma 1.1. Since $f|B\pi_p \simeq 0$ implies $f \simeq 0$ [9], the remaining work is to show that if $\rho$ is not injective, the homomorphism is trivial. A sufficient condition is the hypothesis dealing with weak closures. This hypothesis is related to the fusion problem in group theory, [19]. G. Glauberman points out that there are finite simple groups which do not satisfy the hypothesis. An example is given by the projective group $PSU(3, \mathbb{F}_{p^2})$ of $3 \times 3$ special unitary matrices with $p$ odd, since the center of a $p$-Sylow subgroup is strongly closed. One can show, however, that Corollary 2 holds for this group at the prime $p$. This suggests that Corollary 2 may be true without assumption of the property of a $p$-Sylow subgroup if the finite group $\pi$ is simple.

This paper consists of six sections. In §1, we discuss maps between classifying spaces and prove Theorem 1 and Corollary 2. From §2 to §6, alternating groups, symmetric groups, and finite classical groups are treated. In particular, we show that the hypothesis of $\pi$ in Theorem 1 is satisfied by these groups at a suitable prime $p$.

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1. Maps between classifying spaces

Suppose $H$ is a subgroup of $G$, and $x, y \in H$. We say that $x$ and $y$ are conjugate in $G$, denoted by $x \sim G y$, if $y = gxg^{-1}$ for some $g \in G$. For $g \in G$, the conjugation map $C_g : H \to gHg^{-1}$ is defined by $C_g(x) = gxg^{-1}$ for each $x \in H$. If $g \in H$, then the self-map $BC_g$ of $BH$ is homotopic to the identity map, [18].

Lemma 1.1. Suppose a finite $p$-group $\gamma$ is a subgroup of a compact Lie group $G'$. Let $f: (BG')^\wedge_p \to (BG)^\wedge_p$ with $f|B\gamma \simeq B\rho$ for $\rho \in \text{Hom}(\gamma, G)$. If $x, y \in \gamma$ and $x \sim_G y$, then $\rho(x) \sim_G \rho(y)$.

Proof. Suppose $y = u x u^{-1}$ for $u \in G'$. Since the conjugation map $BC_u$ on $(BG')^\wedge_p$ is homotopic to the identity, we have the homotopy commutative
diagram:

\[
\begin{array}{ccc}
B(p) & \xrightarrow{\lambda} & (BG)_{p}^\wedge \\
\downarrow B_j & & \downarrow (BG)_{p}^\wedge \\
B(uyu^{-1}) & \xrightarrow{B_j} & (BG)_{p}^\wedge \\
\end{array}
\]

In this diagram \(j_1\) and \(j_2\) are the inclusions. We see \(f \circ B j_2 \simeq B p'\) for some \(p' \in \text{Hom}(uyu^{-1}, G)\). Since \(f \circ B j_1 \simeq f \circ B j_2 \circ B C u\), it follows that \(p = p' \circ C u\) up to \(G\)-conjugation. Consequently \(p(x) \sim p'(y)\). Next suppose \(i_1: y \cap uyu^{-1} \to y\) and \(i_2: y \cap uyu^{-1} \to uyu^{-1}\) are the inclusions. We notice that \(y \in y \cap uyu^{-1}\) and \(f \circ B j_1 \circ B i_1 \simeq f \circ B j_2 \circ B i_2\). It follows that \(p(y) \sim p'(y)\) and therefore \(p(x) \sim p(y)\).

We remark here that if \(x \in \ker p\) and \(x \sim y\), then \(y \in \ker p\). Consequently the weak closure of any subset of \(\ker p\) in \(y\) with respect to \(G'\) is included in \(\ker p\).

**Lemma 1.2.** Suppose \(f\) is a map from \((B\pi)^p\) to \((BG)^p\). If \(f|B\pi p \simeq 0\), then \(f \simeq 0\).

**Proof.** Along the line of the proof of result of Friedlander-Mislin [9, Theorem 3.1] we see that if the component of the mapping space \(\text{map}_p(B\gamma, (BG)^p)\) which contains the constant map is weakly contractible for any finite \(p\)-group \(\gamma\), then the map \(f: (B\pi)^p \to (BG)^p\) factors through a \(p\)-cyclic space defined in [13]. From the fibration \(\text{map}_\pi(X, Y)_0 \to \text{map}(X, Y)_0 \to Y\), we see that \(\text{map}_\pi(X, Y)_0\) is weakly contractible if and only if the basepoint evaluation map \(e: \text{map}(X, Y)_0 \to Y\) is a weak equivalence. Suppose \(\lambda: Y \to \text{map}(X, Y)_0\) is the map which sends \(y \in Y\) to the constant map \(\lambda(y)(x) = y\). Note here that the composite \(e \circ \lambda\) is the identity map. Consequently if \(\lambda\) is an equivalence, so is \(e\). To complete the proof, it remains to show that the map \(\lambda: (BG)^p \to \text{map}(B\gamma, (BG)^p)_0\) is weakly equivalent. We use an induction on the order of the finite \(p\)-group \(\gamma\). A result of Lannes [12] implies the case for \(\gamma = \mathbb{Z}/p\), since \(G\) is the centralizer of the trivial homomorphism \(\gamma \to G\). In general, consider a group extension \(1 \to N \to \gamma \to \sigma \to 1\) where \(\sigma = \mathbb{Z}/p\). Recall that the homotopy fixed point space \(\text{map}_\sigma(E\sigma, \text{map}(BN, (BG)^p))\) is homotopy equivalent to \(\text{map}(B\gamma, (BG)^p)\). The \(\sigma\)-action on \(\text{map}(BN, (BG)^p) = \text{map}(E\gamma, (BG)^p)\) is given by the rule \((f \cdot s)(u) = f(ur^{-1}) \cdot r\) where \(f \in \text{map}(E\gamma, (BG)^p), s \in \sigma\) and \(r \in \gamma\) is a preimage of \(s\) under the epimorphism \(\gamma \to \sigma\). Consequently, one has the commutative diagram:

\[
\begin{array}{ccc}
(BG)^p & \xrightarrow{\lambda} & \text{map}(B\gamma, (BG)^p) \\
\downarrow{\lambda_\sigma} & & \uparrow \\
\text{map}(B\sigma, (BG)^p)_0 & \longrightarrow & \text{map}_\sigma(E\sigma, \text{map}(BN, (BG)^p))_0 \\
\end{array}
\]

Since the vertical maps are homotopy equivalences, it remains to show the lower horizontal map is an equivalence. This map is induced by the \(\sigma\)-equivalence.
\((BG)^\wedge_p \xrightarrow{\lambda} \text{map}(BN, X)_0\), where the action of \(\sigma\) on the space \((BG)^\wedge_p\) is trivial. We conclude that \(\lambda\) is a homotopy equivalence. \(\square\)

The result of Friedlander-Mislin implies that Lemma 1.2 is still true when \((BG)^\wedge_p\) is replaced by the \(p\)-completion of a simply connected space whose loop space is homotopy equivalent to a finite dimensional complex. In fact, we have seen that the result holds for a space \(X\) if \(\text{map}_*(B\gamma, X)_0\) is weakly contractible for any finite \(p\)-group \(\gamma\). This condition is satisfied by a simply connected \(p\)-complete space \(X\) where \(H^*(X; \mathbb{F}_p)\) is finitely generated as an algebra. This is due to Dwyer-Wilkerson [6].

**Proof of Theorem 1.** From Lemma 1.2, it suffices to show that if \(f|B\pi_p \simeq B\rho\) and the homomorphism \(\rho : \pi_p \rightarrow G\) is not injective, then \(\rho\) is trivial. Suppose \(\ker \rho \neq 1\). Then \(\ker \rho\) is a nontrivial normal subgroup of a finite \(p\)-group. Hence \(\ker \rho\) must contain a nonidentity element of the center of \(\pi_p\). Lemma 1.1 together with our assumption shows \(\ker \rho = \pi_p\). \(\square\)

**Lemma 1.3.** The evaluation map \(\text{map}((B\pi)^\wedge_p, (BG)^\wedge_p)_0 \rightarrow (BG)^\wedge_p\) is weakly equivalent.

**Proof.** It suffices to show the fibre \(\text{map}_*((B\pi)^\wedge_p, (BG)^\wedge_p)_0\) is weakly contractible. Recall that the natural map

\[
\text{hocolim} \left( E\pi \times_{\pi} \pi/\pi_\alpha \rightarrow B\pi \right)
\]

is a mod \(p\) homology isomorphism, [14, Lemma 3.1]. Here \(\pi_\alpha\) is a \(p\)-subgroup of \(\pi\). Since \(\pi_1(BG)\) is finite, the space \(BG\) is \(\mathbb{Z}/p\)-good [2, p. 215]. Consequently

\[
\pi_i \text{map}_*(X, (BG)^\wedge_p) = \pi_i \text{map}_*(\widehat{X}_p, (BG)^\wedge_p)
\]

for any \(X\) and any \(i \geq 0\). Hence we see the following:

\[
\pi_i \text{map}_*((B\pi)^\wedge_p, (BG)^\wedge_p)_0
\]

\[
= \pi_i \text{map}_* \left( \left( \text{hocolim} B\pi_\alpha \right)^\wedge, (BG)^\wedge_p \right)_0
\]

\[
= \pi_i \text{map}_* \left( \text{hocolim} B\pi_\alpha, (BG)^\wedge_p \right)_0
\]

\[
= \pi_i \text{holim} \text{map}_*(B\pi_\alpha, (BG)^\wedge_p)_0.
\]

In the proof of Lemma 1.2, we have seen that \(\text{map}_*(B\pi_\alpha, (BG)^\wedge_p)\) is weakly contractible for any \(\alpha\). Consequently, so is the cosimplicial replacement \(\prod \{\text{map}_*(B\pi_\alpha, (BG)^\wedge_p)\}\), [2, p. 303]. By [2, Mapping Lemma, p. 285] we see that \(\text{holim} \text{map}_*(B\pi_\alpha, (BG)^\wedge_p)_0\) is weakly contractible and hence so is \(\text{map}_*((B\pi)^\wedge_p, (BG)^\wedge_p)_0\). \(\square\)

**Lemma 1.4.** Let \(f\) be a self-map of \((B\pi)^\wedge_p\) with \(f|B\pi_p \simeq B\rho\). The map \(f\) is a homotopy equivalence if and only if the homomorphism \(\rho\) is injective.
Proof. Suppose \( f \) is a homotopy equivalence. If \( \rho \) is not injective, then we can find a subgroup \( \mathbb{Z}/p \) of \( \ker \rho \). Let \( i: \mathbb{Z}/p \to \pi_p \) and \( j: \pi_p \to \pi \) be the inclusions. We consider the commutative diagram

\[
\begin{array}{ccc}
H^*(B\pi; \mathbb{F}_p) & \xleftarrow{f^*} & H^*(B\pi; \mathbb{F}_p) \\
\downarrow{Bj^*} & & \downarrow{B\rho^*} \\
H^*(B\pi_p; \mathbb{F}_p) & \xleftarrow{Bi^*} & H^*(B\mathbb{Z}/p; \mathbb{F}_p)
\end{array}
\]

Since \( \mathbb{Z}/p \subset \ker \rho \), we see that \( Bi^* \circ B\rho^* = B(\rho \cdot i)^* = 0 \). On the other hand, a result of Lannes [12] implies that the natural map \([B\mathbb{Z}/p, (B\pi)_p^\wedge] \to \text{Hom}_{\mathcal{S}}(H^*(B\pi, H^*B\mathbb{Z}/p))\) is bijective. Consequently \( Bi^* \circ Bj^* \neq 0 \). Since \( f \) is a homotopy equivalence, the map \( f^* \) must be an isomorphism. Thus the composition \( Bi^* \circ B^j \circ f^* \) would not be zero. This is a contradiction, since this composition is equal to the zero map \( Bi^* \circ B\rho^* \).

Next suppose \( \rho \) is injective. Then \( B\rho^* \) is injective by transfer argument. Hence the self-map \( f^* \) is injective on each finite dimensional vector space \( H^n(B\pi; \mathbb{F}_p) \) and hence \( f^* \) is bijective for dimensional reasons. Therefore the self-map \( f \) of \( (B\pi)_p^\wedge \) is a homotopy equivalence. \( \square \)

Proof of Corollary 2. (a) If \( p\text{-rank}(\pi) > p\text{-rank}(G) \) and \( f: (B\pi)_p^\wedge \to (BG)_p^\wedge \) with \( f|B\pi_p \simeq B\rho \), then Theorem 1 shows \( f \simeq 0 \). Thus \( \text{map}((B\pi)_p^\wedge, (BG)_p^\wedge) = \text{map}((B\pi)_p^\wedge, (BG)_p^\wedge)) \). The desired result follows from Lemma 1.3.

(b) Suppose \( X \) is a nontrivial retract of \( (B\pi)_p^\wedge \) with \( X \cong (B\pi)_p^\wedge \) and \( r \circ i \simeq 1_X \). If \( i \circ r|_{B\pi_p} \simeq B\rho \), then Theorem 1 says that \( \rho \) is injective. Lemma 1.4 shows \( i \circ r \) is a homotopy equivalence. Consequently, the epimorphism \( i^* \) is also a monomorphism. Hence the map \( i \) would be a homotopy equivalence. This contradiction completes the proof. \( \square \)

2. Alternating groups and symmetric groups

We will prove that the alternating group \( A_n \) satisfies the hypothesis of Theorem 1. To do so we need to take a close look at the center of a \( p \)-Sylow subgroup. The following lemma will be used for \( A_n \) and other finite groups.

**Lemma 2.1.** Suppose \( V \rtimes H \) is a semidirect product where the center of \( H \), denoted by \( Z(H) \), acts faithfully on the abelian group \( V \). Then the center of the group \( V \rtimes H \) is equal to the set \( \{ v \in V | hv = vh \text{ for any } h \in H \} \).

**Proof.** It is clear that \( Z(V \rtimes H) \) includes this set since \( V \) is abelian. Conversely, if \( v_0 h_0 \in Z(V \rtimes H) \) where \( v_0 \in V \) and \( h_0 \in H \), then we have

\[
(v_0 h_0) h = v_0 \cdot h_0 h, \quad h(v_0 h_0) = hv_0 h^{-1} \cdot h h_0.
\]

Hence \( hv_0 h^{-1} = v_0 \) and \( h_0 h = hh_0 \) for any \( h \in H \). We note that \( h_0 \in Z(H) \). It remains to show \( h_0 = 1 \). Consider the following

\[
(v_0 h_0) v = v_0 h_0 v h_0^{-1} \cdot h_0, \quad v(v_0 h_0) = vv_0 \cdot h_0.
\]
Since $V$ is abelian, we see $v_0h_0^{-1} = v$ for any $v \in V$. According to our assumption, $Z(H)$ acts faithfully on $V$. Consequently $h_0 = 1$, since $h_0 \in Z(H)$. □

**Proposition 2.2.** The alternating group $A_n$ satisfies the hypothesis of Theorem 1 at any prime $p$.

**Proof.** Since any two $p$-Sylow subgroups are conjugate, we may choose one to prove the desired result.

First suppose $p$ is odd. Then a $p$-Sylow subgroup of $A_n$ is also a $p$-Sylow subgroup of the symmetric group $\Sigma_n$. If $n$ is a power of $p$, a $p$-Sylow subgroup of $A_{p^n}$, say $P_i$, can be expressed as the wreath product $P_{i-1} \wr C_p$ where $C_p$ is a cyclic group of order $p$ and $P_1 = \mathbb{Z}/p((1, 2 \cdots p))$. For example, if $i = 2$, the cyclic group $C_p$ is generated by $(1 + 1 \cdots (p - 1)p + 1) \cdots (p - 1)p + (p - 1)$.

In general, if $n = a_0 + a_1p + \cdots + a_kp^k$ with $0 < a_i < p$ for $i = 0, \ldots, k$, then $\prod_{i=1}^{k} (\prod_{j=1}^{a_i} P_i)$ is a $p$-Sylow subgroup of $A_n$.

When $n < 2p$, the $p$-Sylow subgroup is isomorphic to $\mathbb{Z}/p$ if it is not trivial. Obviously the result holds. We now assume $2p \leq n$. Since

$$Z \left( \prod_{i=1}^{k} \prod_{j=1}^{a_i} P_i \right) = \prod_{i=1}^{k} \prod_{j=1}^{a_i} Z(P_i),$$

it suffices to consider the case $n = p^i$ for some $i \geq 2$. Using Lemma 2.1 one can show that (for any $p$) the center of $P_i$ is isomorphic to $\mathbb{Z}/p$ generated by $z = (1 \cdots p)(p+1 \cdots 2p) \cdots (p-1 \cdots p^i)$. Let $z' = (1 \cdots p)(p+1 \cdots 2p)^{-1} \cdots (p-1 \cdots p^i)^{-1}$ so that $z \cdot z' = (1 \cdots p)^2$ and $z' \in P_i$. Note here that $P_i$ contains all of the above $p$-cycles. We claim $z \sim z'$. Notice that if $\sigma$ is a $p$-cycle, then $\sigma^{-1}$ is also a $p$-cycle. Hence there is a $g \in \Sigma_p$ such that $\sigma^{-1} = g \sigma g^{-1}$. Consequently we can find $\hat{g} \in \Sigma_p \times \cdots \times \Sigma_p \subset \Sigma_{p^i}$ such that $z' = \hat{g}z\hat{g}^{-1}$. If $\hat{g} \in A_{p^i}$, we are done. If $\hat{g} \notin A_{p^i}$, let

$$h = (1 \ p + 1)(2 \ p + 2) \cdots (p \ 2p).$$

The conjugation by $h$ switches the first $p$-cycle and the second one. If $\hat{g} = 1 \times g_2 \times \cdots \times g_k \in \Sigma_p \times \cdots \times \Sigma_p$, let $\overline{g} = g_2 \times 1 \times g_3 \times \cdots \times g_k$. It follows that $\overline{g}h \in A_{p^i}$ for $i \geq 2$ and that $(\overline{g}h)z(\overline{g}h)^{-1} = z'$. We now see that the weak closure of $\{z\}$ contains a $p$-cycle since $p$ is odd. Consequently we can show that any generator of $P_i$ is conjugate in $A_{p^i}$ to an element of the elementary $p$-abelian subgroup of $P_i$ generated by the $p$-cycles $(1 \cdots p), \ldots, (p^i-p+1 \cdots p^i)$. Thus the weak closure is equal to the $p$-Sylow subgroup $P_i$.

Next suppose $p = 2$. The argument is similar to the one we just used. Let $A_n(2)$ and $\Sigma_n(2)$ denote a 2-Sylow subgroup of $A_n$ and that of $\Sigma_n$ respectively. We first consider when $n$ is a power of 2. For example, one sees that $A_4(2)$ is $\mathbb{Z}/2 \times \mathbb{Z}/2$ generated by (12)(34) and (13)(24). For $n = 8$, if

$$E = \mathbb{Z}/2((12)(34)) \times \mathbb{Z}/2((12)(56)) \times \mathbb{Z}/2((12)(78))$$

and

$$K = (\mathbb{Z}/2((13)(24)) \times \mathbb{Z}/2((57)(68))) \times \mathbb{Z}/2(\sigma)$$

where $\sigma = (15)(26)(37)(48)$, then $K$ normalizes $E$ and $A_8(2) = E \cdot K$. Inductively $A_2(2) = E_1 \cdot K_1$ where $E_i$ is an elementary abelian 2-group and $K_i$
is a group generated by \( K_{i-1} \times K_{i-1} \) and the element \((1 \ 2^{i-1} + 1) \cdots (2^{i-1} \ 2^i)\). Again using Lemma 2.1 one can show that the center of \( A_{2i}(2) \) is isomorphic to \( \mathbb{Z}/2 \) generated by
\[
Z = (12)(34)(56)(78) \cdots (2^{i-3} \ 2^{i-2}) (2^{i-1} \ 2^i).
\]
Let
\[
z' = (13)(24)(56)(78) \cdots (2^{i-3} \ 2^{i-2}) (2^{i-1} \ 2^i)
\]
so that \( zz' = (14)(23) \), \( z' \in A_{2i} \) and \( z \sim z' \). Thus the weak closure of \( \{z\} \) in \( A_{2i} \) contains \( (14)(23) \). Consequently the group is equal to \( A_{2i}(2) \). Suppose now that \( n = 2^{i_1} + \cdots + 2^{i_k} \) with \( i_1 > \cdots > i_k \) and \( k > 1 \). Then \( A_n(2) = (\Sigma_{2^{i_1}}(2) \times \cdots \times \Sigma_{2^{i_k}}(2)) \cap A_n \). If \( \tau = \tau_1 \times \tau_2 \times \cdots \times \tau_k \) is contained in the center of \( A_n(2) \), then \( \tau_j \in \Sigma(\Sigma_{2^{i_j}}(2)) \). One can show that the weak closure of \( \{\tau\} \) contains an element conjugate to \( (12)(34) \). Thus this group must be \( A_n(2) \). \( \Box \)

Next we consider the space \( (B\Sigma_n)^\wedge_p \), where \( \Sigma_n \) is the symmetric group. For an odd prime \( p \) this group satisfies the hypothesis of Theorem 1. This follows from Proposition 2.2.

**Theorem 2.3.** (a) If \( p \) is odd, then \( (B\Sigma_n)^\wedge_p \) has no nontrivial retracts. The only nontrivial retract of \( (B\Sigma_n)^\wedge_p \) for \( n \geq 4 \) is the space \( \mathbb{Z}/2 \) up to homotopy.

(b) Let \( p \) be odd. If \( n \equiv 0, 1 \mod p \), then \( \{[B\Sigma_n]^\wedge_p, (BA_n)^\wedge_p\} = 0 \). If \( n \equiv 0, 1 \mod p \), then \( [B\Sigma_n]^\wedge_p \simeq (BA_n)^\wedge_p \).

(c) The map \( [B\mathbb{Z}/2, (BA_n)^\wedge_p] \to [([B\Sigma_n]^\wedge_p, (BA_n)^\wedge_p)] \) induced by the projection \( \Sigma_n \to \mathbb{Z}/2 \) is bijective, where the kernel of this projection is \( A_n \).

Here we note related results about the unitary group \( U(n) \) and the orthogonal group \( O(n) \). On the level of classifying spaces we have the fibrations \( BSU(n) \to BU(n) \to B\mathbb{Z}/2 \) and \( BSO(n) \to BO(n) \to B\mathbb{Z}/2 \). From [10 and 11] one can observe the following

**Theorem 2.4.** (a) The nontrivial \( p \)-local retracts of \( BU(n) \) are \( p \)-equivalent to

(i) \( BS^1 \) if \( n \equiv 0 \mod p \),

(ii) \( BS^1 \) or \( BSU(n) \) if \( n \equiv 0 \mod p \).

(b) The nontrivial \( p \)-local retracts of \( BO(n) \) are \( p \)-equivalent to

(i) none if \( p \) is odd,

(ii) \( B\mathbb{Z}/2 \) if \( p = 2 \) and \( n \) is even,

(iii) \( B\mathbb{Z}/2 \) or \( BSO(n) \) if \( p = 2 \) and \( n \) is odd.

We also note that \( BU(n) \simeq BS^1 \times BSU(n) \) when \( n \equiv 0 \mod p \), and that \( BO(2k) \simeq BO(2k + 1) \simeq BSO(2k + 1) \) when \( p \) is odd. It is easy to see that \( O(2k + 1) \simeq \mathbb{Z}/2 \times SO(2k + 1) \) as groups for any \( k \). Consequently \( BO(2k + 1) \simeq B\mathbb{Z}/2 \times BSO(2k + 1) \) without localization. Finally, Theorem 2.3 implies that \( (B\Sigma_n)^\wedge_p \simeq (BA_n)^\wedge_p \) if and only if \( n \equiv 0, 1 \mod p \).

We need the following lemma to prove part (b) of Theorem 2.3.

**Lemma 2.5.** Let \( p \) be odd. The normalizers of the cyclic group \( \mathbb{Z}/p((12 \cdots p)) \) in \( \Sigma_p \) and in \( A_p \) are the following:
\[
N_{\Sigma_p} \mathbb{Z}/p = \mathbb{Z}/p \rtimes \mathbb{Z}/p - 1, \quad N_{A_p} \mathbb{Z}/p = \mathbb{Z}/p \rtimes \mathbb{Z}/p \mathbb{Z}/p - 1
\]
where both \( \mathbb{Z}/p - 1 \) and \( \mathbb{Z}/p^{p - 1} \) act freely on \( (\mathbb{Z}/p)^* \).

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Proof. Let $b$ be a multiplicative generator of the unit group $(\mathbb{Z}/p)^* = \mathbb{Z}/p - 1$. If $a = (12 \cdots p)$, the $b$th power of $a$ is another $p$-cycle. Hence we can find $g \in \Sigma_p$ such that $a^b = gag^{-1}$. If $S(\Sigma_p)$ denotes the set of all $p$-Sylow subgroups of $\Sigma_p$, then $|N_{\Sigma_p}\mathbb{Z}/p| = |\Sigma_p|/|S(\Sigma_p)| = p!/(p - 2)! = p(p - 1)$. Consequently

$$N_{\Sigma_p}\mathbb{Z}/p = \mathbb{Z}/p(a) \rtimes \mathbb{Z}/p - 1(g).$$

Similarly we see

$$N_{\mathbb{Z}/p} = \mathbb{Z}/p(a) \rtimes \mathbb{Z}/p - 1(g^2).$$

Proof of Theorem 2.3. (a) For an odd prime $p$ the desired result is obtained since $\Sigma_p$ satisfies the hypothesis of Theorem 1 at $p$. It remains to show that if $X$ is a nontrivial retract of $(B\Sigma_n)^{\wedge}_r$ for $n \geq 4$, then $X$ is homotopy equivalent to $B\mathbb{Z}/2$. Suppose $X \simeq (B\Sigma_n)^{\wedge}_r$ with $r \circ i \simeq 1_X$. Let $f = i \circ r$. First we claim

$$|f|BA_n \simeq 0.$$  

Let $P$ be a 2-Sylow subgroup of $\Sigma_n$. By Lemma 1.4 it suffices to show that if $|f|BA_n \neq 0$ , then $|f|BP \simeq B\rho$ for some injective homomorphism $\rho: P \to \Sigma_n$. If $|f|BA_n \neq 0$, then $\rho \cap A_n = 1$. Recall that for a 2-Sylow subgroup $Q$ of $A_n$ we have $P = Q \rtimes \mathbb{Z}/2$. We notice that $|Q| \leq |\text{Im } \rho| = |P|/|\ker \rho|$. Consequently $|\ker \rho| \leq 2$. An element $\tau$ of order 2 in $\Sigma_n$ has the form $(a_1 b_1) \cdots (a_k b_k)$ where $a_i$'s and $b_i$'s are mutually distinct. Suppose $\tau \in \ker \rho$. If $k = 1$, then Lemma 1.1 would imply that $\rho$ is trivial, since transpositions generate the symmetric group. If $k \geq 2$, we consider a 2-Sylow subgroup containing the transpositions $(a_1 b_1), (a_2 b_2), \ldots, (a_k b_k)$. One can see that at least two other elements in the 2-Sylow subgroup are conjugate to $\tau$ in $\Sigma_n$. This would imply $|\ker \rho| \geq 3$. Thus $\ker \rho = 1$. Since $X$ is a nontrivial retract, this is a contradiction. Therefore $|f|BA_n \simeq 0$.

We now consider the following commutative diagram

$$
\begin{array}{ccc}
H^*(B\Sigma_n; F_2) & \xrightarrow{r^*} & H^*(X; F_2) \\
\downarrow & & \downarrow \\
H^*(BA_n; F_2) & \xleftarrow{(f|BA_n)^*} & H^*(B\Sigma_n; F_2)
\end{array}
$$

Notice that the image of $f^* = r^* \circ i^*$ is included in the kernel of $H^*(B\Sigma_n; F_2) \to H^*(BA_n; F_2)$. It is known that this kernel is the ideal generated by the generator $w$ of $H^1(B\Sigma_n; F_2)$, [17]. We claim $\text{Im } f^* = F_2[w]$. Since $r^*$ is injective, we may identify $\text{Im } f^*$ with $H^*(X; F_2)$. If $w \notin H^1(X; F_2)$, then $i^*(w) = 0$. This would imply $i^* = i^* \circ f^* = 0$. Thus $H^1(X; F_2) = F_2[w]$ and hence $F_2[w] \subset \text{Im } f^*$. Next suppose $y$ is an element of the set $H^*(X; F_2) - F_2[w]$ with minimal degree. We can write $y = wz$ for some $z \in H^*(B\Sigma_n; F_2)$. Since $y = i^* \circ r^*(y) = i^*(w)i^*(z) = w \cdot i^*(z)$, it follows that $i^*(z) \in H^*(X; F_2) - F_2[w]$. This contradicts the minimality of the degree of $y$ since $\deg w = 1$. Consequently $H^*(X; F_2) \simeq F_2[w]$.

A section $s$ for the group extension $A_n \to \Sigma_n \to \mathbb{Z}/2(t)$ is given by $s(t) = (12)$. If $|f|B\mathbb{Z}/2 \simeq 0$, then Lemma 1.1 implies $f = 0$. Thus $|f|B\mathbb{Z}/2 \simeq 0$. It follows that the retract $X$ is 2-equivalent to $B\mathbb{Z}/2$. Any retract of a $p$-complete space is $p$-complete. We now conclude that $X$ is homotopy equivalent to $B\mathbb{Z}/2$. 

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(b) Suppose \( f : (B\Sigma_n)_p^\wedge \to (BA_n)_p^\wedge \) is a nonzero map. Let \( D \) be a \( p \)-Sylow subgroup of \( \Sigma_n \) containing \( E = \prod_{i=0}^{[(n-p)/p]} \mathbb{Z}/p(\sigma_i) \) with \( \sigma_i = (ip + 1 \cdots ip + p) \). If \( f|_B D \simeq B\rho' \), Theorem 1 says that \( \rho' \) is injective, since \( p \) is odd. Considering the conjugation by an element of \( A_n \), we may assume \( \rho'(D) = D \). Recall here that an element of order \( p \) is a product of distinct \( p \)-cycles. If \( i : A_n \to \Sigma_n \) is the inclusion, the map \( f \circ Bi \) is a homotopy equivalence. Using Lemma 1.1 one can find a nonnegative integer \( k \) such that \( (f \circ Bi)^k \circ f|_B D \simeq B\rho \) where \( \rho \) is an automorphism which sends the \( p \)-cycles to the \( p \)-cycles. Let \( \epsilon_i = \rho(\sigma_i) \) for \( 0 \leq i \leq \lfloor \frac{n-p}{p} \rfloor \). According to Lemma 2.5 there is \( g \in N_{\Sigma_n} \mathbb{Z}/p \) such that \( g\sigma_0 g^{-1} = \sigma_b^0 \) where \( b \) is a multiplicative generator of \( (\mathbb{Z}/p)^* \). If \( \tilde{g} = g \times 1 \cdots 1 \in \Sigma_p \times \cdots \times \Sigma_p \subset \Sigma_n \), then \( \rho|_E = \rho|_E \circ C_{\tilde{g}} \) in \( \text{Rep}(E, A_n) \) since \( BC_{\tilde{g}} \simeq 1_{BS_n} \). Consequently there is \( a \in A_n \) such that \( C_a(\epsilon_0) = e_0^b \) and \( C_a(\epsilon_i) = e_i \) for \( 1 \leq i \leq \lfloor \frac{n-p}{p} \rfloor \). We notice that \( a \in N_{\Sigma_n} \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p \times \cdots \times \Sigma_p \subset \Sigma_n \), since the centralizer of \( \mathbb{Z}/p \) in \( \Sigma_p \) is \( \mathbb{Z}/p \) itself. If \( n \equiv 0, 1 \mod p \), then \( Z/2 = \Sigma_n/A_n \) acts trivially on a \( p \)-Sylow subgroup of \( A_n \). By [3, p. 258] one can show that the map \( H^*(B\Sigma_n; \mathbb{F}_p) \xrightarrow{(Bi)^*} H^*(BA_n; \mathbb{F}_p) \) is an isomorphism. Therefore \( (B\Sigma_n)_p^\wedge \simeq (BA_n)_p^\wedge \).

(c) Let \( f : (B\Sigma_n)_2^\wedge \to (BA_n)_2^\wedge \). If \( f \circ (Bi)_2^\wedge \) is a nonzero self-map of \( (BA_n)_2^\wedge \), Theorem 1 and Lemma 1.5 imply that this map is a homotopy equivalence. This would imply that \( (BA_n)_2^\wedge \) is a retract of \( (B\Sigma_n)_2^\wedge \). According to part (a), this is a contradiction. Thus \( f \circ (Bi)_2^\wedge = 0 \). One can show that \( f \) factors through \( B\mathbb{Z}/2 \) since \( \text{map}((B\Sigma_n)_2^\wedge, (BA_n)_2^\wedge) \simeq \text{map}_{\mathbb{Z}/2}(E\mathbb{Z}/2, \text{map}(BA_n, (BA_n)_2^\wedge)) \) and the map \( \lambda : (BA_n)_2^\wedge \to \text{map}(BA_n, (BA_n)_2^\wedge) \) is weakly equivalent. This proves the诱导 map is onto. Notice next that the map \( H^*(B\mathbb{Z}/2; \mathbb{F}_2) \to H^*(B\Sigma_n; \mathbb{F}_2) \) is induced by the projection is a monomorphism. By a result of Lannes [12] we can show the map \( [B\mathbb{Z}/2, (BA_n)_2^\wedge] \to [(B\Sigma_n)_2^\wedge, (BA_n)_2^\wedge] \) is one-to-one. \( \square \)

3. General linear groups

**Notation.** Let \( e_{ij}(\alpha) \in GL(n, \mathbb{F}_q) \) denote the elementary matrix with entry \( \alpha \in \mathbb{F}_q^* \) in the \((i, j)\)th place. We make a convention that \( e_{ij} = e_{ij}(\alpha) \) for \( \alpha = 1 \). Next \( d_{ij}(\alpha) \in \text{Mat}_n(\mathbb{F}_q) \) denotes the \( n \times n \) matrix with entries 0 except the \((i, j)\)th entry \( \alpha \). Equivalently \( d_{ij}(\alpha) = e_{ij}(\alpha) - I_n \). We write \( d_{ij} = d_{ij}(\alpha) \) for \( \alpha = 1 \), \( d_i(\alpha) = d_{ii}(\alpha) \) for \( i = j \), and \( d_i = d_i(\alpha) \) for \( \alpha = 1 \).

**Lemma 3.1.** The unipotent subgroup \( U_n \) of \( GL(n, \mathbb{F}_q) \), upper triangular matrices with all diagonal entries equal to 1, is generated by the elementary matrices \( \{e_{ii+1}(\alpha) | \alpha \in \mathbb{F}_q^*, 1 \leq i \leq n-1 \} \).

**Proof.** Suppose \( H_n \) is the subgroup of \( U_n \) generated by the above elementary matrices. We will show \( H_n = U_n \) by induction. If \( n = 2 \), it is clear that \( H_2 = U_2 \). Assume \( n \geq 3 \) and the result holds up to \( n - 1 \). By the hypothesis of induction we see \( e_{ij}(\alpha) \in H_n \) for any \( \alpha \in \mathbb{F}_q^* \) unless \( i = 1 \) and \( j = n \):

\[
\begin{pmatrix}
H_{n-1} & 0 \\
0 & 1
\end{pmatrix} = 
\begin{pmatrix}
U_{n-1} & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
p & H_{n-1}
\end{pmatrix} = 
\begin{pmatrix}
1 & 0 \\
0 & U_{n-1}
\end{pmatrix}.
\]
Notice here that \( e_{1n-1}(\alpha) \cdot e_{n-1n}(1) \cdot e_{1n-1}(-\alpha) \cdot e_{n-1n}(-1) = e_{1n}(\alpha) \). Hence \( e_{1n}(\alpha) \in H_n \) for any \( \alpha \in \mathbb{F}_q^* \). Any element in \( U_n \) has the form \( \begin{pmatrix} 1 & B \\ 0 & A \end{pmatrix} \) where \( A \in U_{n-1} \). This matrix decomposes as follows

\[
\begin{pmatrix} 1 & B \\ 0 & A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & I_{n-1} \end{pmatrix}
\]

where \( I_{n-1} \) is the identity matrix. Since \( U_{n-1} = H_{n-1} \), it follows that \( \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in H_n \). If \( B = (b_2, b_3, \ldots, b_n) \), then

\[
\begin{pmatrix} 1 & B \\ 0 & I_{n-1} \end{pmatrix} = \prod_{i=2}^{n} e_{1i}(b_i) \in H_n.
\]

Therefore \( \begin{pmatrix} 1 & B \\ 0 & A \end{pmatrix} \in H_n \) and hence \( H_n = U_n \). \( \square \)

**Lemma 3.2.** Any two of elementary matrices in the unipotent group \( U_n \) are conjugate to each other in \( GL(n, \mathbb{F}_q) \).

**Proof.** We will show that any \( e_{ij}(\alpha) \) is conjugate to \( e_{12} \). If \( D \) is the diagonal matrix \( d_1(\alpha) + \sum_{1=1}^{n} d_i \), then \( De_{12}D^{-1} = e_{12}(\alpha) \). If \( T \) is the permutation \((1 \ 2), \ldots, (n \ 2)\), then \( Te_{12}(\alpha)T^{-1} = e_{ij}(\alpha) \). \( \square \)

**Lemma 3.3.** The center of \( U_n \) is \( \{e_{1n}(\alpha) | \alpha \in \mathbb{F}_q^* \} \cup \{I_n \} \).

**Proof.** Notice that \( A e_{ij} = e_{ij}A \) if and only if \( A d_{ij} = d_{ij}A \) where \( A \in U_n \). If \( A = (a_{ij}) \) we see \( A d_{ij} = \sum_{k=1}^{n} a_{ik} d_{kj} \) and \( d_{ij}A = \sum_{k=1}^{n} d_{ik} a_{jk} \). Hence \( A d_{ij} = d_{ij}A \) if and only if \( a_{ki} = 0 \) for \( 1 \leq i \leq n - 1 \) and \( 1 \leq k \leq i - 1 \), and \( a_{jk} = 0 \) for \( 2 \leq j \leq n \) and \( j + 1 \leq k \leq n \). This implies the desired result. \( \square \)

**Proposition 3.4.** The general linear group \( GL(n, \mathbb{F}_q) \) satisfies the hypothesis of Theorem 1 at \( p \) where \( q \) is a power of \( p \).

**Proof.** First we note that the unipotent group \( U_n \) is a \( p \)-Sylow subgroup \( \pi_\rho \) of \( GL(n, \mathbb{F}_q) \) since \( q \) is a power of \( p \). Lemma 3.3 shows that \( z = e_{1n}(\alpha) \) for some \( \alpha \in \mathbb{F}_q^* \). The desired result follows from Lemma 3.1 and Lemma 3.2. \( \square \)

The following lemma is known.

**Lemma 3.5.** Any element in the finite field \( \mathbb{F}_q \) is written as the sum of two squares.

**Proposition 3.6.** The special linear group \( SL(n, \mathbb{F}_q) \) satisfies the hypothesis of Theorem 1 at \( p \) where \( q \) is a power of \( p \).

**Proof.** If \( n \geq 3 \), Lemma 3.2 is true for \( SL(n, \mathbb{F}_q) \). In the proof the diagonal matrix \( D \) would be \( d_1(\alpha^{-1}) + d_2 + \cdots + d_{n-1} + d_n(\alpha) \) and we can find a suitable \( T \) in \( SL(n, \mathbb{F}_q) \). The rest of the argument is the same. For \( n = 2 \), Lemma 3.2 is false. But, since

\[
\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 1 & x^2 \alpha \\ 0 & 1 \end{pmatrix},
\]

Lemma 3.5 implies the desired result. \( \square \)

Since the kernel of the projection \( SL(n, \mathbb{F}_q) \to PSL(n, \mathbb{F}_q) \) is isomorphic to \( \mathbb{Z}/(n, \ q - 1) \), [2, p. 62] shows \( (BSL(n, \mathbb{F}_q))^\alpha \approx (BPSL(n, \mathbb{F}_q))^\alpha \). Next, we can show \( (BSL(n, \mathbb{F}_q))^\rho \approx (BGL(n, \mathbb{F}_q))^\rho \) if and only if \( (n, \ q - 1) = 1 \). In fact, if
RIGIDITY OF p-COMPLETED SPACES

A(\alpha) = e_{12}(\alpha) + \sum_{i=2}^{n-1} d_{i,i+1}, then A(\alpha) \in U_n and all A(\alpha)'s are conjugate each other in GL(n, F_q). In SL(n, F_q), however, A(1) is not conjugate to A(\beta) unless \beta is the nth power of some element of F^*q. From Theorem 1 and Lemma 1.1 it follows that [(BGL(n, F_q))^\wedge_p, (BSL(n, F_q))^\wedge_p] = 0 if (n, q-1) \neq 1. If (n, q-1) = 1, we see GL(n, F_q) \cong SL(n, F_q) x F^*_q. Because a scalar multiple of the identity \alpha I_n for \alpha \in F^*_q is contained in SL(n, F_q) only if \alpha = 1 in this case.

4. SYMPLECTIC GROUPS

Notation. Let s_{ij}(\alpha) denote the n \times n matrix d_{ij}(\alpha) + d_{ji}(\alpha) for i \neq j. For example, if n = 3, then

$$s_{12}(\alpha) = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad s_{13}(\alpha) = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}.$$  

We write s_{ij} = s_{ij}(\alpha) for \alpha = 1. Next r_{ij}(\alpha) denotes d_{ij}(\alpha) + d_{ji}(-\alpha) \in Mat_n(F_q) for i \neq j. Likewise r_{ij} = r_{ij}(\alpha) for \alpha = 1.

For A \in GL(n, F_q), let [A] denote 2n \times 2n matrix

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \in GL(2n, F_q).$$

For B \in Mat_n(F_q), let

$$\langle B \rangle = \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \in GL(2n, F_q).$$

Lemma 4.1.

(i) $$[A][B][A]^{-1} = \langle AB^tA \rangle,$$

(ii) $$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} \alpha^2 b_1 + \alpha \beta b_2 \\ \alpha \beta b_3 + \beta^2 b_4 \end{pmatrix}.$$

Lemma 4.2. Suppose B = (b_{ij}) \in Mat_n(F_q) for 1 \leq i, j \leq n. If AB^tA = B for any A \in U_n, then b_{ij} = 0 except (i, j) = (1, 1), (1, 2) and (2, 1).

Proof. Suppose k + m = n. The matrix B is partitioned into 9 submatrices

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$$

where, for example, B_{11} is a (k - 2) \times (k - 2) matrix, B_{22} is a 2 \times 2 matrix and B_{33} is an m \times m matrix. If E is a 2 \times 2 matrix and

$$A = \begin{pmatrix} I_{k-2} & 0 \\ 0 & E \end{pmatrix},$$

then

$$AB^tA = \begin{pmatrix} B_{11} & B_{12}^tE & B_{13} \\ EB_{21} & EB_{22}^tE & EB_{23} \\ B_{31} & B_{32}^tE & B_{33} \end{pmatrix}.$$
Suppose $E = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with $a \neq 0$ and $B = \begin{pmatrix} b_{ij} \end{pmatrix}$. Then $B_{12}E = B_{12}$ and $B_{32}E = B_{32}$ imply $b_{ik} = 0$ for $1 \leq i \leq k - 2, k + 1 \leq i \leq n$. Similarly $b_{kj} = 0$ for $1 \leq j \leq k - 2, k + 1 \leq j \leq n$ since $EB_{21} = B_{21}$ and $EB_{23} = B_{23}$. Finally $EB_{22}E = B_{22}$ implies $b_{kk} = 0$ for $2 \leq k \leq n$. □

**Proposition 4.3.** The symplectic group $Sp(2n, \mathbb{F}_q)$ with $(n, q) \neq (2, 2)$ satisfies the hypothesis of Theorem 1 at $p$ where $q$ is a power of $p$.

**Proof.** The subgroup $Sp(2n, \mathbb{F}_q)$ of $GL(2n, \mathbb{F}_q)$ corresponding to the symplectic form $\sum_{i=1}^{n}(X_iY_{n+i}-X_{n+i}Y_i)$ consists of those matrices $M$ with $MJ_1M^{-1} = J_1$ where

$$J_1 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$ 

A $p$-Sylow subgroup is given by the semidirect product

$$\pi_p = \{(B)|B = B \} \rtimes \{[A]|A \in U_n\},$$

[8, p. 192].

The center of $U_n$ acts faithfully on the abelian group of $(B)$'s. Lemma 2.1 and Lemma 4.2 imply that the center $Z(\pi_p)$ is included in the set $\{(d_1(b_1) + s_{12}(b_2))|b_1, b_2 \in \mathbb{F}_q\}$. If $q$ is odd, then $b_2 = 0$. We need to consider three cases.

**Case 1.** Suppose $z = \langle d_1(b_1) \rangle$ for some $b_1 \neq 0$. Lemma 1.1 and Lemma 4.1 show that if $K$ denotes the weak closure of $\{z\}$ in $\pi_p$, then $\langle d_1(\alpha^2b_1) \rangle \in K$ for any $\alpha, \beta \in \mathbb{F}_q$. Since $\langle d_1(\alpha^2b_1) \rangle \cdot \langle d_1(\beta^2b_1) \rangle = \langle d_1((\alpha^2 + \beta^2)b_1) \rangle \in K$ for any $\alpha, \beta \in \mathbb{F}_q$, Lemma 3.5 implies $\langle d_1(b) \rangle \in K$ for any $b \in \mathbb{F}_q$. Note here that if $A \in GL(n, \mathbb{F}_q)$, then $[A] \in Sp(2n, \mathbb{F}_q)$. So $[A]\langle d_1 \rangle[A]^{-1} = \langle d_2 \rangle \in K$ by Lemma 1.1, if $A = s_1 + \sum_{i=3}^{n}d_i$. For $A' = d_1 + s_1 + \sum_{i=3}^{n}d_i$, we see $[A'][d_1 + s_1 + d_2][A']^{-1} = \langle d_1(2) + s_1 + d_2 \rangle \in K$. Consequently $\langle d_1(2) + s_1 + d_2 \rangle \cdot \langle (d_1)^{-1}\cdot d_2 \rangle^{-1} = \langle s_{12} \rangle \in K$. The abelian group $\{\langle B \rangle|B = B \}$ is generated by $\langle d_1 \rangle$ and $\langle s_{12} \rangle$ together with their conjugacy classes in $Sp(2n, \mathbb{F}_q)$. Lemma 1.1 implies $\langle B \rangle \in K$ for any $\langle B \rangle \in \pi_p$. Next we can show that if $R = I_{2n} + r_{2n+2} - d_2 - d_{n+2}$, then $R \in Sp(2n, \mathbb{F}_q)$ and $R(s_{12})R^{-1} = [e_{12}]$. Consequently, using Lemma 1.1, Lemma 3.1, and Lemma 3.2, we can show $[A] \in K$ for any $[A] \in \pi_p$ and therefore $K = \pi_p$.

**Case 2.** Suppose $z = \langle s_{12}(b_2) \rangle$ for some $b_2 \neq 0$. It suffices to show $\langle d_1(b) \rangle \in K$ for some $b \neq 0$ so that the argument is reduced to Case 1. Taking $\alpha = b_2^{-1}$ and $\beta = 1$ in Lemma 4.1(ii) we see $\langle s_{12} \rangle \in K$. Since $[e_{12}]$ is conjugate to $\langle s_{12} \rangle$ in $Sp(2n, \mathbb{F}_q)$, we see $[e_{12}] \in K$. Notice that

$$\langle s_{12} \rangle \cdot [e_{12}] = \begin{pmatrix} e_{12} & d_1(1) + s_{12} \\ 0 & e_{12}^{-1} \end{pmatrix} \in K.$$ 

If $Q = \begin{pmatrix} I_n & 0 \\ d_1 & -I_n \end{pmatrix}$, then $Q \in Sp(2n, \mathbb{F}_q)$ and $Q\langle s_{12} \rangle[e_{12}]Q^{-1} = \langle d_1(-1) + s_{12} \rangle$. Consequently $\langle s_{12} \rangle \cdot \langle d_1(-1) + s_{12} \rangle^{-1} = \langle d_1 \rangle \in K$.

**Case 3.** Suppose $z = \langle d_1(b_1) + s_{12}(b_2) \rangle$ for some $b_1 \neq 0$ and $b_2 \neq 0$. Hence $q$ is assumed to be even. Again, it suffices to show $\langle d_1(b) \rangle \in K$ for some $b \neq 0$ unless $(q, n) = (2, 2)$. If $A = s_{12} + \sum_{i=3}^{n}d_i$, then $[A]\langle d_1(b_1) + s_{12}(b_2) \rangle[A]^{-1}=$
\( \langle d_2(b_1) + s_{12}(b_2) \rangle \in K \). Hence \( \langle d_1(b_1) + s_{12}(b_2) \rangle \cdot \langle d_2(b_1) + s_{12}(b_2) \rangle^{-1} = \langle d_1(b_1) + d_2(-b_1) \rangle \in K \). If \( A' = d_1(x) + \sum_{i=2}^{n} d_i \) with \( x \in \mathbb{F}_q^* \), then

\[
[A'] \langle d_1(b_1) + d_2(-b_1) \rangle \langle A' \rangle^{-1} \cdot \langle d_1(b_1) + d_2(-b_1) \rangle^{-1} = \langle d_1(b_1(x^2 - 1)) \rangle \in K.
\]

If \( q \neq 2 \), there is \( x \in \mathbb{F}_q^* \) such that \( x^2 \neq 1 \). This implies \( \langle d_1(b) \rangle \in K \) for some \( b \neq 0 \).

It remains to consider the case \( q = 2 \) and \( n \geq 3 \). Since \( b_1 = 1 \) and \( b_2 = 1 \) in this case, we can see \( \langle d_1 + s_{12} \rangle \) and \( \langle d_1 + d_2 \rangle \) are contained in \( K \). If \( A = d_1 + d_2 + s_{12} + s_{33} + \sum_{i=4}^{n} d_i \), then \( \langle A \rangle \in Sp(2n, \mathbb{F}_2) \) for \( n \geq 3 \) and \( \langle A \rangle \langle d_1 + d_2 \rangle \langle A \rangle^{-1} = \langle d_3 + s_{13} + s_{23} \rangle \in K \). Since \( \langle d_1 + s_{12} \rangle \) is conjugate to \( \langle d_3 + s_{23} \rangle \) in \( Sp(2n, \mathbb{F}_2) \), we see \( \langle d_3 + s_{23} \rangle \in K \) and hence \( \langle d_3 + s_{13} + s_{23} \rangle \cdot \langle d_3 + s_{23} \rangle^{-1} = \langle s_{13} \rangle \in K \). Consequently \( \langle s_{12} \rangle \in K \) and Case 2 shows \( \langle d_1 \rangle \in K \). This completes the proof. \( \square \)

The kernel of the projection \( Sp(2n, \mathbb{F}_2) \rightarrow PSp(2n, \mathbb{F}_2) \) is \( \mathbb{Z}/2 \) if \( q \) is odd and is trivial if \( q \) is even. Consequently \( (BSp(2n, \mathbb{F}_2))^\wedge \simeq (BPsp(2n, \mathbb{F}_2))^\wedge \) if \( q \) is a power of \( p \). The projective symplectic groups are all simple except for the cases \( (n, q) = (1, 2), (1, 3), (2, 2) \). Note that \( Sp(4, \mathbb{F}_2) \) is isomorphic to the symmetric group \( \Sigma_6 \).

5. Orthogonal groups

**Proposition 5.1.** The orthogonal group \( O(2n, \mathbb{F}_q) \) with \( n \geq 3 \) satisfies the hypothesis of Theorem 1 at \( p \) where \( q \) is a power of \( p \).

**Proof.** Recall that \( O(2n, \mathbb{F}_q) \) can be regarded as the subgroup of \( GL(2n, \mathbb{F}_q) \) which consists of those matrices that preserve the quadratic form \( X_1X_{n+1} + X_2X_{n+2} + \cdots + X_nX_2n \) if \( q \) is even, or \( q \) is odd with \( n \) even or \( 4 \mid q - 1 \). Because the discriminant of this quadratic form is equal to \((-1)^n/2^{2n}\), which is a square under the condition.

Assume first that \( q \) is even. Then a 2-Sylow subgroup is given by the semidirect product \( \pi_2 = \{(B)| B = B \text{ with } b_{ii} = 0 \text{ for any } i\} \rtimes \{[A]| A \in U_n\} \), [8, p. 192]. For \( n \geq 3 \) the center of \( U_n \) acts faithfully on the abelian groups of \( \langle B \rangle \)'s. Using Lemma 2.1 and Lemma 4.2 we can show that \( Z(\pi_2) = \langle s_{12}(b) \rangle | b \in \mathbb{F}_q^* \). Hence \( z = (s_{12}(b)) \) for some \( b \neq 0 \). Taking \( b_1 = 0 \), \( b_2 = b_3 \), and \( b_4 = 0 \) in Lemma 4.1(ii) we can show that any two elements of \( \langle s_{12}(b) \rangle | b \neq 0 \rangle \) are conjugate in \( O(2n, \mathbb{F}_q) \). According to Lemma 1.1, \( \langle s_{12}(b) \rangle \in K \) for any \( b \) where \( K \) is the weak closure of \( \{z\} \) in \( \pi_2 \). For a fixed \( b \in \mathbb{F}_q^* \), all the \( \langle s_{ij}(b) \rangle \)'s are conjugate to each other by the action of the \( \tau \)'s where \( \tau \) is a permutation. Since the abelian group of \( \langle B \rangle \) is generated by \( \langle s_{ij}(b) \rangle \), it follows that \( \langle B \rangle \in K \) for any \( \langle B \rangle \in \pi_2 \). Notice next that if \( T \) is the transposition interchanging \( X_2 \) and \( X_{n+2} \), then \( T \in O(2n, \mathbb{F}_q) \) and \( T[e_{12}]T^{-1} = \langle s_{12} \rangle \). Consequently \( [e_{12}] \in K \) and hence \( [A] \in K \) for any \( A \in U_n \) by Lemma 1.1, Lemma 3.1, and Lemma 3.2. Therefore \( K = \pi_2 \).

Assume next that \( q \) is odd with \( n \) even or \( 4 \mid q - 1 \). A \( p \)-Sylow subgroup is given by the semidirect product

\[ \pi_p = \{(B)| B = -B \} \rtimes \{[A]|A \in U_n\} \].

The center of \( \pi_p \) is \( \langle r_{12}(b) \rangle | b \in \mathbb{F}_q^* \). If \( T \) is the transposition interchanging \( X_2 \) and \( X_{n+2} \), then \( T \in O(2n, \mathbb{F}_q) \) and \( T[e_{12}]T^{-1} = \langle r_{12} \rangle \). We can show \( K = \pi_p \).

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Next we consider the case that \( q \) is odd with \( n \) odd and \( 4 \nmid q - 1 \). The quadratic form \( \sum_{i=1}^{n} X_i X_{n+i} \) is isomorphic to \( \sum_{i=1}^{n-2} X_i X_{n+i} + X_{n-1}^2 - X_{2n-1}^2 + X_n^2 - X_{2n}^2 \). The orthogonal group \( O(2n - 2, \mathbb{F}_q) \) can be regarded as the subgroup of \( GL(2n - 2, \mathbb{F}_q) \) which consists of those matrices that preserve the quadratic form \( \sum_{i=1}^{n-2} X_i X_{n+i} + X_{n-1}^2 + X_{2n-1}^2 \). One can see that the injective map \( O(2n - 2, \mathbb{F}_q) \rightarrow O_-(2n, \mathbb{F}_q) \) sends a \( p \)-Sylow subgroup isomorphically into \( \pi_p \). Namely \( \langle B \rangle [A] \) is contained in the image if and only if \( b_{i,n-1} = a_{i,n-1} \) for \( 1 \leq i \leq n-1 \), \( b_{i,n} = a_{i,n} \) for \( 1 \leq i \leq n-2 \), and \( b_{n-1,n} = 0 = a_{n-1,n} \). Here \( A = (a_{ij}) \) and \( B = (b_{ij}) \). The center of the group is \( \{(r_{12}(b)) | b \in \mathbb{F}_q \} \). When \( n = 3 \) and \( p \) is odd, we see \( (BSL(4, \mathbb{F}_q))^p \sim (B\Omega(6, \mathbb{F}_q))^p \) where \( \Omega(6, q) \) is the commutator subgroup of \( O(6, \mathbb{F}_q) \). Note that the index of \( \Omega(6, \mathbb{F}_q) \) in \( O(6, \mathbb{F}_q) \) is prime to \( p \) since \( p \) is odd. Lemma 1.1 together with the fact that \( SL(4, \mathbb{F}_q) \) satisfies the desired result proves the case \( n = 3 \). An induction completes the proof.

Let \( \Omega(n, \mathbb{F}_q) \) denote the commutator subgroup of \( O(n, \mathbb{F}_q) \). The kernel of the projection \( \Omega(n, \mathbb{F}_q) \rightarrow \Phi\Omega(n, \mathbb{F}_q) \) is at most \( \mathbb{Z}/2 \). Consequently, if \( p \) is odd, we see \( (BSL(2n, \mathbb{F}_q))^p \sim (B\Omega(2n, \mathbb{F}_q))^p \). It is known that \( \Phi\Omega(2n, \mathbb{F}_q) \) is simple if \( n \geq 3 \). It is also known that if \( q \) is even, \( Sp(2n, \mathbb{F}_q) \) is isomorphic to \( O(2n + 1, \mathbb{F}_q) \).

Proposition 5.2. The group \( \Omega(2n, \mathbb{F}_q) \) with \( n \geq 3 \) satisfies the hypothesis of Theorem 1 at \( p \) where \( q \) is a power of the odd prime \( p \).

Proof. Since \( q \) is odd, the commutator subgroup of \( GL(n, \mathbb{F}_q) \) is \( SL(n, \mathbb{F}_q) \). Recall that \( [A] \in O(2n, \mathbb{F}_q) \) for any \( A \in GL(n, \mathbb{F}_q) \) and that \( [e_{12}] \) is conjugate to \( \langle r_{12} \rangle \) in \( O(2n, \mathbb{F}_q) \) when \( n \) is even or \( 4 \) divides \( q - 1 \). One can see the \( p \)-Sylow subgroup \( \pi_p \) of \( O(2n, \mathbb{F}_q) \) is also a \( p \)-Sylow subgroup of \( \Omega(2n, \mathbb{F}_q) \) in this case. In the proof of Proposition 5.1, replace the transposition \( T \) by the permutation \( (2n + 2)(3n + 3) \). We see the permutation is contained in \( \Omega(2n, \mathbb{F}_q) \) since its spinor norm is 1. A similar argument completes the proof for this case. It is not hard to prove the other case.

Proposition 5.3. The orthogonal group \( O(2n - 1, \mathbb{F}_q) \) with \( n \geq 3 \) satisfies the hypothesis of Theorem 1 at \( p \) where \( q \) is a power of the odd prime \( p \).

Proof. First we need to find a suitable quadratic form for \( O(2n - 1, \mathbb{F}_q) \). The quadratic form \( X_1 X_{n+1} + \cdots + X_n X_{2n} \) is isomorphic to \( X_1 X_{n+1} + \cdots + X_{n-1} X_{2n-1} + X_n^2 - X_{2n}^2 \). The group \( O(2n - 1, \mathbb{F}_q) \) can be regarded as the subgroup of \( GL(2n - 1, \mathbb{F}_q) \) which consists of those matrices that preserve the quadratic form \( X_1 X_{n+1} + \cdots + X_{n-1} X_{2n-1} + X_n^2 - X_{2n}^2 \). One can see that the injective map \( O(2n - 1, \mathbb{F}_q) \rightarrow O_-(2n, \mathbb{F}_q) \) sends a \( p \)-Sylow subgroup isomorphically into \( \pi_p \). Namely \( \langle B \rangle [A] \) is contained in the image if and only if \( b_{i,n} = a_{i,n} \) for \( 1 \leq i \leq n-1 \). The center of the group is \( \{(r_{12}(b)) | b \in \mathbb{F}_q \} \). When \( n = 3 \) and \( p \) is odd, we see \( (BSp(4, \mathbb{F}_q))^p \sim (B\Omega(5, \mathbb{F}_q))^p \). Note that the index of \( \Omega(5, \mathbb{F}_q) \) in \( O(5, \mathbb{F}_q) \) is prime to \( p \) since \( p \) is odd. Lemma 1.1 together with the result about \( Sp(4, \mathbb{F}_q) \) proves the desired result for \( n = 3 \). An induction completes the proof.

One can show that the analogous result holds for \( \Omega(2n - 1, \mathbb{F}_q) \) with \( n \geq 3 \).
6. Unitary Groups

Notation. Let \( t_{ij}(\alpha) \) denote the \( n \times n \) matrix \( d_{ij}(\alpha) = d_{ji}(-\alpha^q) \) for \( i \neq j \) and \( \alpha \in \mathbb{F}_{q^2} \). We write \( t_{ij} = t_{ij}(\alpha) \) for \( \alpha = 1 \).

For \( M = (m_{ij}) \in \text{Mat}_n(\mathbb{F}_{q^2}) \), let \( M^{(q)} = (m_{ij}^q) \in \text{Mat}_n(\mathbb{F}_{q^2}) \). Let \( M^* = \iota M^{(q)} \). For example, if \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), then

\[
M^* = \begin{pmatrix} a^q & c^q \\ b^q & d^q \end{pmatrix}.
\]

For \( A \in GL(n, \mathbb{F}_{q^2}) \), let \([A]\) denote

\[
\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix} \in GL(2n, \mathbb{F}_{q^2}).
\]

For \( B \in \text{Mat}_n(\mathbb{F}_{q^2}) \) let \( \langle B \rangle = \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \in GL(2n, \mathbb{F}_{q^2}) \).

Lemma 6.1.

(i) \( [A](B)[A]^{-1} = \langle ABA^* \rangle \),

(ii) \[
\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha^{q+1}b_1 & \alpha b_2 \\ \alpha b_3 & b_4 \end{pmatrix}.
\]

Lemma 6.2. If \( b \in \mathbb{F}_{q^2} \) and \( b^{q-1} + 1 = 0 \), then \( \{ \alpha^{q+1}b | \alpha \in \mathbb{F}_{q^2} \} = \{ x \in \mathbb{F}_{q^2} | x^q + x = 0 \} \).

Proof. First we show \( \alpha^{q+1}b \) is a solution of the equation \( X^q + X = 0 \). Since \( \alpha^2 = \alpha \) and \( b^q + b = 0 \), it follows that \( (\alpha^{q+1}b)^q + \alpha^{q+1}b = \alpha^2 q b^q + \alpha^{q+1}b = \alpha^{q+1}(b^q + b) = 0 \).

It remains to show that if \( c^{q-1} + 1 = 0 \), then \( \frac{c}{b} = \alpha^{q+1} \) for some \( \alpha \in \mathbb{F}_{q^2} \).

Recall that \( \mathbb{F}_{q^2}^* \) is isomorphic to the cyclic group \( \mathbb{Z}/q^2 - 1 \). Suppose \( a \) is a generator of this group. Then \( b = a^k \) and \( c = a^m \) for suitable \( k \) and \( m \).

If \( q \) is even, then \( b^{q-1} = 1 = c^{q-1} \) and hence \( k(q-1) = s(q^2 - 1) \) and \( m(q-1) = t(q^2 - 1) \) for some \( s, t \in \mathbb{Z} \). Consequently \( m-k = (t-s)(q+1) \). Thus \( \frac{c}{b} = a^{m-k} = (a^{t-s})^{q+1} \). In the case \( q \) is odd we have \( k(q-1) = (q^2 - 1)/2 + s(q^2 - 1) \) and \( m(q-1) = (q^2 - 1)/2 + t(q^2 - 1) \) for some \( s, t \in \mathbb{Z} \), since \( b^{q-1} = -1 = c^{q-1} \). Hence \( m-k = (t-s)(q+1) \) and therefore \( \frac{c}{b} = (a^{t-s})^{q+1} \).

Proposition 6.3. The unitary group \( U(2n, \mathbb{F}_{q^2}) \) satisfies the hypothesis of Theorem 1 at \( p \) where \( q \) is a power of \( p \).

Proof. The subgroup \( U(2n, \mathbb{F}_{q^2}) \) of \( GL(2n, \mathbb{F}_{q^2}) \) corresponding to the Hermitian form \( \sum_{i=1}^n (X_i Y_{n+i}^q + X_{n+i} Y_i^q) \) consists of those matrices \( M \) with \( MJ_0 M^* = J_0 \) where \( M^* = \iota M^{(q)} \) and \( J_0 = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \). The semidirect product

\[ \pi_p = \langle \langle B \rangle | (B = -B^{(q)}) \rangle \times \{ [A] | A \in U_n \} \]

is a \( p \)-Sylow subgroup of \( U(2n, \mathbb{F}_{q^2}) \), [8, p. 192]. First we will show that \( Z(\pi_p) = \{ (d_1(b_1)) \} b \in \mathbb{F}_{q^2} \) with \( b^q + b = 0 \}. Along the line of the proof of Lemma 4.2 we can show that if \( \langle B \rangle \) is contained in the center \( Z(\pi_p) \), then \( B = d_1(b_1) + t_{12}(b_2) \) for some \( b_1, b_2 \in \mathbb{F}_{q^2} \). We note that

\[
[e_{12}(a)](d_1(b_1) + t_{12}(b_2))[e_{12}(a)]^{-1} = (d_1(b_1 - ab_2^q + a^qb_2) + t_{12}(b_2)).
\]
Hence $a^q b_2 - ab_2^q = 0$ for any $a \in \mathbb{F}_{q^2}$. Notice that the equation $b_2 X^q - b_2^q X = 0$ has at most $q$ roots in $\mathbb{F}_{q^2}$ if $b_2 \neq 0$. Thus $b_2 = 0$. Since the action of $Z(U_n)$ on the abelian groups of $\langle B \rangle$'s is faithful, one can show the desired result.

Let $K$ be the weak closure of $\{ z \}$ in $\pi_p$ where $z = \langle d_1(b) \rangle$ for some $b \in \mathbb{F}_{q^2}$ with $b^{q-1} + 1 = 0$. Lemma 6.2 together with Lemma 6.1 implies that $\langle d_1(b) \rangle \in K$ for any $b \in \mathbb{F}_{q^2}$ with $b^q + b = 0$. By an argument analogous to a part of the proof of Case 1 for $Sp(2n, \mathbb{F}_q)$ in §4, one can show $\langle s_{12}(b) \rangle \in K$ for such $b \in \mathbb{F}_{q^2}$. Since $b^q + b = 0$, it follows that $\langle s_{12}(b) \rangle = \langle t_{12}(b) \rangle$. Lemma 6.1 implies $\langle t_{12}(b) \rangle \in K$ for any $b \in \mathbb{F}_{q^2}$. The abelian group $\{ \langle B \rangle \mid B = -B^{(q)} \}$ is generated by $\langle d_1(b) \rangle$ with $b^q + b = 0$ and $\langle t_{12}(b) \rangle$ for $b \in \mathbb{F}_{q^2}$ together with their conjugacy classes. Consequently $\langle B \rangle \in K$ for any $\langle B \rangle \in \pi_p$. Notice here that $t_{12} = r_{12}$. Hence, if $T$ is the transposition interchanging $X_2$ and $X_{n+2}$, then $T \in U(2n, \mathbb{F}_{q^2})$ and $T[e_{12}]T^{-1} = \langle t_{12} \rangle$. Thus $[e_{12}] \in K$. This implies $[A] \in K$ for any $A \in U_n$. Consequently $K = \pi_p$. □

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